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UNIONS OF PRIME SUBMODULES

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ABSTRACT. A proper submodule P of a module M over a ring R is said to be prime if $re \in P$ for $r \in R$ and $e \in M$ implies that either $e \in P$ or $r \in P :_R M$. In this paper we investigate the following two topics which are related to unions of prime submodules: *i*) The Prime Avoidance Theorem for modules and *ii*) S-closed subsets of modules.

1. INTRODUCTION

Let M be a module over a ring R. A proper submodule P of M with $P :_R M = p$ is said to be prime or p-prime if $re \in P$ for $r \in R$ and $e \in M$ implies that either $e \in P$ or $r \in p$ (cf. [3], [4], or [5]).

The Prime Avoidance Theorem for rings (in the simplest form) states that if an ideal I of a ring is contained in the union of a finite number of prime ideals, then I must be contained in one of them. In section 2, using the technique of efficient covering of submodules, which is adopted from [1], we prove its generalization to modules (Theorem 2.3). Applying the theorem, in section 3, we characterize the torsion subset $T(M)$ of some type of Noetherian modules M with $T(M) \neq M$ as the set-theoretic union of a finite number of prime submodules, each of which is an annihilator submodule (Theorem 3.6). Section 4 is devoted to an introduction of S-closed subsets of modules, a generalization of multiplicatively closed subsets of rings, and to exploration of various properties of S-closed subsets, particularly, saturated S-closed subsets of modules. We obtain, among other results, a condition under which a submodule of a finitely generated module M maximal with respect to exclusion of an S-closed subset to be a prime submodule of M (Theorem 4.5).

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Every ring in this paper is commutative with identity and every module is unitary.

2. The Prime Avoidance Theorem for Modules

The Prime Avoidance Theorem for rings [2, p.55, Theorem 81] states as follows: Let $J_1, J_2, \cdots, J_n$ be a finite number of ideals in a ring R, and S a subring of R such that $S \subseteq J_1 \bigcup J_2 \bigcup \cdots \bigcup J_n$. Assume that at most two of the J's are not prime. Then $S \subseteq J_k$ for some k. We consider a generalization of this theorem to modules in terms of prime submodules.

Let $L, L_1, L_2, \cdots, L_n$ be submodules of an R-module M. Following [1], we call a covering $L \subseteq L_1 \bigcup L_2 \bigcup \cdots \bigcup L_n$ efficient if no L_k is superfluous. Analogously, we shall say that $L = L_1 \bigcup L_2 \bigcup \cdots \bigcup L_n$ is an efficient union if none of the L_k may be excluded. Any cover or union consisting of submodules of M can be reduced to an efficient one, called an efficient reduction, by deleting any unnecessary terms.

It is well-known that if I, A_1, and A_2 are ideals of a ring such that $I \subseteq A_1 \bigcup A_2$, then $I \subseteq A_1$ or $I \subseteq A_2$. Hence a covering of an ideal by two ideals is never efficient. As McCoy remarked in [6], this result remains valid if I, A_1, and A_2 are subgroups of any arbitrary group. Consequently, a covering of a submodule by two submodules of a module is never efficient. Thus, $L \subseteq L_1 \bigcup L_2 \bigcup \cdots \bigcup L_n$ may possibly be an efficient covering only when $n > 2$ or $n = 1$.

The important Lemma 1 of [1] for ideals, which is frequently used in the paper, is due to McCoy [6, p.634, Lemma]. We can see easily that this result also remains valid if ideals are replaced with subgroups of any group. Thus the module theoretic version of this lemma is

Lemma 2.1. *Let $L = L_1 \bigcup L_2 \bigcup \cdots \bigcup L_n$ be an efficient union of submodules of an R-module M for $n > 1$. Then $\bigcap_{j \neq k} L_j = \bigcap_{j=1}^{n} L_j$ for all k.*

Proposition 2.2. *Let $L \subseteq L_1 \bigcup L_2 \bigcup \cdots \bigcup L_n$ be an efficient covering consisting of submodules of an R-module M where $n > 1$. If $L_j : M \not\subseteq L_k : M$ for every $j \neq k$, then no L_k for $k \in \{1, 2, ..., n\}$ is a prime submodule of M.*

Proof. Since $L \subseteq L_1 \bigcup L_2 \bigcup \cdots \bigcup L_n$ is an efficient covering, $L = (L \bigcap L_1) \bigcup (L \bigcap L_2) \bigcup \cdots \bigcup (L \bigcap L_n)$ is an efficient union. Hence, for every $k \leq n$, there exists an element e_k in $L - L_k$. Moreover, $\bigcap_{j \neq k} (L \bigcap L_k) \subseteq L \bigcap L_k$ by Lemma 2.2. If $j \neq k$, then $L_j : M \not\subseteq L_k : M$ so that there exists an $s_j \in L_j : N$, but $s_j \notin L_k : M$. Now, suppose that some L_k is a prime submodule. Then $L_k : M$ is a prime ideal, therefore, $s = \prod_{j\neq k} s_j \in L_j : M$, but $s \notin L_k : M$. Consequently, $se_k \in L\bigcap L_j$ for every $j \neq k$, but $se_k \notin L\bigcap L_k$, which contradicts to $\bigcap_{j\neq k}(L\bigcap L_j) \subseteq L\bigcap L_k$. Therefore, no L_k is prime. □

Theorem 2.3. [The Prime Avoidance Theorem] *Let M be an R-module, L_1, $L_2, ..., L_n$ a finite number of submodules of M, and L a submodule of M such that $L \subseteq L_1 \bigcup L_2 \bigcup \cdots \bigcup L_n$. Assume that at most two of the L's are not prime, and that $L_j : M \not\subseteq L_k : M$ whenever $j \neq k$. Then $L \subseteq L_k$ for some k.*

PROOF. For the given covering $L \subseteq L_1 \bigcup L_2 \bigcup \cdots \bigcup L_n$, let $L \subseteq L_{i_1} \bigcup L_{i_2} \bigcup \cdots \bigcup L_{i_m}$ be its efficient reduction. Then $1 \leq m \leq n$ and $m \neq 2$. If $m > 2$, then there exists at least one L_{i_j} to be prime. In view of Proposition 2.2, this is impossible as $L_j : M \not\subseteq L_k : M$ if $j \neq k$. Hence $m = 1$, namely, $L \subseteq L_k$ for some k. □

As we can see in the following Example 1, the condition that $L_j : M \not\subseteq L_k : M$ if $j \neq k$ in Theorem 2.3 is essential.

Example 1. Let V be a vector space of dimension > 2 over the field $Z/2Z$. Then every subspace of V is (0)-prime. Let e_1 and e_2 be distinct vectors of a basis for V, $V_1 = e_1F, V_2 = e_2F, V_3 = (e_1 + e_2)F$, and $L = V_1 + V_2$. Then $L = \{0, e_1, e_2, e_1 + e_2\} = V_1 \bigcup V_2 \bigcup V_3$ is an efficient union of three prime submodules V_i with $V_i : V = (0)$, but $L \not\subseteq V_i$ for every $i \in \{1, 2, 3\}$.

In [1], the problem concerning covering of ideals by cosets was studied. Some results of the investigation can also be generalized to modules. The generalizations are counterparts to some of the previous results including The Prime Avoidance Theorem.

Let $L, L_1, L_2, ..., L_n$ be submodules of an R-module M and $L_1 + e_1, L_2 + e_2, ..., L_n+e_n$ cosets in M. We call a covering $L \subseteq (L_1+e_1) \bigcup (L_2+e_2) \bigcup \cdots \bigcup (L_n+e_n)$ efficient if no coset is superfluous. If $e_k = e$ for every $k \in \{1, 2, ..., n\}$, then the above covering is equivalent to $L - e \subseteq L_1 \bigcup L_2 \bigcup \cdots \bigcup L_n$ and this is a coset efficiently covered by a union of submodules.

Lemma 2.4. *Let $L \subseteq (L_1 + e_1) \bigcup (L_2 + e_2) \bigcup \cdots \bigcup (L_n + e_n)$ be an efficient covering of a submodule L by cosets, where $n \geq 2$. Then $L \bigcap (\bigcap_{j\neq k} L_j) \subseteq L_k$, but $L \not\subseteq L_k$ for all k.*

PROOF. Cf. [1, p.3094, Proof of Lemma 10]. □

Proposition 2.5. *Let $L + e \subseteq L_1 \bigcup L_2 \bigcup \cdots \bigcup L_n$ be an efficient covering with $n \geq 2$. If $L_i : M \not\subseteq L_k : M$ for every $j \neq k$, then no L_k is prime.*

PROOF. From Lemma 2.4, $L \bigcap (\bigcap_{j \neq k} L_j) \subseteq L_k$ and $L \not\subseteq L_k$. Put $I = (\bigcap_{j \neq k} L_j) : M$. Then $IL \subseteq L \bigcap (\bigcap_{j \neq k} L_j) \subseteq L_k$. Suppose that L_k is prime for some k. Then either $L \subseteq L_k$ or $I = (\bigcap_{j \neq k} L_j) : M = \bigcap_{j \neq k} (L_j : M) \subseteq L_k : M$ so that $L_j : M \subseteq L_k : M$ for some j. However, both cases are impossible, hence no L_k is prime. □

Theorem 2.6. *Let $L + e \subseteq L_1 \bigcup L_2 \bigcup \cdots \bigcup L_n$ be a covering such that at most one submodule L_i is not prime and that $L_j : M \not\subseteq L_k : M$ if $j \neq k$. Then the submodule $L + eR \subseteq L_k$ for some k.*

PROOF. For the given covering $L + e \subseteq L_1 \bigcup L_2 \bigcup \cdots \bigcup L_n$, let $L + e \subseteq L_{i_1} \bigcup L_{i_2} \bigcup \cdots \bigcup L_{i_m}$ be its efficient reduction. Then $1 \leq m \leq n$. It is immediate from Proposition 2.5 that $m = 1$. Hence $L + e \subseteq L_k$ for some k whence $L + eR \subseteq L_k$ as $e = 0 + e \in L + e \subseteq L_k$. □

3. THE TORSION SUBSET $T(M)$

Let M be an R-module, $Z(M)$ the set of zero divisors on M, and $T(M)$ the torsion subset of M. Clearly, $Z(M) = \bigcup_{0 \neq m \in M} Ann_R m$ and $T(M) = \bigcup_{0 \neq a \in R} Ann_M a$, where $Ann_M a = 0 :_M a = \{e \in M : ae = 0\}$.

If R is an integral domain and $T(M) \neq M$, then $T(M)$ is known to be a submodule of M which is (0)-prime [4, p.62, Result 3]. Furthermore, if $T(M)$ is finitely generated over the integral domain R, then $T(M) = Ann_M a$ for some nonzero element a of R.

It is known that if I is an ideal of a ring R that is maximal among all annihilators $Ann_R m$ of nonzero elements m of any R-module M, then I is a prime ideal. If M is a finitely generated nonzero module over a Noetherian ring R, then $Z(M)$ is also known to be the set-theoretic union of a finite number of prime ideals each of which is $Ann_R m$ for some nonzero element m of M [2, p.4, Theorem 6, p.55, Theorem 80]. In this section, we shall see that $Ann_M a$ and $T(M)$ of some finitely generated modules M over Noetherian rings with $T(M) \neq M$ have similar properties. The Prime Avoidance Theorem proved in section 1 will be applied in proving the main result of this section, Theorem 3.6.

The following notations will be used exclusively:
$X = \{Ann_M a : 0 \neq a \in R\}$, $X' = \{Ann_M b \in X : Ann_M b$ is maximal in $X\}$,

$Y = \{Ann_R a : 0 \neq a \in R\}$, $Y' = \{Ann_R b \in Y : Ann_R b$ is maximal in $Y\}$.

Proposition 3.1. *If M is an R-module with $T(M) \neq M$, then $Ann_M a : M = Ann_R a$ for every nonzero element a of R.*

The proof of Proposition 3.1 is straightforward.

Proposition 3.2. *Let M be an R-module with $T(M) \neq M$ and $0 \neq x \in R$. If $Ann_M x$ is maximal in X, then it is a prime submodule of M.*

PROOF. Since $T(M) \neq M$, $Ann_M x$ is a proper submodule of M. Now let $ye \in Ann_M x$ for $y \in R$ and $e \in M$. Then $xye = 0$. Suppose that $y \notin Ann_M x : M$, that is, $y \notin Ann_R x$ by Proposition 3.1. Then $xy \neq 0$, whence $Ann_M x = Ann_M xy$ by the maximality of $Ann_M x$ in X. It follows that $e \in Ann_M xy = Ann_M x$. Thus $Ann_M x$ is a prime submodule of M. □

Corollary 3.3. *If M is a Noetherian module with $T(M) \neq M$, then $T(M)$ is a union of prime submodules.*

PROOF. We know that $T(M) = \bigcup_{0 \neq a \in R} Ann_M a$. Since M is Noetherian, each $Ann_M a$ of X is contained in a maximal one in X. Hence $T(M)$ is the set theoretic-union of those maximal ones, each of which was proved to be prime in Proposition 3.2. □

An R-module M is called a multiplication module provided that for every submodule N of M there exists an ideal I of R such that $N = IM$.

Lemma 3.4. *Let M be a multiplication R-module with $T(M) \neq M$. Then the mapping $f : X \to Y$ defined by $f(Ann_M a) = Ann_R a$ for every $Ann_M a \in X$ is an order preserving bijection.*

PROOF. That f is both surjective and order preserving stems from the fact that $Ann_R a = Ann_M a : M$ by Proposition 3.1. It is also injective due to that $Ann_M a = (Ann_M a : M)M = (Ann_R a)M$. □

Lemma 3.5. *Let M be a finitely generated module over a Noetherian reduced ring R with $T(M) \neq M$. Let $f : X \to Y$ be defined by $f(Ann_M a) = Ann_R a$ for every $Ann_M a \in X$, and $g = f|X'$, the restriction of f on X'. Then g is an injection from X' into Y'.*

PROOF. Clearly both X' and Y' are not empty. For each $Ann_M a \in X'$, we claim that $g(Ann_M a) = Ann_R a$ belongs to Y'. Assume the contrary and let b

be a nonzero element of R such that $Ann_R a \subsetneq Ann_R b$. Since R is reduced, $ab \neq 0$, for otherwise $ab = 0$ results in $b \in Ann_R a \subset Ann_R b$ whence $b^2 = 0$. By the maximality of $Ann_M a$ in X, we have $Ann_M b \subseteq Ann_M ab = Ann_M a$ which implies a contradiction that $Ann_R b \subseteq Ann_R a$. Thus $Ann_R a$ is maximal in X so that g is a mapping from X' to Y', which is evidently injective by the same arguments as above based on the fact that R is reduced. □

Theorem 3.6. *Let R be a Noetherian ring and M a finitely generated R-module with $T(M) \neq M$. Suppose that either*

i) M is a multiplication module or ii) R is a reduced ring.

Then there are only a finite number of prime submodules which are maximal within $T(M)$, and each is the annihilator submodule $Ann_M a$ for some nonzero element a of R.

Proof. We have seen in the proof of Corollary 3.3 to Proposition 3.2 that $T(M)$ is the union of those annihilator submodules $Ann_M a_i$ which form X'. Since R is Noetherian, the ideal spanned by the set $\{a_i \in R : Ann_M a_i \in X'\}$ is finitely generated, say, by $a_1, a_2, \cdots, a_n$. Let $Ann_M a$ be an arbitrary element of X'. Then $a = r_1 a_1 + r_2 a_2 + \cdots + r_n a_n$ for some $r_1, r_2, \cdots, r_n$ in R, so that $Ann_M a_1 \bigcap \cdots \bigcap Ann_M a_n \subseteq Ann_M a$. Consequently, $(Ann_M a_1 : M) \bigcap \cdots \bigcap (Ann_M a_n : M) \subseteq (Ann_M a : M)$, that is, $Ann_R a_1 \bigcap \cdots \bigcap Ann_R a_n \subseteq Ann_R a$. Since $Ann_R a$ is a prime ideal, $Ann_R a_j \subseteq Ann_R a$ for some $j \in \{1, 2, \cdots, n\}$. We can conclude that $Ann_R a_j = Ann_R a$ because both ideals are maximal in Y. It follows that $Ann_M a_j = Ann_M a$ for each case $i)$ and $ii)$ by Lemma 3.4 and Lemma 3.5, respectively. This proves that $X' = \{Ann_M a_1, ..., Ann_M a_n\}$. To complete the proof of Theorem 3.6, it will suffice to prove that any submodule N contained in $T(M)$ is contained in one of the $Ann_M a_j \in X'$. Since $N \subseteq T(M) = Ann_M a_1 \bigcup \cdots \bigcup Ann_M a_n$, where each $Ann_M a_i \in X'$ is a prime submodule satisfying the property that $Ann_M a_i : M \not\subseteq Ann_M a_j : M$ if $i \neq j$, $N \subseteq Ann_M a_j$ for some $j \in \{1, 2, ..., n\}$ by the Prime Avoidance Theorem. □

4. S-Closed Subsets of Modules

The most fundamental properties of prime ideals p of a ring R are as follows:

i) $R - p$ is a saturated multiplicatively closed subset of R,

ii) the ideal maximal with respect to the exclusion of a multiplicatively closed subset of R is prime, and

iii) a subset S of R is a saturated multiplicatively closed subset of R if and only if the complement of S is a set-theoretic union of prime ideals in R.

In this section, we consider analogue of these and other properties for prime submodules of modules. Throughout this section, we assume that every multiplicatively closed proper subset of R contains 1, but does not contain 0.

Definition 1. Let S be a multiplicatively closed subset of a ring R and M an R-module. (1) A non-empty subset S^* of M is said to be *S-closed* if $se \in S^*$ for every $s \in S$ and $e \in S^*$. (2) An S-closed subset S^* is said to be *saturated* if the following condition is satisfied: whenever $ae \in S^*$ for $a \in R$ and $e \in M$, then $a \in S$ and $e \in S^*$.

Proposition 4.1. *If S^* is a saturated S-closed subset of an R-module M relative to a multiplicatively closed subset S of R, then S is saturated.*

PROOF. Let $ab \in S$ for $a, b \in R$. Then for any $e \in S^*$, $abe \in S^*$. Since S^* is saturated, $a \in S$ and $be \in S^*$, whence $a \in S$ and $b \in S$. □

Evidently, every multiplicatively closed subset S of a ring R is an S-closed subset of the R-module R. However, not every S-closed subset of the R-module R is a multiplicatively closed subset of the ring R. On the other hand, as we shall see in the next proposition, a non-empty subset of the R-module R is a saturated S-closed subset if and only if it is a saturated multiplicatively closed subset of the ring R.

Proposition 4.2. *Let S be a multiplicatively closed subset of a ring R and S^* any nonempty subset of R. Then S^* is a saturated S-closed subset of the R-module R if and only if $S^* = S$ and S^* is a saturated multiplicatively closed subset of R.*

PROOF. The sufficiency is easy to verify. To prove the necessity, we assume that S^* is a saturated S-closed subset of the R-module R. By Proposition 4.1, S is a saturated multiplicatively closed subset of R. Furthermore, for any $a \in S$ and $b \in S^*$, $ab = ba \in S^*$ whence $b \in S$ and $a \in S^*$ as $b \in R$ and $a \in R$, the R-module R. Thus $S^* \subseteq S$ and $S \subseteq S^*$ so that $S = S^*$. □

There are plenty of examples of S-closed subsets of an R-module M. We shall list some of them.

Example 2. Let $\{P_i\}_{i \in I}$ be a collection of prime submodules of M with $P_i : M = p_i$ for every i. Then $S^* = M - \bigcup_{i \in I} P_i$ is a saturated S-closed subset of M, where $S = R - \bigcup_{i \in I} p_i$.

Example 3. For any S-closed subset S^* of M, let $W = \{S_i^* \subseteq M : S_i^*$ is a saturated S_i-closed subset of M relative to a multiplicatively closed subset S_i of R such that $S^* \subseteq S_i^*$ and $S \subseteq S_i\}$. Then W is not empty, for example, $M \in W$. Suppose that W is indexed by a set I and let $\bar{S}^* = \bigcap_{i \in I} S_i^*$. Then $\bar{S}^*$ is a saturated $\bar{S}_0$-closed subset of M containing S^*, where $\bar{S}_0 = \bigcap_{i \in I} S_i$.

Let M be a nonzero module over an integral domain R and $m, m' \in M$. We say that m divides m' and write $m|m'$ if there exists an element $r \in R$ such that $m' = rm$. A nonzero element m is said to be irreducible in M if $m = am'$ for $a \in R$ and $m' \in M$ implies that a is a unit in R. A nonzero element $e \in M$ is said to be primitive in M if, whenever $e|am$ for $0 \neq a \in R$ and $m \in M$, then $e|m$ in M. We remark that every primitive element is irreducible and that an element r of the R-module R is primitive if and only if r is a unit of R. [3, p.126].

Example 4. Let M be a nonzero module over an integral domain R and S the group of units of R. Then the set S^* of all primitive elements (resp. irreducible elements) of M is a saturated S-closed subset of M.

Example 5. Let S be a regular multiplicative system of a ring R and S^* the set of torsion-free elements of an R-module M. Then S^* is a saturated S-closed subset of M if $S^* \neq \emptyset$.

Example 6. Let M be an R-module and $S = R - Z(M)$. Then the set S^* of torsion-free elements of M is S-closed if $S^* \neq \emptyset$. However, S^* is not necessarily saturated as shown below:
Let $N_0 = Z^+ \bigcup \{0\}$ and $\{p_n : n \in Z^+\}$ the set of all prime integers p_n. Let $E(p_n) = \{\alpha_n \in Q/Z : \alpha_n = r/p_n^t + Z$ for some $r \in Z$ and $t \in N_0\}$. Then $E(p_n)$ is a nonzero submodule of the Z-module Q/Z for each $n \in Z^+$. Now, let $M = \prod_{n \in Z^+} E(p_n)$, a Z-module. If $\alpha = (\alpha_n)_{n \in Z^+} \in M$ with $\alpha_n = r/p_n^t + Z \neq 0$ for infinitely many n, then α is torsion free and so is $p_k\alpha$ for any fixed prime integer p_k. Thus $p_k\alpha \in S^*$, but $p_k \notin S = R - Z(M) = Z - Z(M)$, for p_k annihilates $\beta = (\beta_n)_{n \in Z^+}$ with $\beta_k = 1/p_k + Z$ and $\beta_n = 0$ whenever $n \neq k$.

Proposition 4.3. *Let S^* be an S-closed subset of an R-module M, and N a submodule contained in $M - S^*$. Then (1) $(N : M) \bigcap S = \emptyset$, so that $N_S \neq M_S$ if either M is finitely generated or N is a primary submodule. (2) If N is maximal in $M - S^*$, then $N_S \bigcap M = N$.*

PROOF. (1) Suppose that $(N : M) \bigcap S \neq \emptyset$ and let $s \in (N : M) \bigcap S$. Then $sM \subseteq N$ and, for any $e \in S^*$, $se \in S^* \bigcap N = \emptyset$ which is a contradiction. The remaining statement of (1) is due to [5, p.3744, Proposition 2]. (2) According to [7, p.137, Proposition 1], $N_S \bigcap M = \{m \in M : sm \in N \text{ for some } s \in S\}$. Assume that $N \underset{\neq}{\subset} N_S \bigcap M$. Since N is maximal in $M - S^*$, $(N_S \bigcap M) \bigcap S^* \neq \emptyset$ so that there exists an element $e \in S^*$ such that $e \in N_S \bigcap M$. Hence $se \in N$ for some $s \in S$, which is impossible because $se \in N \bigcap S^* = \emptyset$. □

Corollary 4.4. *Let M, S, and S^* be as in Proposition 4.3. Let N be a submodule of M which is maximal in $M - S^*$. Then N is prime in M if and only if N_S is prime in the R_S-module M_S.*

PROOF. By Proposition 4.3, $(N : M) \bigcap S = \emptyset$ and $N = N_S \bigcap M$. Thus the corollary follows from [5, p.3742, Proposition 1]. □

Now, we are ready to prove main results of this section.

Theorem 4.5. *Let S be a multiplicatively closed subset of a ring R and S^* an S-closed subset of a finitely generated R-module M. Let N be a submodule of M which is maximal in $M - S^*$. If the ideal $N : M$ is maximal in $R - S$, then N is a prime submodule of M with $(N : M)_S = N_S : M_S$.*

PROOF. If $p = N : M$ is maximal in $R - S$, then p_s is a maximal ideal of R_S. Since $p_s = (N : M)_S \subseteq N_S : M_S$ and $N_S \neq M_S$ by Proposition 4.3, we have that $N_S : M_S = p_s$. It follows that N_S is a prime submodule of M_S as p_s is a maximal ideal of R_S, whence N is a prime submodule of M by Corollary 4.4 to Proposition 4.3. □

Corollary 4.6. *Let p be a prime ideal of a ring $R, S = R - p$, and S^* an S-closed subset of a finitely generated R-module M. A submodule N of M which is maximal in $M - S^*$ is prime if $N : M = p$.*

The next two theorems characterize saturated S-closed subsets of cyclic modules.

Theorem 4.7. *Let $M = Rm$ be a cyclic R-module over a ring R. Let S^* be an S-closed subset of M relative to a multiplicatively closed subset S of R, and N a submodule of M maximal in $M - S^*$. If S^* is saturated, then the ideal $N : M$ is maximal in $R - S$ so that N is prime in M.*

PROOF. Assume the $J = N : M$ is not maximal in $R - S$. Then there must exist an ideal I in $R - S$ such that $J \underset{\neq}{\subset} I$. Hence $N = JM \underset{\neq}{\subset} IM$ so that there exists $rm \in S^*$ for some $r \in I$ by the maximality of N in $M - S^*$. Since S^* is saturated, $r \in S$ which contradicts to the fact that $I \bigcap S = \emptyset$. Thus J is maximal in $R - S$ and, consequently, N is prime by Theorem 4.5. □

Theorem 4.8. *Let M be a cyclic module over a ring R, S a multiplicatively closed subset of R, and S^* a nonempty subset of M. Then S^* is a saturated S-closed subset of M if and only if the complement of S^* in M is a union of prime submodules P_i, $i \in I$, of M and the complement of S in R is a union of prime ideals $p_i = P_i : M$ for each $i \in I$.*

PROOF. We have seen that the sufficiency holds for any module in Example 3. To prove the necessity, let e be any nonzero element of $M - S^*$. Then $Re \bigcap S^* = \emptyset$ since S^* is saturated. Expand Re to a submodule P maximal with respect to disjointness from S^*. Such a submodule P must exist by Zorn's Lemma and is prime by Theorem 4.7. Hence $M - S^* = \bigcup_{i \in I} P_i$, a union of prime submodules $P_i, i \in I$.

Next, put $S_0 = R - (\bigcup_{i \in I} p_i)$, where $p_i = P_i : M$ for every i; we shall show that $S = S_0$. If $s \in S$ and $m \in S^*$, then $sm \in S^* = M - (\bigcup_{i \in I} P_i)$ so that $sm \notin P_i$ for every i. Since each P_i is a prime submodule and $m \notin P_i$, $s \notin p_i$ for every i whence $s \in S_0$. Therefore $S \subseteq S_0$. On the other hand, if $s' \in S_0$, then $s' \notin p_i$ for every i and $s'm' \in S^*$ for all $m' \in S^* = M - (\bigcup_{i \in I} P_i)$ due to that each P_i is p_i-prime. It follows that $s' \in S$ as S^* is a saturated S-closed subset. We can conclude that $S_0 \subseteq S$ and therefore $S = S_0$. □

In the following example, we show that not all properties of multiplicatively closed subsets S of a ring R are inherited by S-closed subsets S^* of an R-module M even when M is cyclic. In particular, we will demonstrate that a submodule N of M being maximal in $M - S^*$ does not imply, in general, that either N is prime or $N : M$ is maximal in $R - S$. Thus the condition that $N : M$ is maximal in $R - S$ imposed in Theorem 4.5 is essential.

Example 7. Let $R = Z, M = R = Z, S = \{1, -1\}$, and S^* the set of all prime integers. Then S^* is an S-closed subset of the cyclic Z-module $M = Z$, and $M - S^*$ is the set of all composite integers. Now, take $N = 4Z$. Then N is a submodule of M which is maximal in $M - S^*$. However, N is not a prime submodule of M. Moreover, the maximality of N in $M - S^*$ does not imply that of $N : M = (4)$ in $R - S$.

We also remark that S^* in the above example is not saturated and that $M - S^*$ is not a set-theoretic union of prime submodules.

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