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On Prime and Weakly Prime Submodules

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Abstract. A proper submodule N of an R-module M is called a weakly prime [resp. a prime] submodule, if for any elements $a, b \in R$ and $x \in M$, the condition $abx \in N$ [resp. $ax \in N$] implies that $ax \in N$ or $bx \in N$ [resp. $x \in N$ or $aM \subseteq N$]. In this paper the relations between weakly prime submodules of a module M and weakly prime submodules of the localization of M are studied. Some applications of these relations are given. Furthermore, the relations between the intersection of prime submodules and the intersection of weakly prime submodules are discussed.

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1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we denote by R a ring and by M a unitary R-module.

Let N be a prime submodule of M and $P = (N : M) = \{t \in R | \; tM \subseteq N\}$. It is easy to see that P is a prime ideal of R and we say that N is a P-prime submodule of M. Prime submodules have been studied in several papers such as [1-6, 10].

Weakly prime submodules are generalizations of prime submodules and they have been introduced in [6]. Weakly prime submodules also have been studied in [2, 4, 5]. If we consider R as an R-module, then prime submodules and weakly prime submodules are exactly prime ideals of R. For every R-module, it is easy to see that any prime submodule is a weakly prime submodule, but the converse is not always correct. For example let R be an integral domain and P a non-zero prime ideals of R. Then it is easy to see that for the free R-module $R \oplus R$, the submodule $0 \oplus P$ is a weakly prime submodule, which is not a prime submodule.

Recall that the set of *zero divisors* of M, denoted by $Zd_R(M)$ is defined by

$$Zd_R(M) = \{r \in R |\ \exists 0 \neq x \in M, rx = 0\}.$$

In Sec. 2 of this paper, by using the zero divisors, the relations between weakly prime submodules of M and weakly prime submodules of the localization of M are studied. Let S be a multiplicatively closed subset of R. It is proved that there exists an one-to-one correspondence between weakly prime submodules N of M with $Zd_R(\frac{M}{N}) \cap S = \emptyset$ and weakly prime submodules of M_S (see Proposition 2.4).

In Sec. 3 some applications of the localization technique are given (see Theorem 3.5). In [6, Sec. 5], it is proved that if R is a ring with $\dim R < \infty$, then an R-module M has weakly prime submodules if and only if M has a prime submodule. We will relax the condition $\dim R < \infty$, (see Proposition 3.1); furthermore some specific prime submodules are introduced (see Theorem 3.2 and Corollary 3.3).

Let B be a proper submodule of M. The intersection of all prime [resp. weakly prime] submodules of M containing B is denoted by $\mathrm{rad}(B)$ [resp. $\mathrm{wrad}(B)$]. If there does not exist any prime [resp. weakly prime] submodule of M containing B, then we say $\mathrm{rad}(B) = M$ [resp $\mathrm{wrad}(B) = M$]. Evidently $\mathrm{wrad}(B) \subseteq \mathrm{rad}(B)$.

In Sec. 4 we will study the equality $\mathrm{wrad}(B) = \mathrm{rad}(B)$ (see Theorem 4.2).

2. Weakly Prime Submodules and Localization

Let M be an R-module and let N be a proper submodule of M. Obviously,

$$(N : M) \subseteq Zd_R(\frac{M}{N}) = \bigcup_{x \in M \setminus N} (N : x).$$

In this section, we will show that $Zd_R(\frac{M}{N})$ has an important role in studying the relation between weakly prime submodules of a module M and weakly prime submodules of the localization of M.

Lemma 2.1. *Let M be an R-module and let N be a proper submodule of M.*

(i) *N is a weakly prime submodule if and only if for each $x \in M \setminus N$, $(N : x)$ is a prime ideal of R. When this is the case, $\{(N : x)\}_{x \in M \setminus N}$ is a chain of prime ideals of R.*

(ii) *If N is a weakly prime submodule, then $(N : M)$ and $Zd_R(\frac{M}{N})$ are both prime ideals of R.*

Proof. (i) The first part is obvious.

Now let N be a weakly prime submodule and $x, y \in M \setminus N$. Obviously, $(N : x) \cap (N : y) \subseteq (N : x + y)$. By the first part we have $(N : x) \subseteq (N : x + y)$, or $(N : y) \subseteq (N : x + y)$. So $(N : x) = (N : x) \cap (N : x + y) \subseteq (N : y)$, or $(N : y) = (N : y) \cap (N : x + y) \subseteq (N : x)$. Thus $\{(N : x)\}_{x \in M \setminus N}$ is a chain of prime ideals of R.

(ii) The proof is clear by part (i). ■

If N is a weakly prime submodule of an R-module M, then by Lemma 2.1(ii), $(N : M)$ is a prime ideal of R. In this case, if $(N : M) = P$, we say that N is a P-weakly prime submodule.

Lemma 2.2. *Let M be an R-module, N a weakly prime submodule of M, and S a multiplicatively closed subset of R. If $N_S \neq M_S$, then N_S is a weakly prime submodule of M_S as an R_S-module.*

Proof. See [6, Proposition 6.3]. ■

Let M be an R-module, S a multiplicatively closed subset of R, W a submodule of M_S as an R_S-module. We consider W^c to be $W \cap M$, that is, $W^c = \{x \in M | \frac{x}{1} \in W\}$. The following lemma shows that there exists an one-to-one correspondence between prime submodules N of M with $(N : M) \cap S = \emptyset$ and prime submodules N_S of M_S.

Lemma 2.3. *Let M be an R-module and let S be a multiplicatively closed subset of R.*

(i) *If N is a P-prime submodule of M such that $P \cap S = \emptyset$, then N_S is a P_S-prime submodule of M_S as an R_S-module and $(N_S)^c = N$.*

(ii) *If W is a Q-prime submodule of M_S as an R_S-module, then W^c is a Q^c-prime submodule of M, $(W^c)_S = W$ and $Q^c \cap S = \emptyset$.*

Proof. See [10, Proposition 1]. ■

Let M be an R-module, S a multiplicatively closed subset of R. In the following proposition we will show that there exists an one-to-one correspondence between weakly prime submodules N of M with $Zd_R(\frac{M}{N}) \cap S = \emptyset$ and weakly prime submodules of M_S. Comparing Proposition 2.4 with Lemma 2.3 shows that the role of $Zd_R(\frac{M}{N})$ for a weakly prime submodule N is the role of $(N : M)$ for a prime submodule.

Proposition 2.4. *Let M be an R-module and S a multiplicatively closed subset of R.*

(i) *If N is a weakly prime submodule of M, and for some $x \in M \setminus N$, $(N : x) \cap S = \emptyset$, then N_S is a weakly prime submodule of M_S.*

(ii) *If N is a P-weakly prime submodule of M such that $Zd_R(\frac{M}{N}) \cap S = \emptyset$, then N_S is a P_S-weakly prime submodule of M_S and $(N_S)^c = N$. Hence $(N : M)_S = (N_S : M_S)$. Moreover $Zd_{R_S}(\frac{M_S}{N_S}) = (Zd_R(\frac{M}{N}))_S$.*

(iii) *If W is a Q-weakly prime submodule of M_S, then W^c is a Q^c-weakly prime submodule of M and $(W^c)_S = W$. Furthermore $Zd_R(\frac{M}{W^c}) \cap S = \emptyset$ and $Zd_R(\frac{M}{W^c}) = (Zd_{R_S}(\frac{M_S}{W}))^c$.*

Proof. (i) If $P = (N : x)$, then $P_S = (N : x)_S = (N_S : \frac{x}{1})$. By Lemma 2.1(i) the ideal P is prime and $P \cap S = \emptyset$, then P_S is a prime ideal of R_S. So $(N_S : \frac{x}{1}) = P_S \neq R_S$, which implies that $N_S \neq M_S$. Now by Lemma 2.2, N_S is a weakly prime submodule of M_S.

(ii) By part (i), N_S is a weakly prime submodule of M_S. Let $m \in (N_S)^c$. So $\frac{n_1}{s_1} = \frac{m}{1} \in N_S$, for some $n_1 \in N,\ s_1 \in S$. If $m \notin N$, then there exists an $s_2 \in S$ such that $s_2 s_1 \in (N : m) \cap S \subseteq Zd_R(\frac{M}{N}) \cap S$, which is impossible. Therefore $m \in N$, i.e., $(N_S)^c \subseteq N$ and then $(N_S)^c = N$.

If $\frac{x}{s} \in M_S \setminus N_S$, then obviously, $(N_S : \frac{x}{s}) = (N_S : \frac{x}{1})$, and it is easy to see that $\frac{y}{1} \in M_S \setminus N_S$ if and only if $y \in M \setminus N$. Thus $Zd_{R_S}(\frac{M_S}{N_S}) = \cup_{x \in M \setminus N}(N_S : \frac{x}{1}) = \cup_{x \in M \setminus N}(N : x)_S = (\cup_{x \in M \setminus N}(N : x))_S = (Zd_R(\frac{M}{N}))_S$.

Clearly $P_S = (N : M)_S \subseteq (N_S : M_S)$. Let $\frac{r}{t} \in (N_S : M_S) \setminus (N : M)_S$, where $r \in R \setminus (N : M)$ and $t \in S$. Then there exists an element $m_0 \in M$ such that $rm_0 \notin N$. Since $\frac{r}{t} M_S \subseteq N_S$, we have $\frac{rm_0}{t} = \frac{n}{s'}$ for some $n \in N$ and $s' \in S$. Then there exists $s'' \in S$ such that $s'' s' r m_0 = s'' t n \in N$. Since $rm_0 \notin N$, $s'' s' \in (N : m_0) \cap S \subseteq Zd_R(\frac{M}{N}) \cap S$, which is a contradiction.

(iii) It is easy to see that for each $x \in M \setminus W^c$, $(W^c : x)$ is a prime ideal. So by Lemma 2.1(i), W^c is a weakly prime submodule. Evidently $(W^c)_S = W$.

If $s \in Zd_R(\frac{M}{W^c}) \cap S$, then $s \in (W^c : y)$, for some $y \in M \setminus W^c$. So $\frac{sy}{1} \in W$, and then $\frac{y}{1} = \frac{1}{s}\frac{sy}{1} \in W$. Consequently $y \in W^c$, which is a contradiction.

By part (ii), we have, $(W^c : M)_S = ((W^c)_S : M_S) = (W : M_S) = Q$. Note that $(W^c : M)$ is a prime ideal and $(W^c : M) \cap S \subseteq Zd_R(\frac{M}{W^c}) \cap S = \emptyset$. Then $(W^c : M) = ((W^c : M)_S)^c = Q^c$.

Again by part (ii), we get $(Zd_R(\frac{M}{W^c}))_S = Zd_{R_S}(\frac{M_S}{(W^c)_S}) = Zd_{R_S}(\frac{M_S}{W})$, and then $((Zd_R(\frac{M}{W^c}))_S)^c = (Zd_{R_S}(\frac{M_S}{W}))^c$. Since $Zd_R(\frac{M}{W^c})$ is a prime ideal with $Zd_R(\frac{M}{W^c}) \cap S = \emptyset$, $Zd_R(\frac{M}{W^c}) = ((Zd_R(\frac{M}{W^c}))_S)^c$. Hence $Zd_R(\frac{M}{W^c}) = (Zd_{R_S}(\frac{M_S}{W}))^c$. ■

Proposition 2.5. *Let M be an R-module and let S be a multiplicatively closed subset of R.*

(i) *Let M be finitely generated or* $\dim R < \infty$. *If N is a weakly prime submodule of M such that $(N : M) \cap S = \emptyset$, then N_S is a weakly prime submodule of M_S.*

(ii) *If W is a weakly prime submodule of M_S, then W^c is a weakly prime submodule of M and $(W^c)_S = W$. Furthermore $(W^c : M) \cap S = \emptyset$.*

Proof. (i) It is shown that $(N : M) = (N : x)$, for some $x \in M$. Hence the result is given by Proposition 2.4(i).

If M is generated by $x_1, x_2, x_3, \cdots, x_n$, then it is easily checked that $(N : M) = \cap_{i=1}^{n}(N : x_i)$. By Lemma 2.1(ii), $(N : M)$ is a prime ideal, then for some

i, we have $(N : M) = (N : x_i)$.

Now suppose that dim $R < \infty$. By Lemma 2.1(i), the set $\mathcal{P} = \{(N : y)\}_{y \in M \setminus N}$ is a chain of prime ideals of R, and since dim $R < \infty$, the set $\mathcal{P}$ has a minimal element. Suppose $(N : x)$ is a minimal element of $\mathcal{P}$. Then obviously $(N : M) = (N : x)$.

(ii) By Proposition 2.4(iii), W^c is a weakly prime submodule of M, $(W^c)_S = W$ and $Zd_R(\frac{M}{W^c}) \cap S = \emptyset$. Consequently $(W^c : M) \cap S \subseteq Zd_R(\frac{M}{W^c}) \cap S = \emptyset$. ■

Example 1. Let $M = R \oplus R$, $N = P_1 \oplus P_2$, $N' = P_1 \oplus R$, and $S = R \setminus P_1$, where P_1 and P_2 are prime ideals of R such that $P_1 \subset P_2$. It is easy to see that N and N' are weakly prime submodules of M. Also $N_S = (P_1)_S \oplus (P_2)_S = (P_1)_S \oplus R_S = N'_S$ and so $(N_S)^c = P_1 \oplus R \neq N$. Hence even for a free module of finite rank M, the function $N \longrightarrow N_S$ does not define an one-to-one correspondence between weakly prime submodules N of M with $(N : M) \cap S = \emptyset$ and weakly prime submodules of M_S.

3. Some Applications of Localization

In [6, Proposition 5.1], it is proved that if dim $R < \infty$, M is an R-module and M has a weakly prime submodule, then M has a prime submodule. If N is a weakly prime submodule, then by Lemma 2.1(ii), $(N : M)$ is a prime ideal, and if dim $R < \infty$, then by the proof of Proposition 2.5, $(N : M) = (N : x)$, for some $x \in M$. Hence the following result is a generalization of [6, Proposition 5.1].

For the rest of this section, we denote the notation $\subset$ for the proper inclusion.

Proposition 3.1. *Let M be an R-module, N a submodule of M such that $(N : M) = P$ is a prime ideal of R and for some $x \in M$, $(N : x) = P$. Then there exists a P-prime submodule N_0 of M containing N.*

Proof. Consider the following set

$$T = \{C \mid N \subseteq C,\ C \text{ is a submodule of } M \text{ and } (C : x) = P\}.$$

By Zorn's lemma T has a maximal element. Let N_0 be a maximal element of T. We show that N_0 is a P-prime submodule of M. Evidently, $(N_0 : M) = P$.

Let $ra \in N_0$, where $a \in M \setminus N_0$ and $r \in R$. We have $P = (N_0 : M) \subset (N_0 + Ra : x)$. Consider $r_1 \in (N_0 + Ra : x) \setminus P$. Note that $rr_1 x \in rN_0 + Rra \subseteq N_0$, i.e., $rr_1 \in (N_0 : x) = P$, and since $r_1 \notin P$, we have $r \in P = (N_0 : M)$. ■

In the following we will introduce a certain prime submodule containing a weakly prime submodule.

Theorem 3.2. *Let M be an R-module and let N be a P-weakly prime submodule of M such that for some $x \in M$, $(N : x) = P$.*

(i) $((PM)_P)^c$ *is a* P*-prime submodule of* M.

(ii) $(N_P)^c = \{y \in M \setminus N |\ (N : M) \neq (N : y)\} \cup N$, *and* $(N_P)^c$ *is a* P*-prime submodule of* M *and minimal prime over* N.

Proof. (i) By Proposition 2.4(i) and (iii), $(N_P)^c$ is a proper submodule of M. Since $PM \subseteq N$, we have $((PM)_P)^c \subseteq (N_P)^c$. Hence $((PM)_P)^c$ is a proper submodule of M. Now we show that $((PM)_P)^c$ is a P-prime submodule of M.

Let $ra \in ((PM)_P)^c$, where $r \in R$ and $a \in M$. We have $\frac{ra}{1} = \frac{y}{s}$, for some $y \in PM$, and $s \in R \setminus P$. Then $s'sra = s'y$, for some $s' \in R \setminus P$. If $r \in P$, then since $P \subseteq (PM : M) \subseteq (((PM)_P)^c : M)$, we have $r \in (((PM)_P)^c : M)$. If $r \notin P$, then $sr \in R \setminus P$ and $s'sra = s'y$ implies that $\frac{a}{1} = \frac{y}{sr} \in (PM)_P$. Hence $a \in ((PM)_P)^c$.

We have $P \subseteq (PM : M) \subseteq (((PM)_P)^c : M) \subseteq (((PM)_P)^c : x) \subseteq ((N_P)^c : x) = (N : x) = P$. Hence $(((PM)_P)^c : M) = P$.

(ii) Let $N' = \{y \in M \setminus N |\ (N : M) \neq (N : y)\}$ and $y \in (N_P)^c \setminus N$. There exist $n \in N$ and $s \in R \setminus (N : M)$ such that $\frac{y}{1} = \frac{n}{s}$. Then for some $s' \in R \setminus P$, $s's \in (N : y) \setminus P = (N : y) \setminus (N : M)$ and so $y \in N'$.

Obviously, $N \subseteq (N_P)^c$. If $z \in N'$, then $z \in M \setminus N$ and $(N : M) \subset (N : z)$. Let $s_0 \in (N : z) \setminus (N : M)$. We have, $\frac{z}{1} = \frac{s_0 z}{s_0} \in N_P$. So $z \in (N_P)^c$, whence $N = N' \cup N$.

It is easy to see that $P = (N : M) \subseteq ((N_P)^c : M) \subseteq ((N_P)^c : x) = (N : x) = P$, i.e., $((N_P)^c : M) = P$.

To show that $(N_P)^c$ is a prime submodule of M, let $tb \in (N_P)^c$, where $t \in R$ and $b \in M \setminus (N_P)^c$. From $b \notin (N_P)^c = N' \cup N$, we get $(N : b) = (N : M) = P$. Since $tb \in (N_P)^c$, $\frac{tb}{1} = \frac{n'}{t'}$ for some $n' \in N$ and $t' \in R \setminus P$. So for some $t'' \in R \setminus P$, $t''t'tb = t''n \in N$. Then $t''t't \in (N : b) = P$ and since $t''t' \notin P$, we have $t \in P$.

Now let $N \subseteq L \subseteq (N_P)^c$, where L is a prime submodule of M. Then $P = (N : M) \subseteq (L : M) \subseteq ((N_P)^c : M) = P$, that is $(L : M) = P$. If $N = (N_P)^c$, then obviously $L = (N_P)^c$. So suppose that $y' \in (N_P)^c \setminus N$. Then $P = (N : M) \neq (N : y')$. Consider $u \in (N : y') \setminus P$. We have $uy' \in N \subseteq L$, and since L is a P-prime submodule, we have $y' \in L$. Hence $L = (N_P)^c$. Consequently $(N_P)^c$ is a minimal prime submodule over N. ∎

Corollary 3.3. *Let* M *be an* R*-module and let* N *be a* P*-weakly prime submodule of* M*. If dim* $R < \infty$ *or* M *is finitely generated, then* $(N_P)^c$ *is a* P*-prime submodule of* M *and a minimal prime submodule over* N.

Proof. The proof of Proposition 2.5(i) shows that $(N : M) = (N : x)$, for some $x \in M$. Now the result is completed by Theorem 3.1(ii). ∎

A ring R is said to be an *arithmetical* ring, if for any ideals I, J and K of R, $I + (J \cap K) = (I + J) \cap (I + K)$ (see [8]). Obviously Dedekind domains and Prüfer domains are arithmetical rings.

Lemma 3.4. *A ring* R *is arithmetical if and only if for each prime* (*or maximal*) *ideal P of R, any two ideals of the ring R_P are comparable.*

Proof. See [8, Theorem 1]. ∎

Let R be an arithmetical ring and M an R-module with $T(M) = 0$. The following result shows that every Q-weakly prime submodule of M contains a Q-prime submodule of M, and in this case dim R need not be finite (compare with Corollary 3.3). In the following theorem, if $T(M) = \{m \in M |\ \exists r,\ 0 \neq r \in R,\ rm = 0\} = 0$, then the condition $Zd_R(M) = 0 \subseteq (N : M)$ is satisfied.

Theorem 3.5. *Let R be an arithmetical ring, N a Q-weakly prime submodule of an R-module M such that $Zd_R(M) \subseteq Q$. Then there exists a Q-prime submodule of M contained in N.*

Proof. Let $Zd_R(\frac{M}{N}) = P$. By Proposition 2.4(ii), N_P is a Q_P-weakly prime submodule of M_P and $(N_P)^c = N$. We show that there exists a Q_P-prime submodule W of M_P contained in N_P. Then if we consider $N_1 = W^c$, we will have $N_1 \subseteq (N_P)^c = N$ and by Lemma 2.3, N_1 is a Q-prime submodule of M.

We have $Q_P = (N_P : M_P)$, so $Q_P M_P \subseteq N_P$, and consequently $Q_P M_P \neq M_P$.

Since $Zd_R(M) \subseteq Q$, it is easily checked that $Zd_{R_P}(M_P) \subseteq Q_P$. We will show that $Q_P M_P$ is the required prime submodule. Indeed we will show that for any $a \in M_P,\ r \in R_P \setminus Q_P$, if $ra \in Q_P M_P$, then $a \in Q_P M_P$. $(*)$

For some positive number k, $ra = \sum_{j=1}^{k} p_j a_j$, where for each j, $1 \leq j \leq k$, $p_j \in Q_P$ and $a_j \in M_P$. By Lemma 3.4, every two ideals of R_P are comparable and since $R_P r \not\subseteq Q_P$, $Q_P \subseteq R_P r$. Suppose that $p_j = r_j r$, where $r_j \in R_P$, for each j. Then, $r(a - \sum_{j=1}^{k} r_j a_j) = 0$. Since $Zd_{R_P}(M_P) \subseteq Q_P$ and $r \notin Q_P$, we have $a - \sum_{j=1}^{k} r_j a_j = 0$. Note that for each j, $rr_j = p_j \in Q_P$ and $r \notin Q_P$, hence $r_j \in Q_P$. Consequently $a = \sum_{j=1}^{k} r_j a_j \in Q_P M_P$.

Now we show that $(Q_P M_P : M_P) = Q_P$. Obviously $Q_P \subseteq (Q_P M_P : M_P)$. Let $r \in (Q_P M_P : M_P)$. Consider $a \in M_P \setminus Q_P M_P$. Since $ra \in Q_P M_P$ and $a \notin Q_P M_P$, by $(*)$ we have, $r \in Q_P$, that is $(Q_P M_P : M_P) \subseteq Q_P$. ∎

Corollary 3.6. *Let R be an arithmetical ring, and let N be a Q-weakly prime submodule of an R-module M. If $Zd_R(M) \subset Q$, then there exists a non-zero Q-prime submodule of M contained in N.*

Proof. By Theorem 3.5, there exists a Q-prime submodule N_1 of M contained in N. If $N_1 = 0$, then $Q = (N_1 : M) = (0 : M) \subseteq Zd_R(M) \subset Q$, which is a contradiction. ∎

Example 2. Let R be a Dedekind domain, $M = R \oplus R$, P a non-zero prime ideal of R, and $N = 0 \oplus P$. It is easy to see that N is a weakly prime submodule of M. Also clearly, $Zd_R(M) = 0 = (N : M)$. We show that there does not exist any non-zero prime submodule of M contained in N. Hence the condition $Zd_R(M) \subset (N : M)$ in Corollary 3.6 is necessary. Let N_1 be a non-zero prime submodule of M

contained in $N = 0 \oplus P$ and $0 \neq (0, a) \in N_1$. We have $a(0, 1) \in N_1$ and since $(N_1 : M) \subseteq (N : M) = 0$, $a \notin (N_1 : M)$. Therefore $(0, 1) \in N_1 \subseteq 0 \oplus P$, which is impossible.

4. The Equality $\mathrm{wrad}(B) = \mathrm{rad}(B)$

Evidently for any submodule B of a module M we have $\mathrm{wrad}(B) \subseteq \mathrm{rad}(B)$. In this section we study the equality $\mathrm{wrad}(B) = \mathrm{rad}(B)$.

Definition. *We say that the radical equality holds for a module M, if* $\mathrm{wrad}(B) = \mathrm{rad}(B)$, *for every submodule B of M. It will be said that the radical equality holds for a ring R, if the radicals equality holds for every R-module.*

The modules, every weakly prime submodule of which is an intersection of prime submodules, have been studied in [5]. The following lemma shows that these modules are exactly the modules for which the radical equality holds.

Lemma 4.1. *Let M be an R-module. Then the radical equality holds for M if and only if every weakly prime submodule of M is an intersection of prime submodules of M.*

Proof. Suppose that the radical equality holds for M and N a weakly prime submodule of M. Obviously, $N = \mathrm{wrad}(N) = \mathrm{rad}(N)$, that is N is an intersection of prime submodules.

The converse is obvious. ∎

In [5, Proposition 3.2] there claims that the radical equality holds for every projective module. Theorem 4.2(iii) and the next example show that this result is incorrect. Indeed the only result which is proved in [5, Proposition 3.2] is that $\mathrm{wrad}(0) = \mathrm{rad}(0)$, for every projective module. This result is generalized in Theorem 4.2(viii).

Example 3. Let $R = Z[x]$, $P = R2 + Rx$, $N = P(2, x)$. Then it is easy to see that N is a weakly prime submodule of $M = R \oplus R$ and $\mathrm{wrad}(N) = N \neq R(2, x) = \mathrm{rad}(N)$. So the radical equality does not hold for M (or R). Also this example shows that even for a free (and consequently a projective) module of finite rank M over a Noetherian domain, it is not necessary that the radical equality holds. Now if we consider $M = \frac{R \oplus R}{N}$, then $\mathrm{wrad}(0) = 0 \neq \frac{R(2,x)}{N} = \mathrm{rad}(0)$. Thus even for a Noetherian module M over a Noetherian domain, it is not necessary that $\mathrm{wrad}(0) = \mathrm{rad}(0)$ (compare with Theorem 4.2(viii)).

Some generalizations of Dedekind domains such as weak multiplication rings are introduced in [9, Chapter IX].

Theorem 4.2. *Let M be an R-module. Then*
(i) *If R is an arithmetical ring, then the radical equality holds for R if one of*

the following is satisfied.

a) *R has DCC on prime ideals.*

b) $\dim R < +\infty$*;*

c) *R is a Noetherian ring.*

(ii) *If R is an UFD, then the radical equality holds for R if and only if R is a PID.*

(iii) *The radical equality holds for R if and only if the radical equality holds for every free R-module.*

(iv) *If for every maximal ideal m of R containing Ann M, the radical equality holds for the R_m-module M_m, then the radical equality holds for the R-module M.*

(v) *The following are equivalent.*

d) *The radical equality holds for the ring R.*

e) *For any ideal I of R, the radical equality holds for the ring $\frac{R}{I}$.*

f) *For any non-maximal prime ideal P of R, the radical equality holds for the ring $\frac{R}{P}$.*

(vi) *If for every non-maximal prime ideal P of R, $\frac{R}{P}$ is a Prüfer domain with DCC on prime ideals, then the radical equality holds for R.*

(vii) *The radical equality holds for every weak multiplication ring.*

(viii) *If M is a flat R-module, then* $\text{wrad}(0) = \text{rad}(0)$.

(ix) *Let R be an arithmetical ring with $Zd_R(M) \subseteq \mathbb{N}(R)$, where $\mathbb{N}(R)$ is the intersection of all prime ideals of R. Then* $\text{wrad}(0) = \text{rad}(0)$.

Proof. (i)(a) For a submodule B of M, define

$$E(B) = \{x|\ x = ra,\ r^n a \in B,\ \text{for some } r \in R,\ a \in M,\ n \in \mathbb{N}\}.$$

Also we define $E_1(B) = E(B)$, $E_2(B) = E(\langle E_1(B)\rangle)$ and for any positive number n, we define $E_{n+1}(B) = E(\langle E_n(B)\rangle)$ inductively.

Also we set $UE(B) = \bigcup_{n\in\mathbb{N}} \langle E_n(B)\rangle$.

By induction we can show that $\langle E_n(B)\rangle \subseteq \text{wrad } B$, for any positive number n.

Therefore, $UE(B) \subseteq \text{wrad } B$. According to [3, Corollary 2.5], $\text{rad } B = UE(B)$. Hence $\text{rad } B = \text{wrad } B$.

(i)(b) The proof is clear by part (i)(a).

(i)(c) Note that for any prime ideal P of R, the ring $\frac{R}{P}$ is a Noetherian Prüfer domain. Hence $\frac{R}{P}$ is a Dedekind domain or a field, and so *dim* $\frac{R}{P} \leq 1$. Hence *dim* $R \leq 1$. Now the proof is given by part (b).

(ii) Part (i)(b) shows that the radical equality holds for any PID.

By [5, Theorem 3.9], if R is an UFD such that the radical equality holds for R, then R is a Bezout domain. Also we know that any Bezout domain is a Prüfer domain (see [7, p. 278]), and every Prüfer UFD is a PID, by [7, Proposition 23.5].

(iii) Suppose M is an arbitrary R-module. Then M is a homomorphic image of a free R-module F. Let $M \cong \frac{F}{K}$. Assume that $\frac{B}{K}$ is an arbitrary submodule of $\frac{F}{K}$. By our assumption we have $\mathrm{wrad}(\frac{B}{K}) = \frac{\mathrm{wrad}(B)}{K} = \frac{\mathrm{rad}(B)}{K} = \mathrm{rad}(\frac{B}{K})$.

(iv) By Lemma 4.1, it is enough to show that every weakly prime submodule N of M is an intersection of prime submodules. According to Lemma 2.1(ii), $Zd_R(\frac{M}{N})$ is a proper ideal of R. Let m be a maximal ideal of R containing $Zd_R(\frac{M}{N})$. By Proposition 2.4(ii), N_m is a weakly prime submodule of M_m. Now our assumption implies that $(\mathrm{rad}(N))_m \subseteq \mathrm{rad}(N_m) = \mathrm{wrad}(N_m) = N_m$. That is, $(\mathrm{rad}(N))_m \subseteq N_m$. We will show that $rad(N) = N$. If not, consider $x \in rad(N) \setminus N$. Now $\frac{x}{1} \in (rad(N))_m \subseteq N_m$ implies that there exist $n \in N$ and $s \in R \setminus m$ such that $\frac{x}{1} = \frac{n}{s}$. Consequently there exists $s' \in R \setminus m$ with $ss'x = s'n \in N$. Hence $ss' \in (N : x) \subseteq \bigcup_{x \in M \setminus N} (N : x) = Zd_R(\frac{M}{N}) \subseteq m$, which is a contradiction.

(v) $(d) \Longrightarrow (e)$ Suppose that W is a weakly prime submodule of an $\frac{R}{I}$-module M'. Obviously, W is a weakly prime R-submodule of M'. By our assumption $W = rad_R\ W$. It is easy to see that every submodule of M' is a prime R-submodule if and only if it is a prime $\frac{R}{I}$-submodule of M'. Hence $rad_R\ W = rad_{\frac{R}{I}}(W)$, and then $W = rad_{\frac{R}{I}}(W)$.

(v) $(f) \Longrightarrow (d)$ Let N be a weakly P-prime R-submodule of M. If P is a maximal ideal of R, then N is a prime submodule of M, so the proof is clear. Now assume that P is a non-maximal prime ideal of R.

Consider $\frac{M}{N}$ as an $\frac{R}{P}$-module. One can easily see that $\frac{L}{N}$ is a prime $\frac{R}{P}$-submodule of $\frac{M}{N}$ if and only if L is a prime R-submodule of M containing N. Hence by our assumption we have

$$\frac{N}{N} = \mathrm{rad}_{\frac{R}{P}}(\frac{N}{N}) = \bigcap_{\substack{\frac{L}{N}\ prime\ \frac{R}{P}-submodule\ of\ \frac{M}{N} \\ N \subseteq L}} \frac{L}{N} = \frac{\bigcap_{\substack{L\ prime R-submodule \\ N \subseteq L}} L}{N} = \frac{\mathrm{rad}_R N}{N}.$$

Consequently, $N = \mathrm{rad}_R N$.

(vi) The proof is given by parts (v) and (i)(b)

(vii) By [9, p. 224, Exercise 7], for every prime ideal P of R, $\frac{R}{P}$ is a Dedekind domain. Now the proof is given by part (vi).

(viii) Suppose that N is a weakly P-prime submodule of M. Obviously, $PM \subseteq N \subset M$, that is, PM is a proper submodule of M.

According to [1, Corollary 2.6], in a flat R-module M, for any prime ideal P' of R, $P'M$ is a prime submodule of M, or $P'M = M$. Consequently PM is a prime submodule of M, and hence

$$\mathrm{rad}(0) = \bigcap_{T\ prime\ submodule} T \subseteq \bigcap_{N\ weakly\ P-prime\ submodule} PM \subseteq \mathrm{wrad}(0).$$

(ix) Let N be a weakly prime submodule of M. Theorem 3.5 shows that N contains a prime submodule of M. Hence $\mathrm{rad}\ 0 \subseteq \mathrm{wrad}\ 0$. ∎

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