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On non-squashing partitions

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Abstract

A partition $n = p_1 + p_2 + \cdots + p_k$ with $1 \leqslant p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k$ is called *non-squashing* if $p_1 + \cdots + p_j \leqslant p_{j+1}$ for $1 \leqslant j \leqslant k-1$. Hirschhorn and Sellers showed that the number of non-squashing partitions of n is equal to the number of binary partitions of n. Here we exhibit an explicit bijection between the two families, and determine the number of non-squashing partitions with distinct parts, with a specified number of parts, or with a specified maximal part. We use the results to solve a certain box-stacking problem.

MSC: primary 11P81; 05A15

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1. Introduction

A correspondent, Claudio Buffara, recently asked for the solution to the following problem, originally proposed by Telmo Luis Correia Jr. We are given n boxes, labeled $1, 2, \ldots, n$. For $i = 1, \ldots, n$, box i weighs i grams and can support a total weight of i grams. What is $f(n)$, the number of different ways to build a single stack of boxes in which no box will be

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squashed by the weight of the boxes above it? For example, $f(4) = 14$, since we can form the following stacks:

```
                                                  1  1  1
                        1  1  1  2  2  3  2  2  3
Ø,  1,  2,  3,  4,  2,  3,  4,  3,  4,  4,  3,  4,  4.
```

The other two possible stacks:

```
   1
2  2
3  3
4, 4,
```

are excluded, since $2 + 3 > 4$ and the box labeled 4 would collapse in both cases.

To make this more precise, let us say that a partition of a natural number m into k parts is *non-squashing* if when the parts are arranged in nondecreasing order, say

$$m = p_1 + p_2 + \cdots + p_k \quad \text{with } 1 \leqslant p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k, \tag{1}$$

we have

$$p_1 + \cdots + p_j \leqslant p_{j+1} \quad \text{for } 1 \leqslant j \leqslant k-1. \tag{2}$$

If the boxes in a stack are labeled (from the top) $p_1, p_2, \ldots, p_k$, the stack will not collapse if and only if the partition is non-squashing. In the problem as stated, the boxes must also have distinct labels and their sum cannot exceed $\binom{n+1}{2}$. Therefore, $f(n)$ is equal to the total number of partitions of numbers from 0 to $\binom{n+1}{2}$ which are (i) non-squashing, (ii) have distinct parts, and (iii) involve no part greater than n. We will give the solution in Sections 7 and 8. In Sections 2 and 3 we study the numbers of non-squashing partitions and non-squashing partitions with distinct parts. Sections 4, 5 and 6 deal with non-squashing partitions with a given number of parts, a given number of distinct parts, and a specified largest part, respectively. Some of these results are used in the final two sections, others are included because they seem of independent interest.

2. Non-squashing partitions

Let $a(n)$ denote the number of non-squashing partitions of n. It was shown by Hirschhorn and Sellers[1] [1] that $a(n)$ is equal to the number of "binary partitions" of n, that is, the number of partitions of n into powers of 2. See sequences A000123 and A018819 in [2] for properties of the binary partition function and references to the extensive literature.

In fact, Hirschhorn and Sellers prove a more general result. Let $s \geqslant 2$ be an integer. Let us say that a partition (1) is *s-non-squashing* if

$$(s-1)(p_1 + \cdots + p_j) \leqslant p_{j+1} \quad \text{for } 1 \leqslant j \leqslant k-1. \tag{3}$$

[1] Hirschhorn and Sellers regard inequalities (2) as purely arithmetic conditions and do not mention stacking problems.

If the p_j are the labels of the boxes in a stack, not only is no box squashed, no box even comes within a factor of $s-1$ of being squashed. A non-squashing partition as defined in the Introduction is 2-non-squashing.

Theorem 1 (*Hirschhorn and Sellers [1]*). *The number $a_s(n)$ of s-non-squashing partitions of n is equal to the number of "s-ary" partitions of n, that is, the number of partitions of n into powers of s.*

The following is an alternative proof of this result, which leads to a bijection between the two families.

Proof. Let $a'_s(n)$ be the number of partitions of n into powers of s, for some integer $s \geqslant 2$. Suppose

$$n = s^{e_1} + s^{e_2} + \cdots + s^{e_l}$$

is such a partition, where $n \geqslant s$. If at least one of the parts is 1 we can remove it and obtain a partition of $n-1$ into powers of s; if not, all the e_i are greater than 0 and we can also divide by s and obtain a partition of n/s. Therefore, $a'_s(n)$ satisfies the recurrence

$$a'_s(n) = \begin{cases} a'_s(n-1) & \text{if } n \not\equiv 0 \bmod s, \\ a'_s(n-1) + a'_s(n/s) & \text{if } n \equiv 0 \bmod s \end{cases} \tag{4}$$

for $n \geqslant s$. The smallest n for which there is a partition with more than one part is s, so we have the initial conditions

$$a'_s(0) = a'_s(1) = \cdots = a'_s(s-1) = 1. \tag{5}$$

On the other hand, let

$$n = p_1 + p_2 + \cdots + p_k \quad \text{with } 1 \leqslant p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k \tag{6}$$

be an s-non-squashing partition of $n \geqslant s$. If the largest part p_k is strictly greater than $(s-1)n/s$, then the sum of the other parts is strictly less than n/s, and we can subtract 1 from the largest part and obtain an s-non-squashing partition of $n-1$. (We omit the straightforward verification.) If the largest part is equal to $(s-1)n/s$ (implying $n \equiv 0 \bmod s$), we can also delete the largest part and obtain an s-non-squashing partition of n/s. Therefore, $a_s(n)$ satisfies the recurrence

$$a_s(n) = \begin{cases} a_s(n-1) & \text{if } n \not\equiv 0 \bmod s, \\ a_s(n-1) + a_s(n/s) & \text{if } n \equiv 0 \bmod s, \end{cases} \tag{7}$$

for $n \geqslant s$. The smallest n for which there is a partition with more than one part is s (where we have the partition with parts 1 and $s-1$), so we have the initial conditions

$$a_s(0) = a_s(1) = \cdots = a_s(s-1) = 1. \tag{8}$$

Comparing (4), (5) with (7), (8), we conclude that $a'_s(n) = a_s(n)$ for all $n \geqslant 0$ and all $s \geqslant 2$, which is the main result of [1]. □

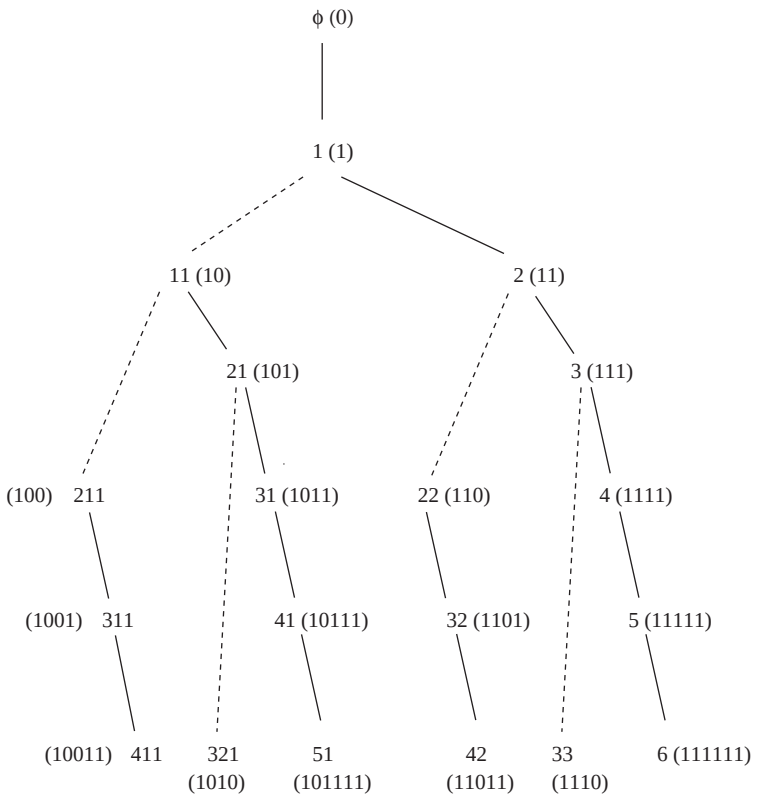

Fig. 1. Non-squashing partitions of the numbers 0, . . . , 6 arranged in tree structure. The binary labels are shown in parentheses. (Every node has out-degree 2, but only edges between partitions of 0, . . . , 6 are shown.)

The above proof associates each partition (from either family) with a unique partition of a smaller number. We can therefore arrange the partitions in each family into a rooted tree, with the empty partition of 0 as the root node. Figs. 1 and 2 show the beginnings of the two trees for the case $s = 2$. (Most of the time we will adopt the standard convention of writing partitions with the parts in nonincreasing order.) Every node has two descendants and (except for the root) one ancestor. We may label the edge leading from a partition of n/s to a partition of n with 0 (such edges are shown as broken lines in Figs. 1 and 2), and the edge leading from a partition of $n - 1$ to a partition of n with 1 (the solid lines in the figures).

This associates a unique binary string with each partition in either tree. A partition of n in one tree receives the same binary string as the corresponding partition of n in the same position in the other tree. In this way we obtain a canonical numbering for the s-non-squashing partitions, a canonical numbering for the partitions into powers of s, and a bijection between them.

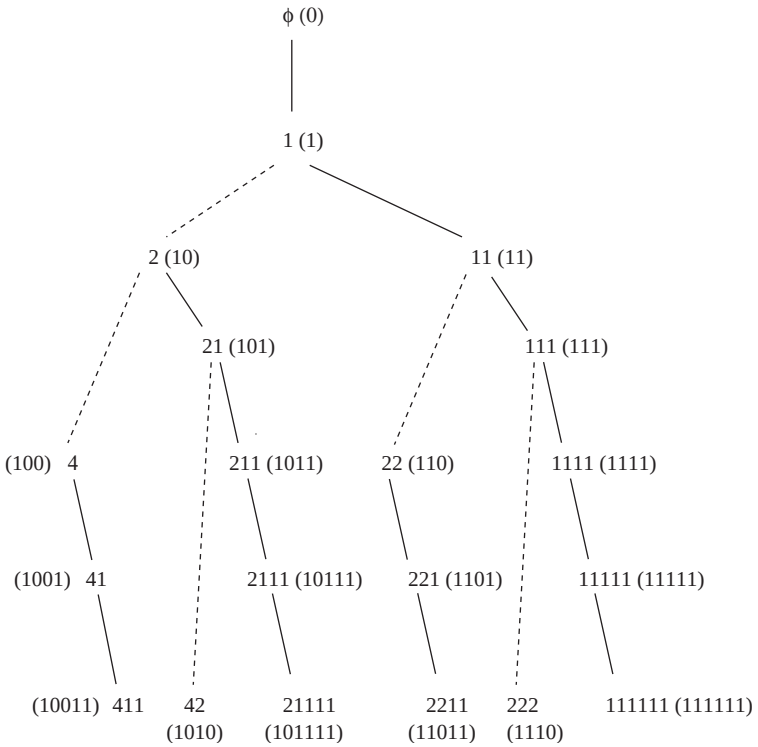

Fig. 2. Binary partitions of the numbers $0, \ldots, 6$ arranged in tree structure. The binary labels are shown in parentheses. (Every node has out-degree 2, but only edges between partitions of $0, \ldots, 6$ are shown.)

Table 1 shows the beginning of the bijection. The first column gives the binary string u, the second column gives the corresponding s-non-squashing partition $P(u)$, the third column gives the corresponding s-ary partition $Q(u)$, and the last column gives the number $n = n(u)$ that is partitioned by both $P(u)$ and $Q(u)$.

We note without proof the following properties of the bijection.

(i) For a nonzero string u, the number of parts in $P(u)$ is equal to 1 plus the number of 0's in u, and the number of parts in $Q(u)$ is equal to the number of 1's in u.

(ii) Thinking of u now as the integer represented by the binary string, the number $n = n(u)$ (given in the last column of the table) that is partitioned by both $P(u)$ and $Q(u)$ is defined by the recurrence

$$n(0) = 0; \quad n(2u) = sn(u) \text{ for } u \geqslant 1, \quad n(2u+1) = n(u) + 1 \quad \text{for } u \geqslant 0$$

(sequences A087808, A090639, etc. in [2]).

(iii) $P(u) = Q(u)$ if and only if $u = (4^k - 1)/3$ for some $k \geqslant 1$ (that is, if u is the binary string $10101\ldots01$).

Table 1
Bijection between s-non-squashing partitions $P(u)$ and s-ary partitions $Q(u)$

u	$P(u)$	$Q(u)$	n
0	$\emptyset$	$\emptyset$	0
1	1	1	1
10	$s-1, 1$	s	s
11	2	1, 1	2
100	$s(s-1), s-1, 1$	s^2	s^2
101	$s, 1$	$s, 1$	$s+1$
110	$2(s-1), 2$	s, s	$2s$
111	3	1, 1, 1	3
1000	$s^2(s-1), s(s-1), s-1, 1$	s^3	s^3
1001	$s^2-s+1, s-1, 1$	$s^2, 1$	s^2+1
1010	$s^2-1, s, 1$	s^2, s	s^2+s
1011	$s+1, 1$	$s, 1, 1$	$s+2$
1100	$2s(s-1), 2(s-1), 2$	s^2, s^2	$2s^2$
1101	$2s-1, 2$	$s, s, 1$	$2s+1$
1110	$3(s-1), 3$	s, s, s	$3s$
1111	4	1, 1, 1, 1	4
10000	$s^3(s-1), s^2(s-1), s(s-1), s-1, 1$	s^4	s^4
10001	$s^3-s^2+1, s(s-1), s-1, 1$	$s^3, 1$	s^3+1
10010	$(s-1)(s^2+1), s^2-s+1, s-1, 1$	s^3, s	s^3+s
10011	$s^2-s+2, s-1, 1$	$s^2, 1, 1$	s^2+2
10100	$s(s^2-1), s^2-1, s, 1$	s^3, s^2	s^3+s^2
10101	$s^2, s, 1$	$s^2, s, 1$	s^2+s+1

u is the index (written as a binary number) and n is the number that is being partitioned.

It is easy to go from the binary vector to the partitions and vice versa. To obtain the s-non-squashing partition $P(u)$ corresponding to the binary vector u, we start with the empty partition $P(u)=\emptyset$, and scan u from left to right (i.e. beginning with the most significant bit):

- if we see a 1, then if $P(u)=\emptyset$ set $P(u)=1$, otherwise add 1 to the largest part of $P(u)$,
- if we see a 0, then if $P(u)=\emptyset$ set $P(u)=0$, otherwise adjoin to $P(u)$ a part equal to $s-1$ times the sum of the parts of $P(u)$.

Example for $s=3$. Suppose $u=10110$. The successive terms in the construction of $P(u)$ are

$$\emptyset, 1, 21, 31, 41, 1041.$$

Likewise, to obtain the partition $Q(u)$ into powers of s, again we start with the empty partition $Q(u)=\emptyset$, and scan u from left to right:

- if we see a 1, then append a part of size 1 to $Q(u)$,
- if we see a 0, then if $Q(u)=\emptyset$ do nothing, otherwise multiply all the parts of $Q(u)$ by s.

Example for $s = 3$. Again we take $u = 10110$. The successive terms in the construction of $Q(u)$ are

$$\emptyset, 1, 3, 31, 311, 933.$$

Thus the bijection associates these two partitions of 15: $P(u) = 1041$ and $Q(u) = 933$.

Finally, we note that the numbers $a_s(n)$ have the generating function

$$\sum_{n=0}^{\infty} a_s(n)\, x^n = \prod_{i=0}^{\infty} \frac{1}{1 - x^{s^i}}. \tag{9}$$

3. Non-squashing partitions into distinct parts

From here on we consider only the case $s = 2$, that is, non-squashing partitions. One of the restrictions in the box-stacking problem mentioned in the Introduction is that the parts be distinct. In this section we investigate the number $b(n)$ of non-squashing partitions of n into distinct parts. The first few values of $b(n)$ for $n = 0, 1, 2, \ldots$ are

$$1, 1, 1, 2, 2, 3, 4, 5, 6, 7, 9, 10, 13, 14, 18, 19, 24, 25, 31, 32, 40, 41, 50, \ldots \tag{10}$$

(this is now sequence A088567 in [2]).

Theorem 2. *The numbers $b(n)$ satisfy the recurrence*

$$\begin{aligned} &b(0) = b(1) = 1,\\ &b(2m) = b(2m-1) + b(m) - 1 \text{ for } m \geqslant 1,\\ &b(2m+1) = b(2m) + 1 \text{ for } m \geqslant 1. \end{aligned} \tag{11}$$

The generating function $B(x) = \sum_{n=0}^{\infty} b(n)x^n$ satisfies

$$B(x) = \frac{1}{1-x}\, B(x^2) - \frac{x^2}{1-x^2}, \tag{12}$$

and is given explicitly by

$$B(x) = 1 + \frac{x}{1-x} + \sum_{i=1}^{\infty} \frac{x^{3\cdot 2^{i-1}}}{\prod_{j=0}^{i}(1 - x^{2^j})}. \tag{13}$$

Proof. We obtain the partitions (non-squashing and with distinct parts being understood) of an odd number $2m + 1$ by adjoining a part of size $2m + 1 - j$ to a partition of j, for some $j = 0, 1, \ldots, m$ (since $2m + 1 - j > j$, these are indeed non-squashing). Likewise, we obtain the partitions of $2m$ by adjoining a part of size $2m - j$ to a partition of j, for some $j = 0, 1, \ldots, m$, except that if $j = m$ we cannot adjoin a part of size m to the partition consisting of a single m. Thus we have

$$\begin{aligned} &b(0) = b(1) = 1,\\ &b(2m+1) = b(0) + b(1) + \cdots + b(m) \text{ for } m \geqslant 1,\\ &b(2m) = b(0) + b(1) + \cdots + b(m) - 1 \text{ for } m \geqslant 1, \end{aligned} \tag{14}$$

from which (11) follows. After some algebra we find that (11) implies

$$B(x) = xB(x) + B(x^2) - \frac{x^2}{1+x}, \tag{15}$$

and after rearranging we obtain (12). Eq. (12) implies

$$B(x^2) = \frac{1}{1-x^2}B(x^4) - \frac{x^4}{1-x^4},$$

and so on, and hence

$$\begin{aligned} B(x) &= \prod_{i=0}^{\infty} \frac{1}{1-x^{2^i}} - \sum_{i=1}^{\infty} \frac{x^{2^i}}{\prod_{j=0}^{i-1}(1-x^{2^j})\,(1+x^{2^{i-1}})} \\ &= \prod_{i=0}^{\infty} \frac{1}{1-x^{2^i}} - \sum_{i=1}^{\infty} \frac{x^{2^i}}{\prod_{j=0}^{i-2}(1-x^{2^j})\,(1-x^{2^i})}. \end{aligned} \tag{16}$$

To simplify this we make use of an identity from [1]: if $m_1 < m_2 < \cdots < m_k$ are positive integers then

$$1 + \sum_{j=1}^{k} \frac{x^{m_j}}{(1-x^{m_1})(1-x^{m_2})\cdots(1-x^{m_j})} = \prod_{j=1}^{k} \frac{1}{1-x^{m_j}}. \tag{17}$$

Applying this to the sum in (16) and simplifying, we eventually obtain (13). □

Corollary 3. (i) *The sequence* $\{b(n)\}$ (see (10)) *has the property that the sequence of partial sums*

$$1, 2, 3, 5, 7, 10, 14, 19, 25, 32, 41, 51, 64, 78, 96, 115, 139, 164, \ldots \tag{18}$$

coincides with the odd-indexed subsequence $b(1), b(3), b(5), \ldots$. *The even-indexed subsequence* $b(2), b(4), b(6), \ldots$ *is obtained by adding* 1 *to the terms of* (18). (ii) $b(n)$, *the number of non-squashing partitions of* n *into distinct parts, is equal to the number of partitions of* n *into powers of* 2 *such that either all the parts are equal to* 1 *or, if the largest part has size* $2^i > 1$, *then there is also at least one part of size* 2^{i-1}.

Proof. (i) The first assertion is equivalent to the algebraic identity

$$\frac{B(x)}{1-x} = \frac{B(\sqrt{x}) - B(-\sqrt{x})}{2\sqrt{x}}, \tag{19}$$

which is easily verified using (15) and (16). The second assertion follows from (11). Property (ii) is an immediate consequence of (16). □

Congruences satisfied by $a_s(n)$ have been studied by many authors (see references in [1]). Here, we record just one such result for $b(n)$.

Corollary 4. *The value of* $b(n) \bmod 2$ *is as follows* (*all congruences are* $\bmod 2$):

$$b(0) \equiv 1, \tag{20}$$

$$\text{if } n \text{ is odd}, \ b(n) \equiv b(n-1) + 1, \tag{21}$$

$$b(8m+2) \equiv 1, b(8m+6) \equiv 0, \tag{22}$$

$$b(16m+4) \equiv 0, b(16m+12) \equiv 1, \tag{23}$$

$$\text{for } m > 0, b(16m) \equiv b(8m), b(32m+8) \equiv 0, b(32m+24) \equiv 1. \tag{24}$$

For $m > 0$, $b(8m)$ *is the value of the bit immediately to the left of the rightmost* 1 *when* m *is written in binary.*

Proof. Eq. (21) follows from (11). To prove the first assertion in (22), we repeatedly apply (11), obtaining

$$\begin{aligned} b(8m+2) &\equiv b(8m+1) + b(4m+1) + 1 \\ &\equiv b(8m) + b(4m) + 1 \\ &\equiv b(8m-1) \\ &\equiv b(8m-2) + 1 \\ &\equiv \ldots \\ &\equiv b(8i-6) \\ &\equiv \ldots \\ &\equiv b(2) = 1. \end{aligned}$$

The other claims in (22)–(24) are established in a similar way. It is easily checked that the final assertion in the corollary is equivalent to (24). (The final assertion was discovered by noticing that the subsequence $\{b(8m)\}$ is, apart from the leading term, the same as sequence A038189 in [2].) □

4. Non-squashing partitions by number of parts

Let $a(n,k)$ be the number of non-squashing partitions of n into exactly k parts. Table 2 shows the initial values of this function.

Theorem 5. *The numbers* $a(n,k)$ *satisfy the recurrence*

$$\begin{aligned} &a(2m,k) = a(2m-1,k) + a(m,k-1) \quad \text{for } m \geqslant 1, k \geqslant 1, \\ &a(2m+1,k) = a(2m,k) \quad \text{for } m \geqslant 1, k \geqslant 1, \end{aligned} \tag{25}$$

with initial conditions

$$\begin{aligned} &a(0,0) = 1, \quad a(n,0) = 0 \text{ for } n \geqslant 1, \quad a(n,k) = 0 \text{ for } k > n, \\ &\quad a(n,1) = 1 \text{ for } n \geqslant 1. \end{aligned}$$

Table 2
Values of $a(n,k)$, the number of non-squashing partitions of n into exactly k parts

n	k				
	0	1	2	3	4
0	1	0	0	0	0
1	0	1	0	0	0
2	0	1	1	0	0
3	0	1	1	0	0
4	0	1	2	1	0
5	0	1	2	1	0
6	0	1	3	2	0
7	0	1	3	2	0
8	0	1	4	4	1
9	0	1	4	4	1
10	0	1	5	6	2
11	0	1	5	6	2
12	0	1	6	9	4
13	0	1	6	9	4

In particular, each odd-indexed row (*except for row* 1) *in Table* 2 *is a copy of the previous row. If the duplicate entries are omitted, the kth column has generating function*

$$\sum_{k=0}^{\infty} a(2m,k)x^m = \frac{x^{2^{k-2}}}{(1-x)\prod_{j=0}^{k-2}(1-x^{2^j})}, \tag{26}$$

while if they are included we get the simpler expression

$$\sum_{k=0}^{\infty} a(m,k)x^m = \frac{x^{2^{k-1}}}{\prod_{j=0}^{k-1}(1-x^{2^j})}. \tag{27}$$

Eq. (27) *implies that the number of non-squashing partitions of n with k parts is equal* (i) *to the number of partitions of* $n-2^{k-1}$ *into powers of* 2 *not exceeding* 2^{k-1}, *and also* (ii) *to the number of binary partitions of n with largest part* 2^{k-1}.

Proof. Recurrence (25) follows at once from the argument used to derive (7). The generating functions then follow from the recurrence; we omit the details. □

For example, the $k=3$ column, omitting the odd-indexed terms, is

$$0, 0, 1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, 42, 49, 56, 64, 72, 81, \ldots,$$

which is the sequence of "quarter-squares", that is,

$$a(2m,3) = \left\lfloor \frac{m}{2} \right\rfloor \left\lceil \frac{m}{2} \right\rceil,$$

with generating function

$$\sum_{k=0}^{\infty} a(2m,3)x^m = \frac{x^2}{(1-x)^2(1-x^2)}$$

(sequence A002620).

5. Non-squashing partitions into k distinct parts

Let $b(n,k)$ be the number of non-squashing partitions of n into exactly k distinct parts. Table 3 shows the initial values of this function.

Comparison of this table with Table 2 suggests that Table 3 is obtained by displacing the kth column of Table 2 (for $k \geqslant 2$) downwards by 2^{k-2} positions. This is true, and we have:

Theorem 6. *The numbers $b(n,k)$ satisfy*

$$\begin{aligned} &b(n,0) = a(n,0) \quad \text{for } n \geqslant 0, \\ &b(n,1) = a(n,1) \quad \text{for } n \geqslant 0, \\ &b(n,k) = a(n-2^{k-2},k) \quad \text{for } n \geqslant 0, k \geqslant 2. \end{aligned} \tag{28}$$

Table 3
Values of $b(n,k)$, the number of non-squashing partitions of n into exactly k distinct parts

n	k				
	0	1	2	3	4
0	1	0	0	0	0
1	0	1	0	0	0
2	0	1	0	0	0
3	0	1	1	0	0
4	0	1	1	0	0
5	0	1	2	0	0
6	0	1	2	1	0
7	0	1	3	1	0
8	0	1	3	2	0
9	0	1	4	2	0
10	0	1	4	4	0
11	0	1	5	4	0
12	0	1	5	6	1
13	0	1	6	6	1
14	0	1	6	9	2
15	0	1	7	9	2
16	0	1	7	12	4
17	0	1	8	12	4

Also

$$\sum_{k=0}^{\infty} b(n,k)x^n = \frac{x^{3\cdot 2^{k-2}}}{\prod_{j=0}^{k-1}(1-x^{2^j})} \quad \text{for } k \geqslant 2. \tag{29}$$

Eq. (29) *implies that the number of non-squashing partitions of* n *with* k *distinct parts is equal to the number of partitions of* $n - 3\cdot 2^{k-2}$ *into powers of* 2 *not exceeding* 2^{k-1}.

Proof. For this discussion we write the parts in nondecreasing order. The non-squashing partition (of some very large number) having the slowest growth begins

$$1, 1, 2, 4, 8, 16, 32, 64, \ldots, \tag{30}$$

while the non-squashing partition with distinct parts and the slowest growth is the sequence $\gamma(1), \gamma(2), \gamma(3), \ldots$ given by

$$1, 2, 3, 6, 12, 24, 48, 96, \ldots \tag{31}$$

with $\gamma(i) = i$ for $i \leqslant 3$, $\gamma(i) = 3\cdot 2^{i-3}$ for $i \geqslant 3$. The difference between (30) and (31) is

$$0, 1, 1, 2, 4, 8, 16, 32, \ldots . \tag{32}$$

One can now verify that adding the initial k terms of (32) term-by-term to the parts of a non-squashing partition of n into k parts provides a bijection with a non-squashing partition of n into k distinct parts, and establishes the relations in (28).

For example, the non-squashing partitions of $n = 4, \ldots, 8$ into $k = 3$ parts are:

$$\begin{aligned} &4 : 112,\\ &5 : 113,\\ &6 : 114, 123,\\ &7 : 115, 124,\\ &8 : 116, 125, 134, 224. \end{aligned} \tag{33}$$

On the other hand, the non-squashing partitions of $n = 6, \ldots, 10$ into $k = 3$ distinct parts are:

$$\begin{aligned} 6 &: 123,\\ 7 &: 124,\\ 8 &: 125, 134,\\ 9 &: 126, 135,\\ 10 &: 127, 136, 145, 235. \end{aligned} \tag{34}$$

Adding 0, 1, 1 term-by-term to the partitions in (33) yields the partitions in (34).

The generating function (29) now follows from (27) and (28). □

6. Non-squashing partitions into distinct parts with largest part m

Let $c(n,k)$ be the number of non-squashing partitions of n into distinct parts of which the greatest is m. Table 4 shows the initial values.

Table 4
Values of $c(n,k)$, the number of non-squashing partitions of n into distinct parts of which the greatest is m (the blank entries are zero)

n	m												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	0	1											
2	0	0	1										
3	0	0	1	1									
4	0	0	0	1	1								
5	0	0	0	1	1	1							
6	0	0	0	1	1	1	1						
7	0	0	0	0	2	1	1	1					
8	0	0	0	0	1	2	1	1	1				
9	0	0	0	0	0	2	2	1	1	1			
10	0	0	0	0	0	2	2	2	1	1	1		
11	0	0	0	0	0	0	3	2	2	1	1	1	
12	0	0	0	0	0	0	3	3	2	2	1	1	1

Theorem 7. (i) *The nonzero values of* $c(n,m)$ *lie within a certain strip*:

$$c(n,m) = 0 \quad \text{if } m < n/2 \text{ or if } n < m.$$

(ii) *For* $m \leqslant n \leqslant 2m$,

$$c(n,m) = \sum_{i=0}^{m-1} c(n-m,i). \tag{35}$$

(iii) *For* $m \leqslant n \leqslant 2m$,

$$c(n,m) = \begin{cases} b(n-m) & \text{if } n < 2m, \\ b(n-m)-1 & \text{if } n = 2m. \end{cases} \tag{36}$$

Proof. (i) The slowest-growing non-squashing partition into distinct parts is (31), so no partition can have $n > 2m$. The second assertion is immediate from the definition of $c(n,m)$.

(ii) This is a consequence of the fact that removing the largest part leaves a partition with largest part $\leqslant m-1$.

(iii) When the largest part is removed, we obtain a non-squashing partition of $n-m$ into distinct parts. Conversely, given a non-squashing partition of $n-m$ into distinct parts, we obtain a partition of n with largest part m by adjoining a part of size m, with the single exception that we cannot adjoin a part of size m to the partition consisting of a single part of size m. □

7. Solution to the box-stacking problem

We can now give the solution to the box-stacking problem mentioned in the Introduction.

Theorem 8. *There is a bijection between non-squashing stacks of boxes in which the largest box has label n and non-squashing partitions of 2n into distinct parts, i.e.*

$$f(n) - f(n-1) = b(2n). \tag{37}$$

Proof. Let

$$1 \leqslant p_1 < p_2 < \cdots < p_k = n$$

be a non-squashing stack of boxes in which the largest box has label n. Let $r = p_1 + \cdots + p_{k-1}$ (take $r = 0$ if $k = 1$). Then $r \leqslant p_k = n$. If we increase the largest part by $n - r$ we obtain a non-squashing partition of $2n$. Conversely, suppose $1 \leqslant p_1 < p_2 < \cdots < p_k$ is a non-squashing partition of $2n$ into distinct parts. Let $r = p_1 + \cdots + p_{k-1}$. Then $r + p_k = 2n$, $r < p_k$, which implies $r < n < p_k$. So we may reduce the largest part to n, obtaining a non-squashing stack with largest part labeled n. □

Eq. (37) could also be derived from the fact that

$$f(n) = \sum_{i=0}^{\binom{n+1}{2}} \sum_{j=0}^{n} c(i,j) = \sum_{i=0}^{2n} \sum_{j=0}^{n} c(i,j).$$

Corollary 9. *The numbers $f(n)$ have generating function*

$$F(x) = \sum_{n=0}^{\infty} f(n)x^n = \frac{B(x) - x}{(1-x)^2}, \tag{38}$$

where $B(x)$ is given in Theorem 2. *Also, $F(x)$ satisfies*

$$F(x) = \frac{(1+x)^2}{1-x} F(x^2) - \frac{x(1-2x^2)}{(1-x)^2(1-x^2)}. \tag{39}$$

Proof. From Theorem 8 we know that

$$f(n) = b(0) + b(2) + \cdots + b(2n),$$

so

$$F(x) = \frac{1}{1-x} \frac{B(\sqrt{x}) + B(-\sqrt{x})}{2}.$$

So (38) will follow if we can show that

$$\frac{2(B(x) - x)}{1-x} = B(\sqrt{x}) + B(-\sqrt{x}).$$

However, from (19) we know that

$$\frac{2\sqrt{x}B(x)}{1-x} = B(\sqrt{x}) - B(-\sqrt{x}).$$

So we must show that

$$B(\sqrt{x}) = \frac{B(x) - x}{1 - x} + \frac{\sqrt{x}B(x)}{1 - x},$$

which follows immediately from (12). Eq. (39) then follows using (15). □

The first few values of $f(n)$ for $n = 0, 1, 2, \ldots$ are

$$1, 2, 4, 8, 14, 23, 36, 54, 78, 109, 149, 199, 262, 339, 434, 548, 686, \ldots \quad (40)$$

(sequence A089054).

The original version of the problem had $n + 1$ boxes labeled $0, 1, \ldots, n$. Since the box labeled 0 may be included in the stack or not, without changing the non-squashing property, the answer to this problem is $2f(n)$.

8. Stacks with a given number of boxes

In this final section we determine the numbers $f(n, k)$, the number of non-squashing stacks of boxes in which the largest box has label $\leqslant n$ and there are exactly k boxes in the stack. Table 5 shows the initial values of this function.

Theorem 10. *We have $f(n, 0) = 1$ for all n, and for $n \geqslant 1$, $k \geqslant 1$,*

$$f(n, k) = \sum_{p=0}^{\min\{k-1, n-\gamma(k)\}} \sum_{m=p}^{n-\gamma(k)} (n - \gamma(k) + 1 - m)\, a(m, p). \quad (41)$$

Table 5
Values of $f(n, k)$, the number of stacks in which there are exactly k boxes and the largest box is $\leqslant n$

n	k					
	0	1	2	3	4	5
0	1	0	0	0	0	0
1	1	1	0	0	0	0
2	1	2	1	0	0	0
3	1	3	3	1	0	0
4	1	4	6	3	0	0
5	1	5	10	7	0	0
6	1	6	15	13	1	0
7	1	7	21	22	3	0
8	1	8	28	34	7	0
9	1	9	36	50	13	0
10	1	10	45	70	23	0
11	1	11	55	95	37	0
12	1	12	66	125	57	1

Proof. We first determine $f(n,k) - f(n-1,k)$, that is, the number of stacks

$$1 \leqslant p_1 < p_2 < \cdots < p_k = n$$

in which the largest box is labeled n. Let $q_i = p_i - \gamma(i)$ for $i = 1, \ldots, k$ (cf. (31)), so that

$$0 \leqslant q_1 \leqslant q_2 \leqslant \cdots \leqslant q_k = n - \gamma(k).$$

Some of the q_i may be zero. The nonzero elements among $q_1, \ldots, q_{k-1}$ (if any) form a non-squashing partition into p parts of some number m between 0 and q_k, where $0 \leqslant p \leqslant k-1$. Hence

$$f(n,k) - f(n-1,k) = \sum_{m \leqslant n-\gamma(k)} \sum_{p \leqslant k-1} a(m,p), \tag{42}$$

and so

$$f(n,k) = \sum_{p=0}^{k-1} \sum_{\tau=k}^{n} \sum_{m=p}^{\tau-\gamma(k)} a(m,p). \tag{43}$$

Eq. (41) follows when we collect terms. □

References

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