TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 357, Number 9, Pages 3425–3442
S 0002-9947(05)03937-1
Article electronically published on April 27, 2005

NOTES ON LIMITS OF SOBOLEV SPACES AND THE CONTINUITY OF INTERPOLATION SCALES

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Abstract. We extend lemmas by Bourgain-Brezis-Mironescu (2001), and Maz'ya-Shaposhnikova (2002), on limits of Sobolev spaces, to the setting of interpolation scales. This is achieved by means of establishing the continuity of real and complex interpolation scales at the end points. A connection to extrapolation theory is developed, and a new application to limits of Sobolev scales is obtained. We also give a new approach to the problem of how to recognize constant functions via Sobolev conditions.

1. Introduction

Bourgain-Brézis-Mironescu [5] (see also [6]) have recently proved, among other results, that for any smooth bounded domain $\Omega \subset \mathbb{R}^n$, $f \in W^{1,p}(\Omega), 1 \leq p < \infty$,

$$\lim_{s \to 1} (1-s)^{1/p} \left\{ \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} dxdy \right\}^{1/p} \sim \|\nabla f\|_{L^p(\Omega)} . \tag{1.1}$$

This result was later complemented by Maz'ya and Shaposhnikova [27] who showed that for $f \in \bigcup_{s \in (0,1)} W_0^{s,p}(\mathbb{R}^n), 1 \leq p < \infty$, we have

$$\lim_{s \downarrow 0} s^{1/p} \|f\|_{W_0^{s,p}(\mathbb{R}^n)} = C \|f\|_{L^p(\mathbb{R}^n)} , \tag{1.2}$$

where C is a constant[1] independent of f. Here $W_0^{s,p}(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ in the norm

$$\|f\|_{W_0^{s,p}(\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} dxdy \right\}^{1/p} , \tag{1.3}$$

while the space $W_0^{1,p}(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ in the norm

$$\|f\|_{W_0^{1,p}(\mathbb{R}^n)} = \|\nabla f\|_{L^p(\mathbb{R}^n)} .$$

The method of [5] relies on mollifiers,[2] while the proof in [27] is based on sharp forms of Hardy inequalities. The purpose of this note is to understand these results from the point of view of interpolation theory. In our setting both (1.1) and (1.2) are simple consequences of a continuity principle for real interpolation scales which we establish in Theorem 1 below.

Received by the editors July 8, 2003.
2000 *Mathematics Subject Classification.* Primary 46E30, 46M35, 26D10.
Key words and phrases. Sobolev spaces, interpolation scales.

[1]Actually they give a precise value $C = 2^{1/p} p^{-1/p} \left|S^{n-1}\right|^{1/p}$.

[2]Another proof of (1.1), using an argument close in spirit to ours, but restricted to Besov spaces, is given in [24].

We think that the interest in the interpolation method we develop here lies in the fact that, in this more general framework, it is easier to formulate and establish similar results not only for more general Sobolev spaces generated using suitable semigroups (cf. (3.8) and (3.9) below), but for other scales of spaces as well. Moreover, connecting these ideas with extrapolation theory [19], we can also consider limits of interpolation spaces with prescribed decay obtaining results of the following type (cf. Example 2 below):

$$\lim_{s\to 0}\left\{\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy\right\}^{1/p}\sim \|f\|_{Dini_p(\Omega)}\,. \tag{1.4}$$

Using our approach we also get a new perspective on recent results in [6][3] on how to recognize constant functions using Sobolev conditions. In our theory these results simply correspond to the statement that certain limiting interpolation spaces are trivial (see section 5).

The paper is organized as follows. In section 2 we establish the continuity of the real method (subsection 2.1) and the complex method (subsection 2.2) at the end points. These results are then used to derive the Bourgain-Brézis-Mironescu-Maz'ya- Shaposhnikova formulae in section 3. The new limiting result (1.4) is then derived in section 4 using extrapolation, and in the final section (section 5) we give our interpretation of the problem on how to recognize constant functions using interpolation scales. A reader mainly interested in (1.1) and (1.2) can move directly from Theorem 1 in subsection 2.1 to sections 3 and 4, while a reader interested mainly in interpolation theory may be especially interested in section 2.

2. Continuity of real and complex interpolation scales at the end points

In this section, which is divided into three parts, we establish the continuity of the real method (subsection 2.1) and the complex method (subsection 2.2) at the end points.[4] In subsection 2.3 we indicate connections with extrapolation theory (cf. [19]) and the problem of computing the distance between interpolation spaces in a given scale (cf. [23]).

2.1. **The real method.** Let $\vec{X}=(X_0,X_1)$ be a given pair of compatible Banach spaces,[5] and let $0<s<1, q\in[1,\infty]$. In the classical literature of interpolation theory (cf. [4], [3], [25]) the real interpolation scale $\vec{X}_{s,q}$ is defined by:

$$\vec{X}_{s,q}=\{f: f\in X_0+X_1 \text{ s.t. } |f|_{\vec{X}_{s,q}}<\infty\},$$

where

$$|f|_{\vec{X}_{s,q}}=\left\{\int_0^\infty \left(t^{-s}K(t,f;\vec{X})\right)^q\frac{dt}{t}\right\}^{1/q}, \tag{2.1}$$

and the "$K-$functional" is defined by

$$K(t,f;\vec{X})=\inf\{\|f_0\|_{X_0}+t\,\|f_1\|_{X_1}: f=f_0+f_1, f_i\in X_i, i=0,1\}. \tag{2.2}$$

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[3]We are very grateful to Haim Brézis for making his preprint [6] available to us.

[4]Although we give a separate treatment for each of these methods, we should note that the method of proof we give for the complex method, combined with the results of [12], give a unified approach to the real (in its $J-$method formulation) and complex methods. See Remark 8 below.

[5]We refer the reader to [4] for background on interpolation theory.

Although the norm (2.1) is, in a suitable sense, continuous for $s \in (0,1)$, it is not continuous at the end points. In fact note that

$$X_0 = (X_0, X_0)_{s,q;K}$$

with

$$\|x\|_{X_0} = (sq(1-s))^{1/q} |x|_{(X_0,X_0)_{s,q;K}}$$

(see [4], Theorem 3.4.1(e), p. 46). To overcome this defect we use the following normalization[6] (cf. [19], p. 19). For $1 \le q < \infty$ let

$$\|f\|_{\vec{X}_{s,q}} = s^{1/q}(1-s)^{1/q} q^{1/q} \left\{ \int_0^\infty \left(t^{-s} K(t,f;\vec{X}) \right)^q \frac{dt}{t} \right\}^{1/q}. \tag{2.3}$$

If $q = \infty$ we let

$$\|f\|_{\vec{X}_{s,\infty}} = \sup_{t>0} t^{-s} K(t,f;\vec{X}). \tag{2.4}$$

In particular, the norm (2.3) has the following monotonicity properties (cf. [19], p. 19):

$$X_0 \cap X_1 \subset \vec{X}_{s,q} \subset X_0 + X_1,\ 0 < s < 1, 1 \le q \le \infty, \tag{2.5}$$

and

$$\vec{X}_{s,q} \subset \vec{X}_{s,p} \text{ for } q \le p. \tag{2.6}$$

Moreover, the embeddings (2.5) and (2.6) have norm one.

In order to compute limits of real interpolation norms at the end points we need one more assumption which holds for $(L^p(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n))$ (see Lemma 2 below) and many familiar pairs of Banach spaces we use in analysis (cf. [7], [3]).

Definition 1. We shall say that a Banach pair $\vec{X}$ is "normal" if the following conditions hold:

$$\lim_{t\to 0} \frac{K(t,f;\vec{X})}{t} = \|f\|_{X_1},\ \text{for } f \in X_1, \tag{2.7}$$

$$\lim_{t\to\infty} K(t,f;\vec{X}) = \|f\|_{X_0},\ \text{for } f \in X_0. \tag{2.8}$$

The Gagliardo closures of X_0 and X_1 are defined by

$$\tilde{X}_0 = \{f \in X_0 + X_1 : \|f\|_{\tilde{X}_0} := \lim_{t\to\infty} K(t,f;\vec{X}) < \infty\}, \tag{2.9}$$

$$\tilde{X}_1 = \{f \in X_0 + X_1 : \|f\|_{\tilde{X}_1} := \lim_{t\to 0} \frac{K(t,f;\vec{X})}{t} < \infty\}. \tag{2.10}$$

We obviously have $X_0 \subset \tilde{X}_0$, $X_1 \subset \tilde{X}_1$, with the norms of the embeddings equal to one. The pair (X_0, X_1) is said to be "mutually closed" if $X_0 = \tilde{X}_0$, $X_1 = \tilde{X}_1$.

Remark 1. In the definition of normal we just need equivalence of norms in (2.7) and (2.8).

Remark 2. Many of the pairs we use in classical analysis are mutually closed, e.g. $(L^p(\mathbb{R}^n), W_0^p(\mathbb{R}^n))$ for $p > 1$; however note that for the pair $(L^1(\mathbb{R}^n), W_0^1(\mathbb{R}^n))$, $(\tilde{W_0^1}) = BV(\mathbb{R}^n)$ (see [9], pp. 217-218). On the other hand $(L^p(\mathbb{R}^n), W_0^p(\mathbb{R}^n))$ is normal for all $p \ge 1$ (cf. [24], Proposition 2.4, and Lemma 2 below).

[6]M. Cwikel pointed out that an analogous normalization was used in [11], p. 252, for the purpose of comparing different methods of interpolation on families of spaces.

Another relevant concept for the study of the continuity of interpolation methods at the end points is given by the category of Banach pairs that are regular.

Definition 2. We shall say that a Banach pair $\vec{X}=(X_0,X_1)$ is a regular pair if

$$X_i = \text{closure of } X_0\cap X_1 \text{ in } X_i, i=0,1.$$

Theorem 1. *Suppose that $\vec{X}=(X_0,X_1)$ is a normal pair. Then*
(i) *For $1\leq q<\infty, f\in X_0\cap X_1$, we have*

$$\lim_{s\to 1}\|f\|_{\vec{X}_{s,q}}=\|f\|_{X_1}.$$

(ii) *For $1\leq q<\infty, f\in X_0\cap X_1$, we have*

$$\lim_{s\to 0}\|f\|_{\vec{X}_{s,q}}=\|f\|_{X_0}.$$

(iii) *If $1\leq q<\infty, f\in X_0\cap\bigcup_{s\in(0,1)}\vec{X}_{s,p}$, then we have*

$$\lim_{s\to 0}\|f\|_{\vec{X}_{s,q}}=\|f\|_{X_0}.$$

Proof. (i) Let $\epsilon>0$. Select $\delta>0$ such that

$$\left|\left(\frac{K(t,f;\vec{X})}{t}\right)^q-\|f\|_{X_1}^q\right|<\epsilon,\ \text{whenever } t<\delta. \tag{2.11}$$

Then

$$\begin{aligned}\left|\|f\|_{\vec{X}_{s,q}}^q-\|f\|_{X_1}^q\right| &= \left|s(1-s)q\left\{\int_0^\infty\left(t^{-s}K(t,f;\vec{X})\right)^q\frac{dt}{t}\right\}-\|f\|_{X_1}^q\right|\\ &= \left|s(1-s)q\left\{\int_0^\infty\left(t^{-s}K(t,f;\vec{X})\right)^q\frac{dt}{t}\right\}\right.\\ &\qquad\left.-\|f\|_{X_1}^q(1-s)q\delta^{-(1-s)q}\int_0^\delta t^{(1-s)q}\frac{dt}{t}\right|.\end{aligned}$$

Now we split the first term

$$\begin{aligned}&s(1-s)q\left\{\int_0^\infty\left(t^{-s}K(t,f;\vec{X})\right)^q\frac{dt}{t}\right\}\\ &\quad= s(1-s)q\left\{\int_0^\delta t^{(1-s)q}\left(\frac{K(t,f;\vec{X})}{t}\right)^q\frac{dt}{t}\right\}\\ &\quad\quad+s(1-s)q\left\{\int_\delta^\infty t^{(1-s)q}\left(\frac{K(t,f;\vec{X})}{t}\right)^q\frac{dt}{t}\right\},\end{aligned}$$

and recombine to get

$$\left|\|f\|_{\vec{X}_{s,q}}^q-\|f\|_{X_1}^q\right|\leq I+II+III,$$

with

$$I = \left| s(1-s)q \int_0^\delta t^{(1-s)q} \left(\left(\frac{K(t,f;\vec{X})}{t} \right)^q - \|f\|_{X_1}^q \right) \frac{dt}{t} \right|,$$

$$II = \left| s(1-s)q \int_0^\delta t^{(1-s)q} \left(\delta^{-(1-s)q} \frac{\|f\|_{X_1}^q}{s} - \|f\|_{X_1}^q \right) \frac{dt}{t} \right|,$$

$$III = \left| s(1-s)q \left\{ \int_\delta^\infty t^{(1-s)q} \left(\frac{K(t,f;\vec{X})}{t} \right)^q \frac{dt}{t} \right\} \right|.$$

We estimate each of these terms. For the first term we use (2.11) to get

$$I \leq \epsilon \delta^{(1-s)q} s. \tag{2.12}$$

The second term is readily seen to be

$$\begin{aligned} II &= \|f\|_{X_1}^q \, s \left| \left(\frac{1}{s} - \delta^{-(1-s)q} \right) \delta^{(1-s)q} \right| \\ &= \|f\|_{X_1}^q \, s \left| \frac{\delta^{(1-s)q}}{s} - 1 \right|. \end{aligned} \tag{2.13}$$

Finally to estimate the third term we use the fact that $K(t,f;\vec{X}) \leq \|f\|_{X_0}$ and find

$$\begin{aligned} III &= s(1-s)q \left\{ \int_\delta^\infty t^{-sq} K(t,f;\vec{X})^q \frac{dt}{t} \right\} \\ &\leq \|f\|_{X_0}^q \, \delta^{-sq}(1-s). \end{aligned} \tag{2.14}$$

From (2.12), (2.13) and (2.14) we see that if s is sufficiently close to 1, we have

$$\left| \|f\|_{\vec{X}_{s,p}}^q - \|f\|_{X_1}^q \right| < \epsilon$$

as we wished to show.

(ii) Consider the pair $R\vec{X} = (X_1, X_0)$. It is well known, and readily seen from the definition, that (cf. [4])

$$K(t,f;R\vec{X}) = tK(\frac{1}{t}, f; \vec{X})$$

and therefore

$$(X_1, X_0)_{1-s,q} = \vec{X}_{s,q}.$$

Moreover, by the symmetry of the definition (2.3),

$$\|f\|_{(X_1,X_0)_{1-s,q}} = \|f\|_{\vec{X}_{s,q}}.$$

Thus

$$\lim_{s\to 0} \|f\|_{\vec{X}_{s,q}} = \lim_{s\to 0} \|f\|_{(X_1,X_0)_{1-s,q}} = \lim_{s\to 1} \|f\|_{(X_1,X_0)_{s,q}} = \|f\|_{X_0}$$

by part (i).

(iii) Let $\epsilon > 0$. Select $\delta > 0$ such that

$$\left| K(t,f;\vec{X})^q - \|f\|_{X_0}^q \right| < \epsilon, \text{ whenever } t > \delta. \tag{2.15}$$

Suppose that $f \in \vec{X}_{s_0,q}$. Write

$$\left| \|f\|_{\vec{X}_{s,q}}^q - \|f\|_{X_0}^q \right| = \left| s(1-s)q \left\{ \int_0^\infty \left(t^{-s} K(t,f;\vec{X}) \right)^q \frac{dt}{t} \right\} - \|f\|_{X_0}^q \right|$$
$$\leq I_1 + I_2 + I_3,$$

where

$$I_1 = s(1-s)q \int_0^\delta (t^{-s}K(t,f;X))^q \frac{dt}{t},$$
$$I_2 = s(1-s)q \int_\delta^\infty t^{-sq} \left| K(t,f;X)^q - \|f\|_{X_0}^q \right| \frac{dt}{t},$$
$$I_3 = \|f\|_{X_0}^q \left((1-s)\delta^{sq} - 1\right).$$

We conclude the proof by showing that each of these three terms converges to zero as $s \to 0$.

To estimate I_1 suppose, as we may, that $s < s_0$, and write

$$\begin{aligned} I_1 &= s(1-s)q\delta^{-sq} \int_0^\delta ((\frac{t}{\delta})^{-s} K(t,f;X))^q \frac{dt}{t} \\ &\leq s(1-s)q\delta^{-sq} \int_0^\delta ((\frac{t}{\delta})^{-s_0} K(t,f;X))^q \frac{dt}{t} \\ &\leq s(1-s)q\delta^{q(s_0-s)} \int_0^\delta (t^{-s_0} K(t,f;X))^q \frac{dt}{t} \\ &\leq s(1-s)q\delta^{q(s_0-s)} \|f\|_{\vec{X}_{s_0,q}}. \end{aligned}$$

It follows that

$$\lim_{s\to 0} I_1 = 0.$$

In view of (2.15) it follows that for s small enough we have

$$I_2 < \epsilon.$$

By simple inspection we see that $I_3 \to 0$ as $s \to 0$. □

Remark 3. If the normality of the pair $\vec{X}$ holds only with equivalence of norms (cf. (1) above), we let $\|f\|_{\tilde{X}_0} = \lim_{t\to\infty} K(t,f,\vec{X})$ and $\|f\|_{\tilde{X}_1} = \lim_{t\to 0} \frac{K(t,f,\vec{X})}{t}$. Then Theorem 1 holds if, in the limiting formulae, we replace the original norms by the new equivalent norms. For example Theorem 1(i) now reads: if $f \in X_0 \cap X_1$, $1 \leq q < \infty$, we have

$$\lim_{s\to 1} \|f\|_{\vec{X}_{s,q}} = \|f\|_{\tilde{X}_1}.$$

Remark 4. Using part (ii), and a sharp version of Holmstedt's reiteration formula, we can derive a slightly less precise version of Theorem 1(iii) with a simpler proof. Indeed, let us show that if $1 \leq q < \infty$, then if $f \in X_0 \cap \vec{X}_{s_0,q}$ for some $s_0 \in (0,1)$, we have

$$\lim_{s\to 0} \|f\|_{\vec{X}_{s,q}} = C \|f\|_{X_0},$$

where C is a constant independent of f.

Proof. (By reiteration) Let $f \in \vec{X}_{s_0,q}$. Recall the sharp form of Holmstedt's reiteration formula (cf. [19], p. 33, [23], Lemma 2.3, [2])

$$K(t, f; X_0, \vec{X}_{s_0,q}) \approx s_0^{1/q}(1-s_0)^{1/q}q^{1/q}t\left\{\int_{t^{1/s_0}}^{\infty}(u^{-s_0}K(u,f;\vec{X}))^q\frac{du}{u}\right\}^{1/q} + K(t^{1/s_0}, f; \vec{X}),$$

with constants of equivalence independent of f. A straightforward, but lengthy, computation then shows that for small s, say $s \leq 1/2$,

$$\vec{X}_{ss_0,q} = (X_0, \vec{X}_{s_0,q})_{s,q} \tag{2.16}$$

with

$$\|g\|_{\vec{X}_{ss_0,q}} \approx C\,\|g\|_{(X_0,\vec{X}_{s_0,q})_{s,q}}\,, \tag{2.17}$$

and $C = C(s_0, p)$ independent of s. Note that $f \in X_0 \cap \vec{X}_{s_0,q}$; therefore, in view of (2.16), we may apply Theorem 1(ii), and use (2.17) to obtain

$$\lim_{s\to 0}\|f\|_{\vec{X}_{ss_0,q}} = C\,\|f\|_{X_0}\,.$$

□

Remark 5. By private communication Georgi Karadzhov observed that by analogous considerations one can show that for $f \in X_0 \cap X_1$,

$$\lim_{s\to 0}\|f\|_{\vec{X}_{s,q}} = \lim_{s\to 0}\|f\|_{\vec{X}_{s,\infty}}\,.$$

Consequently one can also obtain limiting results for Besov spaces with $q = \infty$. We omit the details.

Remark 6. For regular pairs one can use a limiting argument to extend Theorem 1 (cf. Theorem 2 below).

2.2. **The complex method.** We establish the corresponding continuity principle for the complex method of interpolation (cf. [8]). In fact, more generally, we consider interpolation methods $F_s, 0 < s < 1$, that satisfy the following property:

For any Banach pair (X_0, X_1), for all $x \in X_0 \cap X_1$,

$$t^{-s}K(t, x; X_0, X_1) \leq \|x\|_{F_s(X_0,X_1)} \leq \|x\|_{X_0}^{1-s}\|x\|_{X_1}^{s}\,. \tag{2.18}$$

In other words F_s is of "exact of type s" (cf. [4], p. 27, [19], p. 7).

Theorem 2. *Let $\vec{X}$ be a normal Banach pair and let $\{F_s\}_{s\in(0,1)}$ be a family of interpolation methods satisfying* (2.18)*. Let $x \in X_0 \cap X_1$; then*

$$\lim_{s\to j}\|x\|_{F_s(\vec{X})} = \|x\|_{X_j}\,, j = 0, 1. \tag{2.19}$$

Proof. We prove (2.19) for $j = 0$; the corresponding limit for $j = 1$ follows by symmetry. Letting $s \to 0$ in (2.18) and then letting $t \to \infty$, we find

$$\lim_{t\to\infty} K(t, x; X_0, X_1) \leq \underline{\lim}_{s\to 0}\|x\|_{F_s(X_0,X_1)} \leq \overline{\lim}_{s\to 0}\|x\|_{F_s(X_0,X_1)} \leq \|x\|_{X_0}\,.$$

But, since $\vec{X}$ is normal, $\lim_{t\to\infty} K(t, x; X_0, X_1) = \|x\|_{X_0}$, whence (2.19) follows. □

It is well known that the complex method of interpolation of Calderón [8] (cf. also [4]) satisfies (2.18) (cf. [4], p. 102). Therefore we have

Corollary 1. *Let $\vec{X}$ be a normal complex Banach pair. Let $x \in X_0 \cap X_1$; then*

$$\lim_{s \to j} \|x\|_{[X_0,X_1]_s} = \|x\|_{X_j}, j = 0, 1.$$

One can of course state and prove *mutatis mutandis* corresponding limiting results within equivalence.

We now discuss briefly the connection between normality, regularity and Gagliardo completions. In the following discussion we consider a family $\{F_s\}_{s \in (0,1)}$ of interpolation methods of exact type s such that, moreover, for any Banach pair (X_0, X_1) we have

$$F_s(X_0, X_1) = F_s(\tilde{X}_0, \tilde{X}_1), \text{ isometrically.} \tag{2.20}$$

The following corollary holds (the second statement is due to Mastylo by private communication).

Corollary 2. *Let (X_0, X_1) be a Banach pair.*

(i) *Let $x \in X_0 \cap X_1$. Then*

$$\lim_{s \to j} \|x\|_{F_s(\vec{X})} = \|x\|_{\tilde{X}_j}, j = 0, 1.$$

(ii) *Suppose that (X_0, X_1) is regular and such that it holds*

$$\lim_{s \to j} \|x\|_{F_s(\vec{X})} = \|x\|_{X_j}, j = 0, 1, \text{ for all } x \in X_j. \tag{2.21}$$

Then (X_0, X_1) is normal.

Proof. (i) The pair $(\tilde{X}_0, \tilde{X}_1)$ is normal. Therefore, by Theorem 2,

$$\lim_{s \to j} \|x\|_{F_s(\tilde{X}_0, \tilde{X}_1)} = \|x\|_{\tilde{X}_j}, j = 0, 1,$$

and we conclude using (2.20).

(ii) Suppose that (X_0, X_1) is regular and (2.21) holds. We have to show that the pair (X_0, X_1) is normal, that is, $\|x\|_{\tilde{X}_j} = \|x\|_{X_j}$, for all $x \in X_j, j = 0, 1$. By symmetry it is enough to consider the case $j = 0$. Suppose first that $x \in X_0 \cap X_1$. The pair $(\tilde{X}_0, \tilde{X}_1)$ is normal, therefore by Theorem 2 we have

$$\lim_{s \to 0} \|x\|_{F_s(\tilde{X}_0, \tilde{X}_1)} = \|x\|_{\tilde{X}_0}. \tag{2.22}$$

On the other hand, by (2.21) and (2.20),

$$\lim_{s \to 0} \|x\|_{F_s(\tilde{X}_0, \tilde{X}_1)} = \lim_{s \to 0} \|x\|_{F_s(\vec{X})} = \|x\|_{X_0}. \tag{2.23}$$

From (2.22) and (2.23) we see that

$$\|x\|_{\tilde{X}_0} = \|x\|_{X_0}, \text{ for all } x \in X_0 \cap X_1. \tag{2.24}$$

Since (X_0, X_1) is regular we can select $\{x_n\} \subset X_0 \cap X_1$ such that $x_n \to x$ in X_0. Consequently, $x_n \to x$ in $X_0 + X_1$. It follows that $\|x_n - x_m\|_{X_0} = \|x_n - x_m\|_{\tilde{X}_0}$ whence we see that $\{x_n\}$ is Cauchy in $\tilde{X}_0$. By completeness there exists $y \in \tilde{X}_0$ such that $x_n \to y$ in $\tilde{X}_0$. Moreover, since $\tilde{X}_0 \subset X_0 + X_1$, it follows that $x_n \to y$ in $X_0 + X_1$, thus $y = x$. Summarizing, we have

$$\lim_{n \to \infty} \|x_n\|_{X_0} = \|x\|_{X_0}, \ \lim_{n \to \infty} \|x_n\|_{\tilde{X}_0} = \|x\|_{\tilde{X}_0}. \tag{2.25}$$

Therefore since $x_n \in X_0 \cap X_1$, using (2.24) we get that

$$\lim_{n\to\infty} \|x_n\|_{X_0} = \lim_{n\to\infty} \|x_n\|_{\tilde{X}_0}. \tag{2.26}$$

From (2.25) and (2.26) we see that $\|x\|_{\tilde{X}_0} = \|x\|_{X_0}$, for all $x \in X_0$. $\square$

Remark 7. The complex method of interpolation of Calderón satisfies (2.20) (cf. [15])[7] and therefore the previous corollary holds for $F_s(X_0, X_1) = [X_0, X_1]_s$. The second part of the previous corollary admits a converse, but we do not pursue this matter any further here.

The real methods $\vec{X}_{s,q}$ provided with the normalized norms (2.3) are exact interpolation methods of type s (cf. [19]), and they satisfy (2.18) (cf. [9]). The $J-$method of real interpolation, $\vec{X} \to \vec{X}_{s,q;J}$ (cf. [4]), is equivalent to the $(.,.)_{s,q}$ method and thus, in principle, it also satisfies (2.20) and (2.18) with equivalence of norms. We normalize the $J-$method of interpolation by (cf. [19], p. 19)

$$\|.\|_{\vec{X}_{s,q;J}} = ((1-s)sq')^{-1/q'} |.|_{\vec{X}_{s,q;J}}, \tag{2.27}$$

where $|.|_{\vec{X}_{s,q;J}}$ is the usual norm (cf. [4]), and where the value of $((1-s)sq')^{-1/q'}$ at $q = 1$ is, by definition, 1. Provided with the norm (2.27) the $J-$method, $\vec{X} \to \vec{X}_{s,q;J}$, is exact of type s (cf. [19]).

2.3. **Remarks.** We point out further extensions of the results of the previous subsections and make some remarks that could be of interest to interpolation aficionados.

Remark 8. One can also treat in this fashion the interpolation methods introduced in [12]. For the $J-$method it is also easy to check that

$$\vec{X}_{j,1;J} = X_j^\circ = \text{closure of } X_0 \cap X_1 \text{ in } X_0, j = 0, 1$$

(cf. [30]). Therefore, under the assumption that the pair $\vec{X}$ is regular, we have

$$\vec{X}_{j,1;J} = X_j, j = 0, 1. \tag{2.28}$$

It is worthwhile to note that just like the regularity of a pair can be expressed as a limiting reiteration formula for the $J-$method, namely (2.28), the mutual closedness of a pair $\vec{X}$ can be also rewritten as a limiting reiteration formula for the $K-$method,

$$\vec{X}_{j,\infty} = X_j, j = 0, 1. \tag{2.29}$$

Remark 9. In [19] the concept of "complete" interpolation functors was introduced. A family $\{F_s\}_{s\in(0,1)}$ of interpolation functors is complete if for any mutually closed, regular pair $\vec{X}$, and for any linear operator such that $T : F_s(\vec{X}) \to F_s(\vec{X})$ with $\|T\|_{F_s(\vec{X})\to F_s(\vec{X})} \leq 1$ for all $s \in (0, 1)$, we can deduce that $T : \vec{X} \to \vec{X}$ with $\|T\|_{\vec{X}\to\vec{X}} \leq 1$. Complete interpolation scales, and relative complete scales with respect to another scale, were completely characterized in ([19], Theorem 2.5, p. 12). In particular, for each fixed q, $\vec{X} \to \vec{X}_{s,q}$, and $\vec{X} \to \vec{X}_{s,q;J}$ are complete. Theorems 1 and 2 now provide a somewhat different approach to this result. Conversely, we would like to suggest that relative completeness is the underlying theme of the

[7] For the complex method, and other orbital methods of interpolation, (2.20) had been established up to equivalence earlier in [18].

application of Theorem 1 used to derive (1.1) and (1.2) in section 3 below. In particular, the Bourgain et al. limiting formulae could follow from the study of the identity map between the K-method and Lions' method of traces. More precisely, we suggest that the sharp constants of the equivalence of these two methods (cf. [4], p. 73, for a proof of the equivalence, but without paying attention to the sharpness[8] of the constants) and the relative completeness of these scales are underlying these results. The background for this remark comes from the fact that the norm (1.3) can be obtained without constants of equivalence dependent on s using the method of traces (cf. [1], p. 208).

Remark 10. Another interpretation of the results in this section comes from the problem of computing the distance between spaces on a given interpolation scale (cf. [22], [31], [26], and the references therein). For example, in [26] it is established that, with a suitable definition of "distance", for many familiar interpolation functors $\{F_\theta\}_{\theta\in(0,1)}$ (including the methods of [12]) we have

$$d(F_{\theta_0}, F_{\theta_1}) \leq c\,|\theta_0 - \theta_1|\,. \tag{2.30}$$

In [26] no attempt was made to consider the "end points". It is tempting to attempt a proof of Theorem 2 adapting the proof that (2.30) holds for the interpolation method of Cwikel et al., as given in [26], Theorem 16. However the constants involved in the cancellation lemma of [12] (cf. [12], Lemma 3.11, p. 258) blow up as we approach the boundary. Therefore the constants in [26] could also blow up at the boundary. It would be of interest to find the correct extension of (2.30) to the end points. One intriguing question here is if there is a suitable normalization of the Rochberg-Weiss Ω operators (cf. [12], [23], and the references therein) that gives nontrivial commutator results at the end points.[9]

Remark 11. The results in this section extend to quasi-Banach spaces, in a familiar way. Indeed, the real method (in its $K-$formulation) for quasi-Banach spaces requires no changes. For a discussion of the issues that need to be taken care of for the method of the proof of Corollary 1 to work for the complex method in the setting of quasi-Banach spaces, we refer to [21] and also [13].

3. Limiting formulae of Bourgain-Brézis-Mironescu-Maz'ya-Shaposhnikova

Here we discuss in detail our approach to the Bourgain-Brézis-Mironescu-Maz'ya-Shaposhnikova limiting formulae (1.1)-(1.2).

Since C_0^∞ is dense in $L^p(\mathbb{R}^n)$, the pair $(L^p(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n))$ is a Banach pair if we identify functions in $W_0^{1,p}(\mathbb{R}^n)$ differing by a constant. The $K-$functional for the pair $(L^p(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n))$ is well known (cf. [28], p. 286; [3], (4.42), p. 341; [20], especially the discussion after (1.3); [29]):

$$K(t, f; L^p(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n)) \approx \varpi_p(f,t) = \sup_{|h|\leq t} \|f(x+h) - f(x)\|_{L^p(\mathbb{R}^n)}\,,$$

[8] One method to derive sharper constants is to use the sharp constants of equivalence between the K and J methods that can be obtained using the strong form of the fundamental lemma (cf. [19], p. 34), and then the equivalence between the J-method and the method of traces (cf. [4], Theorem 3.12.2, p. 73, and [25], p. 316).

[9] As is well known, without proper normalizations the $\Omega-$commutator theorem fails at the end points.

with constants of equivalence independent of f. Therefore

$$\|f\|_{(L^p(\mathbb{R}^n),W_0^{1,p}(\mathbb{R}^n))_{s,p}} \approx s^{1/p}(1-s)^{1/p}p^{1/p}\{\int_0^\infty [t^{-s}\varpi_p(f,t)]^p \frac{dt}{t}\}^{1/p}. \tag{3.1}$$

The following result is again well known (cf. [4]), but here we strive for precision in the constants of equivalence.

Lemma 1. *Let* $0 < s < 1, 1 \le p < \infty$, *and let*

$$W_0^{p,s}(\mathbb{R}^n) = \textit{closure of } C_0^\infty(\mathbb{R}^n)$$

under the norm

$$\|f\|_{W_0^{s,p}(\mathbb{R}^n)} = \left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy\right\}^{1/p}.$$

Then

$$(L^p(\mathbb{R}^n),W_0^{1,p}(\mathbb{R}^n))_{s,p} = W_0^{s,p}(\mathbb{R}^n), \tag{3.2}$$

with

$$\|f\|_{W_0^{s,p}(\mathbb{R}^n)} \approx (n+sp)^{1/p}(1-s)^{-1/p}s^{-1/p}p^{-1/p}\|f\|_{(L^p(\mathbb{R}^n),W_0^{1,p}(\mathbb{R}^n))_{s,p}}. \tag{3.3}$$

Proof. It will be useful to recall the following well-known fact (cf. [24] and also [32], p. 152):

$$\varpi_p(f,t) \approx \left\{\frac{1}{t^n}\int_{|h|\le t} W_f(h)^p dh\right\}^{1/p}, \tag{3.4}$$

where

$$W_f(h) = \left\{\int_{\mathbb{R}^n} |f(x+h)-f(x)|^p\,dx\right\}^{1/p}.$$

From (3.4) and Fubini we have

$$\int_0^\infty [t^{-s}\varpi_p(f,t)]^p\frac{dt}{t} \approx \frac{1}{n+sp}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy. \tag{3.5}$$

Combining (3.5) and (3.1) gives (3.3). □

The next auxiliary result states that the pair $(L^p(\mathbb{R}^n),W_0^{1,p}(\mathbb{R}^n))$ is normal (see Definition 1 above). The result must surely be known, but since we lack precise references and given that it plays an important role in this note, we include a detailed proof.

Lemma 2. *The pair* $(L^p(\mathbb{R}^n),W_0^{1,p}(\mathbb{R}^n)), p \ge 1$, *is normal.*

Proof. Using a Taylor approximation of order one (in the case of one variable use the mean value theorem), we can easily see that

$$\lim_{t\to 0}\frac{\varpi_{f,p}(t)}{t} = \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

(see [24] for a detailed proof). So it remains to prove (2.8). We actually show that

$$\lim_{t\to\infty}\varpi_{f,p}(t) \approx \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.6}$$

Note that we always have

$$\varpi_{f,p}(t) \leq 2 \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.7}$$

Therefore

$$\lim_{t\to\infty} \varpi_{f,p}(t) \leq 2 \|f\|_{L^p(\mathbb{R}^n)}.$$

To prove the converse inequality suppose first that $f \in C_0^\infty(\mathbb{R}^n)$, and let $\operatorname{supp}(f) \subset \{x \in \mathbb{R}^n : |x| < M\}$. Then, for $|h| \geq 2M$, $|x| < M$, we have $f(x+h) = 0$. It follows that

$$\varpi_{f,p}(t) \geq \|f\|_{L^p(\mathbb{R}^n)} \text{ for } t \geq 2M.$$

Note, moreover, that $\varpi_{f,p}(t)$ is monotone increasing. Therefore, $\lim_{t\to\infty} \varpi_{f,p}(t)$ exists and

$$\|f\|_{L^p(\mathbb{R}^n)} \leq \lim_{t\to\infty} \varpi_{f,p}(t) \leq 2 \|f\|_{L^p(\mathbb{R}^n)}.$$

If $f \in L^p(\mathbb{R}^n)$, we use a standard approximation argument. Select $f_\epsilon \in C_0^\infty(\mathbb{R}^n)$ such that

$$\lim_{\epsilon\to 0} \|f - f_\epsilon\|_{L^p(\mathbb{R}^n)} = 0, \lim_{\epsilon\to 0} \|f_\epsilon\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}.$$

By the first part of the argument

$$\begin{aligned}
\|f_\epsilon\|_{L^p(\mathbb{R}^n)} &\leq \lim_{t\to\infty} \varpi_{f_\epsilon,p}(t) \\
&\leq \lim_{t\to\infty} \varpi_{f_\epsilon - f,p}(t) + \lim_{t\to\infty} \varpi_{f,p}(t) \\
&\leq 2 \|f - f_\epsilon\|_{L^p(\mathbb{R}^n)} + \lim_{t\to\infty} \varpi_{f,p}(t).
\end{aligned}$$

Letting $\epsilon \to 0$, we find

$$\|f\|_{L^p(\mathbb{R}^n)} \leq \lim_{t\to\infty} \varpi_{f,p}(t),$$

which combined with (3.7) gives (3.6). □

3.1. **Bourgain-Brézis-Mironescu formula.** We may now apply Theorem 1(i) to the pair $(L^p(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n))$. Suppose that $f \in W_0^{1,p}(\mathbb{R}^n)$; then

$$\begin{aligned}
&\lim_{s\to 1}(1-s)^{1/p} \left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} dxdy \right\}^{1/p} \\
&\quad = \lim_{s\to 1}(1-s)^{1/p} \|f\|_{W_0^{s,p}(\mathbb{R}^n)} \\
&\quad \approx \lim_{s\to 1}(n+sp)^{1/p} s^{-1/p} p^{-1/p} \|f\|_{(L^p(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n))_{s,p}} \\
&\quad = (n+p)^{1/p} p^{-1/p} \|f\|_{W_0^{1,p}(\mathbb{R}^n)} \\
&\quad = (n+p)^{1/p} p^{-1/p} \|\nabla f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Similar results also hold for Sobolev spaces defined on suitably smooth bounded domains[10] Ω. To be more explicit about the connection with interpolation, recall that in [20] the following formula is given for smooth domains:

$$K(t, f; L^p(\Omega), W^{1,p}(\Omega)) \approx \sup_{|h|\leq t} \left\| (f(x+h) - f(x)) \chi_{\Omega(h)}(x) \right\|,$$

where

$$\|f\|_{W^{1,p}(\Omega)} = \|\nabla f\|_{L^p(\Omega)}$$

[10] An explicit procedure for this deduction is given in detail in [27].

and for $h \in \mathbb{R}^n$,

$$\Omega(h) = \{x \in \Omega : x + th \in \Omega \text{ for } 0 \leq t \leq 1\}.$$

Here $(L^p(\Omega), W^{1,p}(\Omega))$ can be considered a Banach pair if we work modulo constants. Again it is readily seen that $(L^p(\Omega), W^{1,p}(\Omega))$ is normal and

$$(L^p(\Omega), W^{1,p}(\Omega))_{s,p} = W^{s,p}(\Omega).$$

3.2. **Maz'ya-Shaposhnikova formula.** In a similar fashion we can also derive a version of the Maz'ya-Shaposhnikova formula (1.2). In fact let $f \in \bigcup_{s\in(0,1)} W_0^{s,p}(\mathbb{R}^n)$. From (1.3), (3.3) and Theorem 1(iii), we have

$$\begin{aligned}\lim_{s\to 0} s^{1/p} \left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} dxdy \right\}^{1/p} &= \lim_{s\to 0} s^{1/p} \|f\|_{W_0^{s,p}(\mathbb{R}^n)} \\ &\approx p^{-1/p} n^{1/p} \lim_{s\to 0} \|f\|_{(L^p(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n))_{s,p}} \\ &= p^{-1/p} n^{1/p} \|f\|_{L^p(\mathbb{R}^n)} .\end{aligned}$$

3.3. **Application.** There are several possible generalizations of (1.1) and (1.2) to the setting of semigroups. Such results rely on the real method and follow the pattern of subsections 3.1 and 3.2. There is a set of different applications to semigroups which are connected with the complex method. In this vein we now consider a limiting formula for fractional powers.

Let X be a reflexive Banach space X and let A be a densely defined positive operator on X, i.e. $A : D(A) \subset X \to X$, and

$$\left\|(A+\lambda)^{-1}\right\|_X \leq \frac{C}{1+\lambda}, \text{ for } \lambda \geq 0.$$

Moreover, assume that there exists $\epsilon > 0$ such that for $t \in (-\epsilon, \epsilon)$ the operators A^{it} are well-defined bounded operators with $\sup_{t\in(-\epsilon,\epsilon)} \left\|A^{it}\right\| < \infty$. Let $\|x\|_{D(A)} = \|Ax\|_X$. Then it is well known (cf. [33], 1.15.3, p. 103) that

$$[X, D(A)]_s = D(A^s)$$

with

$$\|x\|_{[X,D(A)]_s} \approx \|A^s x\|_X ,$$

and (cf. [33], (6), p. 100)

$$A^s x = \frac{1}{\Gamma(s)\Gamma(1-s)} \int_0^\infty \lambda^s A(A+\lambda)^{-1} x \frac{d\lambda}{\lambda}, x \in D(A).$$

Since A is a densely defined invertible operator, it follows that $(X, D(A))$ is a normal, regular pair (cf. [4], p. 159). Therefore, from Corollary 1, we find that for $x \in D(A)$,

$$\lim_{s\to 1} \left\| \frac{1}{\Gamma(s)\Gamma(1-s)} \int_0^\infty \lambda^s A(A+\lambda)^{-1} x \frac{d\lambda}{\lambda} \right\|_A = \|x\|_{D(A)} , \tag{3.8}$$

$$\lim_{s\to 0} \left\| \frac{1}{\Gamma(s)\Gamma(1-s)} \int_0^\infty \lambda^s A(A+\lambda)^{-1} x \frac{d\lambda}{\lambda} \right\|_A = \|x\|_X . \tag{3.9}$$

4. Extrapolation

We now wish to indicate further applications of our ideas and consider limits of norms with "decay". The limits in this case will turn out to be "extrapolation spaces" (introduced in [17] and [19]). Since we are mainly interested in showing the connections, we will not consider here the most general results.[11] On the other hand we discuss a limiting result for Sobolev spaces in detail.

The results take a particularly simple form if we make an additional assumption. We consider "ordered pairs" $\vec{X} = (X_0, X_1)$, that is, we assume that $X_1 \subset X_0$. Furthermore it will be convenient to assume that the norms of the spaces have been normalized so that the norm of the embedding $X_1 \subset X_0$ is less than or equal to one.

For ordered pairs, a different normalization, that also comes from [19], will be useful to deal with limits of the corresponding real interpolation spaces. Let $s \in (0,1), q \in [1,\infty]$; then we let

$$[f]_{\vec{X}_{s,q}} = s^{1/q}(1-s)^{1/q}q^{1/q}\left\{\int_0^1 (t^{-s}K(t,f;\vec{X}))^q \frac{dt}{t}\right\}^{1/q}. \tag{4.1}$$

For $s = 0$ we let

$$[f]_{\vec{X}_{0,q}} := q^{1/q}\left\{\int_0^1 K(t,f;\vec{X})^q \frac{dt}{t}\right\}^{1/q}. \tag{4.2}$$

Lemma 3 (cf. [19], [23])**.** *Suppose that $\vec{X}$ is an ordered pair. Then $\vec{X}_{s,q}$ can be equivalently renormed using $[f]_{\vec{X}_{s,q}}$; in fact we have*

$$[f]_{\vec{X}_{s,q}} \leq \|f\|_{\vec{X}_{s,q}} \leq (1 + s^{-1/q}(1-s)^{1/q})\,[f]_{\vec{X}_{s,q}}. \tag{4.3}$$

Proof. It follows directly from the definition of the $K-$functional (cf. (2.2)) that

$$K(t,f;\vec{X}) \equiv \|f\|_{X_0}, \text{ for } t \geq 1. \tag{4.4}$$

Therefore

$$\|f\|_{\vec{X}_{s,q}} = [f]_{\vec{X}_{s,q}} + \|f\|_{X_0}(1-s)^{1/q}.$$

By (4.4) we can write $\|f\|_{X_0} = \frac{K(1,f;\vec{X})}{1}$. Then, using the fact that $K(t,f;\vec{X})/t$ is decreasing, we get

$$\begin{aligned}[f]_{\vec{X}_{s,q}} &\geq \|f\|_{X_0} s^{1/q}(1-s)^{1/q}q^{1/q}\left\{\int_0^1 t^{q-sq}\frac{dt}{t}\right\}^{1/q} \\ &= \|f\|_{X_0} s^{1/q}.\end{aligned}$$

Therefore

$$\|f\|_{\vec{X}_{s,q}} \leq [f]_{\vec{X}_{s,q}} + s^{-1/q}(1-s)^{1/q}\,[f]_{\vec{X}_{s,q}},$$

as we wished to show. □

With the normalization[12] (4.1) in hand we can now show the following prototype limit theorem with "decay".

[11] We will develop this point in detail elsewhere.

[12] It is important to note that, since we are using a new normalization, Theorem 3 does not follow directly from Theorem 1.

Theorem 3. *Suppose that $\vec{X}$ is an ordered pair, let $1 \leq q < \infty$. Then, for all $f \in \bigcup_{s\in(0,1)} \vec{X}_{s,q}$,*

$$\lim_{s\to 0} \frac{1}{s^{1/q}} [f]_{\vec{X}_{s,q}} = [f]_{\vec{X}_{0,q}}. \tag{4.5}$$

Proof. By (4.1) we have

$$\lim_{s\to 0} \frac{1}{s^{1/q}} [f]_{\vec{X}_{s,q}} = \lim_{s\to 0} (1-s)^{1/q} q^{1/q} \left\{ \int_0^1 [t^{-s} K(t,f;\vec{X})]^q \frac{dt}{t} \right\}^{1/q}. \tag{4.6}$$

Suppose that $f \in \bigcup_{s\in(0,1)} \vec{X}_{s,q}$. Therefore there exists $s_0 \in (0,1)$ such that

$$\int_0^1 [t^{-s_0} K(t,f;\vec{X})]^q \frac{dt}{t} < \infty.$$

Consequently we have $t^{-s}K(t,f;\vec{X}) \in L^q(\frac{dt}{t},(0,1))$ for all $s \in [0,s_0]$. We may thus apply dominated convergence,

$$\lim_{s\to 0} (1-s)^{1/q} q^{1/q} \left\{ \int_0^1 [t^{-s} K(t,f;\vec{X})]^q \frac{dt}{t} \right\}^{1/q} = q^{1/q} \left\{ \int_0^1 K(t,f;\vec{X})^q \frac{dt}{t} \right\}^{1/q},$$

which, combined with (4.6), gives the desired result. □

Remark 12. It is important to stress the important role that the normalizations (4.1)-(4.2) play in this computation. Indeed, while (4.1) is equivalent to usual interpolation norm, the constants associated with this equivalence blow up as $s \to 0$ (cf. (4.3)). In fact, a limiting condition of the form

$$\int_0^\infty K(t,f;\vec{X})^q \frac{dt}{t} < \infty$$

can only hold for $f = 0$, while the generalized Dini condition[13]

$$\int_0^1 K(t,f;\vec{X})^q \frac{dt}{t} < \infty$$

is not trivial. Concerning this point see also section 5 below.

Example 1. Let $X_0 = L^1(\Omega), X_1 = L^\infty(\Omega)$, with $|\Omega| = 1$. It is well known that (cf. [3])

$$K(t,f;L^1,L^\infty) = tf^{**}(t) = \int_0^t f^*(s)ds,$$

where f^* is the nonincreasing rearrangement of f. Therefore,

$$\lim_{s\to 0} \frac{[f]_{(L^1(\Omega),L^\infty(\Omega))_{s,1}}}{s} = \|f\|_{LLogL}.$$

Proof. We have

$$\begin{aligned} \lim_{s\to 0} \frac{[f]_{(L^1(\Omega),L^\infty(\Omega))_{s,1}}}{s} &= \lim_{s\to 0} (1-s) \int_0^1 t^{1-s} f^{**}(t) \frac{dt}{t} \\ &= \int_0^1 f^{**}(t)dt \\ &= \|f\|_{L(LogL)(\Omega)}. \end{aligned}$$

□

[13] This generalized Dini condition was actually introduced by Peetre many years ago.

Example 2. Let Ω be a smooth open domain in $\mathbb{R}^n$ with $|\Omega| = 1$. Let $X_0 = L^p(\Omega), X_1 = W^{1,p}(\Omega)$. Then,

$$\lim_{s\to 0}\left\{\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy\right\}^{1/p} \approx \left\{\int_0^1 \varpi_p(f,t)^p\frac{dt}{t}\right\}^{1/p}. \tag{4.7}$$

Proof. The analogue of (3.5) for domains is

$$\int_0^1 [t^{-s}\varpi_p(f,t)]^p\frac{dt}{t} \approx \frac{1}{n+sp}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy.$$

Therefore

$$\begin{aligned}\lim_{s\to 0}\left\{\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy\right\}^{1/p} &= \lim_{s\to 0} s^{-1/p}(n+sp)^{1/p}(1-s)^{-1/p}p^{-1/p}\,[f]_{\vec{X}_{s,p}} \\ &= n^{1/p}p^{-1/p}\,[f]_{\vec{X}_{0,p}}.\end{aligned}$$

The condition

$$\int_0^1 \varpi_p(f,t)^p\frac{dt}{t} < \infty$$

is known as a Dini condition. □

Remark 13. To understand the previous example one should compare the resulting formula with the Maz'ya-Shaposhnikova formula (1.2). Our result can be written as

$$\lim_{s\downarrow 0}\|f\|_{W^{s,p}(\Omega)} = C\,\|f\|_{Dini_p(\Omega)}.$$

Now $\|.\|_{W^{s,p}(\Omega)}$ and the interpolation norm differ by the decay factor $s^{-1/p}$. In terms of the interpolation norms the extra decay $s^{-1/p}$ forces some extra regularity to compensate, and our limiting result reflects this fact. Thus we must replace the L^p norm with a stronger one but still weaker than all the Lip conditions in the scale!

Remark 14. Of course more general decay rates can be considered. For example, the rate of decay $s^{-\alpha}$ leads to Dini-type conditions with logarithmic weights. We refer to [19] and [23] for descriptions of the spaces that will then appear as limiting spaces.

5. On how to recognize constant functions

The problem treated in [6], on how to recognize constant functions using Sobolev conditions, is closely related to the material treated in this paper. Again we only illustrate the ideas here by means of treating the easiest example in [6]. Suppose that Ω is an open bounded connected set in $\mathbb{R}^n$, and let f be such that

$$\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+p}}dxdy < \infty. \tag{5.1}$$

Then f is constant. To see this note that (5.1) is equivalent[14] (cf. (3.5)) to

$$\int_0^1 t^{-p}\varpi_p(f,t)^p\frac{dt}{t} < \infty. \tag{5.2}$$

[14]Suppose without loss of generality that $|\Omega| = 1$.

Now this condition cannot hold unless $\varpi_p(f,t) = 0$ for all t, that is,

$$(5.2) \text{ holds if and only if } f = \text{ constant}.$$

In fact, if ϖ is any positive function such that $\varpi(t)/t \downarrow$, then $\int_0^1 t^{-p}\varpi(t)^p \frac{dt}{t} < \infty$ implies (formally) that for any $a \in (0,1)$

$$\begin{aligned} \infty &> \int_0^a t^{-p}\varpi(t)^p \frac{dt}{t} \\ &\geq \frac{\varpi(a)^p}{a^p} \int_0^a \frac{dt}{t}. \end{aligned}$$

More generally, this discussion corresponds to the fact that, in general, the interpolation scale $\vec{X}_{1,p}$ is trivial unless, of course, $p = \infty$, in which case we are back to the situation described in (2.29), namely, if $\vec{X}$ is mutually closed, then

$$\vec{X}_{1,\infty} = X_1.$$

One can now formulate conditions under which the interpolation spaces $\vec{X}_{w,p}$, which are defined by means of replacing the power weight t^{-s} in (2.1) by a general weight $w(t)$, are trivial. Such results, when specialized to the Sobolev scale treated in this paper, would yield conditions allowing us to recognize constant functions in the Sobolev scale. We hope to return to this point elsewhere.

Acknowledgement

I am grateful to Georgi Karadzhov, Mieczysław Mastyło and Jie Xiao for their careful reading of the manuscript and for a number of suggestions and corrections. In particular, one of Georgi's suggestions led to a streamlined version of Theorem 1, and Mieczysław pointed out a gap (and a fix!) in the original proof of Theorem 2.

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