Decentralized Receding Horizon Control for Large Scale Dynamically Decoupled Systems ⋆

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Abstract

We present a detailed study on the design of decentralized Receding Horizon Control (RHC) schemes for decoupled systems. We formulate an optimal control problem for a set of dynamically decoupled systems where the cost function and constraints couple the dynamical behavior of the systems. The coupling is described through a graph where each system is a node and, cost and constraints of the optimization problem associated with each node are only function of its state and the states of its neighbors. The complexity of the problem is addressed by breaking a centralized RHC controller into distinct RHC controllers of smaller sizes. Each RHC controller is associated with a different node and computes the local control inputs based only on the states of the node and of its neighbors. We analyze the properties of the proposed scheme and introduce sufficient stability conditions based on prediction errors. Finally, we focus on linear systems and show how to recast the stability conditions into a set of matrix semi-definiteness tests.

Key words: Receding horizon control; Decentralized predictive control; Decoupled subsystems.

1 Introduction

Research on decentralized control dates back to the pioneering work of Wang and Davison in [26] and since then, the interest has grown significantly. Decentralized control techniques today can be found in a broad spectrum of applications ranging from robotics and formation flight to civil engineering. Such a wide interest makes a survey of all the approaches that have appeared in the literature very difficult and goes also beyond the scope of this paper. Approaches to decentralized control design differ from each other in the assumptions they make on: (*i*) the kind of interaction between different systems or different components of the same system (dynamics, constraints, objective), (*ii*) the model of the system (linear, nonlinear, constrained, continuous-time, discrete-time), (*iii*) the model of information exchange between the systems, and (*iv*) the control design technique used.

Dynamically coupled systems have been the most studied [6, 8, 23, 25, 26]. In particular the authors in [6, 11] consider distributed min-max model predictive control problems for large scale systems with coupled linear time-invariant dynamics and propose a scheme using stability-constraints and assuming a one-step communication delay. Sufficient conditions for stability with information exchange between the local controllers are given by treating neighboring subsystem states as diminishing disturbances.

In this paper, we focus on *decoupled systems*. Our interest in decentralized control for dynamically decoupled systems arises from the abundance of networks of independently actuated systems and the necessity of avoiding centralized design when this becomes computationally prohibitive. Networks of vehicles in formation, production units in a power plant, network of cameras at an airport, mechanical actuators for deforming surface are just a few examples.

In a descriptive way, the problem of decentralized control for decoupled systems can be formulated as follows. A dynamical system is composed of (or can be decomposed into) distinct dynamical subsystems that can be independently actuated. The subsystems are dynamically decoupled but have common objectives and con-

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straints which make them interact between each other. Typically the *interaction* is local, i.e., the objective and the constraints of a subsystem are function of only a subset of other subsystems' states. The interaction will be represented by an "interaction graph", where the nodes represent the subsystems and an edge between two nodes denotes a coupling term in the objectives and/or in the constraints associated with the nodes. Also, typically it is assumed that the *exchange of information* has a special structure, i.e., it is assumed that each subsystem can sense and/or exchange information with only a subset of other subsystems. Often the *interaction graph* and the *information exchange graph* coincide. A decentralized control scheme consists of distinct controllers, one for each subsystem, where the inputs to each subsystem are computed only based on local information, i.e., on the states of the subsystem and its neighbors.

In this paper we make use of Receding Horizon Control (RHC) schemes. The main idea of RHC is to use the *model* of the plant to *predict* the future evolution of the system [19]. Based on this prediction, at each time step t a certain performance index is optimized under operating constraints with respect to a sequence of future input moves. The first of such optimal moves is the *control* action applied to the plant at time t. At time $t+1$, a new optimization is solved over a shifted prediction horizon.

A decentralized RHC scheme for decoupled systems has recently appeared in [9], which uses a continuous time formulation with unconstrained subsystems (decoupled input constraints only). The authors make use of information exchange between subsystems and establish lower bounds on the update rate and upper bounds on the mismatch between actual and predicted state evolutions, which are sufficient for stability of the overall system. A different, sequential information exchange scheme can be found in [21, 22], which is valid for a special graph structure based on a leader-follower architecture. In this latter work, authors formulate their problem in discrete time using robust RHC and assume the presence of coupling constraints between subsystems.

In this manuscript we propose a rigorous mathematical framework for designing decentralized receding horizon controllers in discrete time. In our framework a centralized RHC controller is broken into distinct RHC controllers of smaller sizes. Each RHC controller is associated with a different node and computes the local control inputs based only on the states of the node and of its neighbors. We take explicitly into account constraints and use the model of the neighbors to predict their behavior. In the second part of the paper, we analyze the properties of the proposed scheme and introduce sufficient stability conditions. Such conditions *(i)* highlight the role of prediction errors between neighbors in the stability of the overall system, *(ii)* are local to each node and function only of neighboring nodes that can be reached trough at most two edges, thus leading to complexity reduction for interconnection graphs of large diameter, and *(iii)* help understand the importance of information exchange between neighbors and its role in stabilizing the entire system. Finally, in the last part of the paper we focus on linear systems and show how to recast the stability conditions into a set of semi-definiteness tests.

2 Problem formulation

We consider a set of N_v linear decoupled dynamical systems, the i-th system being described by the discrete-time time-invariant state equation:

$$x^i_{k+1} = f^i(x^i_k, u^i_k), \tag{1}$$

where $x^i_k \in \mathbb{R}^{n^i}$, $u^i_k \in \mathbb{R}^{m^i}$, $f^i : \mathbb{R}^{n^i} \times \mathbb{R}^{m^i} \to \mathbb{R}^{n^i}$ are state, input and state update function of the i-th system, respectively. Let $\mathcal{X}^i \subseteq \mathbb{R}^{n^i}$ and $\mathcal{U}^i \subseteq \mathbb{R}^{m^i}$ denote the set of feasible states and inputs of the i-th system, respectively:

$$x^i_k \in \mathcal{X}^i, \;\; u^i_k \in \mathcal{U}^i, \; k \geq 0, \tag{2}$$

where $\mathcal{X}^i$ and $\mathcal{U}^i$ are given polytopes.

We will refer to the set of N_v constrained systems as the *overall system*. Let $\tilde{x}_k \in \mathbb{R}^{\tilde{n}}$ with $\tilde{n} = \sum_i n^i$ and $\tilde{u}_k \in \mathbb{R}^{\tilde{m}}$ with $\tilde{m} = \sum_i m^i$ be the vectors which collect the states and inputs of the overall system at time k, i.e., $\tilde{x}_k = [x^1_k, \ldots, x^{N_v}_k]$, $\tilde{u}_k = [u^1_k, \ldots, u^{N_v}_k]$, with

$$\tilde{x}_{k+1} = f(\tilde{x}_k, \tilde{u}_k), \tag{3}$$

where $f(\tilde{x}_k, \tilde{u}_k) = [f^1(x^1_k, u^1_k), \ldots, f^{N_v}(x^{N_v}_k, u^{N_v}_k)]$ and $f : \mathbb{R}^{\tilde{n}} \times \mathbb{R}^{\tilde{m}} \to \mathbb{R}^{\tilde{n}}$. We denote by (x^i_e, u^i_e) the equilibrium pair of the i-th system and $(\tilde{x}_e, \tilde{u}_e)$ the corresponding equilibrium for the overall system. The state update functions f^i are assumed to be continuous and stabilizable at the equilibrium for all $i = 1, \ldots, N_v$.

So far the subsystems belonging to the overall system are completely decoupled. We consider an optimal control problem for the overall system where cost function and constraints couple the dynamic behavior of individual systems. We use a graph structure to represent the coupling in the following way. We associate the i-th system to the i-th node of the graph, and if an edge (i, j) connecting the i-th and j-th node is present, then the cost and the constraints of the optimal control problem will have coupling terms, which are functions of both x^i and x^j. The graph will be defined as

$$\mathcal{G} = \{\mathcal{V}, \mathcal{A}\}, \tag{4a}$$

$$\mathbf{A}(i, j) = \begin{cases} 1, \text{ if } (i, j) \in \mathcal{A} \\ 0, \text{ otherwise} \end{cases} \tag{4b}$$

where $\mathcal{V}$ is the set of nodes $\mathcal{V} = \{1, \ldots, N_v\}$, $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$ is the sets of edges (i, j) with $i \in \mathcal{V},\ j \in \mathcal{V}$ and $\mathbf{A} \in \mathbb{R}^{N_v \times N_v}$ is the adjacency matrix of the graph.

Remark 1 *Often the graph edges are chosen to be time-varying, based on a particular neighbor selection policy. For instance in the case of formation flight, the interconnection graph is full due to collision avoidance (since each vehicle has to avoid every other), but it is usually replaced with a time-varying "closest spatial neighbor" relationship. For simplicity, a time-invariant graph structure is assumed throughout the paper. References to decentralized RHC using time-varying graphs can be found in [5].*

Once the graph structure has been fixed, the optimization problem is formulated as follows. Denote with $\tilde{x}^i$ the states of all neighboring systems of the i-th system, i.e., $\tilde{x}^i = \{x^j \in \mathbb{R}^{n^j} | (i,j) \in \mathcal{A}\}$, $\tilde{x}^i \in \mathbb{R}^{\tilde{n}^i}$ with $\tilde{n}^i = \sum_{j|(i,j)\in\mathcal{A}} n^j$. Analogously, $\tilde{u}^i \in \mathbb{R}^{\tilde{m}^i}$ denotes the inputs to all the neighboring systems of the i-th system, and $(\tilde{x}^i_e, \tilde{u}^i_e)$ represent their equilibria. Let

$$g^{i,j}(x^i, x^j) \leq 0 \tag{5}$$

define the coupling constraints between the i-th and the j-th systems, where $(i,j) \in \mathcal{A}$, with $g^{i,j} : \mathbb{R}^{n^i} \times \mathbb{R}^{n^j} \to \mathbb{R}^{nc^{i,j}}$ being a continuous function. We will often use the following shorter form of the coupling constraints defined between the i-th system and all its neighbors:

$$g^i(x^i, \tilde{x}^i) \leq 0, \tag{6}$$

with $g^i : \mathbb{R}^{n^i} \times \mathbb{R}^{\tilde{n}^i} \to \mathbb{R}^{nc^i}$, where $nc^i = \sum_{j|(i,j)\in\mathcal{A}} nc^{i,j}$.

Consider the following cost:

$$l(\tilde{x}, \tilde{u}) = \sum_{i=1}^{N_v} l^i(x^i, u^i, \tilde{x}^i, \tilde{u}^i), \tag{7}$$

where $l^i : \mathbb{R}^{n^i} \times \mathbb{R}^{m^i} \times \mathbb{R}^{\tilde{n}^i} \times \mathbb{R}^{\tilde{m}^i} \to \mathbb{R}$ is the cost associated with the i-th system and is a function only of its states and the states of its neighbor nodes.

$$l^i(x^i, u^i, \tilde{x}^i, \tilde{u}^i) = l^{i,i}(x^i, u^i) + \sum_{(i,j)\in\mathcal{A}} l^{i,j}(x^i, u^i, x^j, u^j) \tag{8}$$

where $l^{i,i} : \mathbb{R}^{n^i} \times \mathbb{R}^{m^i} \to \mathbb{R}$ and $l^{i,j} : \mathbb{R}^{n^i} \times \mathbb{R}^{m^i} \times \mathbb{R}^{n^j} \times \mathbb{R}^{m^j} \to \mathbb{R}$ is the cost function for two adjacent nodes. We assume throughout the paper that $l^{i,i}$ and $l^{i,j}$ are positive convex continuous functions such that $l^{i,j}(x^i, u^i, \tilde{x}^i, \tilde{u}^i) \geq c\|(x^i, u^i, \tilde{x}^i, \tilde{u}^i)\|_2$, $c > 0$ and that $l^i(x^i_e, u^i_e, \tilde{x}^i_e, \tilde{u}^i_e) = 0$.

We design a controller by repeatedly solving finite time optimal control problems in a receding horizon fashion as described next. At each sampling time, starting at the current state, an open-loop optimal control problem is solved over a finite horizon. The optimal command signal is applied to the process only during the following sampling interval. At the next time step a new optimal control problem based on new measurements of the state is solved over a shifted horizon. The resultant controller is often referred to as Receding Horizon Controller (RHC). Assume at time t the current state $\tilde{x}_t$ to be available. Consider the following constrained finite time optimal control problem:

$$\tilde{J}^*_N(\tilde{x}_t) \triangleq \min_{\tilde{U}_t} \sum_{k=0}^{N-1} l(\tilde{x}_{k,t}, \tilde{u}_{k,t}) + l_N(\tilde{x}_{N,t}) \tag{9a}$$

$$\text{subj. to } x^i_{k+1,t} = f^i(x^i_{k,t}, u^i_{k,t}), \quad i = 1, \ldots, N_v, \quad k \geq 0 \tag{9b}$$

$$x^i_{k,t} \in \mathcal{X}^i, \quad u^i_{k,t} \in \mathcal{U}^i, \quad i = 1, \ldots, N_v, \quad k = 1, \ldots, N-1 \tag{9c}$$

$$g^{i,j}(x^i_{k,t}, x^j_{k,t}) \leq 0, \quad i = 1, \ldots, N_v, \quad (i,j) \in \mathcal{A}, \quad k = 1, \ldots, N-1 \tag{9d}$$

$$\tilde{x}_{N,t} \in \mathcal{X}_f, \tag{9e}$$

$$\tilde{x}_{0,t} = \tilde{x}_t, \tag{9f}$$

where N is the prediction horizon, $\mathcal{X}_f \subseteq \mathbb{R}^{\tilde{n}}$ is a terminal region, l_N is the cost on the terminal state. In (9) $\tilde{U}_t \triangleq [\tilde{u}_{0,t}, \ldots, \tilde{u}_{N-1,t}] \in \mathbb{R}^s$, $s \triangleq \tilde{m}N$ denotes the optimization vector, $x^i_{k,t}$ denotes the state vector of the i-th node predicted at time $t + k$ obtained by starting from the state x^i_t and applying to system (1) the input sequence $u^i_{0,t}, \ldots, u^i_{k-1,t}$.

Let $\tilde{U}^*_t = [\tilde{u}^*_{0,t}, \ldots, \tilde{u}^*_{N-1,t}]$ be an optimal solution of (9) at time t. Then, the first sample of $\tilde{U}^*_t$ is applied to the overall system (3)

$$\tilde{u}_t = \tilde{u}^*_{0,t}. \tag{10}$$

The optimization (9) is repeated at time $t+1$, based on the new state $\tilde{x}_{t+1}$.

It is well known that stability is not ensured by the RHC law (9)–(10). Usually the terminal cost l_N and the terminal constraint set $\mathcal{X}_f$ are chosen to ensure closed-loop stability. A treatment of sufficient stability conditions goes beyond the scope of this work and can be found in the surveys [7, 18, 19]. We assume that the reader is familiar with the basic concept of RHC and its main issues, we refer to the above references for a comprehensive treatment of the topic. In general, the optimal input u^i_t to the i-th system computed by solving (9) at time t, will be a function of the overall state information $\tilde{x}_t$. In the next section we propose a way to decentralize the RHC problem defined in (9)-(10).

3 Decentralized control scheme

We address the complexity associated with a centralized optimal control design for the class of large scale decoupled systems described in the previous section by formulating N_v decentralized finite time optimal control problems, each one associated with a different node. Each node has information about its current states and its neighbors' current states. Based on such information, each node computes its optimal inputs and its neighbors' optimal inputs. The input to the neighbors will only be used to predict their trajectories and then discarded, while the first component of the optimal input to the node will be implemented where it was computed.

A more formal description follows. Let the following finite time optimal control problem $\mathcal{P}_i$ with optimal value function $J_N^{i*}(x_t^i, \tilde{x}_t^i)$ be associated with the i-th system at time t:

$$\min_{\tilde{U}_t^i} \sum_{k=0}^{N-1} l^i(x_{k,t}^i, u_{k,t}^i, \tilde{x}_{k,t}^i, \tilde{u}_{k,t}^i) + l_N^i(x_{N,t}^i, \tilde{x}_{N,t}^i)$$

$$\text{subj. to } x_{k+1,t}^i = f^i(x_{k,t}^i, u_{k,t}^i), \tag{11a}$$

$$x_{k,t}^i \in \mathcal{X}^i, \quad u_{k,t}^i \in \mathcal{U}^i, \quad k = 1, \ldots, N-1 \tag{11b}$$

$$x_{k+1,t}^j = f^j(x_{k,t}^j, u_{k,t}^j), \quad (i,j) \in \mathcal{A}, \tag{11c}$$

$$x_{k,t}^j \in \mathcal{X}^j, \quad u_{k,t}^j \in \mathcal{U}^j, \quad (i,j) \in \mathcal{A}, \quad k = 1, \ldots, N-1 \tag{11d}$$

$$g^{i,j}(x_{k,t}^i, u_{k,t}^i, x_{k,t}^j, u_{k,t}^j) \leq 0, \quad (i,j) \in \mathcal{A}, k = 1, \ldots, N-1 \tag{11e}$$

$$x_{N,t}^i \in \mathcal{X}_f^i, \quad x_{N,t}^j \in \mathcal{X}_f^j, \quad (i,j) \in \mathcal{A} \tag{11f}$$

$$x_{0,t}^i = x_t^i, \quad \tilde{x}_{0,t}^i = \tilde{x}_t^i, \tag{11g}$$

where $\tilde{U}_t^i \triangleq [u_{0,t}^i, \tilde{u}_{0,t}^i, \ldots, u_{N-1,t}^i, \tilde{u}_{N-1,t}^i] \in \mathbb{R}^{s^i}$, $s^i \triangleq (\tilde{m}^i + m^i)N$ denotes the optimization vector, $x_{k,t}^i$ denotes the state vector of the i-th node predicted at time $t+k$ obtained by starting from the state x_t^i and applying to system (1) the input sequence $u_{0,t}^i, \ldots, u_{k-1,t}^i$. The tilded vectors denote the prediction vectors associated with the neighboring systems by starting from their states $\tilde{x}_t^i$ and applying to their models the input sequence $\tilde{u}_{0,t}^i, \ldots, \tilde{u}_{k-1,t}^i$. Denote by $\tilde{U}_t^{i*} = [u_{0,t}^{*i}, \tilde{u}_{0,t}^{*i}, \ldots, u_{N-1,t}^{*i}, \tilde{u}_{N-1,t}^{*i}]$ an optimizer of problem $\mathcal{P}_i$.

Note that problem $\mathcal{P}_i$ involves only the state and input variables of the i-th node and its neighbors at time t. We will define the following decentralized RHC scheme. At time t

(1) Each node i solves problem $\mathcal{P}_i$ based on measurements of its state x_t^i and the states of all its neighbors $\tilde{x}_t^i$.

(2) Each node i implements the first sample of $\tilde{U}_t^{i*}$

$$u_t^i = u_{0,t}^{*i}. \tag{12}$$

(3) Each node repeats steps 1 to 3 at time $t+1$, based on the new state information x_{t+1}^i, $\tilde{x}_{t+1}^i$.

In order to solve problem $\mathcal{P}_i$ each node needs to know its current states, its neighbors' current states, its terminal region, its neighbors' terminal regions and models and constraints of its neighbors. Based on such information each node computes its optimal inputs and its neighbors' optimal inputs. The input to the neighbors will only be used to predict their trajectories and then discarded, while the first component of the i-th optimal input of problem $\mathcal{P}_i$ will be implemented on the i-th node. The solution of the i-th subproblem will yield a control policy for the i-th node of the form $u_t^i = c^i(x_t^i, \tilde{x}_t^i)$, where $c^i : \mathbb{R}^{n^i} \times \mathbb{R}^{\tilde{n}^i} \rightarrow \mathbb{R}^{m^i}$ is a time-invariant feedback control law implicitly defined by the optimization problem $\mathcal{P}_i$.

Remark 2 *In the formulation above, a priori knowledge of the overall system equilibrium $(\tilde{x}_e, \tilde{u}_e)$ is assumed. The equilibrium could be defined in several other different ways. For instance, in a problem involving mobile agents we can assume that there is a leader (real or virtual) which is moving and the equilibrium is given in terms of distances of each agent from the leader. Also, it is possible to formulate the equilibrium by using relative distances between agents and signed areas.The approach of this paper does not depend on the way the overall system equilibrium is defined, as long as this is known a priori. In some decentralized control schemes, the equilibrium is not known a priori, but is the result of the evolution of decentralized control laws such as in [10]. The approach of this paper is not applicable to such schemes.*

Remark 3 *The problem formulation of (11) lends itself to generalization and is flexible enough to describe additional characteristics of a particular application. For instance, delayed information and additional communication between neighbors can be incorporated in the formulation as well. Additional terms that represent coupling between any two neighbors of a particular node can be included in the local cost function (8) as well. This leads to a better representation of the centralized problem and based on our numerical simulations, more accurate predictions regarding the behavior of neighbors. The derivation of results presented in this paper can be found in [16] for such case. One can also assume that terminal set constraints of neighbors are not known exactly. For the sake of simplicity, we will not consider these possible modifications and focus on problem (11).*

Even if we assume N to be infinite, the decentralized RHC approach described so far does not guarantee that solutions computed locally are globally feasible and stable. The reason is simple: at the i-th node the prediction of the neighboring state x^j is done independently from the prediction of problem $\mathcal{P}_j$. Therefore, the trajectory of x^j predicted by problem $\mathcal{P}_i$ and the one predicted by problem $\mathcal{P}_j$, based on the same initial conditions, are different (since in general, $\mathcal{P}_i$ and $\mathcal{P}_j$ will be different). This will imply that constraint fulfillment will be ensured by the optimizer u_t^{*i} for problem $\mathcal{P}_i$ but not for the centralized problem involving the states of all nodes.

Stability and feasibility of decentralized RHC schemes are currently active research areas [6, 9, 15, 21]. In the following section the stability of the decentralized RHC scheme given in (11) and (12) is analyzed in detail.

4 Stability analysis

Without loss of generality, we assume the origin to be an equilibrium for the overall system. In this section, we rely on the general problem formulation introduced in Section 2 and focus on systems *with* input and state constraints, *no* coupling constraints and terminal point constraint to the origin $\mathcal{X}_f^i = \mathbf{0}$. Thus the decentralized finite time optimal control problem associated with the i-th node at time t will have the following form:

$$\min_{\tilde{U}_t^i} \sum_{k=0}^{N-1} l^i(x_{k,t}^i, u_{k,t}^i, \tilde{x}_{k,t}^i, \tilde{u}_{k,t}^i)$$

$$\text{subj. to } x_{k+1,t}^i = f^i(x_{k,t}^i, u_{k,t}^i), \tag{13a}$$

$$x_{k,t}^i \in \mathcal{X}^i, \quad u_{k,t}^i \in \mathcal{U}^i, \quad k = 1, \ldots, N-1 \tag{13b}$$

$$x_{k+1,t}^j = f^j(x_{k,t}^j, u_{k,t}^j), \quad (i,j) \in \mathcal{A}, \tag{13c}$$

$$x_{k,t}^j \in \mathcal{X}^j, \quad u_{k,t}^j \in \mathcal{U}^j, \quad (i,j) \in \mathcal{A}, \quad k = 1, \ldots, N-1 \tag{13d}$$

$$x_{N,t}^i = \mathbf{0}, \quad x_{N,t}^j = \mathbf{0}, \quad (i,j) \in \mathcal{A} \tag{13e}$$

$$x_{0,t}^i = x_t^i, \quad \tilde{x}_{0,t}^i = \tilde{x}_t^i. \tag{13f}$$

We will make the following assumption on the structure of individual cost functions:

Assumption 1 *The cost term l^i in (8) associated with the i-th system can be written as follows*

$$l^i(x^i, u^i, \tilde{x}^i, \tilde{u}^i) = \|Qx^i\|_p + \|Ru^i\|_p + \sum_{j|(i,j)\in\mathcal{A}} \|Qx^j\|_p + \sum_{j|(i,j)\in\mathcal{A}} \|Ru^j\|_p + \sum_{j|(i,j)\in\mathcal{A}} \|Q(x^i - x^j)\|_p. \tag{14}$$

where $\|Mx\|_p$ denotes the p-norm of the vector Mx if $p = 1, \infty$ or $x'Mx$ if $p = 2$.

Remark 4 *The cost function structure in Assumption 1 can be used to describe several practical applications including formation flight, paper machine control and monitoring network of cameras [5]. The relative state term in (14) also assumes $n^i = \tilde{n}/N_v$ for all $i = 1, \ldots, N_v$.*

In classical RHC schemes, stability and feasibility is proven by using the value function as a Lyapunov function. We will investigate three different approaches to analyzing and ensuring stability of the overall system:

(1) Use of individual cost functions as Lyapunov functions for each node (Sections 4.1, 4.2 and 4.3).
(2) Use of the sum of individual cost functions as Lyapunov function for the entire system (Section 4.4).
(3) Exchange of optimal solutions between neighbors (Section 4.5).

The following notation will be used to describe state and input signals. For a particular variable, the first superscript refers to the index of the corresponding system, the second superscript refers to the location where it is computed. For instance the input $u^{i,j}$ represents the input to the i-th system calculated by solving problem $\mathcal{P}_j$. Similarly, the state variable $x^{i,j}$ stands for the states of system i predicted by solving $\mathcal{P}_j$. The lower indices conform to the standard time notation of RHC schemes. For example, variable $x_{k,t}$ denotes the k-step ahead prediction of the states made at time instant t.

4.1 Individual value functions as Lyapunov functions

In order to illustrate the fundamental issues regarding stability in a simple way, we first consider two systems ($N_v = 2$). The general formulation for an arbitrary number of nodes is treated later in Section 4.2. We consider two decentralized RHC problems $\mathcal{P}_1$ and $\mathcal{P}_2$ according to (13). In order to simplify notation, we define

$$\ell^1(x_t^1, U_t^{1,1}, x_t^2, U_t^{2,1}) = \sum_{k=0}^{N-1} l^1(x_{k,t}^{1,1}, u_{k,t}^{1,1}, x_{k,t}^{2,1}, u_{k,t}^{2,1}), \tag{15}$$

where x_t^1 and x_t^2 are the initial states of systems 1 and 2 at time t, and $U_t^{1,1}$, $U_t^{2,1}$ are the control sequences for node 1 and 2, respectively, calculated by node 1. Let $[U_t^{1,1*}, U_t^{2,1*}]$ be an optimizer of problem $\mathcal{P}_1$ at time t:

$$U_t^{1,1*} = \left[u_{0,t}^{1,1}, \ldots, u_{N-1,t}^{1,1}\right], \; U_t^{2,1*} = \left[u_{0,t}^{2,1}, \ldots, u_{N-1,t}^{2,1}\right], \tag{16}$$

and

$$\mathbf{x}_t^{1,1} = \left[x_{0,t}^{1,1}, \ldots, x_{N,t}^{1,1}\right], \quad \mathbf{x}_t^{2,1} = \left[x_{0,t}^{2,1}, \ldots, x_{N,t}^{2,1}\right],$$

be the corresponding optimal state trajectories of node 1 and 2 predicted at node 1 by $\mathcal{P}_1$.

Analogously, let $[U_t^{1,2*}, U_t^{2,2*}]$ be an optimizer of problem $\mathcal{P}_2$ at time t:

$$U_t^{1,2*} = \left[u_{0,t}^{1,2}, \ldots, u_{N-1,t}^{1,2}\right], \; U_t^{2,2*} = \left[u_{0,t}^{2,2}, \ldots, u_{N-1,t}^{2,2}\right], \tag{17}$$

and

$$\mathbf{x}_t^{1,2} = \left[x_{0,t}^{1,2}, \ldots, x_{N,t}^{1,2}\right], \quad \mathbf{x}_t^{2,2} = \left[x_{0,t}^{2,2}, \ldots, x_{N,t}^{2,2}\right],$$

be the corresponding optimal state trajectories of node 1 and 2 predicted at node 2 by $\mathcal{P}_2$. By hypothesis, neighboring systems either measure or exchange state information, so the initial states for both problems are the same at each time step, i.e., $x_{0,t}^{1,1} = x_{0,t}^{1,2}$ and $x_{0,t}^{2,1} = x_{0,t}^{2,2}$.

Remark 5 *It should be noted that although the two problems $\mathcal{P}_1$ and $\mathcal{P}_2$ involve the same subsystems, multiple optima can arise from non-strictly convex cost functions. Furthermore, in a more general setting, for larger number of nodes with an arbitrary graph interconnection, adjacent nodes have different set of neighbors and thus are solving different subproblems $\mathcal{P}_i$. Non-convex coupling constraints would be a source of multiple optimal solutions as well. These factors lead to different optimal solutions for neighboring problems and warrant distinguishing between $U^{1,1*}, U^{1,2*}$ and $U^{2,1*}, U^{2,2*}$ in (16) and (17).*

We denote the set of states of node i at time k feasible for problem $\mathcal{P}_i$ by

$$\mathcal{X}_k^i = \left\{x^i \mid \exists u^i \in \mathcal{U}^i \text{ such that } f^i(x^i, u^i) \in \mathcal{X}_{k+1}^i\right\} \cap \mathcal{X}^i, \quad \text{with } \mathcal{X}_N^i = \mathcal{X}_f^i. \tag{18}$$

Since we are neglecting coupling constraints, the set of feasible states for the decentralized RHC scheme described by (13) and (12) applied to the overall system is the cross product of the feasible set of states associated with each node:

$$\mathcal{X}_k = \prod_{i=1}^{N_v} \mathcal{X}_k^i, \tag{19}$$

where the symbol $\prod$ denotes the standard Cartesian product of sets.

Denote with

$$c(\tilde{x}_k) = \left[u_{0,k}^{1,1*}(\tilde{x}_k), u_{0,k}^{2,2*}(\tilde{x}_k)\right], \tag{20}$$

the control law obtained by applying the decentralized RHC policy in (13) and (12) with cost function (15), when the current state is $\tilde{x}_k = [x_k^1, x_k^2]$. Consider the overall system model (3) consisting of two nodes ($N_v = 2$), and denote with

$$\tilde{x}_{k+1} = f\left(\tilde{x}_k, c(\tilde{x}_k)\right), \tag{21}$$

the closed-loop dynamics of the entire system. In the following theorem we state sufficient conditions for the asymptotic stability of the closed-loop system.

Theorem 1 *Assume that*

- **(A0)** *$Q = Q' \succ 0, R = R' \succ 0$ if $p = 2$ and Q, R are full column rank matrices if $p = 1, \infty$.*
- **(A1)** *The state and input constraint sets $\mathcal{X}^1, \mathcal{X}^2$ and $\mathcal{U}^1, \mathcal{U}^2$ contain the origin in their interior.*
- **(A2)** *The following inequality is satisfied for all $x_t^i \in \mathcal{X}_0^i$, $x_t^j \in \mathcal{X}_0^j$ with $i = 1$, $j = 2$ and $i = 2$, $j = 1$:*

$$\varepsilon \leq \|Qx_t^i\|_p + \|Qx_t^j\|_p + \|Q(x_t^i - x_t^j)\|_p + \|Ru_{0,t}^{i,i}\|_p + \|Ru_{0,t}^{j,i}\|_p, \tag{22}$$

where

$$\varepsilon = \sum_{k=1}^{N-1} \left(2\|Q(x_{k,t}^{j,j} - x_{k,t}^{j,i})\|_p + \|R(u_{k,t}^{j,j} - u_{k,t}^{j,i})\|_p\right). \tag{23}$$

Then, the origin of the closed loop system (21) is asymptotically stable with domain of attraction $\mathcal{X}_0^1 \times \mathcal{X}_0^2$.

Proof: We will show that the value function of each individual node is a Lyapunov function, which decreases along the closed-loop trajectories at each time step $J_N^{i*}(x_{t+1}^i, \tilde{x}_{t+1}^i) \leq J_N^{i*}(x_t^i, \tilde{x}_t^i)$, if the assumptions of the theorem hold. Although this condition involves the closed-loop evolution of only local variables, it is required to hold for each individual node, leading to $J_N^*(\tilde{x}_{t+1}) \leq J_N^*(\tilde{x}_t)$ and the stability of the overall system. Without loss of generality we will consider problem $\mathcal{P}_1$ first and its optimal solution $U_t^{1,1*}$ and $U_t^{2,1*}$ at time t. The shifted sequences $U_{t+1}^{1,1} = [u_{1,t}^{1,1}, \ldots, u_{N-1,t}^{1,1}, \mathbf{0}]$ and $U_{t+1}^{2,1} = [u_{1,t}^{2,1}, \ldots, u_{N-1,t}^{2,1}, \mathbf{0}]$ of problem $\mathcal{P}_1$, are not necessarily feasible at the next time step $t+1$ since the state of system 2 at time $t+1$ is $x_{1,t}^{2,2}$ and not $x_{1,t}^{2,1}$, even assuming no model uncertainty (see Remark 5). This means that elements of the state sequence $\mathbf{x}_{t+1}^{2,1}$ may not be in $\mathcal{X}^2$ and $\mathcal{X}_f^2 = \mathbf{0}$, respectively, even though they were produced by the input sequence $U_{t+1}^{2,1}$ whose elements belong to $\mathcal{U}^2$. However, one can construct a feasible shifted sequence for problem $\mathcal{P}_1$ by using the optimizer of problem $\mathcal{P}_2$:

$$U_{t+1}^{2,2} = \left[u_{1,t}^{2,2}, \ldots, u_{N-1,t}^{2,2}, \mathbf{0}\right]. \tag{24}$$

This is possible, since the dynamics of both subsystems are decoupled. Furthermore, we have assumed no coupling constraints, which implies that $U_{t+1}^{1,1}$ and $U_{t+1}^{2,2}$ will be feasible at time $t+1$ for problem $\mathcal{P}_1$.

At the next time step $(t+1)$, the current states of the two systems are denoted by $x_{0,t+1}^{1,1}$ and $x_{0,t+1}^{2,2}$. Since the neighboring state information is exchanged between nodes, or assumed to be measured, we have $x_{0,t+1}^{1,2} = x_{0,t+1}^{1,1}$ and $x_{0,t+1}^{2,1} = x_{0,t+1}^{2,2}$ as well. We use the following notation:

$$\begin{aligned} x_t^1 &= x_{0,t}^{1,1} = x_{0,t}^{1,2}, & x_{t+1}^1 &= x_{0,t+1}^{1,1} = x_{0,t+1}^{1,2}, \\ x_t^2 &= x_{0,t}^{2,2} = x_{0,t}^{2,1}, & x_{t+1}^2 &= x_{0,t+1}^{2,2} = x_{0,t+1}^{2,1}, \\ \tilde{x}_t &= (x_t^1, x_t^2), & \tilde{x}_{t+1} &= (x_{t+1}^1, x_{t+1}^2). \end{aligned}$$

We can compute a bound on the value function as follows:

$$J_N^{1*}(\tilde{x}_{t+1}) \le \ell^1(x_{t+1}^1, U_{t+1}^{1,1}, x_{t+1}^2, U_{t+1}^{2,2}) \quad (25a)$$

$$= J_N^{1*}(\tilde{x}_t) - \|Qx_t^1\|_p - \|Qx_t^2\|_p - \|Q(x_t^1 - x_t^2)\|_p - \|Ru_{0,t}^{1,1}\|_p - \|Ru_{0,t}^{2,1}\|_p \quad (25b)$$

$$- \sum_{k=1}^{N-1} (\|Qx_{k,t}^{2,1}\|_p - \|Qx_{k,t}^{2,2}\|_p) \quad (25c)$$

$$- \sum_{k=1}^{N-1} (\|Ru_{k,t}^{2,1}\|_p - \|Ru_{k,t}^{2,2}\|_p) \quad (25d)$$

$$- \sum_{k=1}^{N-1} (\|Q(x_{k,t}^{1,1} - x_{k,t}^{2,1})\|_p - \|Q(x_{k,t}^{1,1} - x_{k,t}^{2,2})\|_p). \quad (25e)$$

It should be emphasized that in (25a) the cost function ℓ^1 of problem $\mathcal{P}_1$ is evaluated using the feasible shifted input sequence $U_{t+1}^{2,2}$ for node 2 and the corresponding state trajectory.

The cost function $J_N^{1*}(\tilde{x}_t)$ in (25b) is associated with the optimal control solution $U_t^{2,1*}$ of $\mathcal{P}_1$. The cost ℓ^1 in (25a) instead is evaluated at the sequence $U_{t+1}^{2,2}$ associated with $\mathcal{P}_2$. The mismatch between the two control sequences $U_{t+1}^{2,2}$ in (24) and $U_t^{2,1*}$ in (16) generates the terms in (25d). The difference between these control sequences generates also a mismatch between the state trajectories of node 2 predicted at node 1 and predicted at node 2. These are represented by the terms in (25c) and (25e).

Using the homogenity axiom of vector norms and applying $\|\alpha\|_p - \|\beta\|_p \le \|\alpha - \beta\|_p$ leads to

$$J_N^{1*}(\tilde{x}_{t+1}) \le J_N^{1*}(\tilde{x}_t) - (\text{terms in (25b)}) \quad (26a)$$

$$+ \sum_{k=1}^{N-1} \left(2\|Q(x_{k,t}^{2,2} - x_{k,t}^{2,1})\|_p + \|R(u_{k,t}^{2,2} - u_{k,t}^{2,1})\|_p \right). \quad (26b)$$

Notice that the term (26b) arises from the control solution mismatch between $\mathcal{P}_1$ and $\mathcal{P}_2$, and it represents $\varepsilon = \sum_{k=1}^{N-1} (2\|Q(x_{k,t}^{j,j} - x_{k,t}^{j,i})\|_p + \|R(u_{k,t}^{j,j} - u_{k,t}^{j,i})\|_p)$ defined in (23) for $i = 1$, $j = 2$. It follows that if inequality (22) holds, then $J_N^{1*}(\tilde{x}_{t+1}) \le J_N^{1*}(\tilde{x}_t)$. This implies that under the assumptions of Theorem 1, $J_N^{1*}(\tilde{x})$ is positive and non-increasing along the closed-loop trajectories, thus can be used as a Lyapunov function for node 1. The same derivation applies to node 2 and its associated cost function.

The rest of the proof follows from Lyapunov arguments, close in spirit to the arguments of [13] where it is established that the value function $J_N^*(\cdot)$ of the receding horizon problem is a Lyapunov function for the closed-loop system. Based on the hypothesis (A0) on the matrices Q and R, inequality (22) is sufficient to ensure that the state of the closed-loop system (21) converges to zero as $k \to \infty$. Stability follows from the fact that $J_N^{1*}(x)$ and $J_N^{2*}(x)$ can be lower and upper bounded by functions $\alpha(\|\tilde{x}\|)$ and $\beta(\|\tilde{x}\|)$, where $\alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ are of class K [19]. □

Theorem 1 highlights the relationship between the stability of the decentralized scheme given in (13) and (12), and the allowable prediction mismatch at all points in the state space of the overall system. The term ε in inequality (22) is a function of the error between the trajectories of node 2 predicted by node 1 and the one predicted by node 2 itself. The smaller the error, the larger the set of initial states for which the value function will decrease along the overall system trajectories.

Similar ideas can be used if instead of a terminal point constraint, nonzero terminal cost and terminal set constraints $\mathcal{X}_f \neq \mathbf{0}$ are used. In this case, the terminal set has to be control invariant and the terminal cost is chosen as a control Lyapunov function [19]. In the following section we show how to extend the previous arguments to a graph with more than two nodes.

4.2 Generalization to arbitrary number of nodes and graph

The development of Section 4.1 carries over to any number of nodes and general graph structure. Let us denote the decentralized receding horizon control law for the overall system with

$$c(\tilde{x}_k) = \left[u_{0,k}^{1,1*}(x_k^1, \tilde{x}_k^1), \ldots, u_{0,k}^{N_v,N_v*}(x_k^{N_v}, \tilde{x}_k^{N_v}) \right], \quad (27)$$

obtained by applying the decentralized RHC policy of each subproblem $\mathcal{P}_i$ described in (13) and (12) when the current state is $\tilde{x}_k = [x_k^1, \ldots, x_k^{N_v}]$. Note that since there are no coupling constraints, the feasible states for the overall system is the cross product of the feasible states associated with each node as defined in (18) and (19). Consider the system model (3) and denote by

$$\tilde{x}_{k+1} = f\left(\tilde{x}_k, c(\tilde{x}_k)\right), \quad (28)$$

the closed-loop dynamics of the overall system. Sufficient conditions for asymptotic stability of the closed-loop system are given next.

Theorem 2 *Assume*

(A0) $Q = Q' \succ 0, R = R' \succ 0$ *if* $p = 2$ *and* Q, R *are full column rank matrices if* $p = 1, \infty$.

(A1) *The state and input constraint sets* $\mathcal{X}^i$ *and* $\mathcal{U}^i$ *contain the origin for each node in their interior.*

(A2) *The following inequality is satisfied for each node and all* $x_t^i \in \mathcal{X}_0^i$:

$$\sum_{j|(i,j)\in\mathcal{A}} \varepsilon^{i,j} \leq J_0^{i*}, \tag{29}$$

where

$$\varepsilon^{i,j} = \sum_{k=1}^{N-1} \left(2\|Q(x_{k,t}^{j,j} - x_{k,t}^{j,i})\|_p + \|R(u_{k,t}^{j,j} - u_{k,t}^{j,i})\|_p\right), \tag{30}$$

and

$$J_0^{i*} = \|Qx_t^i\|_p + \|Ru_{0,t}^{i,i}\|_p + \sum_{j|(i,j)\in\mathcal{A}} (\|Qx_t^j\|_p + \|Ru_{0,t}^{j,i}\|_p) + \sum_{j|(i,j)\in\mathcal{A}} \|Q(x_t^i - x_t^j)\|_p. \tag{31}$$

Then, the origin of the closed loop system (28) is asymptotically stable with domain of attraction $\prod_{i=1}^{N_v} \mathcal{X}_0^i$.

Proof: The proof follows along the lines of Theorem 1. The difference is the derivation of stability condition (29) for any particular node within an arbitrary graph interconnection $\mathcal{A}$. This is given next.

Consider the cost function (14) in Assumption 1 for any node i. For more compact notation, we can define $l^{i,j}$ as

$$l^{i,j}(x^i, u^i, x^j, u^j) = \|Qx^j\|_p + \|Q(x^i - x^j)\|_p + \|Ru^j\|_p, \tag{32}$$

and construct $\ell^{i,j}$ as

$$\ell^{i,j}(x_t^i, U_t^{i,i}, x_t^j, U_t^{j,i}) = \sum_{k=0}^{N-1} l^{i,j}(x_{k,t}^{i,i}, u_{k,t}^{i,i}, x_{k,t}^{j,i}, u_{k,t}^{j,i}). \tag{33}$$

The value function $J_N^i(x_t^i, \tilde{x}_t^i) = \ell^i(x_t^i, U_t^{i,i}, \tilde{x}_t^i, \tilde{U}_t^{\tilde{i},\tilde{i}})$ of node i will then have the following form:

$$\ell^i(x_t^i, U_t^{i,i}, \tilde{x}_t^i, \tilde{U}_t^{\tilde{i},\tilde{i}}) = \sum_{k=0}^{N-1} (\|Qx_{k,t}^{i,i}\|_p + \|Ru_{k,t}^{i,i}\|_p) + \sum_{j|(i,j)\in\mathcal{A}} \ell^{i,j}(x_t^i, U_t^{i,i}, x_t^j, U_t^{j,j}), \tag{34}$$

where $\tilde{U}_t^{\tilde{i},\tilde{i}} = \{U_t^{j,j} | (i,j) \in \mathcal{A}\}$ and for a neighboring node j, the shifted feasible solution sequence of problem $\mathcal{P}_j$ is denoted by $U_t^{j,j} = [u_{1,t-1}^{j,j}, \ldots, u_{N-1,t-1}^{j,j}, \mathbf{0}]$.

Using the notation introduced above, we can construct the following upper bound on the cost function:

$$\begin{aligned} J_N^{i*}(x_{t+1}^i, \tilde{x}_{t+1}^i) &\leq \ell^i(x_{t+1}^i, U_{t+1}^{i,i}, \tilde{x}_{t+1}^i, \tilde{U}_{t+1}^{\tilde{i},\tilde{i}}) \\ &= J_N^{i*}(x_t^i, \tilde{x}_t^i) \\ &- \|Qx_t^i\|_p - \|Ru_{0,t}^{i,i}\|_p - \sum_{j|(i,j)\in\mathcal{A}} \|Qx_t^j\|_p - \sum_{j|(i,j)\in\mathcal{A}} \|Ru_{0,t}^{j,i}\|_p \quad (35a) \\ &- \sum_{j|(i,j)\in\mathcal{A}} \|Q(x_t^i - x_t^j)\|_p \quad (35b) \\ &- \sum_{k=1}^{N-1} \sum_{j|(i,j)\in\mathcal{A}} (\|Qx_{k,t}^{j,i}\|_p - \|Qx_{k,t}^{j,j}\|_p) \quad (35c) \\ &- \sum_{k=1}^{N-1} \sum_{j|(i,j)\in\mathcal{A}} (\|Ru_{k,t}^{j,i}\|_p - \|Ru_{k,t}^{j,j}\|_p) \quad (35d) \\ &- \sum_{k=1}^{N-1} \sum_{j|(i,j)\in\mathcal{A}} (\|Q(x_{k,t}^{i,i} - x_{k,t}^{j,i})\|_p - \|Q(x_{k,t}^{i,i} - x_{k,t}^{j,j})\|_p). \quad (35e) \end{aligned}$$

Using the homogenity axiom of vector norms and applying $\|\alpha\|_p - \|\beta\|_p \leq \|\alpha - \beta\|_p$ leads to

$$\begin{aligned} J_N^{i*}(x_{t+1}^i, \tilde{x}_{t+1}^i) \leq\ & J_N^{i*}(x_t^i, \tilde{x}_t^i) \quad (36a) \\ & - \|Qx_t^i\|_p - \|Ru_{0,t}^{i,i}\|_p \\ & - \sum_{j|(i,j)\in\mathcal{A}} \|Qx_t^j\|_p - \sum_{j|(i,j)\in\mathcal{A}} \|Ru_{0,t}^{j,i}\|_p \quad (36b) \\ & - \sum_{j|(i,j)\in\mathcal{A}} \|Q(x_t^i - x_t^j)\|_p \quad (36c) \\ & + \sum_{j|(i,j)\in\mathcal{A}} \varepsilon^{i,j}, \quad (36d) \end{aligned}$$

where $\varepsilon^{i,j}$ was defined in (30).

The positive value function $J_N^{i*}(x^i, \tilde{x}^i)$ is non-increasing along the closed-loop trajectories and thus can be used as a Lyapunov function for node i if the sum of terms in (36b)+(36c)+(36d) is nonpositive:

$$\sum_{j|(i,j)\in\mathcal{A}} \varepsilon^{i,j} + \underbrace{(\text{terms in (36b) and (36c)})}_{-J_0^{i*}} \leq 0, \tag{37}$$

which leads to the inequality $\sum_{j|(i,j)\in\mathcal{A}} \varepsilon^{i,j} \leq J_0^{i*}$ shown in the stability condition (29) of Theorem 2. □

Remark 6 *Assumption* (A2) *of Theorem 2 extends Assumption* (A2) *of Theorem 1 to arbitrary graphs and number of nodes. In general, nodes may have multiple neighbors, which lead to additional terms in inequality (29) compared to (22).*

Theorem 2 presents a sufficient condition for testing the stability of the decentralized scheme introduced in Section 3. It involves local conditions to be tested at each individual node and requires bounding the prediction mismatch between neighboring subsystems. These results follow in the footsteps of one of the stability analysis methods presented in the survey paper [12], where stability tests for large-scale interconnected systems are formulated in terms of the individual Lyapunov functions and bounds on subsystem interconnections. In our work, the concept of these bounds has an exact relationship with the prediction mismatch between neighboring subsystems. It is clear that for large scale systems, the stability condition (29) leads to complexity reduction. The formulation of these local stability tests is highlighted in the next section for the case of linear systems.

4.3 Heterogeneous unconstrained LTI subsystems

For heterogeneous unconstrained linear time-invariant (LTI) subsystems, with state matrices $A^i \in \mathbb{R}^{n\times n}, B^i \in \mathbb{R}^{n\times m}, i = 1, \dots, N_v$, the stability condition (29) when $p = 2$ is used in Assumption 1 leads to testing the semi-definiteness of N_v matrices. The stability condition (29) for a particular node i involves the states of its neighbors $x^{j,j}_{k,t}$ predicted at time t by the neighbor j itself, $(i,j) \in \mathcal{A}$. A predicted neighboring state $x^{j,j}_{k,t}$ is a function of the input sequence $u^{j,j}_{k,t}$ computed at node j, which is then a function of the *initial* states of all the neighbors of node j. This implies that the test for node i involves the states of node i, all its neighbors' states, and the states of the neighbors of neighbors. Thus the *dimension of the local stability tests are limited by the maximum size of any subgraph with diameter less than or equal to four.* In this section, we will describe the local semi-definiteness tests deriving from (29). An example for the case of identical subsystems can be found in [16].

Since the local RHC problems are time-invariant, without loss of generality we set the generic initial time to $t = 0$ for notational simplicity. Consider node i with initial state x^i_0. As defined in Section 2, $\tilde{x}^i_0$ denotes the states of its neighbors at the same time instant. We denote the states of the neighbors of neighbors to node i (which are not connected to the i-th node) by $\breve{x}^i_0 = \{x^q \in \mathbb{R}^{n^q} | \exists j \ (j,q) \in \mathcal{A}, (i,j) \in \mathcal{A}, (i,q) \notin \mathcal{A}\}$. We use $\bar{x}^i_0 = [x^i_0, \ \tilde{x}^i_0, \ \breve{x}^i_0]$ to denote the collection of self, neighboring and two-step neighboring states of node i. Using the notation $u^{i,i}_{[0,N-1],0} = [u^{i,i}_{0,0}, \dots, u^{i,i}_{N-1,0}]$ for time sequences, the solution to problem (13) associated with node i can be expressed as

$$\begin{bmatrix} u^{i,i}_{[0,N-1],0} \\ \tilde{u}^{i,i}_{[0,N-1],0} \\ \breve{u}^{i,i}_{[0,N-1],0} \end{bmatrix} = \begin{bmatrix} K^i_{11} & K^i_{12} & \mathbf{0} \\ K^i_{21} & K^i_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x^i_0 \\ \tilde{x}^i_0 \\ \breve{x}^i_0 \end{bmatrix} = \begin{bmatrix} K^i_1 \\ K^i_2 \\ \mathbf{0} \end{bmatrix} \bar{x}^i_0, \quad (38)$$

when the subsystems (1) are unconstrained LTI systems. Based on (38), we will use the following notation for k-step ahead predicted input values:

$$u^{i,i}_{k,0} = \underbrace{\begin{bmatrix} K^i_{11,k} & K^i_{12,k} & \mathbf{0} \end{bmatrix}}_{K^i_{1,k}} \bar{x}^i_0, \quad u^{j,i}_{k,0} = \underbrace{\begin{bmatrix} K^{j,i}_{21,k} & K^{j,i}_{22,k} & \mathbf{0} \end{bmatrix}}_{K^{j,i}_{2,k}} \bar{x}^i_0, \quad (39)$$

where $K^i_{1,k}$ and $K^{j,i}_{2,k}$ are submatrices of K^i_1 and K^i_2, respectively.

We use the above notation to express the solution $u^{j,j}_{[0,N-1],0}$ to problem (13) associated with node $j, (i,j) \in \mathcal{A}$ as an explicit function of the initial states:

$$u^{j,j}_{[0,N-1],0} = \begin{bmatrix} {}^iK^j_{11} & {}^iK^j_{12} & {}^iK^j_{13} \end{bmatrix} \bar{x}^i_0, \quad (40)$$

and thus

$$u^{j,j}_{k,0} = \underbrace{\begin{bmatrix} {}^iK^j_{11,k} & {}^iK^j_{12,k} & {}^iK^j_{13,k} \end{bmatrix}}_{{}^iK^j_{1,k}} \bar{x}^i_0. \quad (41)$$

Note that the control input $u^{j,j}_{k,0}$ can also be expressed as a function of $\bar{x}^j_0$ (e.g. in the local stability condition associated with node j), thus the upper left index i is needed to distinguish the above controller gain matrix entries.

Using (39) and (41), we can express predicted states for any node $j, (i,j) \in \mathcal{A}$ as

$$x^{j,j}_{k,0} = \Psi^{j,j}_k \bar{x}^i_0, \quad x^{j,i}_{k,0} = \Psi^{j,i}_k \bar{x}^i_0, \quad (42)$$

where the matrix $\Psi^{j,j}_k$ is a function of $A^i, \{A^j | (i,j) \in \mathcal{A}\}, B^i, \{B^j | (i,j) \in \mathcal{A}\}$ and ${}^iK^j_1$. Similarly, matrix $\Psi^{j,i}_k$ is a function of $A^i, \{A^j | (i,j) \in \mathcal{A}\}, B^i, \{B^j | (i,j) \in \mathcal{A}\}$ and $K^{j,i}_2$.

Based on equations (39), (41) and (42), all the terms in the stability condition (29) can be expressed as a quadratic form of $\bar{x}^i_0$. The terms on the left side of (29) can be expressed using the following two matrices:

$$\Theta^i_N = \sum_{k=1}^{N-1} \sum_{j|(i,j)\in\mathcal{A}} 2(\Psi^{j,j}_k - \Psi^{j,i}_k)' Q (\Psi^{j,j}_k - \Psi^{j,i}_k), \quad (43a)$$

$$\Gamma^i_N = \sum_{k=1}^{N-1} \sum_{j|(i,j)\in\mathcal{A}} ({}^iK^j_{1,k} - K^{j,i}_{2,k})' R ({}^iK^j_{1,k} - K^{j,i}_{2,k}). \quad (43b)$$

Denoting the number of neighbors of node i by N^i_v, the following matrices are used to express terms on the right side of (29):

$$R_1^i = (K_{1,0}^i)' R (K_{1,0}^i), \tag{44a}$$

$$R_2^i = (K_{2,0}^i)' (I_{N_v^i} \otimes R)(K_{2,0}^i), \tag{44b}$$

$$Q_1 = \begin{bmatrix} Q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad Q_2^i = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{N_v^i} \otimes Q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{44c}$$

$$Q_3^i = D_1' \left(I_{N_v^i} \otimes Q \right) D_1, \tag{44d}$$

$$D_1 = \begin{bmatrix} \left[\mathbf{1}_{N_v^i} \ -I_{N_v^i} \right] \otimes I_n \\ \mathbf{0} \end{bmatrix}, \tag{44e}$$

where $\mathbf{1}_{N_v^i}$ denotes a column vector of '1'-s of size N_v^i.

Using the matrices in (43)-(44), the stability condition (29) for node i is equivalent to testing whether

$$\Theta_N^i + \Gamma_N^i \leq Q_1 + R_1^i + Q_2^i + R_2^i + Q_3^i. \tag{45}$$

Stability of the overall system can be concluded if (45) holds for all $i \in \{1, \ldots, N_v\}$.

4.4 *Sum of value functions as Lyapunov function*

If we consider the sum of individual cost functions as a Lyapunov function for the entire system, the value function inequality such as the one in (26) will involve significantly more terms than the case presented in the previous sections [14]. In fact, this condition might be less restrictive than the one presented in (29). Even if the individual inequalities (26) presented in the previous section do not hold for every subproblem $\mathcal{P}_i$, the sum of individual value functions could still be used as a Lyapunov function for the entire system. This will be the case if $\sum_{i \in \mathcal{I}} \sum_{j|(i,j) \in \mathcal{A}} \varepsilon^{i,j} < -\sum_{i \in \bar{\mathcal{I}}} \sum_{j|(i,j) \in \mathcal{A}} \varepsilon^{i,j}$, where $\mathcal{I}$ is the set of nodes for which the Lyapunov function $J_N^{i*}(x^i, \tilde{x}^i)$ is decreasing and $\bar{\mathcal{I}}$ is its complement.

4.5 *Exchange of information*

Stability conditions derived in the previous sections show that it is the mismatch between the predicted and actual control solutions of neighbors that plays a central role in the stability problem. Therefore we are prompted to investigate how sufficient conditions for stability could be improved by allowing the exchange of optimal solutions between neighbors. Examining condition (22) from this standpoint, we can immediately make two general observations:

(1) Using bounds on the mismatch between the predicted and actual inputs and states of neighbors, the stability condition (22) could be made less restrictive by reducing the size of positive terms in (26b), which adversely affect the value function variation of (26). In other words, using a coordination scheme based on information exchange, it may be possible to reduce the size of ε to decrease the left side of inequality (22).

(2) Also, one can observe that as each node is getting closer to its equilibrium (in our example the origin) the right side of inequality (22) starts to diminish, which leads to more stringent restrictions on the allowable prediction mismatch between neighbors, represented by the left side of the inequality.

These observations suggest that information exchange between neighboring nodes has a beneficial effect in proving stability, *if it leads to reduced prediction mismatch.* As each system converges to its equilibrium, assumptions on the behavior of neighboring systems should get more and more accurate to satisfy the stability condition (22). In fact, as system (21) approaches its equilibrium the right hand side of inequality (22) decreases. In turn, the left hand side of inequality (22) has to diminish as well. This leads to the counter-intuitive conclusion that an increasing information exchange rate between neighbors might be needed when approaching the equilibrium. These conclusions are in agreement with the stability conditions of a distributed RHC scheme proposed in [9], where it is shown that convergence to a smaller neighborhood of the system equilibrium requires more frequent updates. However, our simulation examples [5, 15, 20] suggest that the prediction errors between neighbors tend to disappear as each node approaches its equilibrium, and the prediction mismatch converges to zero at a faster rate than the decay in the right hand side. A different, sequential information exchange scheme can be found in [21, 22], which is valid for a special graph structure based on a leader-follower architecture.

Remark 7 *If coupling constraints are present, ensuring feasibility in a decentralized receding horizon control scheme without introducing excessively conservative assumptions is a challenging problem. We refer the reader to the works [4–6, 11, 14, 17, 21, 24] and references therein for a detailed discussion on various approaches to constraint fulfillment in such decentralized schemes.*

5 Conclusions

A decentralized receding horizon control scheme for decoupled systems has been proposed and its stability investigated. We have highlighted how the derived stability conditions lead to complexity reduction in stability analysis and as an example, local matrix semi-definiteness tests have been provided for the case of heterogeneous unconstrained LTI systems. Each test involves the states of as many nodes as are included in the "one-neighbor-expansion" of the subgraph associated with each subproblem. This means that the size of these local tests are limited by the maximum size of any subgraph with diameter less than or equal to four. Thus the largest reduction in complexity can be expected when the diameter of the overall interconnection graph is large.

Systematic design of decentralized RHC controllers and the appropriate choice of weighting matrices is a topic of current research. As a first step in this direction, the work in [3] studies properties of stabilizing distributed LQR control solutions for decoupled systems. Using algorithms described in [1, 2], the decentralized receding horizon framework proposed in this paper has been applied in simulation to a number of large scale control problems with success. Different methodologies of handling the feasibility issue were implemented on numerous examples. References to formation flight application examples and to a paper machine control problem can be found in [5, 14].

Acknowledgments

This work was funded in part by the Defense Advanced Research Projects Agency under the Software Enabled Control Program, Dr. John Bay Program Manager, contract USAF/AFMC F33615-99-C-1497. Dale Van Cleave was the technical contract monitor. Support for theoretical development of the algorithms was also provided under NASA Langley Cooperative Agreement No. NCC-1-337, Dr. Christine Belcastro is the technical contract monitor. The work of Francesco Borrelli was funded by Honeywell Laboratories in Minneapolis, Minnesota.

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