# Analysis of a Class of Decentralized Decision Processes: Quantized Progressive Second Price Auctions

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***Abstract*— A progressive second price (PSP) auction mechanism was proposed in [1] for network bandwidth allocation. In this paper a quantized version of this mechanism (QPSP) is analyzed where the agents have similar demand functions and submit bids synchronously. It is shown that the non-linear dynamics induced by this mechanism are such that the prices bid by the various agents and the quantities allocated to these agents converge in at most five iterations or oscillate indefinitely, independently of the number of agents involved.**

## I. Introduction

Progressive second price auctions (PSPs) were proposed in [1] for dynamic network service market-pricing with the objective of providing consistent services when the so-called DiffServ customer access control protocols [2] are in use. In particular, it was shown that for differentiated services allocated between multiple agents there exist Nash market equilibria when all players bid their real marginal valuation of the bandwidth resource. In [3] and [4], an accelerated convergence version of PSP was derived which avoids signaling bursts but this is at the cost of multi-dimensional bidding.

The PSP dynamical auction mechanism introduced and analyzed in [5] and [6] was defined in such a manner that agents compute the $\epsilon$-best response to the current strategy profile of their opponents as their bids. Each agent's bid consists of (i) a required quantity and (ii) a unit-price (calculated using its own demand functions). All agents submit bids cyclically until an ($\epsilon$-Nash) equilibrium is reached where $\epsilon$ corresponds to a bid fee. It was proved in [5], [6] that PSP has the desirable properties of incentive compatibility and efficiency (i.e. bids correspond to the actual level of demand at a given price and the sum of all utility functions are optimized); however, the rate of convergence is inversely proportional to the bid fee $\epsilon$.

In this paper, a quantized version of this mechanism, Q-PSP, is analyzed where all agents have similar demand functions. In Q-PSP, each agent submits a bid which consists of both a unit-price and a quantity as in PSP. Then it uses the following quantized strategy: (i) it computes its best quantity response with respect to the previous strategy profile of its opponents; (ii) chooses a lowest quantized price as the unit bid price; (iii) then calculates the bid quantity based on the unit bid price and the agent's own demand function (see Sect. II B below for details). In Q-PSP auctions all agents submit bids synchronously until a (Nash) equilibrium in the quantized framework is reached. It is generally believed that PSP and Q-PSP constitute interesting classes of decentralized dynamical optimization procedures.

We will show that the nonlinear dynamics induced by Q-PSP are such that the prices bid by the various agents and the quantities allocated to these agents converge in at most five iterations or oscillate indefinitely, independently of the number of agents involved.

Lazar and Semret suggested in [5] that each agent bids successively its $\epsilon$-best reply with respect to the current bids of its opponents, whereas here we assume that all agents make the quantized bids simultaneously with respect to the previous bids of their opponents, these being the best response dynamics described in [7] and [8]. Both the cyclic and the simultaneous update rules are widely used in the theory of learning in games [7], [8]. For PSP, one of main disadvantages is its slow convergence, which brings about signal bursts, i.e. a part of channel capacity has to be taken for communication between agents and sellers. Here we prove, in comparison with the successive bid system in [5], that Q-PSP systems may converge in at most five steps when all agents share similar demand functions. Such convergence is independent of the number of agents. Hence synchronous Q-PSP avoids the signaling overhead of PSP.

In the work of Maille and Tuffin (see [3] and [4]), a multi-bid auction was constructed to achieve one step convergence of PSP systems. In multi-bid auctions it is assumed that each agent submits multiple bids simultaneously once and only once; then the market clearing price [3] and the allocations are calculated. It is known that the precision and the efficiency of this mechanism depends upon the dimension of each agent's bid set: the more bids each agent submits, the more efficient the equilibrium is. That is to say, in order to achieve a satisfactory approximation of each agent's own valuation function (so that the final state is close to a (Nash) Equilibrium), the dimension of each agent's bid set must be large. This latter fact engenders high computational, communication, and transmission costs.

We summarize the distinctions between the PSP and Q-PSP mechanisms as follows: (i) In Q-PSP auctions, all agents submit their bids simultaneously, while the cyclic bid algorithm is applied in PSP. (ii) The strategies in Q-PSP are quantized and all agents' bids are based upon a set of quantized prices, while there is no quantization in PSP. To support the quantization assumption we note that bidding with quantized prices often occurs in real auctions due to standard institutional rules. [9] (iii) 5-step (absolute) convergence to $\mu$-Nash equilibrium is achieved in Q-PSP (where

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$\mu$ is a measure of the quantization level and the divergence between the demand functions), but, in PSP, convergence to an $\epsilon$-Nash equilibrium is shown to be inversely proportional to $\epsilon$. (iv) there is no bid fee in Q-PSP, i.e. $\epsilon = 0$, but in PSP $\epsilon > 0$.

## II. The Quantized Progressive Second Price Auction and the Associated Dynamical System

### A. Progressive Second Price Auctions

To begin with we give a summary of the PSP auction first introduced in [5] which forms a part of the overall market based bandwidth allocation model.

Consider a noncooperative game where $N$ agents buy the fixed amount of bandwidth $C$ from one seller. Suppose each agent $A_i, 1 \le i \le N$, makes a *bid* $s_i = (p_i, q_i)$ to the seller, where $p_i$ is the unit-price the agent is willing to pay and $q_i$ is the quantity the agent desires. $s \equiv [s_i]_{1 \le i \le N}$ is the *bidding profile* and $s_{-i} \equiv [s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_N]$ is the profile of Agent $A_i$'s opponents. The *market price function (MPF)* of Agent $A_i$ is defined as:

$$P_i(z, s_{-i}) = \inf \left\{ y \ge 0 : C - \sum_{p_k > y, k \ne i} q_k \ge z \right\}, \quad \text{(II.1)}$$

which is interpreted as the minimum price an agent bids in order to obtain the bandwidth $z$ given the opponents' profile $s_{-i}$. Its inverse function $Q_i$ is defined as follows:

$$Q_i(y, s_{-i}) = \left[ C - \sum_{p_k > y, k \ne i} q_k \right]^+ ,$$

which means the maximum available quantity at a bid price of $y$ given $s_{-i}$. With this notation, the PSP allocation rule [10] is defined as

$$a_i(s) = \min\{q_i, \frac{q_i}{\sum_{k:p_k = p_i} q_k} Q_i(p_i, s_{-i})\}, \quad \text{(II.2)}$$

$$c_i(s) = \sum_{j \ne i} p_j \left[ a_j(0; s_{-i}) - a_j(s_i; s_{-i}) \right], \quad \text{(II.3)}$$

where $a_i$ denotes the quantity Agent $A_i$ obtains by a bid price $p_i$ (when the opponents bid $s_{-i}$) and the charge to Agent $A_i$ by the seller is denoted $c_i$. $c_i$ is interpreted to be the total cost incurred in the system if Agent $A_i$ is removed from the auction.

**Definition 1** (see Assumption 1 of [5]).

A real valued function $\theta(\cdot)$ is an *(elastic) valuation function* on $[0, C]$ if

- $\theta(0) = 0$;
- $\theta$ is differentiable;
- $\theta' \ge 0$, and $\theta'$ is non-increasing and continuous;
- There exists $\gamma, \gamma > 0$, such that for all $z, z \in [0, C], \theta'(z) > 0$ implies that for all $\eta \in [0, z)$, $\theta'(z) \le \theta'(\eta) - \gamma(z - \eta)$.

The function $\theta'(\cdot)$ on $[0, C]$ is called an *(elastic) demand function*. □

Agent $A_i$'s *utility* is defined as

$$u_i(s) = \theta_i(a_i(s)) - c_i(s),$$

which implies the agent's preferences.

Under the PSP rule above, it is shown in [5], [10] that given the opponent bids $s_{-i}$, Agent $A_i$'s $\epsilon$-best response $s_i = (w_i, v_i)$ in the sense of a Nash move (i.e. where $s_i$ is chosen to maximize its utility with $s_{-i}$ held constant) is given by:

$$v_i = \sup \left\{ z \ge 0 : \theta_i'(z) > P_i(z), \int_0^z P_i(\eta) d\eta \le b_i \right\} - \frac{\epsilon}{\theta_i'(0)} \quad \text{(best quantity reply)} \quad \text{(II.4a)}$$

$$w_i = \theta_i'(v_i) \quad \text{(best unit-price reply)}, \quad \text{(II.4b)}$$

where $\epsilon > 0$ is the bid fee, $b_i$ is Agent $A_i$'s budget, and every agent has an elastic demand function. Further it is shown in [5] that in case the bidding iterations converge they will converge at a rate inversely proportional to $\epsilon$ to an $\epsilon$-Nash equilibrium.

### B. Quantized Progressive Second Price Auctions (Q-PSP)

We now analyze the Quantized Progressive Second Price Auction introduced in [12]. Adopting a similar framework to the original PSP scheme, we assume for Q-PSP that all agents follow quantized strategies where their bids are based upon a set of quantized prices as described below.

Let us define

$$horP_i = \{(x, y); 0 \le x \le C, y = P_i(x)\};$$
$$vertP_i = \{(x, y); P_i(x) < y \text{ and for all } \delta > 0$$
$$\text{sufficiently small, } y' = P_i(x + \delta) \text{ satisfies } y \le y'\}.$$

(see the horizontal and vertical segments in Fig. 1). Further define the *market price curve* $P_i^{cv}$ (sometimes written $P_i$ for simplicity) to be the disjoint union $\{horP_i \cup vertP_i\}$. A relation $R$ on $\Re' \times \Re'$ is said to be increasing if for all $(x, y) \in R$, $(x', y') \in R$ and $x < x'$ imply $y \le y'$. Taken together these definitions make $P_i^{cv}$ an increasing relation in the $(x, y) \equiv (quantity, price)$ space such that the relation is a piece-wise constant function at all except a finite number of points where it is given by a vertical segment. When $\epsilon = 0$, the intersection in (II.4) between any agent's demand curve and its market price curve has the interpretation that it would be the best reply of each agent once the agent senses, via (II.1), how much the other agents are collectively bidding. In the context of the equations (II.4), we shall adopt the following hypotheses:

(1) All bid quantities $q$ are bounded by $C$, i.e. $0 \le q \le C$,
(2) There is no bid fee, i.e. $\epsilon = 0$,
(3) The budget $b_i$ of each agent is sufficiently large that the condition $\int_0^z P_i(\eta) d\eta \le b_i$ in (II.4) is always satisfied.

An intersection between a demand curve and a market price curve can occur in two distinct ways, namely the demand curve may either intersect $horP_i$ or $vertP_i$. Under the quantization assumption, in the first case, agents make normal bid as in (II.4), i.e. the values of the price and the quantity at the intersection. In the second case, the unique point of intersection $(z^*, p^{**})$ lies in $vertP_i$. Agents are then assumed to take the price $p_i^*$ corresponding to the value of $P_i$ at the limit $z^*$ from the left of $\{z \geq 0, \theta_i^{'}(z) > P_i(z)\}$ and the corresponding quantized price satisfies $p_i^* = P_i(z^*) \in B_p$. Finally the corresponding quantity $q_i^*$ is defined by $q_i^* = \theta_i^{'-1}(p_i^*)$. Here $B_p$ is the basic quantized set of prices. These two cases are shown in Fig. 1 and Fig. 2 respectively, where three agents are considered.

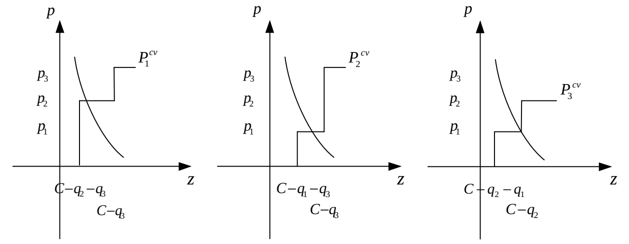


Fig. 1. Market price curves $P^{cv}$ and demand functions $\theta'$ of agents $A_1$, $A_2$, $A_3$. $w_1 = p_2^* = p_2$, $w_2 = p_2^* = p_1$ and $w_3 = p_3^* = p_1$.

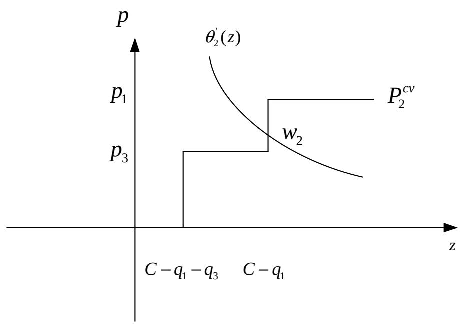


Fig. 2. Demand curve of agent 2 intersecting the market price curve $P_2^{cv}$ on a vertical segment and $w_2 \neq p_2^* = p_3$.

The result of applying the hypotheses (1)-(3) leads to the recursively defined market price functions (MPFs) $\{P_i^{k+1}; 1 \leq i \leq N, k \geq 0\}$ (see (II.1)), and the following *quantized PSP (Q-PSP) dynamical (state space) system* (with state $(v, p, q)$) equations.

$$
\begin{aligned}
v_i^{k+1} &= \sup\left\{z \geq 0 : \theta_i^{'}(z) > P_i^{k+1}(z, s_{-i}^k)\right\} && \text{(II.5a)}\\
p_i^{k+1} &= P_i^{k+1}(v_i^{k+1}, s_{-i}^k) && \text{(II.5b)}\\
q_i^{k+1} &= \theta_i^{'-1}(p_i^{k+1}), && \text{(II.5c)}
\end{aligned}
$$

with the initial conditions $p_i^0 \in B_p^0$, $q_i^0 = \theta_i^{'-1}(p_i^0)$, $0 \leq k < \infty$, $1 \leq i \leq N$. One may verify that $\{(p_i^k, q_i^k); 1 \leq i \leq N, k \geq 0\}$ constitutes a minimum dimension state process for the dynamical system (II.5) and for all $k$, $\{p_i^k; 1 \leq i \leq N\} \subset B_p^0$.

### C. Best Reply Bids for Q-PSP

In this subsection we analyze the difference between the best quantized strategy of each agent (which is effectively uncomputable) and the dynamical recursion prescribed by (II.5). The best quantized strategy of each agent should be the quantized price $p_i \in B_p^0$ such that the agent's utility is maximized given $s_{-i}$. If the demand function $\theta_i^{'}$ intersects the corresponding market price curve $P_i^{cv}$ on $(w_i^{k+1}, v_i^{k+1})$ at the $(k+1)$th iteration and $p_n \leq w_i^{k+1} < p_{n+1}$ with two adjacent quantized prices $p_n, p_{n+1} \in B_p^0$, the best quantized reply for Agent $A_i$ is $(p_{bst}^{k+1}, \theta_i^{'-1}(p_{bst}^{k+1}))$ with

$$
p_{bst}^{k+1} = arg \max_{p \in \{p_n, p_{n+1}\}} a_i^{k+1}(p, s_{-i}^k) \leq Q_i(p_n, s_{-i}^k).
$$

More specifically, the allocated quantity within the best strategy is

$$
a_i^{k+1} = v_i^{k+1};
$$

the allocated quantity within the best quantized strategy is

$$
\begin{aligned}
a_i^{k+1} &= \max\{\theta_i^{'-1}(p_{n+1}),\\
&\quad \frac{\theta_i^{'-1}(p_n)}{\theta_i^{'-1}(p_n) + \sum_{l: p_l^k = p_n, l \neq i} q_l^k} Q_i(p_n, s_{-i}^k)\};
\end{aligned}
$$

and the quantized strategy in (II.5) brings about the allocated quantity as

$$
a_i^{k+1} = \frac{\theta_i^{'-1}(p_i^{k+1})}{\theta_i^{'-1}(p_i^{k+1}) + \sum_{l: p_l^k = p_i^{k+1}, l \neq i} q_l^k} Q_i(p_i^{k+1}, s_{-i}^k),
$$

where $p_i^{k+1}$ is calculated from (II.5b). Here the quantized strategy in (II.5) may not be the best (quantized) strategy for each agent, but it is the strategy that provides the lowest bid price such that the maximum available quantity is greater than the desired quantity $v_i$. Hence the quantized strategy is a $\gamma$-best reply with

$$
\gamma = u_i(v_i^{k+1}, s_{-i}^k) - u_i(a^{k+1}(p_i^{k+1}, s_{-i}^k), s_{-i}^k).
$$

On the other hand, if a Q-PSP dynamical system converges to a quantized price $p^*$, then $s^*$ is a $\delta$-Nash equilibrium in the quantized framework with $s_i^* = (p^*, \theta_i^{-1}(p^*))$ in the sense that:

$$
u_i(s_i^*, s_{-i}^*) \geq \sup_{p_i \in B_p^0} u_i(s_i, s_{-i}^*) - \delta \qquad \text{(II.6)}
$$

where $\delta$ is such that

$$
\begin{aligned}
\delta &> \max_i |u_i(s^*) - u_i((p^{**}, \theta_i^{-1}(p^{**})), s_{-i}^*)| && \text{(II.7)}\\
p^{**} &= \min\{p : p > p^*, p \in B_p^0\}.
\end{aligned}
$$

It is to be noted that in PSP and Q-PSP it is assumed that each agent $A_i$ makes bids only based upon its own knowledge, i.e. only based upon its own demand function and the bidding profiles of the other agents.

## III. Example of Fast Convergence

It will be shown in Sect. IV that a Q-PSP dynamical system converges to a limit or settles into an oscillation in a limited number of iterations, regardless of the number of agents. To illustrate this, assume there are $N$ agents and initially,

$$
\begin{aligned}
p_i^0 &= p_i = \frac{i}{N} && \text{(III.8a)}\\
q_i^0 &= q_i = -p_i + \eta = -\frac{i}{N} + \eta. && \text{(III.8b)}
\end{aligned}
$$

For simplicity the demand curve is taken to be linear and identical for all agents. Most adjacent steps of the market price curve of Agent $A_i$, $P_i$, are of equal height $\frac{i}{N}$ except where a discontinuity of more than a single price difference of the form $\{p_i, 1 \leq i \leq N\}$ takes place; this discontinuity will be termed a $jump$. A jump in $P_i$ will occur when both $p_{i-1}$ and $p_{i+1}$ are present in the set $\{0, p_1, p_2, p_3, ..., p_N\}$. Consequently, $P_N$ contains no price jump. In other words, the price jump is due to the fact that Agent $A_i$'s price is necessarily absent in its own MPF.

Let $N = 5$ and initial conditions are based on (III.8). Thus $p_1 < p_2 < p_3 < p_4 < p_5$, and $q_1 > q_2 > q_3 > q_4 > q_5$. The market price curves for all 5 agents are shown in Fig. 3.

Assume the demand curve is linear and identical for all 5 agents, and assume that it passes through their market price curves in the middle range as shown in Fig. 3 Thus there are cases where the price jump is above, below and close to the intersection point respectively. To illustrate what takes place at the first iteration, all 5 market price curves and the demand curve are drawn on the same plot.

Consequently, at $k = 1$, each agent would change to a new bid in the next iteration as follows:

Agent 1 : from $(p_1, q_1)$ to $(p_2, q_2)$
Agent 2 : from $(p_2, q_2)$ to $(p_1, q_1)$
Agent 3 : from $(p_3, q_3)$ to $(p_2, q_2)$
Agent 4 : from $(p_4, q_4)$ to $(p_2, q_2)$
Agent 5 : from $(p_5, q_5)$ to $(p_2, q_2)$

which is shown in Fig. 3.

Thus, at $k = 1$, the resulting market price curves and the demand curve are shown in Fig. 4. At $k = 2$, all 5 agents would settle down to $(p_2, q_2)$. Therefore, with one identical demand curve for all 5 agents, the top prices get cleared out in the first iteration, whereas the bottom prices are eliminated in the following iteration.

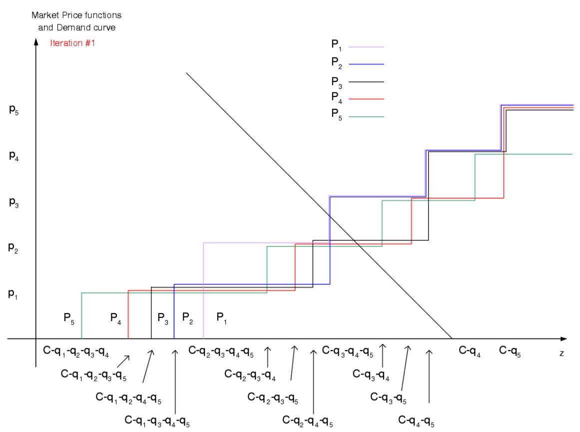


Fig. 3. At $k = 0$: The market price curves of 5 agents with one linear demand curve

## IV. Rapid Convergence for Multiple Users

### *A. Convergence Analysis for Multiple Users with Identical Linear Demand Functions*

Consider the hypotheses:

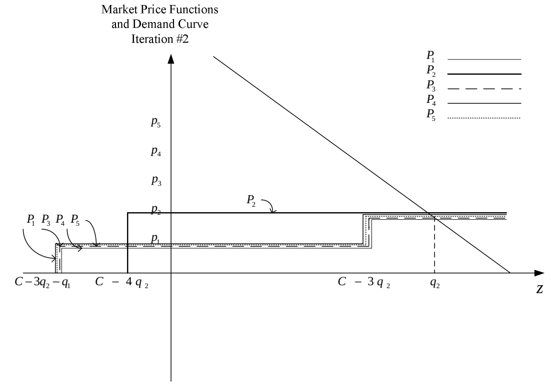


Fig. 4. At $k = 1$: The market price curves of 5 agents with one linear demand curve

**H1.** Let the initial condition for the Q-PSP system:

$$s \equiv (p, q) = [(p_1, q_1), ..., (p_N, q_N)], \qquad N > 2$$

be such that $0 < p_i$, $0 < q_i$, $1 \leq i \leq N$, where $p_i < p_{i+1}$, $1 \leq i \leq N-1$, and set $B_p^0 = \{p_i; 1 \leq i \leq N\} \cup \{0\}$. □

**H2.** Let all agents $A_i, 1 \leq i \leq N$, have the single demand function

$$q = \theta^{'-1}(p) = -\alpha p + \eta, \qquad p \in [0, \theta^{'}(0)].$$

□

**Theorem 1.** ([11], [12])

Subject to H1 and H2, the Q-PSP system trajectories $s^k = [s_i^k] = [(p_i^k, q_i^k)], 1 \leq i \leq N, 1 \leq k < \infty$, exhibit one of four distinct characteristics, namely:

(1) Convergence to a vector of (price, quantity) bids for all agents $s = ((p, q), \ldots, (p, q))$ in at most three iterations, where $p \in (p_1, ...p_N)$ and $q = \theta^{'-1}(p)$.
(2) Convergence to $((0, \eta), \ldots, (0, \eta))$ in at most three iterations.
(3) Convergence in at most three iterations to a non-trivial order-two orbit (i.e. an order two sustained oscillation) such that at Iteration $2k+1$ ($k \geq 1$), $r$ agents have the (price, quantity) pair $(p^*, \theta^{'-1}(p^*))$ and $N - r$ agents have the pair of $(p^{**}, \theta^{'-1}(p^{**}))$; at Iteration $2k$ ($k \geq 2$), $r$ agents have the (price, quantity) pair of $(p^{**}, \theta^{'-1}(p^{**}))$ and $N - r$ agents have the pair $(p^*, \theta^{'-1}(p^*))$, where $q^* < q^{**}$, $p^* = p_{j+1}$, and $p^{**} = p_j$ for some $j$ with $1 \leq j \leq N-1$.
(4) Convergence in at most three iterations to an order two sustained oscillation between $0$ and $p^{**}$.

□

**Outline of proof:**

The principle of the proof argument is summarized in the following steps: first we show that all prices strictly above and strictly below the (at most two) intersection prices $\{p^*, p^{**}\}$ given by $\{\text{demand curve}\} \cap \{\cup_{i=1}^N P_{i,0}^{cv}\}$ are eliminated at $k = 1$; then we recompute the market price curves of all agents at $k = 1$ and we show that the resulting market price functions' domains have at most three prices $\{p^*, p^{**}, 0\}$. The set of new market price curves at $k = 1$

has five distinguished zones, and the intersections $\{\text{demand curve}\} \cap \{\cup_{i=1}^{N} P_{i,1}^{cv}\}$ give rise to convergence in three cases; at $k = 2$ we show that the five possible cases repeat the situations at $k = 1$; finally at $k = 3$, there is either one price left on which all agents have converged, or oscillations initiate between $p^*, p^{**}$ or between $0, p^{**}$. Here $p^*, p^{**}$ are time-invariant and the quantized price set at $k \geq 1$ includes at most $0, p^*$, and $p^{**}$. □

In fact we may show [12] that under the hypotheses of Theorem 1, oscillations between $p^*$ and $p^{**}$ (or, respectively, $p^{**}$ and 0) will happen if and only if $N$ is even and $C$ satisfies

$$(\frac{N}{2} + 1)q^* > C \geq \frac{N}{2}q^*,$$

$$\left( \text{or, respectively, } (\frac{N}{2} + 1)q^{**} > C \geq \frac{N}{2}q^{**} \right).$$

Fig. 5 displays a dynamical quantized PSP system with 20 agents and one demand curve which converges at $k = 1$.

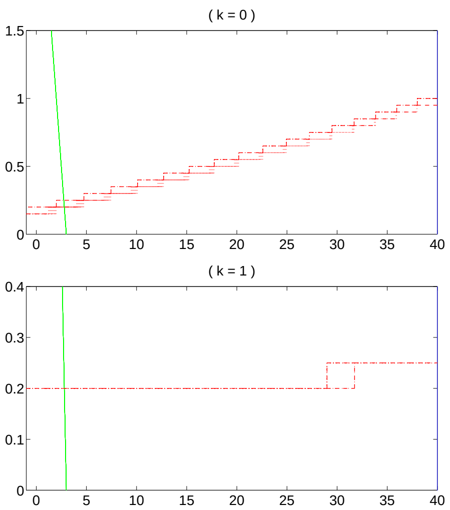


Fig. 5. Rapid convergence of a Q-PSP system with 20 agents and a single demand curve, for $k = 0, 1$.

**Efficiency**

In those cases where the Q-PSP system converges to a quantized price $p_\infty \neq 0$, each agent obtains the return $\frac{C}{N}$ based on the allocation rule (II.2) and the equilibrium is efficient, i.e. $\sum_{i=1}^{N} \theta_i(a_i)$ is maximized. This is clear, since by the decreasing property H2, $\theta_i(\cdot)$ is convex upwards. Hence, for all $i, q_1, q_2$, and $q_3$ satisfying $\theta_i^{'-1}(0) > q_1 > q_2 > q_3 > 0$ and $q_1 - q_2 = q_2 - q_3$, it implies $2\theta_i(q_2) > \theta_i(q_1) + \theta_i(q_3)$. The oscillatory case is discussed in [12].

### *B. Convergence Analysis for Multiple Users with a Family of $L_\infty$ Perturbations of a Given Demand Function*

We let $\Phi$ be the family of (elastic) demand functions on $[0, C]$. We observe that any function $\theta' \in \Phi$ is continuous on the compact set $[0, C]$ and is $1 : 1$ on $[0, C]$; it follows that $\theta'^{-1}$ is continuous and $1 : 1$ on $\theta'([0, C])$.

**Definition 2.**

The *$\delta$-neighborhood* of an inverse demand function $\theta'^{-1}, \theta' \in \Phi$, is the set of $\phi'^{-1}, \phi' \in \Phi$, satisfying

$$\sup_{z \geq 0} |\theta'^{-1}(z) - \phi'^{-1}(z)| < \delta,$$

i.e.

$$||\theta'^{-1}(\cdot) - \phi'^{-1}(\cdot)||_{L_\infty} < \delta.$$

□

**H$^\delta$ ($\delta$-neighborhood hypothesis).**

There exists a function $\theta' \in \Phi$ such that $\{\theta_i'^{-1}; 0 < i \leq N, \theta_i' \in \Phi\}$ lie in a $\delta$-neighborhood of $\theta'^{-1}$ with

$$\delta = \frac{1}{2N} \min_{0<m,n \leq N, m \neq n} |\theta'^{-1}(p_m) - \theta'^{-1}(p_n)|,$$

where $\{p_m, p_n\} \subset B_p^0$. □

This hypothesis guarantees that the family of demand curves will intersect the corresponding market price functions sufficiently closely that at most three quantized prices result after the first iteration.

**Theorem 2.**

Subject to H1 and the $\delta$-neighborhood hypothesis H$^\delta$, the Q-PSP system trajectories $s^k = [s_i^k] = [(p_i^k, q_i^k)], 1 \leq i \leq N, 1 \leq k < \infty$, exhibit the properties (1)-(4) of Theorem 1, but (i) with convergence taking place in at most five iterations, or (ii) with the oscillations of properties (3) and (4) being established in at most five iterations. □

**Outline of Proof**:

The proof is summarized by the following sequential steps: first we prove all prices strictly above and strictly below the (at most three) intersection prices $\{p_n, p_{n-1}, p_{n-2}\}$ given by $\{\text{demand curves}\} \cap \{\cup_{i=1}^{N} P_{i,0}^{cv}\}$ are eliminated at $k = 1$; then we recompute the market price curves of all agents at $k = 1$ and we show that the resulting market price functions' domains have at most three prices $\{p_n, p_{n-1}, 0\}$ based on the relations between $p_n$, $p_{n-1}$, $N$, and $C$; next the intersections $\{\text{demand curve}\} \cap \{\cup_{i=1}^{N} P_{i,1}^{cv}\}$ give convergence in two cases at $k = 1$; at $k = 2$, we recompute the market price curves under the non-convergence condition of $k = 1$, and show that the three possible cases repeat the situations at $k = 1$; this is continued until $k = 5$, when all possible relations between $p_n$, $p_{n-1}$, $N$, and $C$ are proved to satisfy either converge conditions or oscillation conditions. Here $p_n, p_{n-1}$ are time-invariant and the quantized price set at $k \geq 1$ includes at most $0, p_n$, and $p_{n-1}$. See [12] for the detailed proof. □

Theorem 1 is evidently a special case of Theorem 2, and Fig. 6 illustrates the convergence in the case where 8 agents share similar demand curves.

**Efficiency**

In those cases where the Q-PSP system converges to a quantized price $p_\infty \neq 0$ and the quantity allocation is $a^*$, the steady state is a $\delta$-Nash equilibrium in the quantized framework as described in Sect. II. Applying Proposition 3 in [5], we obtain

$$\max_{a \in A} \sum_i \theta_i(a_i) - \sum_i \theta_i(a_i^*) = O(\sqrt{\delta \kappa})$$

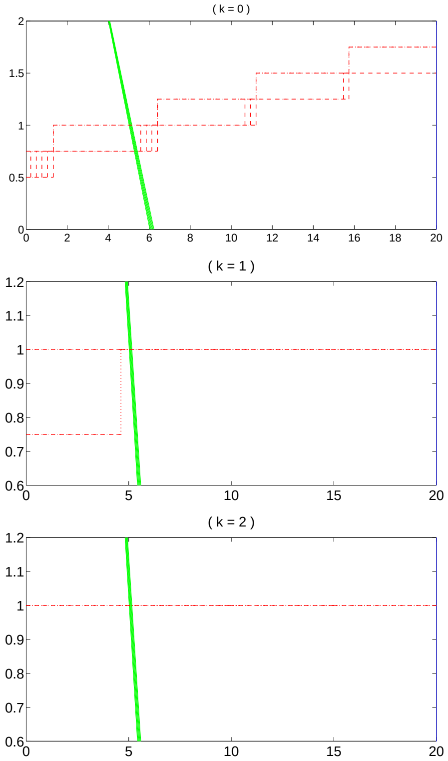


Fig. 6. Rapid convergence of a Q-PSP system with 8 agents and similar demand curves, for $k = 0, 1, 2$.

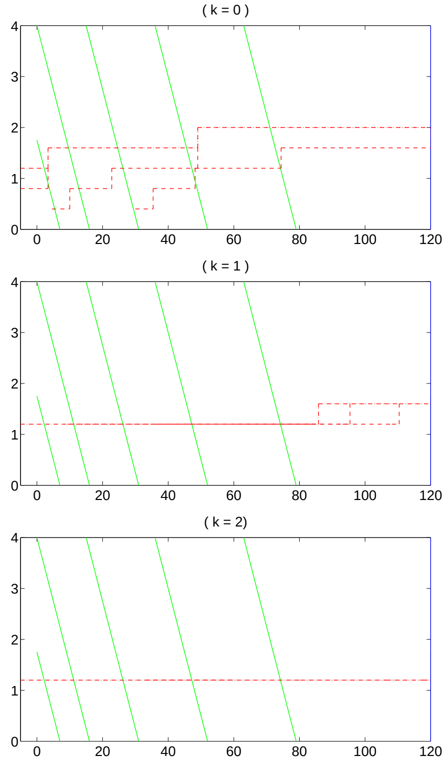


Fig. 7. Rapid convergence of a Q-PSP system with 5 agents and significantly distinct demand curves, for $k = 0, 1, 2$.

where $A$ describes the set of all possible quantity allocations under the quantization assumption, and it is assumed that for all $i, 0 < i \leq N$, the elastic demand functions $\theta_i^{'}$ satisfy

$$\theta_i^{'}(z) - \theta_i^{'}(z') > -\kappa(z - z'),$$

whenever $z > z' \geq 0$ (see Assumption 2 in [5]).

## V. Future Work

1) Simulations appear to demonstrate that the rapid convergence property still holds for dynamical Q-PSP systems in cases where the agents have significantly different demand functions (see Fig. 7). This is the subject of current research [12].
2) The bids of the agents in a Q-PSP system may be viewed as decentralized feedback controls. In this context, a current topic of study is the extent to which one can further control dynamical Q-PSP systems so as to avoid oscillatory behaviour and to manipulate the value of the social welfare function. The three principal control methods under analysis are: (i) manipulation of $C$, (ii) manipulation of a bid fee $\epsilon$, and (iii) the existence of an independent control agent.

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