Connectivity and Separating sets of Cages

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Abstract

A $(k;g)$-*cage* is a graph of minimum order among k-regular graphs with girth g. We show that for every cutset S of a $(k;g)$-cage G, the induced subgraph $G[S]$ has diameter at least $\lfloor g/2 \rfloor$, with equality only when distance $\lfloor g/2 \rfloor$ occurs for at least two pairs of vertices in $G[S]$. This structural property is used to prove that every $(k;g)$-cage with $k \geq 3$ is 3-connected. This result supports the conjecture of Fu, Huang, and Rodger that every $(k;g)$-cage is k-connected.

A nonseparating g-cycle C in a graph G is a cycle of length g such that $G-V(C)$ is connected. We prove that every $(k;g)$-cage contains a nonseparating g-cycle. For even g, we prove that every g-cycle in a $(k;g)$-cage is nonseparating.

1 Introduction

The *girth* of a graph G is the length of a shortest cycle in G. A $(k;g)$-*graph* is a k-regular graph with girth g. A $(k;g)$-*cage* G is a $(k;g)$-graph of minimum order. Let $f(k;g)$ be the order of a $(k;g)$-cage. Cages were introduced by Tutte [2]; results are surveyed in [3]. Recently, Fu, Huang, and Rodger [1] proved that all $(k;g)$-cages are 2-connected and that all $(3;g)$-cages are 3-connected. They further conjectured that every $(k;g)$-cage is k-connected. In this paper, we strengthen their results by showing for $k \geq 3$ that every $(k;g)$-cage is 3-connected. In order to prove this result, we develop a structural property of separating sets in cages. We show that every separating set of a $(k;g)$-cage induces a subgraph with diameter at least $\lfloor g/2 \rfloor$, with equality only when distance $\lfloor g/2 \rfloor$ occurs for at least two pairs of vertices in the induced subgraph.

A *nonseparating* g-cycle in a graph G is a cycle C of length g such that $G-V(C)$ is connected. We use the structural property about separating sets in cages to prove that every $(k;g)$-cage contains at least one nonseparating g-cycle. For even g, we prove that every g-cycle in a $(k;g)$-cage is nonseparating. We conjecture that the same is true for odd g.

Conjecture. *Every g-cycle in a $(k;g)$-cage is nonseparating.*

We consider only simple graphs. We denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. The neighborhood of a vertex $v \in V(G)$, written $N(v)$, is the set of all vertices adjacent to v. The degree of a vertex $v \in V(G)$ is $d(v) = |N(v)|$. A graph is k-regular if every vertex has degree k. For $u, v \in V(G)$, let $d_G(u,v)$ denote the distance between u and v in G; if u and v are in different components of G , then let $d_G(u,v) = \infty$. The diameter of G, written diam(G), is the maximum distance over all pairs of vertices in G. If $S \subseteq V(G)$, then $N(S) = \bigcup_{v \in S} N(v)$. If

$H \subseteq G$, then $N_H(S) = N(S) \cap V(H)$. The subgraph of G induced by S is denoted by $G[S]$. If $S \subseteq V(G)$ and $G - S$ is not connected, then S is a *cutset* or *separating set.*

Since the unique $(k; g)$-cages for $k = 2$, $g = 3$, and $g = 4$ are C_g, K_{k+1}, and $K_{k,k}$, respectively, we henceforth assume that $k \geq 3$ and $g \geq 5$ unless otherwise specified.

2 Cages are 3-connected

The following theorem about the monotonicity of cages is proved in [1].

Theorem 1. *If $k \geq 3$ and $3 \leq g_1 \leq g_2$, then $f(k; g_1) \leq f(k; g_2)$.*

Definition. Suppose that G is a $(k; g)$-cage and H is an induced subgraph of G with minimum degree $k - 1$. Let B be the set of vertices of degree $k - 1$ in H. For each permutation σ of B, let

$$D_\sigma(x, y) = d_H(x, y) + d_H(\sigma(x), \sigma(y)) + 2.$$

We say that H is a *special* subgraph of G if there exists a σ such that $d_H(x, y) \geq \lceil g/4 \rceil - 1$ and $D_\sigma(x, y) \geq g$ for all distinct $x, y \in B$.

Lemma 2. *If G is a $(k; g)$-cage and H is a special subgraph of G, then $|V(H)| \geq |V(G)|/2$.*

Proof. Let H' be a copy of H. For every $x \in V(H)$, let x' denote its image in H', and let B' be the copy of B in H'. Let G' be the graph that consists of the disjoint union of H and H' plus the edge $u\sigma(u)'$ for each $u \in B$. Since the edges between H and H' form a perfect matching between B and B', G' is k-regular and $|V(G')| = 2|V(H)|$. It suffices to show that G' has girth at least g.

Let C be a cycle in G'. If C lies completely in H or H', then it is a copy of a cycle in G and hence has length at least g. Hence we may assume that C uses some edges between H and H'. If C uses at least four edges between H and H', then it has length at least $4(\lceil g/4 \rceil - 1) + 4 \geq g$. If C uses exactly two edges $u\sigma(u)'$ and $v\sigma(v)'$ between H and H', where $u, v \in V(H)$, then it has length at least

$$d_H(u, v) + d_{H'}(\sigma(u)', \sigma(v)') + 2 = D_\sigma(u, v) \geq g. \qquad \text{(See Fig. 1)}$$

Hence G' has girth at least g. By Theorem 1, $|V(G')| \geq |V(G)|$. Thus $|V(H)| \geq |V(G)|/2$. $\square$

The following corollary is a simple application of Lemma 2 and appears in [1].

Corollary 3. *Every $(k; g)$-cage is 2-connected.*

Proof. Suppose u is a cut-vertex in a $(k; g)$-cage G and H is a component in $G - u$ of minimum order. Since G has girth g, the distance in H between two neighbors of u in H is at least $g - 2$. Letting σ be the identity permutation on $N_H(u)$, we have $D_\sigma(x, y) \geq g$ for $x, y \in N_H(u)$. Thus H is a special subgraph of G which contradicts $|V(H)| < |V(G)|/2$. $\square$

Lemma 4. *If G is a graph with girth $g \geq 3$, S is a subset of $V(G)$ with $\operatorname{diam}(G[S]) = d < g - 2$, and H is subgraph of G with $V(H) \cap S = \emptyset$, then every vertex of $N_H(S)$ has exactly one neighbor in S, and the distance in H between distinct neighbors of S is at least $g - d - 2$, with equality only if they are neighbors of a pair of vertices with distance d in $G[S]$.*

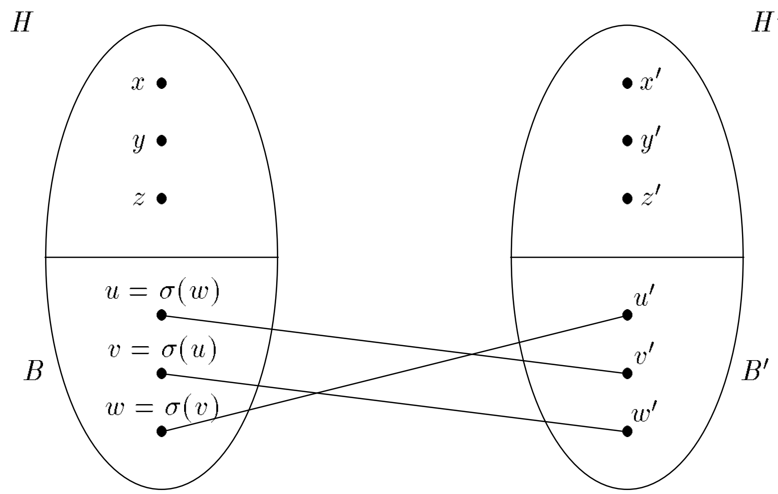

Fig. 1: Construction of G'

Proof. Let uu', vv' be distinct edges such that $u, v \in S$ and $u', v' \in V(H)$. We may assume that u', v' are in the same component of H. These edges together with a shortest u, v-path in $G[S]$ and a shortest u', v'-path in H form a cycle which must have length at least g. Hence $d_{G[S]}(u, v) + d_H(u', v') + 2 \geq g$. Since $d_{G[S]}(u, v) \leq d$, we conclude that $d_H(u', v') \geq g - 2 - d > 0$, with equality only if $d_{G[S]}(u, v) = d$. In particular, $u' \neq v'$ and no vertex in H can have two neighbors in S. □

Corollary 5. *Suppose G is a $(k; g)$-cage and S is a cutset of G with* $\operatorname{diam}(G[S]) < g - 2$. *If H is a smallest component in $G - S$, then $N_H(S)$ contains a pair of vertices u, v with $d_H(u, v) < \lceil g/2 \rceil - 1$.*

Proof. Since $\operatorname{diam}(G[S]) < g - 2$, by Lemma 4 every vertex of $N_H(S)$ has exactly one neighbor in S and hence has degree $k - 1$ in H. All other vertices in H have degree k in H. Suppose that $d_H(u, v) \geq \lceil g/2 \rceil - 1$ for every pair of vertices $u, v \in N_H(S)$. Let σ be the identity permutation of $N_H(S)$. We have

$$D_\sigma(u, v) = 2d_H(u, v) + 2 \geq 2(\lceil g/2 \rceil - 1) + 2 \geq g$$

for every pair of vertices $u, v \in N_H(S)$. By Lemma 2, $|V(H)| \geq |V(G)|/2$. This contradicts H being a smallest component of $G - S$, so $d_H(u, v) < \lceil g/2 \rceil - 1$ for some pair $u, v \in N_H(S)$. □

Theorem 6. *Suppose G is a $(k; g)$-cage and S is a cutset of G. Then* $\operatorname{diam}(G[S]) \geq \lfloor g/2 \rfloor$. *Furthermore, the inequality is strict if $d_{G[S]}(u, v)$ is maximized for exactly one pair of vertices.*

Proof. Suppose $\operatorname{diam}(G[S]) \leq \lfloor g/2 \rfloor - 1$. Let H be a smallest component in $G - S$. Since $g \geq 5$, we have $\lfloor g/2 \rfloor < g - 2$. If $N_H(S) = \{x\}$, then because G is 2-connected, x has at least 2 neighbors in S, which contradicts Lemma 4. We therefore suppose that $|N_H(S)| > 1$. Now Lemma 4 implies that the distance in H between distinct neighbors of S is at least $\lceil g/2 \rceil - 1$, contradicting Corollary 5.

Now suppose that $\operatorname{diam}(G[S]) = d = \lfloor g/2 \rfloor$ and u, v is the only pair in S with $d_{G[S]}(u, v) = \lfloor g/2 \rfloor$. By Lemma 4, every vertex of $N_H(S)$ has exactly one neighbor in S and $d_H(x, y) \geq g - 2 - d = \lceil g/2 \rceil -$

2 for all $x, y \in N_H(S)$. Since distance d in $G[S]$ occurs only for u, v, we have $d_H(x, y) = \lceil g/2 \rceil - 2$ only if x, y are neighbors of u, v in H, respectively. In particular, this implies that both u and v have neighbors in H; otherwise we have $d_H(x, y) \geq \lceil g/2 \rceil - 1$ for all $x, y \in N_H(S)$, contradicting Corollary 5. Suppose that $N_H(u) = \{u_1, \ldots, u_r\}$ and $N_H(v) = \{v_1, \ldots, v_s\}$, where $r, s \geq 1$. Without loss of generality, assume that $r \geq s$.

If we can show there exists a permutation σ of $N_H(S)$ such that $D_\sigma(x, y) \geq g$ for every pair $x, y \in N_H(S)$, then we can conclude from Lemma 2 that $V(H) \geq |V(G)|/2$ to get a contradiction.

Case 1 $r \geq 2$.

In this case, we define $\sigma(u_i) = u_{i+1}$, for $i = 1, \ldots, r$, where indices are taken modulo r. Also, define $\sigma(w) = w$ for all $w \in N_H(S) - N_H(u)$. Given a pair $x, y \in N_H(S)$, if both $d_H(x, y) \geq \lceil g/2 \rceil - 1$ and $d_H(\sigma(x), \sigma(y)) \geq \lceil g/2 \rceil - 1$, then $D_\sigma(x, y) \geq g$. By our previous discussion and our definition of σ, $d_H(x, y) = \lceil g/2 \rceil - 2$ or $d_H(\sigma(x), \sigma(y)) = \lceil g/2 \rceil - 2$ only if x, y are neighbors of u, v in H respectively. Hence we may assume that $x = u_i$ and $y = v_j$ for some i, j. The union of a shortest u_i, v_j-path and a shortest v_j, u_{i+1}-path in H contains a u_i, u_{i+1}-path in H. Together with the edges uu_i and uu_{i+1}, this forms a cycle in G, which must have length at least g. Therefore, $D_\sigma(x, y) \geq g$.

Case 2 $r = s = 1$ and $u_1 \leftrightarrow v_1$.

We have shown that in this case, u_1, v_1 is the only pair in $N_H(S)$ with distance $\lceil g/2 \rceil - 2$ in H; all other pairs have larger distance.

Since $1 = d_H(u_1, v_1) = \lceil g/2 \rceil - 2$, this case requires $g \in \{5, 6\}$. There is a vertex $w \in N_H(S) - u_1 - v_1$; otherwise $\{u_1, v_1\}$ is a cutset of G inducing a subgraph with diameter 1. We wish to show that $d_H(w, v_1)$ or $d_H(w, u_1)$ is at least $\lceil (g-1)/2 \rceil$. Because u, v is the only pair with distance $\lfloor g/2 \rfloor$ in S, both $d_H(w, v_1)$ and $d_H(w, u_1)$ are at least $\lfloor g/2 \rfloor - 1 \geq \lceil (g-1)/2 \rceil - 1$. If equality holds for both then the union of a shortest u_1, w-path and a shortest w, v_1-path contains a u_1, v_1-path of length between 2 and $2(\lfloor g/2 \rfloor - 1) < g - 1$. Together with $u_1 v_1$ this forms a cycle in G of length at most $g - 1$, contradicting $girth(G) = g$. Thus we may assume without loss of generality that $d_H(w, v_1) \geq \lceil (g-1)/2 \rceil$. Let $\sigma(u_1) = w$, $\sigma(w) = u_1$ and $\sigma(z) = z$ for all $z \in N_H(S) - u_1 - w$. If neither $\{x, y\}$ nor $\{\sigma(x), \sigma(y)\}$ is $\{u_1, v_1\}$, then $D_\sigma(x, y) \geq 2(\lceil g/2 \rceil - 1) + 2 \geq g$. Otherwise

$$D_\sigma(x, y) = d_H(u_1, v_1) + d_H(w, v_1) + 2 \geq (\lceil g/2 \rceil - 2) + \lceil (g-1)/2 \rceil + 2 = g.$$

Case 3 $r = s = 1$ and $u_1 \not\leftrightarrow v_1$.

Since $d_H(u_1, v_1) = \lceil g/2 \rceil - 2$, it follows that $g \geq 7$ in this case. We show that $H - u_1$ is a special subgraph of G to get a contradiction. The following discussion applies even if $H - u_1$ is not connected.

The distance in H between distinct vertices of $N_H(S)$ is at least $\lceil g/2 \rceil - 2$, which is at least 2 for $g \geq 7$. Thus u_1 has no neighbors in $N_H(S)$, so $H - u_1$ has minimum degree $k - 1$. Suppose that $\{z_1, \ldots, z_{k-1}\} = N_H(u_1)$. We have $d_{H-u_1}(z_i, w) \geq d_H(u_1, w) - 1 \geq \lceil g/2 \rceil - 3$ for all $w \in N_H(S) - u_1$ and $1 \leq i \leq k - 1$. Since a shortest path in $H - u_1$ from z_i to z_j (if z_i and z_j lie in the same component of $H - u_1$) forms a cycle of length at least g with the edges $u_1 z_i$ and $u_1 z_j$, we have $d_{H-u_1}(z_i, z_j) \geq g - 2$. Trivially $d_{H-u_1}(x, y) \geq d_H(x, y) \geq \lceil g/2 \rceil - 1$ for all pairs $\{x, y\}$ in $N_H(S) - u_1$. Now in $H - u_1$, vertices in $B = (N_H(S) - u_1) \cup \{z_1, \ldots, z_k\}$ have degree $k - 1$ and all other vertices have degree k. By the discussion above, the distance in $H - u_1$ between distinct vertices of B is at least $\lceil g/2 \rceil - 3$, which is at least $\lceil g/4 \rceil - 1$ for $g \geq 7$.

It remains to construct a permutation σ of B such that $D_\sigma(x, y) \geq g$ for all pairs x, y in B. Let $\sigma(z_i) = z_{i+1}$ for $i = 1, \ldots, k - 1$, where indices are taken modulo k, and let $\sigma(w) = w$ for all $w \in N_H(S) - u_1 - N_(u_1)$. If $x, y \in \{z_1, \ldots, z_{k-1}\}$ then $D_\sigma(x, y) \geq 2(g-2) + 2 > g$. If $x, y \in N_H(S) - u_1 - N(u_1)$ then $D_\sigma(x, y) \geq 2(\lceil g/2 \rceil - 1) + 2 \geq g$. Hence we may assume that $x = z_i$ and $y \in N_H(S)$.

The union of a shortest z_i, y-path and a shortest y, z_{i+1}-path in $H - u_1$ contains a z_i, z_{i+1}-path in $H - u_1$. Together with the edges $u_1 z_i$ and $z_{i+1} u_1$, this forms a cycle in G, which must have length at least g. Therefore, $D_\sigma(x, y) \geq g$. □

Remarks: A *separating path* of a graph G is an induced path P such that $G - V(P)$ is disconnected. By Theorem 6, a separating path of a $(k; g)$-cage has length at least $\lfloor g/2 \rfloor + 1$.

A *star cutset* is a cutset that contains a vertex adjacent to all the other vertices in the cutset. Since a star cutset induces a subgraph of diameter at most two, a $(k; g)$-cage contains no star cutset if $g \geq 6$. Thus if $k \geq 3$, $g \geq 6$ and G is a $(k; g)$-cage, then the removal of any vertex $v \in V(G)$ and any subset of vertices adjacent to v leaves G connected.

Now we can strengthen the results in [1] and prove that for $k \geq 3$ and $g \geq 5$, every $(k; g)$-cage is 3-connected.

Theorem 7. *If $k \geq 3$ and G is a $(k; g)$-cage, then G is 3-connected.*

Proof. Since the unique $(k; g)$-cages for $g = 3$ and $g = 4$ are K_{k+1} and $K_{k,k}$, respectively, and each of these is 3-connected, we may assume that $g \geq 5$. Suppose to the contrary that G has a cutset S of size 2. Among all such sets, choose $S = \{u, v\}$ to minimize the size of the smallest component of $G - S$. Let H be a smallest component of $G - S$. Since G is 2-connected, both $N_H(u)$ and $N_H(v)$ are nonempty. Furthermore, we have $|N_H(u)| \geq 2$ and $|N_H(v)| \geq 2$, otherwise either $N_H(v) \cup \{u\}$ or $N_H(u) \cup \{v\}$ would be a cutset of size 2 whose deletion leaves a component smaller than H. Let $N_H(u) = \{u_1, \ldots, u_r\}$ and $N_H(v) = \{v_1, \ldots, v_s\}$, where $r, s \geq 2$. Since $\{u, v\}$ is a cutset of G and $r, s \geq 2$, a shortest u, v-path through H is a separating path of G. By the remark after Theorem 6, it has length at least $\lfloor g/2 \rfloor + 1$ which is at least 3. This implies that $N_H(u) \cap N_H(v) = \emptyset$ and that $d_H(u_i, v_j) \geq \lfloor g/2 \rfloor - 1$ for all i, j. Furthermore, the distance in H between two distinct neighbors of u or two distinct neighbors of v is at least $g - 2$. The above discussions show that vertices of $N_H(S)$ have degree $k - 1$ in H and all other vertices in H have degree k in H. Also, the distance between distinct vertices of $N_H(S)$ is at least $\lfloor g/2 \rfloor - 1$, which is at least $\lceil g/4 \rceil - 1$.

We construct a permutation σ of $N_H(S)$ such that $D_\sigma(x, y) \geq g$. By Lemma 2 this contradicts $|V(H)| < |V(G)|/2$ and completes the proof. Let $\sigma(u_i) = u_{i+1}$ for $i = 1, \ldots, r$, where indices are taken modulo r, and let $\sigma(v_j) = v_j$ for $j = 1, \ldots, s$. Consider $x, y \in N_H(S)$. If both x and y are neighbors of u or neighbors of v, then the same is true for $\sigma(x)$ and $\sigma(y)$ and we have $D_\sigma(x, y) \geq 2(g-2) + 2 > g$. Hence we may assume that $x = u_i$ and $y = v_j$ for some i, j. The union of a shortest u_i, v_j-path and a shortest v_j, u_{i+1}-path in H contains a u_i, u_{i+1}-path in H. Together with the edges uu_i and $u_{i+1}u$, this forms a cycle in G, which must have length at least g. Therefore, $D_\sigma(x, y) \geq g$. □

3 Nonseparating cycles

Theorem 6 enables us to prove the following result about g-cycles in $(k; g)$-cages.

Theorem 8. *For $g \geq 5$, every $(k; g)$-cage contains a nonseparating g-cycle.*

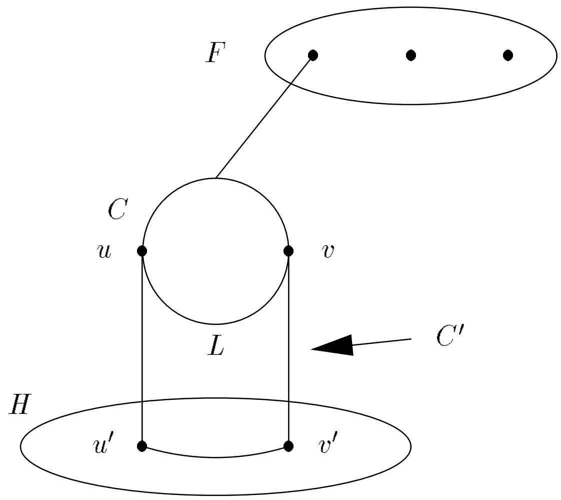

Fig. 2: A nonseparating cycle

Proof. Among the set $\mathcal{C}$ of g-cycles in G such that $G - V(C)$ is disconnected, choose $C \in \mathcal{C}$ to minimize the order of the smallest component H of $G - V(C)$.

Since C has diameter $\lfloor g/2 \rfloor$, Lemma 4 implies that every vertex $u \in N_H(C)$ has exactly one neighbor in C, and the distance in H between distinct neighbors of $V(C)$ is at least $\lceil g/2 \rceil - 2$. By Corollary 5, there are $u', v' \in N_H(C)$ with $d_H(u', v') = \lceil g/2 \rceil - 2$.

Suppose u, v are the neighbors of u', v' in C respectively. The cycle C' formed by the shorter piece L of C between u and v, a shortest u', v'-path in H and the edges uu', vv' has length at most $\lfloor g/2 \rfloor + 2 + (\lceil g/2 \rceil - 2) = g$. Since G has girth g, C' has length exactly g, and this occurs only when $d_C(u, v) = \lfloor g/2 \rfloor$. We claim that C' is nonseparating. Suppose this is not the case and that $G - V(C')$ is disconnected. Since L is a path of length at most $\lfloor g/2 \rfloor$, the remark after Theorem 6 implies that it is not separating. Hence every component in $G - V(C)$ has at least one neighbor in $C - L$. Consequently, the disconnected graph $G - V(C')$ has a component containing $V(C - L)$ and all the vertices in the components other than H in $G - V(C)$. All other components in $G - V(C')$ are smaller than H, contradicting our choice of C. $\square$

With more effort, we can strengthen this result for even g. We prove that for even g, *every* g-cycle in a $(k; g)$-cage is nonseparating. To do so, we need another lemma.

Given a cycle C of even length g and a vertex u on C, there is a unique vertex v on C with $d_C(u, v) = g/2$. We call v the *antipodal vertex* of u on C, and we call $\{u, v\}$ an *antipodal pair* on C. Note that distinct antipodal pairs on C are pairwise disjoint.

Lemma 9. *Let C be a cycle of even length $g \geq 6$, and let $\{u_1, v_1\}, \ldots, \{u_m, v_m\}$ be m antipodal pairs on C, not necessarily all distinct. Then there exists a cyclic list of m vertices consisting of one from $\{u_i, v_i\}$ for each i, such that any two consecutive vertices in the list are identical or nonadjacent on C.*

Proof. First suppose that all the pairs are distinct. Assume without loss of generality that these $2m$ distinct vertices appear along C in the order $u_1, \ldots, u_m, v_1, \ldots, v_m$. If $m = 1$, choose either vertex. If $m = 2$, then $g \geq 6$ implies that u_2 is not adjacent to both u_1 and v_1 on C. Select u_2 and one of $\{u_1, v_1\}$ not adjacent to it. For $m \geq 3$ the requirement is satisfied by the cyclic listing $u_1, u_3, \ldots, u_{2\lceil m/2\rceil -1}, v_{2\lfloor m/2\rfloor}, v_{2\lfloor m/2\rfloor -2}, \ldots, v_2$.

If some pairs are repeated, make the choice indicated above for the list obtained by deleting extra copies, and then expand the selection for each pair into successive copies of that selection for all copies of the pair. □

Theorem 10. *If $k \geq 3$ and $g \geq 4$ is even, then every g-cycle in a $(k;g)$-cage is nonseparating.*

Proof. Since the unique $(k;g)$-cage for $g = 4$ is $K_{k,k}$ for which the claim holds trivially, we may assume that $g \geq 6$. Let G be a $(k;g)$-cage, and let C be a g-cycle in G. Suppose that $G - V(C)$ is disconnected. Let H be a smallest component in $G - V(C)$. Since $\text{diam}(C) = g/2$, by Lemma 4 every vertex w in $N_H(C)$ has a unique neighbor, which we denote by w', in C. Hence H has minimum degree $k-1$, with $N_H(C)$ being the set of vertices with degree $k-1$. For $x, y \in N_H(C)$, a shortest x,y-path in H, a shortest x', y'-path in C and the edges xx', yy' form a cycle in G of length at least g. Thus

$$d_H(x, y) \geq g - 2 - d_C(x', y'), \qquad (*).$$

Since $d_C(x', y') \leq g/2$, we have $d_H(x, y) \geq g/2 - 2$, with equality only if $d_C(x', y') = g/2$. We shall call a pair $\{x, y\}$ in $N_H(C)$ a *bad pair* if $d_H(x, y) = g/2 - 2$. As we have seen, if $\{x, y\}$ is a bad pair, then $\{x', y'\}$ is an antipodal pair on C. A pair $\{x, y\}$ in $N_H(C)$ that is not a bad pair satisfies $d_H(x, y) \geq g/2 - 1$. Now suppose $\{x, y\}$ and $\{x, z\}$ are two distinct bad pairs. Then both y' and z' are antipodal vertices of x' on C, hence $y' = z'$. A shortest x, y-path and a shortest x, z-path in H together contain a y, z-path in H of length at most $g - 4$. Adding the edges $y'y$ and $y'z$ yields a cycle in G of length less than g. This is impossible, so distinct bad pairs are pairwise disjoint.

If we can show that there exists a permutation σ of $N_H(C)$ such that $D_\sigma(x, y) \geq g$ for all pairs $\{x, y\}$ in $N_H(C)$, we can conclude from Lemma 2 that $|V(H)| \geq |V(G)|/2$ to obtain a contradiction. If neither $\{x, y\}$ nor $\{\sigma(x), \sigma(y)\}$ is a bad pair, then $D_\sigma(x, y) \geq g$. It suffices to construct σ such that $D_\sigma(x, y) \geq g$ and $D_\sigma(\sigma^{-1}(x), \sigma^{-1}(y)) \geq g$ whenever $\{x, y\}$ is a bad pair.

If there is no bad pair, then any permutation of $N_H(C)$ works. Hence we may assume that a bad pair exists. Suppose $\{x_1, y_1\}, \ldots, \{x_m, y_m\}$ are all the distinct bad pairs in $N_H(C)$. Correspondingly, $\{x'_1, y'_1\}, \ldots, \{x'_m, y'_m\}$ are m antipodal pairs on C.

Case 1 $m \geq 2$.

By Lemma 9, there is a cyclic listing of m vertices, one from each pair in $\{x'_1, y'_1\}, \ldots, \{x'_m, y'_m\}$, such that two consecutive vertices on the listing, if distinct, are nonadjacent on C. By relabeling if necessary, we may assume that $x'_1, \ldots, x'_m$ is the desired cyclic listing. Define σ on $N_H(C)$ as follows: For $i = 1, \ldots, m$, let $\sigma(x_i) = x_{i+1}$, where indices are taken modulo m; for $z \in N_H(C) - \{x_1, \ldots, x_m\}$, let $\sigma(z) = z$. The permutation σ maps the bad pair $\{x_i, y_i\}$ to $\{x_{i+1}, y_i\}$. If $x'_i = x'_{i+1}$, then a shortest x_i, y_i-path and a shortest x_{i+1}, y_i-path in H together contain an x_i, x_{i+1} path, which forms a cycle in G with the edges $x_i x'_i$ and $x_{i+1} x'_i$ (see Fig. 3). Since G has girth g, we have $D_\sigma(x_i, y_i) \geq g$.

If $x'_i \neq x'_{i+1}$, then these vertices are nonadjacent on C, which implies that $d_C(x'_{i+1}, y'_i) \leq g/2 - 2$. By $(*)$, we have $d_H(x_{i+1}, y_i) \geq g/2$, and thus $D_\sigma(x_i, y_i) \geq g$. Similar arguments apply to $\{\sigma^{-1}(x), \sigma^{-1}(y)\} = \{x_{i-1}, y_i\}$.

Case 2 $m = 1$

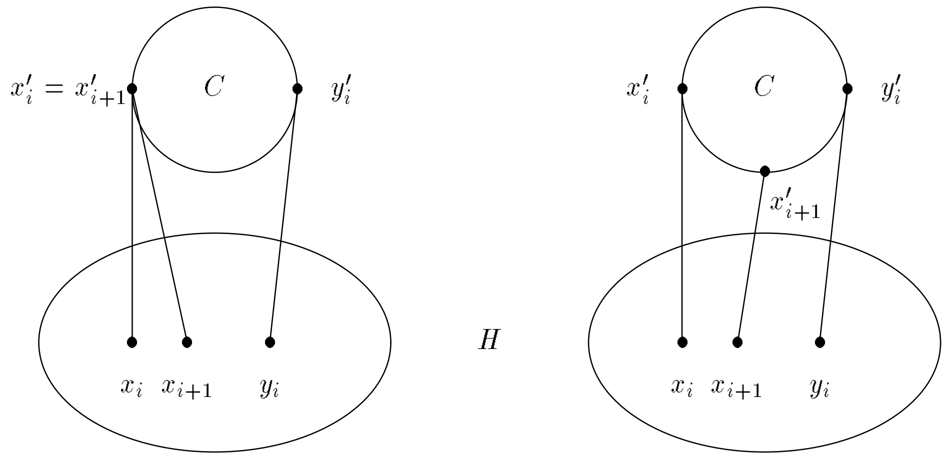

Fig. 3: Bad pairs and antipodal pairs of vertices

In this case, $\{x_1, y_1\}$ is the only bad pair. If no vertex in C other than x'_1 and y'_1 has a neighbor in H, then a shortest x'_1, y'_1-path in C is a separating path of length $g/2$, contradicting the remark after Theorem 6. Hence there exists $w \in N_H(C) - x_1 - y_1$ such that its neighbor w' in C is distinct from x_1 and y_1. Since $g \geq 6$, w' is not adjacent to both x'_1 and y'_1 on C. We may assume that w' is not adjacent to x'_1 on C. Then $d_C(w', y'_1) \leq g/2 - 2$. By $(*)$, we have $d_H(w, y_1) \geq g/2$. Now let $\sigma(x_1) = w$, $\sigma(w) = x_1$ and $\sigma(z) = z$ for $z \in N_H(C) - x_1 - w$. Both σ and σ^{-1} map the only bad pair $\{x_1, y_1\}$ to $\{w, y_1\}$, so $D_\sigma(x, y) \geq g$ for all $x, y \in N_H(C)$. This completes the proof. $\square$

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