Performance Analysis of Pairwise Amplify-and-Forward Multi-Way Relay Networks

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***Abstract*—For the first time, the performance of pairwise amplify-and-forward multi-way relay networks (MWRNs) is studied. To this end, new end-to-end signal-to-noise ratio (e2e SNR) expression at an arbitrary source is first derived in closed-form, and thereby, an insightful statistical characterization is developed. In this context, tight closed-form approximations are derived for the cumulative distribution function, probability density function and moment generating function of the e2e SNR. Specifically, the conditional outage probability and the average bit error rate conditioned on error-free back-propagated successive interference cancellation are also derived in closed-form. Moreover, valuable insights into practical MWRN system-designing is obtained by deriving the fundamental diversity-multiplexing trade-off by employing the high SNR outage probability approximation.**

***Index Terms*—Relay networks, amplify-and-forward.**

I. Introduction

TWO-WAY relay networks (TWRNs) are being studied for next generation wireless networks with half-duplex terminals as they are as twice spectral efficient as one-way relay networks (OWRNs) [1], [2]. Multi-way relay networks (MWRNs) are the natural generalization of TWRNs in which more than two sources exchange their information bearing signals by using intermediate relays. While the achievable rates have been quantified in [3], [4], a comprehensive performance analysis of MWRNs is lacking right now. In this letter, the performance of pairwise amplify-and-forward (AF) MWRNs is studied.

Although multi-way communication channels have been studied as early as 1970s, their practical significance has only been fully exploited after the emergence of research in modern cooperative relay networks. To this end, multi-way communication channels have been exploited with the aid of relays, and consequently, resulted a flurry of recent research in the context of MWRNs [3]–[5]. To be more specific, in [3], the achievable symmetric rate of MWRNs are studied for several relay processing strategies. In [4], the capacity region of a class of multi-way relay channels, where the channel inputs and outputs take values over finite fields, is derived. Moreover, in [5], a pairwise decode-and-forward[1] (DF) transmission strategy is proposed and studied for MWRNs. It is also shown that this pairwise DF transmission strategy achieves the common-rate capacity for binary MWRNs. Recently, in [6], pairwise DF MWRNs based on deterministic broadcasting with side information have been shown to be optimal in the sense of sum-capacity. All the aforementioned transmission strategies employ the DF protocol, and exploit inherent benefits of physical layer network coding.

In this letter, the pairwise AF transmission is studied for MWRNs[2]. Similar to [5], in our system set-up, $M \geq 2$ sources exchange M independent symbols with the aid of a relay in two consecutive multiple-access (MAC) and broadcast (BC) phases each having $M-1$ time-slots. Unlike [5], the relay employs the AF protocol in the BC phase, and each source decodes the symbols belonging to the remaining $M-1$ sources by employing back-propagated successive interference cancellation (SIC).

In this work, we establish basic performance metrics of pairwise AF MWRNs. To this end, tight closed-form approximations for the cumulative distribution function (CDF), probability density function (PDF) and the moment generating function (MGF) of the end-to-end signal-to-noise ratio (e2e SNR) are derived. Moreover, the conditional outage probability and the average bit error rate (BER) conditioned on error-free back-propagated SIC are also derived in closed-form. Useful insights are obtained by quantifying the achievable diversity-multiplexing trade-off (DMT) by employing the high SNR approximation of the outage probability.

Notations: $\mathcal{K}_\nu(z)$ is the Modified Bessel function of the second kind of order ν [7, Eq. (8.407.1)]. ${}_2\mathcal{F}_1(\alpha,\phi;\gamma;z)$ is the Gauss Hypergeometric function [7, Eq. (9.14.1)]. $\mathcal{W}_{\nu,\mu}(z)$ is the Whittaker-W function [7, Eq. (9.222.1)]. $\mathcal{U}(\mu,\nu,z)$ is the HypergeometricU function [8]. $\Gamma(\nu,z)$ is the upper Incomplete Gamma function [7, Eq. (8.350.2)]. $\mathcal{Q}(z)$ denotes the Gaussian Q-function [9, Eq. (26.2.3)].

II. System model

We consider a pairwise AF MWRN consisting of M sources (S_m) for $m \in \{1,\cdots,M\}$, and one relay node (R). Each terminal is equipped with a single-antenna and operates in half-duplex mode. All the channel amplitudes are assumed to be independently distributed frequency-flat Rayleigh fading, and are assumed to be remain constant over $2(M-1)$ time-slots [5]. The direct channel between S_i and S_j for $i \neq j$ is assumed to be unavailable due to heavy path-loss and shadowing [5].

In this protocol, all M sources exchange their information-bearing signals, x_m, satisfying $\mathcal{E}\left[|x_m|^2\right] = 1$ for $m \in \{1,\cdots,M\}$, each other in two consecutive MAC and BC transmission phases each having $M-1$ time-slots. In the ith time-slot of the MAC phase, S_i and S_{i+1} source pair, where $i \in \{1,\cdots,M-1\}$, transmits $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ simultaneously to R. Consequently, the superimposed-signal received at R in the ith time-slot of MAC phase is given by

$$y_R^{(i)} = \sqrt{\mathcal{P}_i} h_{i,R}\mathbf{x}_i + \sqrt{\mathcal{P}_{i+1}} h_{i+1,R}\mathbf{x}_{i+1} + n_R^{(i)}, \qquad (1)$$

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[1]This pairwise decode-and-forward transmission strategy is also known as functional decode-and-forward (FDF).

[2]It is worth noticing that all the benefits of AF relaying over DF relaying in both OWRNs and TWRNs are applicable to MWRNs as well.

where $\mathcal{P}_i$ is the transmit power at S_i and $h_{i,R} \sim \mathcal{CN}(0, \phi_i)$ is the channel coefficient[3] from S_i to R. Moreover, $n_R^{(i)} \sim \mathcal{CN}(0, \sigma_R^2)$ is additive white Gaussian noise (AWGN) at R. The aforementioned MAC phase continues until the last source pair, S_{M-1} and S_M, completes their transmission. In this context, R has now received M–1 pairwise transmissions.

During the BC phase, R broadcasts an amplified version of the $M-1$ superimposed-signals back to all M sources sequentially in $M-1$ consecutive time-slots. The received signal at the mth source in the jth time-slot of the BC phase is thus given by

$$y_{S_m}^{(j)} = G_j h_{R,i} y_R^{(i)} + n_m^{(j)}, \quad \text{for } j \in \{1, \cdots, M-1\}, \quad (2)$$

where $G_j = \sqrt{\mathcal{P}_R/(\mathcal{P}_i|h_{i,R}|^2 + \mathcal{P}_{i+1}|h_{i+1,R}|^2 + \sigma_R^2)}$ is the power normalizing factor designed to constraint the instantaneous power at R. In (2), $h_{R,i}$ is the channel coefficient from R to S_i, and is assumed to be equal to $h_{i,R}$ due to the reciprocity of wireless channels [1], [4], [5]. Further, P_R is the transmit power at R and $\mathbf{n}_m^{(j)} \sim \mathcal{CN}(0, \sigma_m^2)$ is the AWGN at S_m for $m \in \{1, \cdots, M\}$.

After the BC phase, an arbitrary source, S_m has now received $M-1$ independent signals from which the signals belonging to the remaining $M-1$ sources can readily be decoded by using back-propagated SIC.

The e2e SNR of the symbol belonging to S_n received at S_m assuming no error propagations for $m \in \{1, \cdots, M\}$, $n \in \{1, \cdots, M\}$ and $m \neq n$ is given by

$$\gamma_{S_m}^{(n)} = \begin{cases} \frac{\bar{\gamma}_{R,m}\bar{\gamma}_{n,R}|h_m|^2|h_n|^2}{(\bar{\gamma}_{R,m}+\bar{\gamma}_{m,R})|h_m|^2+\bar{\gamma}_{n,R}|h_n|^2+1}, & n = m \pm 1 \\ \frac{\bar{\gamma}_{R,m}\bar{\gamma}_{n,R}|h_m|^2|h_n|^2}{\bar{\gamma}_{R,m}|h_m|^2+\bar{\gamma}_{n,R}|h_n|^2+\bar{\gamma}_{n',R}|h_{n'}|^2+1}, & n \neq m \pm 1, \end{cases} \quad (3)$$

where $\bar{\gamma}_{R,m} = \mathcal{P}_R/\sigma_m^2$, $\bar{\gamma}_{n,R} = \mathcal{P}_n/\sigma_R^2$, and $h_m = h_{R,m} = h_{m,R}$. In (3), $n' = n-1$ if $n > m$ and $n' = n+1$ otherwise.

III. STATISTICAL CHARACTERIZATION OF THE E2E SNR

In this section, the CDF of the e2e SNR, $\gamma_{S_m}^{(n)}$ is first derived, and then used to derive the PDF and MGF in closed-form.

A. The CDF of the e2e SNR

The e2e SNR for $n = m \pm 1$ case has already appeared in the context of AF TWRNs, and consequently, its exact CDF has already been derived in closed-form in [2].

On the contrary, to the best of our knowledge, the CDF of $\gamma_{S_m}^{(n)}$ for $n \neq m \pm 1$ case has not yet been derived in the open literature, and this derivation is one of the main contribution of this work. In this context, the exact CDF of $\gamma_{S_m}^{(n)}$ for $n \neq m \pm 1$ case can be derived as (see Appendix I for the proof)

$$F_{\gamma_{S_m}^{(n)}}(w) = 1 - \frac{1}{\alpha\beta} \int_0^\infty \int_0^\infty \bar{F}_X\left(\frac{w}{\alpha\beta t}\left[\mu\left(t + \eta w\right) + \alpha\beta\left(\nu\lambda + c\right)\right]\right) \times f_Y\left(\frac{t + \eta w}{\alpha\beta}\right) f_Z(\lambda)\, \mathrm{d}t \mathrm{d}\lambda, \quad (4)$$

where $X = |h_m|^2$, $Y = |h_n|^2$, and $Z = |h_{n'}|^2$. In (4), $\alpha = \bar{\gamma}_{R,m}$, $\beta = \bar{\gamma}_{n,R}$, $\eta = \bar{\gamma}_{R,m}$, $\mu = \bar{\gamma}_{n,R}$, $\nu = \bar{\gamma}_{n',R}$, and $c = 1$. Moreover, $\bar{F}_X(x)$ is the complementary CDF (CCDF) of X.

[3]Here, ϕ_i accounts for the path-loss effect and is modeled as $\phi_i \propto (d_{S_i,R})^{-\zeta_i}$, where $d_{S_i,R}$ is the distance between S_i and R, and ζ_i is the path-loss exponent.

By first noting that X, Y and Z are independent exponentially distributed, and then by substituting the corresponding CDFs and PDFs into (4), $F_{\gamma_{S_m}^{(n)}}(w)$ can be further simplified as (see Appendix I for the proof)

$$F_{\gamma_{S_m}^{(n)}}(w) = 1 - \frac{\mathrm{e}^{-\frac{w}{\alpha\beta}\left(\frac{\mu}{\phi_x} + \frac{\eta}{\phi_y}\right)}}{\alpha\beta\sqrt{\phi_x\phi_y}} \int_0^\infty \mathcal{J}(\lambda)\mathrm{e}^{-\lambda}\mathrm{d}\lambda, \quad (5)$$

where the function $\mathcal{J}(\lambda)$ is given by

$$\mathcal{J}(\lambda) = \sqrt{\mu\eta w^2 + \alpha\beta w\left(\nu\phi_z\lambda + c\right)} \times \mathcal{K}_1\left(\frac{2}{\alpha\beta}\sqrt{\frac{\mu\eta w^2 + \alpha\beta w\left(\nu\phi_z\lambda + c\right)}{\phi_x\phi_y}}\right). \quad (6)$$

In (5) and (6), $\phi_x = \phi_m$, $\phi_y = \phi_n$, and $\phi_z = \phi_{n'}$. The exact closed-form evaluation of (5) appears mathematically intractable. Interestingly, the integral of (5) is in the form of Gauss-Laguerre quadrature (GLQ) [9, Eq. (25.4.45)], and thus can readily be evaluated as

$$F_{\gamma_{S_m}^{(n)}}(w) = 1 - \frac{\mathrm{e}^{-\frac{w}{\alpha\beta}\left(\frac{\mu}{\phi_x} + \frac{\eta}{\phi_y}\right)}}{\alpha\beta\sqrt{\phi_x\phi_y}} \sum_{t=1}^{T_g} \Delta_t\mathcal{J}(\lambda_t) + \mathcal{R}_{T_g}, \quad (7)$$

In (7), $\lambda_t|_{t=1}^{T_g}$ and $\Delta_t|_{t=1}^{T_g}$ are the abscissas and weights of the GLQ, respectively [9, Eq. (25.4.45)]. Specifically, λ_t is the tth root of the Laguerre polynomial $\mathcal{L}_n(x)$ [9, Chap. 22], and the corresponding tth weight is given by $\Delta_t = \frac{(t!)^2 x_t}{(t+1)^2[\mathcal{L}_{t+1}(x_t)]^2}$. Both λ_t and Δ_t can be efficiently computed by using the classical algorithm proposed in [10]. In particular, T_g is the number of terms used in the GLQ summation, and $\mathcal{R}_{T_g}$ is the residue term, which readily diminishes as T_g approaches as small as 20 [10].

Next, in order to obtain valuable insights into practical AF MWRN designs, a computationally efficient and simple approximation of $F_{\gamma_{S_m}^{(n)}}(w)$ is also derived in closed-form as follows (see Appendix II for the proof):

$$F_{\gamma_{S_m}^{(n)}}(w) \approx 1 - \Omega w \mathrm{e}^{-w(\Psi - \Omega/2)}\Gamma(-1, \Omega w), \quad (8)$$

where $\Omega = \frac{\nu\phi_z}{\alpha\beta\phi_x\phi_y}$ and $\Psi = \frac{\mu\phi_y + \eta\phi_x - \nu\phi_z/2}{\alpha\beta\phi_x\phi_y}$.

Remarks III.1: The PDF and MGF of $\gamma_{S_m}^{(n)}$ for the case $n = m \pm 1$ and the corresponding performance metrics have been extensively studied in the open literature [2]. Henceforth only the results related to $n \neq m \pm 1$ are provided in the sequel.

B. The PDF of the e2e SNR

The PDF of $\gamma_{S_m}^{(n)}$ for $n \neq m \pm 1$ can be approximately derived by differentiating (8) as

$$f_{\gamma_{S_m}^{(n)}}(w) \approx \Omega\left(\left(\Psi - \Omega/2\right)w - 1\right)\mathrm{e}^{-w(\Psi - \Omega/2)}\Gamma(-1, \Omega w) + \frac{1}{w}\mathrm{e}^{-w(\Psi + \Omega/2)}. \quad (9)$$

C. The MGF of the e2e SNR

The MGF is an useful statistical metric, which can be employed to yield unified analysis. Thus, the MGF of $\gamma_{S_m}^{(n)}$ for $n \neq m \pm 1$ can be approximately derived by substituting (9) into $\mathcal{M}_\Lambda(x) = \int_0^\infty \mathrm{e}^{-s} f_\Lambda(x)\, \mathrm{d}x$ and evaluating the resulting integral by using [7, Eq. (6.455.1)] as

$$\mathcal{M}_{\gamma_{S_m}^{(n)}}(s) \approx 1 - \frac{s}{2\Psi + \Omega}\, {}_2\mathcal{F}_1\left(1, 1; 3; \frac{2s + 2\Psi - \Omega}{2s + 2\Psi + \Omega}\right). \quad (10)$$

IV. Performance Analysis

In this section, the basic performance metrics of pairwise AF MWRNs are derived. To this end, the outage probability and average BER are derived by using the statistical characterization of e2e SNR presented in Section III.

A. Outage probability

The conditional outage probability of decoding the symbol sent by S_n at S_m conditioned on error-free back-propagated SIC can be readily derived by employing either (7) or (8) as:

$$P_{\text{out},S_m^{(n)}} = \Pr\left(\gamma_{S_m}^{(n)} \leq \gamma_{th}\right) = F_{\gamma_{S_m}^{(n)}}\left(\gamma_{th}\right), \tag{11}$$

where γ_{th} is the preset SNR threshold, which is determined to satisfy the minimum service-rate constraint; $\gamma_{th} = 2^{\mathcal{R}_{th}} - 1$, where $\mathcal{R}_{th}$ is the target rate.

The high SNR outage probability approximation provides useful insights into valuable system design parameters such as the achievable diversity order and DMT. By first substituting the Maclaurin series expansions of exponential and upper incomplete gamma functions into (8), and then evaluating the resulting first order expansion at γ_{th}, the outage probability approximation at high SNRs is derived as

$$P_{\text{out},S_m^{(n)}}^{\infty} \approx \left[\Psi' + \frac{\Omega'}{2} - \Omega'\gamma - \Omega' \ln\left(\frac{\Omega'\gamma_{th}}{\bar{\gamma}_{n,R}}\right)\right]\left(\frac{\gamma_{th}}{\bar{\gamma}_{n,R}}\right), \tag{12}$$

where $\Omega' = \frac{k_1\phi_z}{\phi_x\phi_y}$, $\Psi' = \frac{k_1\phi_y + k_2\phi_x - k_1k_2\phi_z/2}{k_2\phi_x\phi_y}$, $k_1 = \frac{\bar{\gamma}_{n,R}}{\bar{\gamma}_{R,m}}$, and $k_2 = \frac{\bar{\gamma}_{n_R}}{\bar{\gamma}_{n',R}}$. In (12), γ is the Euler-Mascheroni constant and given by $\gamma = -\psi(1) = 0.577215665...$ [7, Eq. (8.367.1)].

Now, one can argue that $P_{\text{out},S_m}^{\infty}$ in (12) is proportional to $(\bar{\gamma}_{n,R})^{-1}$ at high SNRs, and hence, the achievable diversity order is quantified to be unity.

The overall outage probability at S_m is defined as

$$P_{\text{out},S_m}^{\text{overall}} = \Pr\left(\min_{n\in\{1,\cdots M\}\cap n\neq m} \gamma_{S_m}^{(n)} \leq \gamma_{th}\right). \tag{13}$$

Finally, the overall outage probability of the whole system is defined as

$$P_{\text{out}}^{\text{overall}} = \Pr\left(\min_{m\in\{1,\cdots M\}}\left[\min_{n\in\{1,\cdots M\}\cap n\neq m} \gamma_{S_m}^{(n)}\right] \leq \gamma_{th}\right). \tag{14}$$

Unfortunately, the exact analysis of $P_{\text{out},S_m}^{\text{overall}}$ or $P_{\text{out}}^{\text{overall}}$ in (14) appears mathematically intractable due to the statistical dependence of $\gamma_{S_m}^{(n)}$ for $n \in \{1, \cdots M\} \cap n \neq m$, and hence, we leave it as an open problem. However, the overall diversity order of the whole system can be intuitively determined to be unity, as each of data stream of any particular source is no larger than unity and taking the minimum in (14) would not increase the achievable diversity gains.

B. Average BER

The conditional average BER of decoding the symbol belonging to S_n at S_m conditioned on error-free back-propagated SIC can be derived by averaging the instantaneous BER, $P_{e,S_m}^{(n)} = a\mathcal{Q}\left(\sqrt{b\gamma_{S_m}^{(n)}}\right)$, over the PDF of $\gamma_{S_m}^{(n)}$ as follows:

$$\bar{P}_{e,S_m}^{(n)} \approx \frac{a}{2} - \frac{a}{3}\sqrt{\frac{b}{b+2\Psi+\Omega}}\, {}_2\mathcal{F}_1\left(1, \frac{1}{2}; \frac{5}{2}; \frac{b+2\Psi-\Omega}{b+2\Psi+\Omega}\right), \tag{15}$$

where a and b are modulation dependent constants.

The instantaneous block error rate (BLER) at S_m, which is the probability of having at least one error at the detected symbols sent by the remaining $M-1$ sources, can be derived assuming errors are independent at each step of SIC as

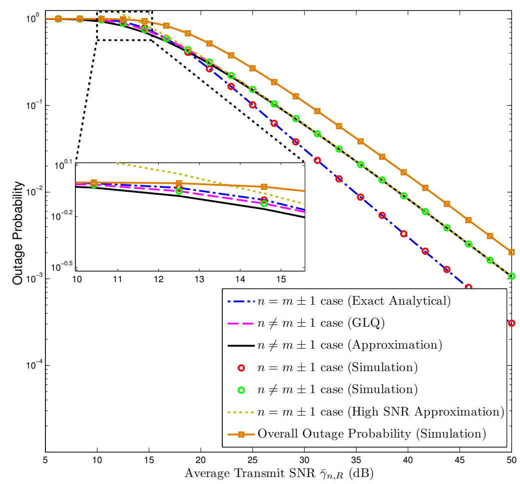

Fig. 1. The outage probability of a pairwise AF four-way relay network with $\gamma_{th} = 5.00\,\text{dB}$.

$$P_{B,S_m} = 1 - \prod_{n=1}^{M-1}\left(1 - P_{e,S_m}^{(n)}\right) = \sum_{j=1}^{M-1} P_{e,S_m}^{(n)} \prod_{n=1}^{j-1}\left(1 - P_{e,S_m}^{(n)}\right). \tag{16}$$

However, the average BLER can not be expressed as $\bar{P}_{B,S_m} = \sum_{j=1}^{M-1} \bar{P}_{e,S_m}^{(n)} \prod_{n=1}^{j-1}\left(1 - \bar{P}_{e,S_m}^{(n)}\right)$ because of the statistical dependence of $\gamma_{S_m}^{(n)}$ for $n \in \{1, \cdots, M\}$. Again, we leave the derivation of the average BLER as an open problem.

C. Diversity-multiplexing trade-off

In this pairwise AF MWRN, M independent symbols are exchanged in $2(M-1)$ time-slots among M participating sources. In this context, the overall DMT of the whole system can be derived by employing the results in Section IV-A as

$$d(r) = (1 - 2(M-1)r/M). \tag{17}$$

It is worth noticing that the maximum achievable multiplexing gain is given by $r = \frac{M}{2(M-1)}$. Interestingly, r is maximized when $M = 2$, i.e., $r_{\max} = \lim_{M\to 2} \frac{MN_R}{2(M-1)} = 1$. Counterintuitively, as M becomes large, r approaches $1/2$, i.e., $r_{\min} = \lim_{M\to\infty} \frac{M}{2(M-1)} = \frac{1}{2}$, which can readily be achieved by AF OWRNs. This result leads us to one important insight into practical implementation of MWRNs with pairwise transmissions; i.e., the multiplexing gain of AF MWRNs gradually reduces to $1/2$ as the number of sources increases, and consequently, the asymptotic multiplexing gain approaches that of AF OWRNs.

V. Numerical Results

In Fig. 1, both conditional and overall outage probability curves of S_1 for a pairwise AF four-way relay network are plotted. The analytical curves for $n = m \pm 1$ are plotted by using the outage probability analysis of [2]. Fig. 1 clearly reveals that the outage probability corresponding to the case $n = m \pm 1$ is considerably lower[4] than that of the case $n \neq m \pm 1$. In particular, our outage approximation (11) is notably tight to the exact curves in almost entire SNR regime. In addition, the overall outage probability at S_1 is plotted by using Monte-Carlo simulations.

[4]This behavior is evident from the high SNR outage approximation (12) as it contains $\bar{\gamma}_{n,R}^{-1}\log\left(\bar{\gamma}_{n,R}^{-1}\right)$ term.

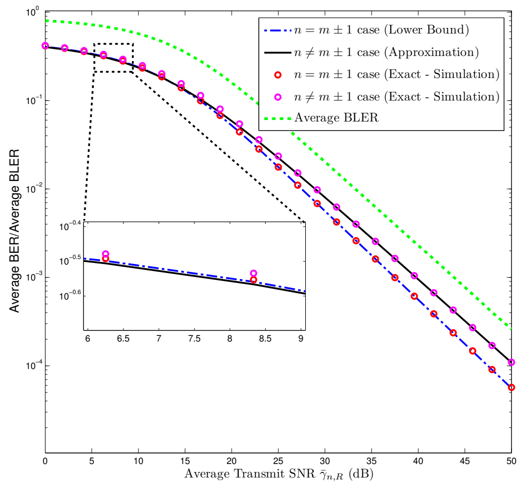

Fig. 2. The average BER of a pairwise AF four-way relay network.

In Fig. 2, the conditional average BER of binary phase shift keying and the average BLER are plotted for a pairwise AF four-way relay network by letting $a = 1$ and $b = 2$ in (15). The analytical BER curve of $n = m \pm 1$ case is plotted by using the BER lower bound of [2]. Again, Fig. 2 clearly reveals that the BER of $n = m \pm 1$ case is lower than that of $n \neq m \pm 1$ case. Specifically, our BER approximation (15) is considerably tight to the exact BER curves in entire SNR regime. However, Fig. 2 clearly shows that both $n = m \pm 1$ and $n \neq m \pm 1$ cases achieve the same diversity order of one.

VI. Conclusion

The performance of pairwise AF MWRNs was studied. To this end, tight approximations for the CDF, PDF, and MGF of e2e SNR were derived in closed-form. In particular, the conditional outage probability and the average bit error rate conditioned on error-free back-propagated SIC were derived in closed-form. The fundamental DMT was also quantified by using the high SNR approximation of the outage probability. Our analysis reveals that the pairwise AF MWRNs achieves the maximum multiplexing gain whenever the number of participating sources are limited to two.

Appendix I : Proof of $F_{\gamma_{S_m}^{(n)}}(w)$

In this appendix, the proof of the CDF of $\gamma_{S_m}^{(n)}$ for $n \neq m \pm 1$ is sketched. To this end, $\gamma_{S_m}^{(n)}$ in (3) is rewritten as

$$\gamma_{S_m}^{(n)} = \frac{\alpha\beta XY}{\eta X + \mu Y + \nu Z + c}, \tag{18}$$

where α, β, η, μ, and ν are the average transmit SNRs, and are already defined in (4). In addition, $X = |h_m|^2$, $Y = |h_n|^2$, and $Z = |h_{n'}|^2$ are independently distributed exponential random variables with means ϕ_x, ϕ_y, and ϕ_z, respectively. Thus, the CDF of $\gamma_{S_m}^{(n)}$ can be derived as

$$\begin{aligned} F_{\gamma_{S_m}^{(n)}}(w) &= \Pr\left(\frac{\alpha\beta XY}{\eta X + \mu Y + \nu Z + c} \leq w\right) \\ &= \int_0^\infty\int_0^\infty \Pr\left(x[\alpha\beta\theta - \eta w] \leq w[\mu\theta + \nu\lambda + c]\right) f_Y(\theta) f_Z(\lambda)\, \mathrm{d}\theta \mathrm{d}\lambda \\ &= \int_0^\infty \left[F_Y\left(\frac{\eta w}{\alpha\beta}\right) + \int_{\frac{\eta w}{\alpha\beta}}^\infty F_X\left(\frac{w(\mu\theta + \nu\lambda + c)}{\alpha\beta\theta - \eta w}\right) f_Y(\theta)\, \mathrm{d}\theta\right] f_Z(\lambda)\, \mathrm{d}\lambda \\ &= F_Y\left(\frac{\eta w}{\alpha\beta}\right) + \frac{1}{\alpha\beta}\int_0^\infty\int_0^\infty F_X\left(\frac{w\left(\mu(t + \eta w) + \alpha\beta(\nu\lambda + c)\right)}{\alpha\beta t}\right) \\ &\quad \times f_Y\left(\frac{t + \eta w}{\alpha\beta}\right) f_Z(\lambda)\, \mathrm{d}t\, \mathrm{d}\lambda. \end{aligned} \tag{19}$$

The last equality of (19) is obtained by changing the dummy variable as $\theta = \frac{t + \eta w}{\alpha\beta}$. Next, by substituting $F_X(x) = 1 - \bar{F}_X(x)$ into (19), the desired integral expression of $F_{\gamma_{S_m}^{(n)}}(w)$ can readily be derived as in (4).

Now, by substituting $\bar{F}_X(x) = \mathrm{e}^{-\frac{x}{\phi_x}}$, $f_Y(y) = \frac{1}{\phi_y}\mathrm{e}^{-\frac{y}{\phi_y}}$, and $f_Z(z) = \frac{1}{\phi_z}\mathrm{e}^{-\frac{z}{\phi_z}}$ into (4), and after some mathematical manipulations, $F_{\gamma_{S_m}^{(n)}}(w)$ can be written as

$$F_{\gamma_{S_m}^{(n)}}(w) = 1 - \int_0^\infty \frac{\mathrm{e}^{-\frac{w}{\alpha\beta}\left(\frac{\mu}{\phi_x} + \frac{\eta}{\phi_y} + \frac{\lambda}{\phi_z}\right)}}{\alpha\beta\phi_y\phi_z}\mathcal{I}(\lambda)\, \mathrm{d}\lambda, \tag{20}$$

where the function $\mathcal{I}(\lambda)$ is defined as

$$\mathcal{I}(\lambda) = \int_0^\infty \mathrm{e}^{-\left(\frac{t}{\alpha\beta\phi_y} + \frac{\mu\eta w^2 + \alpha\beta w(\nu\lambda + c)}{\alpha\beta\phi_x t}\right)}\mathrm{d}t. \tag{21}$$

Next, by first evaluating $\mathcal{I}(\lambda)$ in closed-form by using [7, Eq. (3.471.9)], and then changing the dummy variable $\lambda \to \phi_z\lambda$, the desired single integral expression for $F_{\gamma_{S_m}^{(n)}}(w)$ can be derived as given in (5).

Appendix II : Proof of Approximation of $F_{\gamma_{S_m}^{(n)}}(w)$

The exact closed-form evaluation of $F_{\gamma_{S_m}^{(n)}}(w)$ in (5) appears mathematically intractable. However, in order to evaluate (5) in closed-form, $\mathcal{J}(\lambda)$ can be tightly approximated as

$$J(\lambda) \approx J'(\lambda) = \sqrt{\alpha\beta w\nu\phi_z\lambda}\ \mathcal{K}_1\left(\frac{2}{\alpha\beta}\sqrt{\frac{\alpha\beta w\nu\phi_z\lambda}{\phi_x\phi_y}}\right). \tag{22}$$

By substituting (22) into (5), and evaluating the resulting integral by using [7, Eq. (6.631.3)], a tight approximation of $F_{\gamma_{S_m}^{(n)}}(w)$ can be derived as

$$F_{\gamma_{S_m}^{(n)}}(w) \approx 1 - \mathrm{e}^{-\Psi w}\ \mathcal{W}_{-1,\frac{1}{2}}(\Omega w), \tag{23}$$

where Ψ and Ω are already defined in (8). By substituting the identities $\mathcal{W}_{a,b}(z) = z^{a+1/2}\mathrm{e}^{-z/2}\mathcal{U}(a - b + 1/2, 2a + 1, z)$ [11] and $\mathcal{U}(k, k, z) = \mathrm{e}^z\Gamma(1 - k, z)$ [12] successively into (23), the desired result can be obtained as in (8).

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