RESIDUAL QUOTIENT FUZZY SUBSET IN NEAR-RINGS

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For any fuzzy subsets λ and μ, we introduce the notion of residual quotient fuzzy subset $(\lambda : \mu)$ and we have characterized residual quotient fuzzy subset in near-rings.

1. Introduction

In 1965, Zadeh [14] introduced the concept of fuzzy subsets and studied their properties on the parallel lines to set theory. In 1971, Rosenfeld [10] defined the fuzzy subgroup and gave some of its properties. Rosenfeld's definition of a fuzzy group is a turning point for pure mathematicians. Since then, the study of fuzzy algebraic structure has been pursued in many directions such as groups, rings, modules, vector spaces, and so on. In 1981, Das [2] explained the interrelationship between the fuzzy subgroups and their t-level subsets. Fuzzy subrings and ideals were first introduced by Wang-jin Liu [5] in 1982. Subsequently, Mukherjee and Sen [7], Swamy and Swamy [13], Dixit et al. [3], and Rajesh Kumar [4] applied some basic concepts pertaining to ideals from classical ring theory and developed a theory of fuzzy. The notions of fuzzy subnear-ring and ideal were first introduced by Abou-Zaid [1] in 1991.

Wang-jin Liu [6] introduced residual quotient fuzzy subset $(\lambda : \mu)$ for any two fuzzy ideals in rings. In this paper, we introduce residual quotient fuzzy subset $(\lambda : \mu)$ for any two fuzzy subsets, which is different from [6], and we characterize some related results in near-rings.

2. Preliminaries

We would like to reproduce some definitions and results proposed by the pioneers in this field earlier for the sake of completeness.

Definition 2.1. A *near-ring* N is a system with two binary operations $+$ and $\cdot$ such that

(1) $(N,+)$ is a group, not necessarily abelian;
(2) $(N,\cdot)$ is a semigroup;
(3) $(x+y)z = xz + yz$ for all $x,y,z \in N$.

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We will use the word "near-ring" to mean "right distributive near-ring." We denote xy instead of $x \cdot y$. Note that $0 \cdot x = 0$ and $(-x)y = -xy$ but in general $x \cdot 0 \neq 0$ for some $x \in N$.

Definition 2.2. Let $(N, +, \cdot)$ be a near-ring. A subset I of N is said to be an *ideal* of N if
(1) $(I, +)$ is a normal subgroup of $(N, +)$;
(2) $IN \subseteq I$;
(3) $n_1(n_2 + i) - n_1 n_2 \in I$ for all $i \in I$ and $n_1, n_2 \in N$.

If I satisfies (1) and (2), then it is called a *right ideal* of N. If I satisfies (1) and (3), then it is called a *left ideal* of N.

Let N be a near-ring. Given two subsets A and B of N, the product $AB = \{ab \mid a \in A,\ b \in B\}$ and $A * B = \{a(a' + b) - aa' \mid a,\ a' \in A,\ b \in B\}$. From now on, throughout this paper N will denote right distributive near-ring, unless otherwise specified. For the basic terminology and notation, we refer to Pilz [9] and Abou-Zaid [1].

Definition 2.3. A mapping $\mu : N \to [0, 1]$ is called a *fuzzy subset* of N.

A fuzzy subset $\mu : N \to [0, 1]$ is nonempty if μ is not the constant map which assumes the value 0. For any two fuzzy subsets λ and μ of N, $\lambda \leq \mu$ means that $\lambda(a) \leq \mu(a)$ for all $a \in N$. The characteristic function of N is denoted by $\mathbf{N}$ and, of its subset A is denoted by f_A. The image of a fuzzy subset μ is denoted by $Im(\mu) = \{\mu(n) \mid n \in N\}$. Hereafter, we consider only nonempty fuzzy subsets of N.

Definition 2.4. Let E be an N-group and let μ be a fuzzy subset of E. Then μ is called a *fuzzy N-subgroup* of N if for all $x, y \in E$ and $n \in N$,
(1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$;
(2) $\mu(nx) \geq \mu(x)$.

Definition 2.5. Let μ be any fuzzy subset of N. For $t \in [0, 1]$, the set $\mu_t = \{x \in N \mid \mu(x) \geq t\}$ is called a *level subset* of μ.

Definition 2.6. Let f and g be any two fuzzy subsets of N. Then $f \cap g$, $f \cup g$, $f + g$, fg, and $f * g$ are fuzzy subsets of N defined by

$$
\begin{gathered}
(f \cap g)(x) = \min\{f(x), g(x)\}, \\
(f \cup g)(x) = \max\{f(x), g(x)\}, \\
(f + g)(x) = \begin{cases} \sup\limits_{x = y + z} \{\min\{f(y), g(z)\}\}, & \text{if } x \text{ is expressed as } x = y + z, \\ 0, & \text{otherwise}, \end{cases} \\
(fg)(x) = \begin{cases} \sup\limits_{x = yz} \{\min\{f(y), g(z)\}\}, & \text{if } x \text{ is expressed as } x = yz, \\ 0, & \text{otherwise}, \end{cases} \\
(f * g)(x) = \begin{cases} \sup\limits_{x = a(b+c) - ab} \{\min\{f(a), g(c)\}\}, & \text{if } x = a(b + c) - ab, \\ 0, & \text{otherwise}. \end{cases}
\end{gathered}
\tag{2.1}
$$

Definition 2.7. For any $x \in N$ and $t \in (0,1]$, define a *fuzzy point* x_t as

$$x_t(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases} \tag{2.2}$$

If x_t is a fuzzy point and μ is any fuzzy subset of N and $x_t \leq \mu$, then we write $x_t \in \mu$. Note that $x_t \in \mu$ if and only if $x \in \mu_t$ where μ_t is a level subset of μ. If x_r and y_s are fuzzy points, then $x_r y_s = (xy)_{\min\{r,s\}}$.

Definition 2.8. Let μ be a nonempty fuzzy subset of N. μ is a *fuzzy ideal* of N if for all $x, y, i \in N$,

(1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$;
(2) $\mu(x + y) = \mu(y + x)$;
(3) $\mu(xy) \geq \mu(x)$;
(4) $\mu(x(y + i) - xy) \geq \mu(i)$.

If μ satisfies (1), (2), and (3), then it is called a *fuzzy right ideal* of N. If μ satisfies (1), (2), and (4), then it is called a *fuzzy left ideal* of N. If μ is both fuzzy right and fuzzy left ideal of N, then μ is called a *fuzzy ideal* of N.

Definition 2.9. An ideal P of N is called *completely prime* if any two elements a, b of N such that $ab \in P$ implies either $a \in P$ or $b \in P$.

Definition 2.10. An ideal P of N is called *completely semiprime* if any element a of N such that $a^2 \in P$ implies $a \in P$ for all $x \in N$.

Definition 2.11. A fuzzy ideal μ of N is called *completely fuzzy prime ideal* if any two fuzzy points x_r, y_s of N such that $x_r y_s \in \mu$ implies either $x_r \in \mu$ or $y_s \in \mu$ for all $x, y \in N$ and for all $r, s \in [0,1]$.

Definition 2.12. A fuzzy ideal μ of N is called *completely fuzzy semiprime ideal* if any fuzzy point x_r of N such that $(x_r)^2 \in \mu$ implies $x_r \in \mu$ for all $x \in N$ and $r \in [0,1]$.

Lemma 2.13 [11]. *Let I be a nonempty subset of N. I is an N-subgroup of N if and only if f_I is a fuzzy N-subgroup of N.*

Lemma 2.14 [11]. *Let μ be a fuzzy subset of N. μ is a fuzzy N-subgroup of N if and only if the level subset μ_t, $t \in Im(\mu)$, is an N-subgroup of N.*

Lemma 2.15 [1]. *Let I be a subset of N. I is an (left or right) ideal of N if and only if f_I is a fuzzy (left or right) ideal of N.*

Lemma 2.16 [1]. *Let μ be a fuzzy subset of N. μ is a fuzzy (left or right) ideal of N if and only if the level subset μ_t, $t \in Im(\mu)$, is an ideal of N.*

Definition 2.17. A subgroup Q of $(N, +)$ is said to be a *quasi-ideal* of N if $QN \cap NQ \cap N * Q \subseteq Q$.

A subgroup B of $(N, +)$ is said to be a *bi-ideal* of N if $BNB \cap (BN) * B \subseteq B$.

Now we introduce the notion of fuzzy bi-ideal of N. We characterize fuzzy quasi-ideal and fuzzy bi-ideal of N.

Definition 2.18 [8]. A fuzzy subgroup μ of N is called a *fuzzy quasi-ideal* of N if $(\mu\mathbf{N}) \cap (\mathbf{N}\mu) \cap (\mathbf{N} * \mu) \leq \mu$.

Definition 2.19. A fuzzy subgroup μ of N is called a *fuzzy bi-ideal* of N if $(\mu\mathbf{N}\mu) \cap (\mu\mathbf{N} * \mu) \leq \mu$.

Note that

$$\begin{aligned}(\mu\mathbf{N} * \mu)(z) &= \sup_{z=x(y+c)-xy}\{\min\{(\mu\mathbf{N})(x),\mu(c)\}\}\\ &= \sup_{z=x(y+c)-xy}\left\{\min\left\{\sup_{x=x_1x_2}\{\mu(x_1),\mu(c)\}\right\}\right\}\\ &= \sup_{z=x_1x_2(y+c)-x_1x_2y}\{\min\{\mu(x_1),\mu(c)\}\}\\ &= 0, \quad \text{otherwise.}\end{aligned} \tag{2.3}$$

It is very clear that if N is a zero-symmetric near-ring, then $\mu\mathbf{N}\mu \leq \mu$ for every fuzzy bi-ideal μ.

Lemma 2.20 [8]. *Let μ be a fuzzy subset of N. If μ is a fuzzy left ideal (right ideal, N-subgroup, subnear-ring) of N, then μ is a fuzzy quasi-ideal of N.*

Proof. Let μ be a fuzzy left ideal of N. Let $x \in N$ and $x = ab = n_1(n_2+c) - n_1n_2$, where a, b, n_1, n_2, and c are in N. Consider

$$\begin{aligned}&(\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x)\\ &\quad= \min\{(\mu\mathbf{N})(x),(\mathbf{N}\mu)(x),(\mathrm{N} * \mu)(x)\}\\ &\quad= \min\left\{\sup_{x=ab}\{\mu(a)\},\sup_{x=ab}\{\mu(b)\},\sup_{x=n_1(n_2+c)-n_1n_2)}\{\mu(c)\}\right\}\\ &\quad\leq \min\left\{1,1,\sup_{x=n_1(n_2+c)-n_1n_2)}\{\mu(n_1(n_2+c)-n_1n_2)\}\right\}\\ &\qquad(\text{as } \mu \text{ is a fuzzy left ideal}, \mu(n_1(n_2+c)-n_1n_2) \geq \mu(c))\\ &\quad\leq \mu(x).\end{aligned} \tag{2.4}$$

We remark that if x is not expressed as $x = ab = n_1(n_2+c) - n_1n_2$, then $(\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x) = 0 \leq \mu(x)$. Thus $\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu \leq \mu$. Hence μ is a fuzzy quasi-ideal of N. □

Lemma 2.21. *For any nonempty subsets A and B of N,*

(1) $f_Af_B = f_{AB}$;
(2) $f_A \cap f_B = f_{A\cap B}$;
(3) $f_A * f_B = f_{A*B}$.

Proof. Proof is straight forward. □

Lemma 2.22. *Let Q be a subgroup of N.*
(1) *Q is a quasi-ideal of N if and only if f_Q is a fuzzy quasi-ideal of N.*
(2) *Q is a bi-ideal of N if and only if f_Q is a fuzzy bi-ideal of N.*

Proof. Proof of (1) can easily be seen in [8].

Proof of (2). Assume that Q is a bi-ideal of N. Then f_Q is a fuzzy subgroup of N. $f_Q f_N f_Q \cap f_Q f_N * f_Q \le f_{QNQ\cap QN*Q} \le f_Q$. This means that f_Q is a fuzzy bi-ideal of N.

Conversely, let us assume that f_Q is a fuzzy quasi-ideal of N. Let y be any element of $QNQ \cap QN * Q$. Then, we have

$$f_Q(y) \ge (f_Q f_N f_Q \cap f_Q f_N * f_B)(y) = f_{QNQ\cap QN*Q}(y) = 1. \tag{2.5}$$

Thus $y \in Q$ and $QNQ \cap QN * Q \subseteq Q$. Hence Q is a bi-ideal of N. □

Lemma 2.23. *Any fuzzy quasi-ideal of N is a fuzzy bi-ideal of N.*

Proof. Let μ be any fuzzy quasi-ideal of N. Then, we have

$$\begin{gathered}\mu\mathbf{N}\mu \subseteq \mu(\mathbf{N}\,\mathbf{N}) \subseteq \mu\mathbf{N},\\ \mu\mathbf{N}\mu \subseteq (\mathbf{N}\,\mathbf{N})\mu \subseteq \mathbf{N}\mu,\\ \mu\mathbf{N} * \mu \subseteq (\mathbf{N}\,\mathbf{N}) * \mu \subseteq \mathbf{N} * \mu,\\ \mu\mathbf{N}\mu \cap \mu\mathbf{N} * \mu \subseteq \mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu \subseteq \mu.\end{gathered} \tag{2.6}$$

Hence, μ is a fuzzy bi-ideal of N. □

However, the converse of Lemma 2.23 is not true.

Example 2.24. Let $N = \{0, a, b, c\}$ be Klein's four group. Define multiplication in N as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

•	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

(2.7)

Then, $(N, +, \bullet)$ is a near-ring (see [9, Page 408, Scheme 15]).

Define $\mu : N \to [0, 1]$ by

$$\mu(x) = \begin{cases} 1, & \text{if } x = 0, a,\\ 0, & \text{otherwise.}\end{cases} \tag{2.8}$$

For any $t \in [0,1]$, $\mu_t = \{0,a\}$ or $\{0,a,b,c\}$. Since $\{0,a\}$ and $\{0,a,b,c\}$ are bi-ideal in N, μ_t is the bi-ideal in N for all t. Hence μ is a fuzzy bi-ideal of N. Now,

$$\begin{aligned}
(\mu\mathbf{N})(b) &= \sup_{b=xy}\{\min\{\mu(x),\mathbf{N}(y)\}\} \\
&= \sup\{\min\{\mu(a),\mathbf{N}(a)\},\min\{\mu(c),\mathbf{N}(c)\},\min\{\mu(a),\mathbf{N}(c)\}, \\
&\qquad \min\{\mu(c),\mathbf{N}(a)\}\}, \quad \text{as } b = a\cdot a = c\cdot c = a\cdot c = c\cdot a, \\
&= \sup\{\min\{1,1\},\min\{0,1\},\min\{1,1\},\min\{0,1\}\} \\
&= \sup\{1,0,1,0\} \\
&= 1.
\end{aligned} \tag{2.9}$$

Similarly, we have $(\mathbf{N}\mu)(b) = 1$. Thus,

$$(\mu\mathbf{N}\cap\mathbf{N}\mu)(b) = \min\{(\mu\mathbf{N})(b),(\mathbf{N}\mu)(b)\} = \min\{1,1\} = 1. \tag{2.10}$$

But $\mu(b) = 0$. Thus, $(\mu\mathbf{N}\cap\mathbf{N})(b) = 1 \nleq \mu(b) = 0$. Therefore, μ is not a fuzzy quasi-ideal of N.

Lemma 2.25. *Let μ be a fuzzy subset of N. If μ is a fuzzy left ideal (right ideal, N-subgroup, subnear-ring) of N, then μ is a fuzzy bi-ideal of N.*

Proof. As μ is a left ideal of N and Lemma 2.20, μ is a fuzzy quasi-ideal of N. Hence by Lemma 2.23, μ is a fuzzy bi-ideal of N. □

Theorem 2.26. *Let μ be a fuzzy subset of N. If μ is a fuzzy quasi-ideal of N, if and only if μ_t is a quasi-ideal of N, for all $t \in Im(\mu)$.*

Proof. Let μ be a fuzzy quasi-ideal of N. Let $t \in Im(\mu)$. Suppose $x,y \in N$ such that $x,y \in \mu_t$. Then $\mu(x) \ge t$, $\mu(y) \ge t$, and $\min\{\mu(x),\mu(y)\} \ge t$. As μ is a fuzzy quasi-ideal, $\mu(x-y) \ge t$ and hence $x-y \in \mu_t$. Let $x \in N$. Suppose $x \in \mu_t N \cap N\mu_t \cap N * \mu_t$. Then, there exist $a,b,c \in \mu_t$ and $n_1,n_2,n_3,n_4 \in N$ such that $x = an_1 = n_2b = n_3(n_4+c) - n_3n_4$. Thus, $\mu(a) \ge t$, $\mu(b) \ge t$, and $\mu(c) \ge t$. Then,

$$\begin{aligned}
(\mu\mathbf{N}\cap\mathbf{N}\mu\cap\mathbf{N}*\mu)(x) &= \min\{(\mu\mathbf{N})(x),(\mathbf{N}\mu)(x),(\mathbf{N}*\mu)(x)\} \\
&= \min\Big\{\sup_{x=an_1}\mu(a), \sup_{x=n_2b}\mu(b), \sup_{x=n_3(n_4+c)-n_3n_4}\mu(c)\Big\} \\
&\ge t.
\end{aligned} \tag{2.11}$$

As μ is the fuzzy quasi-ideal of N, $\mu(x) \ge t$. Thus, $x \in \mu_t$ and hence μ_t is a quasi-ideal of N.

Conversely, let us assume that $\mu_t, t \in Im(\mu)$, is a quasi-ideal of N. Let $x \in N$. Consider

$$\begin{aligned}(\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x) &= \min\{(\mu\mathbf{N})(x), (\mathbf{N}\mu)(x), (\mathbf{N} * \mu)(x)\} \\ &= \min\Big\{\sup_{x=ab}\{\min\{\mu(a), \mathbf{N}(b)\}\}, \sup_{x=ab}\{\min\{\mathbf{N}(a), \mu(b)\}\}, \\ &\qquad \sup_{x=n_1(n_2+c)-n_1n_2}\{\min\{\mathbf{N}(n_1), \mu(c)\}\}\Big\} \\ &= \min\Big\{\sup_{x=ab}\{\mu(a)\}, \sup_{x=ab}\{\mu(b)\}, \sup_{x=n_1(n_2+c)-n_1n_2}\{\mu(c)\}\Big\}.\end{aligned} \tag{2.12}$$

Let $\sup_{x=ab}\{\mu(a)\} = t_1$, $\sup_{x=ab}\{\mu(b)\} = t_2$, and $\sup_{x=n_1(n_2+c)-n_1n_2}\{\mu(c)\} = t_3$ for any a, b, n_1, n_2, and c in N. Assume that $\min\{t_1, t_2, t_3\} = t_1$. Then $a, b, c \in \mu_{t_1}$. Since μ_{t_1} is a quasi-ideal of N, then $x = ab \in N\mu_{t_1}$, $x = ab \in \mu_{t_1}N$, and $x = n_1(n_2 + c) - n_1n_2 \in N * \mu_{t_1}$. This implies $x \in \mu_{t_1}N \cap N\mu_{t_1} \cap N * \mu_{t_1} \subseteq \mu_{t_1}$. Thus, $\mu(x) \geq t_1 = \min\{t_1, t_2, t_3\}$. Hence, $(\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x) \leq t_1 \leq \mu(x)$. Similarly, if we take $\min\{t_1, t_2, t_3\} = t_2$ or t_3, we can prove that $(\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x) \leq t_2$ or $t_3 \leq \mu(x)$. Thus $(\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x) \leq \mu(x)$, for all $x \in N$. This shows that μ is a fuzzy quasi-ideal of N. □

Theorem 2.27. *Let μ be a fuzzy subset of N. If μ is a fuzzy bi-ideal of N, if and only if μ_t is a bi-ideal of N, for all $t \in Im(\mu)$.*

Proof. Let μ be a fuzzy bi-ideal of N. Let $t \in Im(\mu)$. Suppose $x, y \in N$ such that $x, y \in \mu_t$. Then, $\mu(x) \geq t$, $\mu(y) \geq t$, and $\min\{\mu(x), \mu(y)\} \geq t$. As μ is a fuzzy bi-ideal, $\mu(x - y) \geq t$ and thus $x - y \in \mu_t$. Let $z \in N$. Suppose $z \in \mu_tN\mu_t \cap \mu_tN * \mu_t$. Then there exist $x, y, a_1, a_2, b \in \mu_t$ and $n_1, n_2, n_3 \in N$ such that $z = xn_1y = a_1n_2(a_2n_3 + b) - a_1n_2a_2n_3$. Then, $(\mu\mathbf{N}\mu \cap \mu\mathbf{N} * \mu)(z) = \min\{(\mu\mathbf{N}\mu)(z), (\mu\mathbf{N} * \mu)(z)\}$.

Now,

$$\begin{aligned}(\mu\mathbf{N}\mu)(z) &= \sup_{z=xn_1y}\{\min\{\mu(x), \mu(y)\}\} \geq t, \\ (\mu\mathbf{N} * \mu)(z) &= \sup_{z=a_1n_2(a_2n_3+b)-a_1n_2a_2n_3}\{\min\{\mu(a_1), \mu(b)\}\} \geq t.\end{aligned} \tag{2.13}$$

Therefore, $\min\{(\mu\mathbf{N}\mu)(z), (\mu\mathbf{N} * \mu)(z)\} \geq t$ and thus $(\mu\mathbf{N}\mu \cap \mu\mathbf{N} * \mu)(z) \geq t$. As μ is a bi-ideal of N, $\mu(z) \geq t$ implies $z \in \mu_t$. Hence μ_t is a bi-ideal in N.

Conversely, let us assume that μ_t is a bi-ideal of $N, t \in Im(\mu)$. Let $p \in N$. Consider

$$\begin{aligned}(\mu\mathbf{N}\mu \cap \mu\mathbf{N} * \mu)(p) &= \min\{(\mu\mathbf{N}\mu)(p), (\mu\mathbf{N} * \mu)(p)\} \\ &= \min\Big\{\sup_{p=xny}\{\min\{\mu(x), \mu(y)\}\}, \sup_{p=a_1n_1(b+c)-a_1n_1b}\{\min\{\mu(a_1), \mu(c)\}\}\Big\} \\ &= \sup_{p=xny=a_1n_1(b+c)-a_1n_1b}\{\min\{\mu(x), \mu(y), \mu(a_1), \mu(c)\}\}.\end{aligned} \tag{2.14}$$

Let $\mu(x) = t_1 < \mu(y) = t_2 < \mu(a_1) = t_3 < \mu(c) = t_4$. Then, $\mu_{t_1} \supseteq \mu_{t_2} \supseteq \mu_{t_3} \supseteq \mu_{t_4}$. Thus, $x, y, a_1, c \in \mu_{t_1}$, $p = xny \in \mu_{t_1}\mathbf{N}\mu_{t_1}$, and $p = a_1n_1(b+c) - a_1n_1b \in \mu_{t_1}\mathbf{N} * \mu_{t_1}$. Thus, $p \in \mu_{t_1}\mathbf{N}\mu_{t_1} \cap \mu_{t_1}\mathbf{N} * \mu_{t_1} \subseteq \mu_{t_1}$. This implies $\mu(p) \geq t_1$ and hence $\mu\mathbf{N}\mu \cap \mu\mathbf{N} * \mu \leq \mu$. Therefore, μ is a fuzzy bi-ideal of N. □

3. Residual quotient fuzzy subset in N

Lemma 3.1. *If μ is a fuzzy left ideal of N, then $\mu(n_0x) \geq \mu(x)$ for all $x \in N$ and $n_0 \in N_0$.*

Proof. As μ is a fuzzy left ideal of N, $\mu(n_1(n_2+c) - n_1n_2) \geq \mu(c)$ for all $n_1, n_2, c \in N$. Taking $n_1 = n_0 \in N_0$ and $n_2 = 0$, we have $\mu(n_0c) \geq \mu(c)$. □

Lemma 3.2. *N is zero-symmetric near-ring if and only if each fuzzy left ideal of N is a fuzzy N-subgroup of N.*

Proof. Assume that $N = N_0$. Let μ be a fuzzy left ideal of N. As μ is a fuzzy left ideal of N, by Lemma 3.1, $\mu(n_0x) \geq \mu(x)$ for all $x \in N$ and $n_0 \in N_0 = N$. Thus, μ is a fuzzy N-subgroup of N.

Conversely, let us assume that each fuzzy left ideal of N is a fuzzy N-subgroup of N. Let L be a left ideal of N. Then, f_L is a fuzzy N-subgroup of N. This implies $f_L(nx) \geq f_L(x)$ for all $n, x \in N$. In particular, $x \in L$ and $n \in N$, then $NL \subseteq L$. Taking L as $\{0\}$, we have $N\{0\} \subseteq \{0\}$. This implies $N \cdot 0 = \{0\}$ and hence $N = N_0$. □

Now we introduce the notion of residual quotient fuzzy subset $(\lambda : \mu)$ for any two fuzzy subsets λ and μ and annihilator, $\operatorname{ann}(\mu)$, of fuzzy subset μ of N.

Definition 3.3. Let λ and μ be any two fuzzy subsets of N. The *residual quotient fuzzy subset* $(\lambda : \mu)$ of N is defined as

$$\begin{aligned}(\lambda : \mu)(x) &= \vee_{t \in Im(\lambda)}\{t : x \in (\lambda_t : \mu_\alpha), \text{ where } \alpha = \sup\{Im(\mu)\}\} \\ &= 0, \quad \text{otherwise,}\end{aligned} \tag{3.1}$$

where $(\lambda_t : \mu_\alpha) = \{x \in N \mid x\mu_\alpha \subseteq \lambda_t\}$.

Definition 3.4. Let O be a fuzzy subset defined as $O(0) = 1$ and $O(x) = 0$ for all $x \neq 0 \in N$. Then $(O : \mu)$ is the *annihilator* of μ and it is denoted by $\operatorname{ann}(\mu)$.

Definition 3.4 is different from the definition for $\operatorname{ann}(\mu)$ given in [12]. It is clear that, for any $t_1, t_2 \in Im(\lambda)$ and $\alpha = \sup\{Im(\mu)\}$ with $t_1 < t_2$, we have $(\lambda_{t_2} : \mu_\alpha) \subseteq (\lambda_{t_1} : \mu_\alpha)$. For, let $x \in (\lambda_{t_2} : \mu_\alpha)$. Then, $x\mu_\alpha \subseteq \lambda_{t_2} \subseteq \lambda_{t_1}$. Thus, $x \in (\lambda_{t_1} : \mu_\alpha)$ and therefore $(\lambda_{t_2} : \mu_\alpha) \subseteq (\lambda_{t_1} : \mu_\alpha)$.

Theorem 3.5. *Let λ and μ be any two fuzzy subsets of N. If λ is a fuzzy left ideal of N, then $(\lambda : \mu)$ is a fuzzy left ideal of N.*

Proof. Let $x, y \in N$ and $\alpha = \sup\{Im(\mu)\}$. Suppose $(\lambda : \mu)(x) = t_1$ and $(\lambda : \mu)(y) = t_2$, where $t_1, t_2 \neq 0 \in Im(\lambda)$. Assume that $t_1 < t_2$. Then, $(\lambda_{t_2} : \mu_\alpha) \subseteq (\lambda_{t_1} : \mu_\alpha)$. Thus, $x, y \in (\lambda_{t_1} : \mu_\alpha)$ implies for any $b \in \mu_\alpha$; we have $(x-y)b = xb - yb \in \lambda_{t_1}$. Then, $(x-y)\mu_\alpha \subseteq \lambda_{t_1}$. Hence

$(x-y) \in (\lambda_{t_1} : \mu_\alpha)$ which implies that $(\lambda : \mu)(x-y) \geq t_1$ and $(\lambda : \mu)(x-y) \geq t_1 = \min\{t_1, t_2\}$. Similarly, if $t_1 > t_2$, then $(\lambda : \mu)(x-y) \geq t_2 = \min\{t_1, t_2\}$. Thus, $(\lambda : \mu)(x-y) \geq \min\{(\lambda : \mu)(x), (\lambda : \mu)(y)\}$. For other choices of t_1 and t_2, it can be verified that $(\lambda : \mu)(x-y) \geq \min\{(\lambda : \mu)(x), (\lambda : \mu)(y)\}$. Suppose $(\lambda : \mu)(x+y) = t$. Then, $x+y \in (\lambda_t : \mu_\alpha)$. As λ_t is normal subgroup of N, $(\lambda_t : \mu_\alpha)$ is also normal subgroup of N. Then, $y+x \in (\lambda_t : \mu_\alpha)$. Thus, $(\lambda : \mu)(y+x) \geq t = (\lambda : \mu)(x+y)$. Similarly, $(\lambda : \mu)(x+y) \geq (\lambda : \mu)(y+x)$. Hence, $(\lambda : \mu)(x+y) = (\lambda : \mu)(y+x)$. Let $(\lambda : \mu)(x) = t$. Now, $x \in (\lambda_t : \mu_\alpha)$. Thus for any $a, b \in N$, $a(b+x) - ab \in (\lambda_t : \mu_\alpha)$. This implies $(\lambda : \mu)(a(b+x) - ab) \geq t = (\lambda : \mu)(x)$. Thus $(\lambda : \mu)$ is a fuzzy left ideal of N. □

Corollary 3.6. *Let μ be any fuzzy subset of N, then* $\operatorname{ann}(\mu)$ *is a fuzzy left ideal of N.*

Remark 3.7. Let λ and μ be any two fuzzy subsets of N. If λ is a fuzzy left ideal of N, then $(\lambda : \mu)$ is not necessarily fuzzy ideal of N as the following example shows.

Example 3.8. Let $N = \{0, a, b, c\}$ be Klein's four group. Define multiplication in N as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

•	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	a	b	c
c	a	0	c	b

(3.2)

Then $(N, +, \bullet)$ is a near-ring (see [9, Page 408, Scheme 20]). Let λ be a fuzzy subset of N. Define $\lambda : N \to [0,1]$ by $\lambda(0) = \lambda(b) = 1$ and $\lambda(a) = \lambda(c) = 0$. For any $t \in [0,1]$, we have only two level sets, viz, $\{0, b\}$ and $\{0, a, b, c\}$. Since $\{0, b\}$ is the zero-symmetric part of N, by [9, Proposition 1.32(a)], $\{0, b\}$ is a left ideal in N. By Lemma 2.16, λ is a fuzzy left ideal of N. Let μ be any fuzzy subset of N. Define $\mu : N \to [0,1]$ by $\mu(c) = 1$ and $\mu(0) = \mu(a) = \mu(b) = 0$. Now we have $(\lambda : \mu)(0) = 1$, $(\lambda : \mu)(a) = 0$, $(\lambda : \mu)(b) = 0$, and $(\lambda : \mu)(c) = 1$, as $(\lambda : \mu)(c.c) = (\lambda : \mu)(b) = 0 \ngeq (\lambda : \mu)(c)$. Hence, $(\lambda : \mu)$ is not a fuzzy ideal of N.

Now we find the conditions under which $(\lambda : \mu)$ is a fuzzy ideal of N.

Theorem 3.9. *Let λ and μ be any two fuzzy subsets of N. If λ is a fuzzy left ideal and μ is a fuzzy N-subgroup of N, then $(\lambda : \mu)$ is a fuzzy ideal of N.*

Proof. By Theorem 3.5, $(\lambda : \mu)$ is a fuzzy left ideal of N. Let $(\lambda : \mu)(x) = t$. Then, $x \in (\lambda_t : \mu_\alpha)$, where $\alpha = \sup\{Im(\mu)\}$. Now, $x\mu_\alpha \subseteq \lambda_t$. Let $n \in N$. Consider $xnb = x(nb) \in x\mu_\alpha \subseteq \lambda_t$. Since μ_α is N-subgroup of N, $nb \in \mu_\alpha$ for all $n \in N$ and $b \in \mu_\alpha$, $xn\mu_\alpha \subseteq \lambda_t$. This implies $xn \in (\lambda_t : \mu_\alpha)$. Thus $(\lambda : \mu)(xn) \geq t = (\lambda : \mu)(x)$. Hence, $(\lambda : \mu)$ is a fuzzy right ideal of N. Therefore, $(\lambda : \mu)$ is a fuzzy ideal of N. □

Example 3.10. Let $N = \{0,a,b,c\}$ be Klein's four group. Define multiplication in N as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

•	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

(3.3)

Then $(N,+,\bullet)$ is a near-ring (see [9, Page 408, Scheme 13]).

Define $\lambda : N \to [0,1]$ by

$$\lambda(x) = \begin{cases} 1, & \text{if } x = 0,a, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Clearly λ is a fuzzy left ideal of N. But λ is not a fuzzy right ideal of N, since $0 = \lambda(ac) \ngeq \lambda(a) = 1$. Define $\mu : N \to [0,1]$ by

$$\mu(x) = \begin{cases} 1, & x = 0,c, \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

Clearly μ is a fuzzy N-subgroup of N. Now for any $x \in N$,

$$\begin{aligned} (\lambda : \mu)(x) = 1 &\Longleftrightarrow x \in (\lambda_1 : \mu_1) \\ &\Longleftrightarrow x \in (\{0,a\} : \{0,c\}) \\ &\Longleftrightarrow x \in \{0,b\}. \end{aligned} \tag{3.6}$$

Thus, $(\lambda : \mu)(0) = (\lambda : \mu)(b) = 1$ and $(\lambda : \mu)(a) = (\lambda : \mu)(c) = 0$. Hence, $(\lambda : \mu)$ is a fuzzy ideal of N. Note that λ is only a fuzzy left ideal but not a fuzzy ideal of N. Now, $\text{ann}(\mu)(0) = \text{ann}(\mu)(b) = 1$ and $\text{ann}(\mu)(a) = \text{ann}(\mu)(c) = 0$. Clearly $\text{ann}(\mu)$ is a fuzzy ideal of N.

Corollary 3.11. *If μ is a fuzzy N-subgroup of N, then* $\text{ann}(\mu)$ *is a fuzzy ideal of N.*

Theorem 3.12. *Let λ and μ be any two fuzzy subset of N. If λ and μ are fuzzy ideals of zero-symmetric near-ring N, then $(\lambda : \mu)$ is a fuzzy ideal of N.*

Proof. $(\lambda : \mu)$ is a fuzzy left ideal of N, by Theorem 3.5. Now μ is a fuzzy ideal of N, then by Lemma 3.2, μ is a fuzzy N-subgroup of N. Thus by the Theorem 3.9, $(\lambda : \mu)$ is a fuzzy ideal of N. □

Lemma 3.13. *Let λ be a fuzzy ideal of N. λ is completely fuzzy semiprime ideal of N if and only if $\lambda_t, t \in Im(\lambda)$ is a completely semiprime ideal of N.*

Proof. Let λ be a fuzzy completely semiprime ideal of N. Let $t \neq 0 \in Im(\lambda)$. Let $x \in N$ such that $x^2 \in \lambda_t$. This implies $(x_t)^2 \in \lambda$. As λ is a fuzzy completely semiprime ideal of N, $x_t \in \lambda$. This implies $x \in \lambda_t$.

Conversely, let us assume that λ_t is a completely semiprime ideal of N, $t \in Im(\lambda)$. Suppose ${y_s}^2 \in \lambda$. Then, $y^2 \in \lambda_s$ and $y_s \in \lambda$. Thus, λ is a fuzzy completely semiprime ideal of N. □

Theorem 3.14. *Let λ be a fuzzy ideal of N. If $a_r b_s \in \lambda$ implies $b_s a_r \in \lambda$ for any fuzzy points a_r, b_s of N, then for any fuzzy subset μ of N, $(\lambda : \mu)$ is a fuzzy ideal of N.*

Proof. By Theorem 3.5, $(\lambda : \mu)$ is a fuzzy left ideal. Let $(\lambda : \mu)(x) = t$ and $\alpha = \sup\{Im(\mu)\}$. This implies that $x \in (\lambda_t : \mu_\alpha)$ and $xb \in \lambda_t$, for all $b \in \mu_\alpha$. Then, $(xb)_t = x_t b_t = b_t x_t \in \lambda$. Hence, $bx \in \lambda_t$. As λ_t is an ideal of N, $nbx \in \lambda_t$ and hence $(nbx)_t \in \lambda$, for all $n \in N$. Now, $(xnb)_t = x_t(n_t b_t) = (n_t b_t)x_t = (nbx)_t \in \lambda$ and hence $xnb \in \lambda_t$. This implies that $xn \in (\mu_t : \mu_\alpha)$. Thus, $(\lambda : \mu)(xn) \geq t = (\lambda : \mu)(x)$. Hence, $(\lambda : \mu)$ is a fuzzy right ideal of N. Therefore, $(\lambda : \mu)$ is a fuzzy ideal of N. □

Theorem 3.15. *Let λ and μ be any fuzzy subsets of N. If λ is a fuzzy completely semiprime ideal of N, then $(\lambda : \mu)$ is a fuzzy ideal of N.*

Proof. By Theorem 3.5, $(\lambda : \mu)$ is a fuzzy left ideal of N. Let $(\lambda : \mu)(x) = t$ and $\alpha = \sup\{Im(\mu)\}$. Then, $x \in (\lambda_t : \mu_\alpha)$. This implies $x\mu_\alpha \subseteq \lambda_t$. For any $n \in N$, let us show that $xn \in (\lambda_t : \mu_\alpha)$. As $x\mu_\alpha \subseteq \lambda_t$, $xb \in \lambda_t$ for all $b \in \mu_\alpha$. Now, $(bx)^2 = bxbx = b(xb)x \in \lambda_t$. As λ_t is a completely semiprime ideal in N, $bx \in \lambda_t$. Consider $(xnb)^2 = xnbxnb = xn(bx)nb \in \lambda_t$. This implies $xnb \in \lambda_t$ for every $b \in \mu_t$. Thus, $xn \in (\lambda_t : \mu_\alpha)$. Hence, $(\lambda : \mu)(xn) > t = (\lambda : \mu)(x)$. Therefore, $(\lambda : \mu)$ is a fuzzy ideal of N. □

Theorem 3.16. *If λ is a fuzzy bi-ideal and let μ is a fuzzy N-subgroup of zero-symmetric near-ring N, then $(\lambda : \mu)$ is a fuzzy bi-ideal of N.*

Proof. Let λ be a fuzzy bi-ideal and μ be a fuzzy N-subgroup of zero-symmetric near-ring N. Clearly $(\lambda : \mu)$ is a fuzzy subgroup of N. Next we prove that $(\lambda : \mu)$ is a fuzzy bi-ideal of N. Let $t \in N$ and $a, n, b \in N$ such that $t =$ anb. Consider

$$\begin{aligned} ((\lambda : \mu)\mathbf{N}(\lambda : \mu))(t) &= \sup_{t=\text{anb}} \{\min\{(\lambda : \mu)(a), \mathbf{N}(n), (\lambda : \mu)(b)\}\} \\ &= \sup_{t=\text{anb}} \{\min\{(\lambda : \mu)(a), (\lambda : \mu)(b)\}\}. \end{aligned} \tag{3.7}$$

Let $\min\{(\lambda : \mu)(a), (\lambda : \mu)(b)\} = t$. This implies that $(\lambda : \mu)(a) \geq t$ and $(\lambda : \mu)(b) \geq t$. Then, $a, b \in (\lambda_t : \mu_\alpha)$. As λ is the fuzzy bi-ideal and μ is the fuzzy N-subgroup, $(\lambda_t : \mu_\alpha)$ is a bi-ideal of N. Hence, anb $\in (\lambda_t : \mu_\alpha)$. This implies $(\lambda : \mu)(\text{anb}) \geq t = \min\{(\lambda : \mu)(a), (\lambda : \mu)(b)\}$. Thus, $\min\{(\lambda : \mu)(a), (\lambda : \mu)(b)\} \leq (\lambda : \mu)(\text{anb})$. This shows that $\sup_{t=\text{anb}} \min\{(\lambda : \mu)(a), (\lambda : \mu)(b)\} \leq (\lambda : \mu)(\text{anb})$. Thus, we have $((\lambda : \mu)\mathbf{N}(\lambda : \mu))(t) \leq (\lambda : \mu)(t)$. Hence, $(\lambda : \mu)\mathbf{N}(\lambda : \mu) \leq (\lambda : \mu)$ and $(\lambda : \mu)$ is a fuzzy bi-ideal of N. □

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