THE PICARD GROUP OF TOPOLOGICAL MODULAR FORMS VIA DESCENT THEORY

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Abstract. This paper starts with an exposition of descent-theoretic techniques in the study of Picard groups of $\mathbf{E}_\infty$-ring spectra, which naturally lead to the study of Picard spectra. We then develop tools for efficient and explicit determination of differentials in the associated descent spectral sequences for the Picard spectra thus obtained. As a major application, we calculate the Picard groups of the periodic spectrum of topological modular forms TMF, as well as the non-periodic and non-connective Tmf. We find that $\mathrm{Pic}(TMF)$ is cyclic of order 576, generated by the suspension ΣTMF (a result originally due to Hopkins), while $\mathrm{Pic}(Tmf) = \mathbb{Z} \oplus \mathbb{Z}/24$, so that in particular, there exists an invertible Tmf-module which is not equivalent to a suspension of Tmf.

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1. Introduction

Elliptic curves and modular forms occupy a central role in modern stable homotopy theory in the guise of the variants of *topological modular forms*: the connective tmf, the periodic TMF, and Tmf which interpolates between them. These are structured ring spectra which have demonstrated surprising connections between the arithmetic of elliptic curves and v_2-periodicity in stable homotopy. For example, tmf detects a number of 2-torsion and 3-torsion classes in the stable homotopy groups of spheres through the Hurewicz image. Even more interestingly, the more geometric-natured TMF can be used to detect and describe, using congruences between modular forms, the 2-line of the Adams-Novikov spectral sequence at primes $p \geq 5$, according to [Beh09].

From a different perspective, the structure of topological modular forms as $\mathbf{E}_\infty$-ring spectra leads to well-behaved symmetric monoidal ∞-categories of modules which give rise to well-behaved invariants of algebraic or algebro-geometric type. For instance, [Mei12] has studied TMF-modules which become free when certain level structures are introduced; these can be thought of as locally free sheaves with respect to a pre-determined cover.

Our goal in this paper is to understand another such invariant, the Picard group. Any symmetric monoidal category has an associated group of objects invertible under the tensor product, which is called the *Picard group*. The classical examples are the Picard group $\mathrm{Pic}(R)$ of a ring R, i.e., of the category $\mathrm{Mod}(R)$ of its modules, or the Picard group of a scheme X, i.e., of the category $\mathrm{Mod}(\mathcal{O}_X)$ of quasi-coherent modules over its structure sheaf. In homotopy theory, the interest in Picard groups arose when Mike Hopkins made the observation that the homotopy categories of E_n-local and $K(n)$-local spectra have interesting Picard groups, particularly when the prime at hand is small in comparison with n. Here, E_n is the Lubin-Tate spectrum and $K(n)$ is the Morava K-theory spectrum at height n. In the few existing computations of such groups, notably [HMS94, HS99, KS07, GHMR12, Hea14], one often uses that an invertible E_n-module must be a suspension of E_n itself.

The $K(2)$-localization of either of the three versions of topological modular forms gives a spectrum closely related to the Lubin-Tate spectrum E_2; namely, this localization is the homotopy fixed point spectrum of a finite group action on E_2. More generally, each E_n is an $\mathbf{E}_\infty$-ring spectrum with an action, through $\mathbf{E}_\infty$-ring maps, by a profinite group $\mathbb{G}_n$ called the Morava stabilizer group (see [Rez98] for the $\mathbf{E}_1$-ring case). The $K(n)$-local sphere is obtained then as $E_n^{h\mathbb{G}_n}$. However, $\mathbb{G}_n$ also has interesting finite subgroups when the prime is relatively small with respect to n. If G is such a subgroup, the homotopy fixed points E_n^{hG} are an $\mathbf{E}_\infty$-ring spectrum, which is in theory easier to study than the $K(n)$-local sphere, but hopefully contains a lot of information about the $K(n)$-local sphere. For instance, Hopkins has observed that in all known examples, the Picard group of E_n^{hG} (unlike that of the $K(n)$-local category) is very simple as it only contains suspensions of E_n^{hG}, and raised the following natural question.

Question (Hopkins)**.** *Let G be a finite subgroup of the Morava stabilizer group $\mathbb{G}_n$ at height n. Is it true that any invertible $K(n)$-local module over E_n^{hG} is a suspension of E_n^{hG}?*

The periodic TMF is closer to its $K(2)$-localization than Tmf, and this is demonstrated by the following result, originally due to Hopkins but unpublished.

Theorem A (Hopkins)**.** *The Picard group of TMF is isomorphic to $\mathbb{Z}/576$, generated by the suspension ΣTMF.*

In the paper at hand, we prove Theorem A using a descent-theoretic approach. In particular, our method is different from Hopkins's. The descent-theoretic approach also enables us to prove that, nonetheless, the non-connective, non-periodic flavor of topological modular forms Tmf behaves differently and has a more interesting Picard group.

Theorem B. *The Picard group of Tmf is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/24$, generated by the suspension ΣTmf and a certain 24-torsion invertible object.*

In addition, we explicitly construct the 24-torsion module in 8.4.2.

We hope that our method of proof of Theorem A and Theorem B, which is very general, will also be of interest to those not directly concerned with TMF. Our method is inspired by and analogous to the forthcoming work of Gepner-Lawson [GL] on Galois descent of Brauer as well as Picard groups, though the key ideas are classical.

Take, for example, the periodic variant TMF. Its essential property is that it arises as the global sections of the structure sheaf $\mathcal{O}^{\mathrm{top}}$ of a regular "derived stack" $(\mathfrak{M}_{ell}, \mathcal{O}^{\mathrm{top}})$ over the moduli stack of elliptic curves M_{ell}. Thus

$$TMF = \Gamma(\mathfrak{M}_{ell}, \mathcal{O}^{\mathrm{top}}) = \varprojlim_{\mathrm{Spec} R \to M_{ell}} \Gamma(\mathrm{Spec} R, \mathcal{O}^{\mathrm{top}}),$$

where the maps $\mathrm{Spec} R \to M_{ell}$ range over all étale maps from affine schemes to M_{ell}. Moreover, the $\mathbf{E}_\infty$-ring spectra $\Gamma(\mathrm{Spec} R, \mathcal{O}^{\mathrm{top}})$ are even periodic; thus we have TMF as the homotopy limit of a diagram of even periodic $\mathbf{E}_\infty$-rings. It follows by the main result of [MM13] that the *module category* of TMF can also be represented as the inverse limit of the module categories $\mathrm{Mod}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec} R))$, that is, as quasi-coherent sheaves on the derived stack. In any analogous situation, our descent techniques for calculating Picard groups apply.

Over an affine chart $\mathrm{Spec} R \to M_{ell}$, the Picard group of $\Gamma(\mathrm{Spec} R, \mathcal{O}^{\mathrm{top}})$ (i.e. that of an *elliptic spectrum*) is purely algebraic, by a classical argument of [HMS94, BR05] with "residue fields." This results from the fact that the ring $\pi_*\Gamma(\mathrm{Spec} R, \mathcal{O}^{\mathrm{top}})$ is *homologically* simple: in particular, it has finite global dimension, which makes the study of $\Gamma(\mathrm{Spec} R, \mathcal{O}^{\mathrm{top}})$-modules much easier. One attempts to use this together with descent theory to compute the Picard group of TMF itself; however, doing so necessitates the consideration of higher homotopy coherences. For this, it is important to work with Picard *spectra* rather than Picard groups, as they have a better formal theory of descent.

The Picard spectrum $\mathfrak{pic}(A)$ of an $\mathbf{E}_\infty$-ring A is a delooping of the unit spectrum $\mathfrak{gl}_1(A)$ of [May77][1]: it is connective, its π_0 is the Picard group of A, and its 1-connective cover $\tau_{\geq 1}\mathfrak{pic}(A)$ is equivalent to $\Sigma\mathfrak{gl}_1(A)$. We find that the *Picard spectrum* of TMF is the connective cover of the homotopy limit of $\mathfrak{pic}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec}R))$, taken over étale maps $\mathrm{Spec}R \to M_{ell}$. This statement is a homotopy-theoretic expression of the descent theory that we need. Thus, we get a *descent spectral sequence* for the homotopy groups of $\mathfrak{pic}(TMF)$, which is a computational tool for understanding the aforementioned homotopy coherences concretely. We use this technique to compute $\pi_0(\mathfrak{pic}(TMF))$, the group we are after.

The descent spectral sequence has many consequences in cases where it degenerates simply for dimensional reasons, or in cases where the information sought is coarse. For instance, in a specific example (Proposition 2.4.8), we show that the Picard group of the $\mathbf{E}_\infty$-ring $C^*(S^1; \mathbb{Q}[\epsilon]/\epsilon^2)$ is given by $\mathbb{Z} \times \mathbb{Q}$, which yields a counterexample to a general conjecture of Balmer [Bal10, Conjecture 74] on the Picard groups of tensor-triangulated categories with local spectrum. We also prove the following general results in Sections 4 and 5.

Theorem C. *Let A be an even periodic, Landweber exact $\mathbf{E}_\infty$-ring with $\pi_0(A)$ regular. Let $n \geq 1$ be an integer, and let L_n denote localization with respect to the Lubin-Tate spectrum E_n. The Picard group of L_nA is*

$$\mathrm{Pic}(L_nA) = \mathrm{Pic}(\pi_0(A)) \times \mathbb{Z}/2 \times \pi_{-1}(L_nA),$$

where $\mathrm{Pic}(\pi_0(A))$ refers to the (algebraic) Picard group of the ordinary commutative ring $\pi_0(A)$.

Theorem D. *Let A be an $\mathbf{E}_\infty$-ring such that $\pi_0(A)$ is a field of characteristic zero and such that $\pi_i(A) = 0$ for $i > 0$. Then $\mathrm{Pic}(A)$ is infinite cyclic, generated by ΣA.*

Theorem E. *Let G be a finite group, and let $A \to B$ be a faithful G-Galois extension of $\mathbf{E}_\infty$-rings in the sense of Rognes [Rog08]. Then the relative Picard group of B/A (i.e., the kernel of $\mathrm{Pic}(A) \to \mathrm{Pic}(B)$) is $|G|$-power torsion of finite exponent.*

For TMF, the descent spectral sequence does not degenerate so nicely, and we need to work further to obtain our main results. The homotopy groups of the Picard spectrum of an $\mathbf{E}_\infty$-ring A, starting with π_2, are simply those of A: in fact, we have a natural equivalence of *spaces*

$$\Omega^{\infty+2}\mathfrak{pic}(A) \simeq \Omega^{\infty+1}A.$$

This determines the E_2-page and many of the differentials in the descent spectral sequence for $\mathrm{Pic}(TMF)$, but not the ones that affect π_0. A key step in our argument is the identification of the differentials of the descent spectral sequence for the Picard spectra, in a certain *range of dimensions*, with that of the (known) descent spectral sequence for $\pi_*(TMF)$. We prove this in a general setting in Section 5 below.

At the prime 2, this technique is not sufficient to determine all the differentials in the descent spectral sequence, and we need to determine in addition the first "unstable" differential in the Picard spectral sequence (in comparison to the usual descent spectral sequence). We give a "universal" formula for this first differential in Theorem 6.1.1, which we hope will have further applications.

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[1]See [ABG+08] for a very important application.

Part I. Generalities

2. Picard groups

We begin by giving an introduction to Picard groups in stable homotopy theory. General references here include [HMS94, May01].

2.1. Generalities.

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category.

Definition 2.1.1. The *Picard group* of $\mathcal{C}$ is the group of isomorphism classes of objects $x \in \mathcal{C}$ which are *invertible*, i.e. such that there exists an object $y \in \mathcal{C}$ such that $x \otimes y \simeq \mathbf{1}$. We will denote this group by $\mathrm{Pic}(\mathcal{C})$.

When $\mathcal{C}$ is the category of quasi-coherent sheaves on a scheme (or stack) X, then this recovers the usual Picard group of X: line bundles are precisely the invertible objects. The goal of this paper is to compute a Picard group in a homotopy-theoretic setting.

We will repeatedly use the following simple principle, which follows from the observation that tensoring with an invertible object induces an autoequivalence of categories:

Proposition 2.1.2. *Let $\mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory that is preserved under any autoequivalence of $\mathcal{C}$. Suppose the unit object $\mathbf{1} \in \mathcal{C}$ belongs to $\mathcal{C}_0$. Then any $x \in \mathrm{Pic}(\mathcal{C})$ belongs to $\mathcal{C}_0$ as well.*

For example, if $\mathbf{1}$ is a compact object (that is, if $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \cdot)$ commutes with filtered colimits), then so is x.

Suppose now that, more generally, $\mathcal{C}$ is a symmetric monoidal ∞-category in the sense of [Lur12], which is the setting that we will be most interested in. Then we can still define the Picard group $\mathrm{Pic}(\mathcal{C})$ of $\mathcal{C}$, which is the same as $\mathrm{Pic}(\mathrm{Ho}(\mathcal{C}))$. Moreover, Proposition 2.1.2 is valid, but where one is allowed to (and often should) use ∞-categorical properties.

Example 2.1.3. Suppose that $\mathcal{C}$ is a symmetric monoidal stable ∞-category such that the tensor product commutes with finite colimits. Then one has a natural homomorphism

$$\mathbb{Z} \to \mathrm{Pic}(\mathcal{C}),$$

sending $n \mapsto \Sigma^n \mathbf{1}$.

Example 2.1.4. Let Sp be the ∞-category of spectra with the smash product. Then it is a classical result [HMS94, p. 90] that $\mathrm{Pic}(\mathcal{C}) \simeq \mathbb{Z}$, generated by the sphere S^1. A quick proof based on the above principle (which simplifies the argument in [HMS94] slightly) is as follows. If $T \in \mathrm{Sp}$ is invertible, so that there exists a spectrum T' such that $T \wedge T' \simeq S^0$, then we need to show that T is a suspension of S^0.

Since the unit object $S^0 \in \mathrm{Sp}$ is compact (that is, $\mathrm{Hom}_{\mathrm{Sp}}(S^0, \cdot)$ commutes with filtered ∞-categorical colimits), it follows that T is compact: that is, it is a finite spectrum. By suspending or desuspending, we may assume that T is connective, and that $\pi_0 T \neq 0$. By the Künneth formula, it follows easily that $H_*(T; F)$ is concentrated in one dimension for each field F. Since $H_*(T; \mathbb{Z})$ is finitely generated, an argument with the universal coefficient theorem implies that $H_*(T; \mathbb{Z})$ is torsion-free of rank one and is concentrated in dimension zero: i.e. $H_0(T; \mathbb{Z}) \simeq \mathbb{Z}$. By the Hurewicz theorem, $T \simeq S^0$.

Example 2.1.5. Other variants of the stable homotopy category can have more complicated Picard groups. For instance, if $E \in \mathrm{Sp}$, one can consider the ∞-category $L_E\mathrm{Sp}$ of *E-local spectra,* with the symmetric monoidal structure given by the E-localized smash product $(X, Y) \mapsto L_E(X \wedge Y)$. The Picard group of $L_E\mathrm{Sp}$ is generally much more complicated than $\mathbb{Z}$. When E is given by the Morava E-theories E_n or the Morava K-theories $K(n)$, the resulting Picard groups have been studied in [HMS94] and [HS99], among other references.

Another important example of this construction arises for R an $\mathbf{E}_\infty$-ring, when we can consider the symmetric monoidal ∞-category $\mathrm{Mod}(R)$ of R-modules.

Definition 2.1.6. Given an $\mathbf{E}_\infty$-ring R, we write $\mathrm{Pic}(R)$ for the Picard group $\mathrm{Pic}(\mathrm{Mod}(R))$.

Using the same argument as in Example 2.1.4, it follows that any invertible R-module is necessarily compact (i.e. perfect): in particular, the invertible modules actually form a set rather than a proper class. Note that if R is simply an $\mathbf{E}_2$-ring spectrum, then $\mathrm{Mod}(R)$ is a monoidal ∞-category, so one can still define a Picard group. This raises the following natural question.

Question 2.1.7. Is there an example of an $\mathbf{E}_2$-ring whose Picard group is nonabelian?

We will only work with $\mathbf{E}_\infty$-rings in the future, as it is for these highly commutative multiplications that we will be able to obtain good (from the point of view of descent theory) infinite loop spaces that realize $\mathrm{Pic}(R)$ on π_0.

2.2. **Picard ∞-groupoids.** If $(\mathcal{C}, \otimes, \mathbf{1})$ is a symmetric monoidal ∞-category, we reviewed in the previous section the *Picard group* of $\mathcal{C}$. There is, however, a more fundamental invariant of $\mathcal{C}$, where we remember all isomorphisms (and higher isomorphisms), and which behaves better with respect to descent processes.

Definition 2.2.1. Let $\mathcal{P}\mathrm{ic}(\mathcal{C})$ denote the ∞-groupoid (i.e. space) of *invertible objects* in $\mathcal{C}$ and equivalences between them. We will refer to this as the *Picard ∞-groupoid* of $\mathcal{C}$; it is a group-like $\mathbf{E}_\infty$-space, and thus the delooping of a connective *Picard spectrum* $\mathfrak{pic}(\mathcal{C})$.

We have in particular
$$\pi_0 \mathcal{P}\mathrm{ic}(\mathcal{C}) \simeq \mathrm{Pic}(\mathcal{C}).$$
However, we can also describe the higher homotopy groups of $\mathcal{P}\mathrm{ic}(\mathcal{C})$. Recall that since $\mathcal{C}$ is symmetric monoidal, $\mathrm{End}(\mathbf{1})$ is canonically an $\mathbf{E}_\infty$-space and $\mathrm{Aut}(\mathbf{1})$ consists of the grouplike components. Since
$$\Omega \mathcal{P}\mathrm{ic}(\mathcal{C}) \simeq \mathrm{Aut}(\mathbf{1}),$$
we get the relations
$$\pi_1 \mathcal{P}\mathrm{ic}(\mathcal{C}) = (\pi_0 \mathrm{End}(\mathbf{1}))^\times, \quad \pi_i \mathcal{P}\mathrm{ic}(\mathcal{C}) = \pi_{i-1}\mathrm{End}(\mathbf{1}) \text{ for } i \geq 2.$$

Example 2.2.2. Let R be an $\mathbf{E}_\infty$-ring. We will write
$$\mathcal{P}\mathrm{ic}(R) \stackrel{\text{def}}{=} \mathcal{P}\mathrm{ic}(\mathrm{Mod}(R)), \quad \mathfrak{pic}(R) \stackrel{\text{def}}{=} \mathfrak{pic}(\mathrm{Mod}(R)).$$
Then $\mathcal{P}\mathrm{ic}(R)$ is a delooping of the *space of units* $GL_1(R)$ studied in [May77] and more recently using ∞-categorical techniques in [ABG⁺08]. In particular, the homotopy groups of $\mathcal{P}\mathrm{ic}(R)$ look very much like those of R (with a shift), starting at π_2. In fact, if we take the connected components at the basepoint, we have a natural equivalence of spaces
$$\tau_{\geq 1}(GL_1 R) \simeq \tau_{\geq 1}(\Omega \mathcal{P}\mathrm{ic}(R)) \simeq \tau_{\geq 1}(\Omega^\infty R),$$
given by subtracting 1. Nonetheless, the *spectra* $\mathfrak{pic}(R)$ and R are generally very different: that is, the infinite loop structure on $\mathcal{P}\mathrm{ic}(R)$ behaves very differently from that of $\Omega^\infty R$.

Unlike the group-valued functor Pic, $\mathcal{P}\mathrm{ic}$ (as well as $\mathfrak{pic}$) has the fundamental property, upon which the calculations in this paper are based, that it commutes with homotopy limits.

Proposition 2.2.3. *The functor*
$$\mathfrak{pic}\colon \mathrm{Cat}^{\otimes} \to \mathrm{Sp}_{\geq 0},$$
from the ∞-category $\mathrm{Cat}^{\otimes}$ of symmetric monoidal ∞-categories to the ∞-category $\mathrm{Sp}_{\geq 0}$ of connective spectra, commutes with limits and filtered colimits, and $\mathcal{P}\mathrm{ic} = \Omega^\infty \circ \mathfrak{pic}\colon \mathrm{Cat}^{\otimes} \to \mathcal{S}_*$ does as well.

Proof. We will treat the case of limits; the case of filtered colimits is similar and easier. It suffices to show that $\mathcal{P}\mathrm{ic}$ commutes with homotopy limits, since $\Omega^\infty : \mathrm{Sp}_{\geq 0} \to \mathcal{S}_*$ creates limits. Let $\mathrm{CAlg}(\mathcal{S})$ be the ∞-category of $\mathbf{E}_\infty$-spaces. Now, $\mathcal{P}\mathrm{ic}$ is the composite $\mathrm{inv} \circ \bar{\iota}$ where:

(1) $\bar{\iota} : \mathrm{Cat}^{\otimes} \to \mathrm{CAlg}(\mathcal{S})$ sends a symmetric monoidal ∞-category to the symmetric monoidal ∞-groupoid (i.e. $\mathbf{E}_\infty$-space) obtained by excluding all non-invertible morphisms.
(2) $\mathrm{inv} : \mathrm{CAlg}(\mathcal{S}) \to \mathcal{S}_*$ sends an $\mathbf{E}_\infty$-space X to the union of those connected components which are invertible in the commutative monoid $\pi_0 X$, with basepoint given by the identity.

It thus suffices to show that $\bar{\iota}$ and inv both commute with limits.

(1) The functor $\iota : \mathrm{Cat} \to \mathcal{S}$ that sends an ∞-category $\mathcal{C}$ to its *core* $\iota\mathcal{C}$ commutes with limits: in fact, it is right adjoint to the inclusion $\mathcal{S} \to \mathrm{Cat}$ that regards a space as an ∞-groupoid. See for instance [Rie14, §17.2]. Now, to see that $\bar{\iota}$ commutes with limits, we observe that limits either in $\mathrm{Cat}^{\otimes}$ or in $\mathrm{CAlg}(\mathcal{S})$ are calculated at the level of the underlying spaces (resp. ∞-categories), so the fact that ι commutes with limits implies that $\bar{\iota}$ does too.
(2) It is easy to see that inv commutes with arbitrary products. Therefore, we need to show that inv turns pullbacks in $\mathrm{CAlg}(\mathcal{S})$ into pullbacks in $\mathcal{S}_*$. Suppose given a homotopy pullback

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in $\mathrm{CAlg}(\mathcal{S})$; we need to show that

$$\begin{array}{ccc} \mathrm{inv}(A) & \longrightarrow & \mathrm{inv}(B) \\ \downarrow & & \downarrow \\ \mathrm{inv}(C) & \longrightarrow & \mathrm{inv}(D) \end{array}$$

is one too, in $\mathcal{S}_*$. Given the construction of inv as a union of connected components, it suffices to show that if $x \in \pi_0(A)$ has the property that x maps to invertible elements in the monoids $\pi_0(B), \pi_0(C)$, then x itself is invertible.

Consider the union of the connected components in B, C, D that are multiples of the image of x, to get new $\mathbf{E}_\infty$-spaces B_0, C_0, D_0. The relevant connected component of A (given by x) is actually a connected component of $B_0 \times_{C_0} D_0$. But B_0, C_0, D_0 are grouplike $\mathbf{E}_\infty$-spaces, and therefore the deloopings of connective spectra [Seg74]. Thus, $B_0 \times_{C_0} D_0$ is also the delooping of a connective spectrum (given by the connective cover of the fiber product of the deloopings of B_0, C_0, D_0) and x must be invertible.

□

2.3. **Descent.** Let $R \to R'$ be a morphism of $\mathbf{E}_\infty$-rings. Recall the *cobar construction*, a cosimplicial $\mathbf{E}_\infty - R$-algebra

$$R' \rightrightarrows R' \otimes_R R' \substack{\to\\\to\\\to} \dots,$$

important in descent procedures, which receives an augmentation from R. The cobar construction is the *Čech nerve* (see [Lur09, 6.1.2]) of $R \to R'$, in the opposite ∞-category.

Definition 2.3.1. We say that $R \to R'$ is *faithfully flat* if $\pi_0 R \to \pi_0 R'$ is faithfully flat and the natural map $\pi_* R \otimes_{\pi_0 R} \pi_0 R' \to \pi_* R'$ is an isomorphism.

In this case, the theory of faithfully flat descent goes into effect. We have:

Theorem 2.3.2 ([Lur11b, Theorem 6.1])**.** *Suppose $R \to R'$ is a faithfully flat morphism of $\mathbf{E}_\infty$-rings. Then the symmetric monoidal ∞-category* $\mathrm{Mod}(R)$ *can be recovered as the limit of the cosimplicial diagram of symmetric monoidal ∞-categories*

$$\mathrm{Mod}(R') \rightrightarrows \mathrm{Mod}(R' \otimes_R R') \substack{\to\\ \to\\ \to} \dots .$$

As a result, by Proposition 2.2.3, $\mathcal{P}\mathrm{ic}(R)$ can be recovered as a totalization of spaces,

$$\mathcal{P}\mathrm{ic}(R) \simeq \mathrm{Tot}(\mathcal{P}\mathrm{ic}(R'^{\otimes(\bullet+1)})). \tag{2.1}$$

Equivalently, one has an equivalence of connective spectra

$$\mathfrak{pic}(R) \simeq \tau_{\geq 0}\mathrm{Tot}(\mathfrak{pic}(R'^{\otimes(\bullet+1)})). \tag{2.2}$$

In this paper, we will apply a version of this, except that we will work with morphisms of ring spectra that are not faithfully flat on the level of homotopy groups. As we will see, the descent spectral sequences given by (2.1) and (2.2) are not very useful in the faithfully flat case for our purposes.

Example 2.3.3. A more classical example of this technique (e.g. [Har77, Exercise 6.9]) is as follows. Let X be a nodal cubic curve over the complex numbers $\mathbb{C}$. Then X can be obtained from its normalization $\mathbb{P}^1$ by gluing together 0 and ∞. There is a pushout diagram of schemes

$$\begin{array}{ccc} \{0,\infty\} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \longrightarrow & X. \end{array}$$

Therefore, one would *like* to say that the $\mathrm{QCoh}(X)$ of on X fits into a homotopy pullback square

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \longrightarrow & \mathrm{QCoh}(*) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathbb{P}^1) & \longrightarrow & \mathrm{QCoh}(* \sqcup *), \end{array} \tag{2.3}$$

and that therefore, therefore, the Picard *groupoid* of X fits into the homotopy cartesian square

$$\begin{array}{ccc} \mathcal{P}\mathrm{ic}(X) & \longrightarrow & \mathcal{P}\mathrm{ic}(*) \\ \downarrow & & \downarrow \\ \mathcal{P}\mathrm{ic}(\mathbb{P}^1) & \longrightarrow & \mathcal{P}\mathrm{ic}(*) \times \mathcal{P}\mathrm{ic}(*). \end{array} \tag{2.4}$$

Unfortunately, (2.3) is not a pull-back square of categories, because restricting to a closed subscheme is not an exact functor. It is possible to remedy this (up to connectivity issues) by working with derived ∞-categories [Lur11a, Theorem 7.1], or by noting that we are working with locally free sheaves and applying a version of [Mil71, Theorems 2.1–2.3]. In any event, one can argue that (2.4) is homotopy cartesian.

Alternatively, we obtain a homotopy pullback diagram of *connective* spectra. Using the long exact sequence on π_*, it follows that we have a natural short exact sequence

$$0 \to \mathbb{C}^\times \to \mathrm{Pic}(X) \to \mathrm{Pic}(\mathbb{P}^1) \simeq \mathbb{Z} \to 0.$$

The approach of this paper is essentially an elaboration of this example.

2.4. **Picard groups of $\mathbf{E}_\infty$-rings.** We now specialize to the case of interest to us in this paper. Let R be an $\mathbf{E}_\infty$-ring, and consider the Picard group $\mathrm{Pic}(R)$, and better yet, the Picard ∞-groupoid $\mathcal{P}\mathrm{ic}(R)$ and the Picard spectrum $\mathfrak{pic}(R)$. This situation has been studied by Baker-Richter in the paper [BR05], and we start by recalling some of their results.

We start with the following useful property.

Proposition 2.4.1. *The functor $R \mapsto \mathrm{Pic}(R)$ commutes with filtered colimits in R.*

Proof. This is a consequence of a form of "noetherian descent" [Gro66, §8]. Given an $\mathbf{E}_\infty$-ring T, let $\mathrm{Mod}^\omega(T)$ denote the ∞-category of perfect T-modules. If I is a filtered ∞-category and $\{R_i\}_{i\in I}$ is a filtered system of $\mathbf{E}_\infty$-rings indexed by I, then the functor of symmetric monoidal ∞-categories

$$\varinjlim_{i\in I} \mathrm{Mod}^\omega(R_i) \to \mathrm{Mod}^\omega(\varinjlim_I R_i) \tag{2.5}$$

is an equivalence. We outline the proof of this below.

Assume without loss of generality that I is a filtered partially ordered set and write $R = \varinjlim_I R_i$. To see that (2.5) is an equivalence, observe that the ∞-category $\varinjlim_{i\in I} \mathrm{Mod}^\omega(R_i)$ has objects given by pairs (M, i) where $i \in I$ and $M \in \mathrm{Mod}^\omega(R_i)$. The space of maps between (M, i) and (N, j) is given by $\varinjlim_{k\geq i,j} \mathrm{Hom}_{\mathrm{Mod}(R_k)}(R_k \otimes_{R_i} M, R_k \otimes_{R_j} N)$. For instance, this implies that if $i' \geq i$, the pair (M, i) is (canonically) equivalent to the pair $(R_{i'} \otimes_{R_i} M, i')$. Thus, the assertion that (2.5) is fully faithful is equivalent to the assertion that if $M, N \in \mathrm{Mod}^\omega(R_i)$ for some i, then the natural map

$$\varinjlim_{j\geq i} \mathrm{Hom}_{\mathrm{Mod}^\omega(R_j)}(R_j \otimes_{R_i} M, R_j \otimes_{R_i} N) \to \mathrm{Hom}_{\mathrm{Mod}^\omega(R)}(R \otimes_{R_i} M, R \otimes_{R_i} N) \tag{2.6}$$

is an equivalence. But (2.6) is clearly an equivalence if $M = R_i$ for *any* N. The collection of $M \in \mathrm{Mod}^\omega(R_i)$ such that (2.6) is an equivalence is closed under finite colimits, desuspensions, and retracts, and therefore it is all of $\mathrm{Mod}^\omega(R_i)$. It therefore follows that (2.5) is fully faithful.

Moreover, the image of (2.5) contains $R \in \mathrm{Mod}^\omega(R)$ and is closed under desuspensions and cofibers (thus finite colimits). Let $\mathcal{C} \subset \mathrm{Mod}^\omega(R)$ be the subcategory generated by R under finite colimits and desuspensions. We have shown the image of the fully faithful functor (2.5) contains $\mathcal{C}$. Any object $M \in \mathrm{Mod}^\omega(R)$ is a retract of an object $X \in \mathcal{C}$, associated to an idempotent map $e\colon X \to X$. We can "descend" X to some $X_i \in \mathrm{Mod}^\omega(R_i)$ and the map e to a self-map $e_i : X_i \to X_i$ such that e_i^2 is homotopic to e_i. Now form the filtered colimit Y_i of $X_i \xrightarrow{e_i} X_i \xrightarrow{e_i} \dots$. Lifting our way up the tower $\cdots \to \mathrm{Hom}(X_i, X_i) \xrightarrow{e_i} \mathrm{Hom}(X_i, X_i)$, we can form a map $Y_i \to X_i$ such that the natural composite $X_i \to Y_i \to X_i$ is given by e_i, and it follows that Y_i is a direct summand of X_i and in particular belongs to $\mathrm{Mod}^\omega(R_i)$. The tensor product $R \otimes_{R_i} Y_i$ is the direct summand of X given by the idempotent e and is therefore equivalent to M.

The association $\mathcal{C} \mapsto \mathcal{P}\mathrm{ic}(\mathcal{C})$ commutes with filtered colimits of symmetric monoidal ∞-categories by Proposition 2.2.3. Taking Picard groups in the equivalence (2.5), the proposition follows. □

To begin getting a hold of $\mathrm{Pic}(R)$, purely algebraic information can be used. Let $\mathrm{Pic}(R_*)$ be the Picard group of the symmetric monoidal category of *graded* R_*-modules. The starting point of [BR05] is the following.

Construction 2.4.2. There is a monomorphism

$$\Phi\colon \mathrm{Pic}(R_*) \to \mathrm{Pic}(R),$$

constructed as follows. If M_* is an invertible R_*-module, it has to be finitely generated and projective of rank one. Consequently, there is a finitely generated free R_*-module F_* of which M_* is a direct summand, i.e. there is a projection p_* with a section s_*, $F_* \underset{p_*}{\overset{s_*}{\rightleftarrows}} M_*$.

Clearly, F_* can be realized as an R-module F which is a finite wedge sum of copies of R or its suspensions. Let e_* be the idempotent given by composition $s_* \circ p_*$. Since F is free over R, e_* can be realized as an R-module map $e : F \to F$ which must be idempotent. Define M to be the colimit of the sequence

$$F \xrightarrow{e} F \xrightarrow{e} \dots,$$

i.e. the image of the idempotent e. Observe that the homotopy groups of M are given by M_*, as desired. If M'_* is the inverse to M_* in the category of graded R_*-modules, we can construct an analogous R-module M', and clearly $M \otimes_R M' \simeq R$ by the degeneration of the Künneth spectral sequence. Thus, $M \in \mathrm{Pic}(R)$. The association $M_* \mapsto M$ defines Φ, and the Künneth spectral sequence again shows that Φ is a homomorphism.

Note that Φ is clearly a monomorphism as isomorphisms of R-modules are detected on homotopy groups.

Definition 2.4.3. When Φ is an isomorphism, we say that $\mathrm{Pic}(R)$ is *algebraic*.

Baker-Richter [BR05] determine certain conditions which imply algebraicity. There are, in particular, two fundamental examples. The first one generalizes Example 2.1.4.

Theorem 2.4.4 (Baker-Richter [BR05])**.** *Suppose R is a connective $\mathbf{E}_\infty$-ring. Then the Picard group of R is algebraic.*

Proof. Since the formulation in [BR05, Theorem 5.3] assumed a coherence hypothesis on $\pi_*(R)$, we explain briefly how this (slightly stronger) version can be deduced from the theory of flatness of [Lur12, §8.2.2]. Recall that an R-module M is *flat* if $\pi_0 M$ is a flat $\pi_0 R$-module and the natural map

$$\pi_* R \otimes_{\pi_0 R} \pi_0 M \to \pi_*(M),$$

is an isomorphism.

Since the Picard group commutes with filtered colimits in R, we may assume that R is finitely presented in the ∞-category of connective $\mathbf{E}_\infty$-rings: in particular, by [Lur12, Proposition 8.2.5.31], $\pi_0(R)$ is a finitely generated $\mathbb{Z}$-algebra and in particular noetherian; moreover, each $\pi_j(R)$ is a finitely generated $\pi_0(R)$-module. These are the properties that will be critical for us.

Let M be an invertible R-module. We will show that $\pi_*(M)$ is a flat module over $\pi_*(R)$, which immediately implies the claim of the theorem. Localizing at a prime ideal of $\pi_0(R)$, we may assume that $\pi_0(R)$ is a noetherian local ring; in this case we will show the Picard group is $\mathbb{Z}$ generated by the suspension of the unit. We saw that M is perfect, so we can assume by shifting that M is connective and that $\pi_0(M) \neq 0$. Now for every map $R \to k$, for k a field, we have that $M \otimes_R k$ is necessarily concentrated in a single dimension: in fact, it is an invertible object in $\mathrm{Mod}(k)$ and one can apply the Künneth formula to see that $\mathrm{Pic}(\mathrm{Mod}(k)) \simeq \mathbb{Z}$ generated by Σk. By Nakayama's lemma, since $\pi_0(M) \neq 0$, the homotopy groups of $M \otimes_R k$ must be concentrated in degree zero. Thus, $M \otimes_R k \simeq k$ itself. Using Lemma 2.4.5 below, it follows that M is equivalent to R as an R-module, so we are done. □

Lemma 2.4.5. *Let R be a connective $\mathbf{E}_\infty$-ring with $\pi_0(R)$ noetherian local with residue field k. Suppose moreover each $\pi_i(R)$ is a finitely generated $\pi_0(R)$-module. Suppose M is a connective, perfect R-module. Then, for $n \geq 0$, the following are equivalent:*

(1) $M \simeq R^n$.
(2) $M \otimes_R k \simeq k^n$.

Proof. Suppose $M \otimes_R k$ is isomorphic to k^n and concentrated in degree zero. Note that $\pi_0(M \otimes_R k) \simeq \pi_0(M) \otimes_{\pi_0(R)} k$. Choose a basis $\overline{x_1}, \dots, \overline{x_n}$ of this k-vector space and lift these elements to $x_1, \dots, x_n \in \pi_0(M)$. These define a map $R^n \to M$ which induces an equivalence after tensoring with k, since $M \otimes_R k \simeq k^n$.

Now consider the cofiber C of $R^n \to M$. It follows that $C \otimes_R k$ is contractible. Suppose C itself is not contractible. The hypotheses on $\pi_*(R)$ imply that C is connective and each $\pi_j(C)$ is a finitely generated module over the noetherian local ring $\pi_0(R)$. If j is chosen minimal such that $\pi_j(C) \neq 0$, then

$$0 = \pi_j(C \otimes_R k) \simeq \pi_j(C) \otimes_{\pi_0(R)} k,$$

and Nakayama's lemma implies that $\pi_j(C) = 0$, a contradiction. □

Some of our analyses in the computational sections will rest upon the next result about the Picard groups of *periodic* ring spectra.

Theorem 2.4.6 (Baker-Richter [BR05, Theorem 8.8])**.** *Suppose R is an even periodic $\mathbf{E}_\infty$-ring with $\pi_0(R)$ regular noetherian. Then the Picard group of R is algebraic.*

The result in [BR05, Theorem 8.8] actually assumes that $\pi_0(R)$ is a complete regular local ring. However, one can remove the hypotheses by replacing R with the localization $R_\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Spec}\pi_0 R$ and then by forming the completion at the maximal ideal.

We will need a slight strengthening of Theorem 2.4.6, though.

Corollary 2.4.7. *Suppose R is an $\mathbf{E}_\infty$-ring satisfying the following assumptions.*

1. *$\pi_0(R)$ is regular noetherian.*
2. *$\pi_i(R) = 0$ if $i \not\equiv 0 \mod 2k$.*
3. *There exists a unit in $\pi_{2k}(R)$ for some $k > 0$.*

Then the Picard group of R is algebraic.

Proof. Using the obstruction theory of [Ang04] (as well as localization), we can construct "residue fields" in R as $\mathbf{E}_1$-algebras in $\mathrm{Mod}(R)$ (which will be $2k$-periodic rather than 2-periodic). After this, the same argument as in Theorem 2.4.6 goes through. □

An example of a non-algebraic Picard group, based on [Mat14b, Example 6.1], is as follows.

Proposition 2.4.8. *The Picard group of the rational $\mathbf{E}_\infty$-ring $R = \mathbb{Q}[\epsilon_0, \epsilon_{-1}]/\epsilon_0^2$ (free on two generators $\epsilon_0, \epsilon_{-1}$ of degree 0 and -1, and with the relation $\epsilon_0^2 = 0$) is given by $\mathbb{Z} \times \mathbb{Q}$.*

Proof. The key observation is that R is equivalent, as an $\mathbf{E}_\infty$-ring, to cochains over S^1 on the (discrete) $\mathbf{E}_\infty$-ring $\mathbb{Q}[\epsilon_0]/\epsilon_0^2$, because $C^*(S^1; \mathbb{Q})$ is equivalent to $\mathbb{Q}[\epsilon_{-1}]$. By [Mat14a, Remark 7.9], we have a fully faithful, symmetric monoidal imbedding $\mathrm{Mod}(R) \subset \mathrm{Loc}_{S^1}(\mathrm{Mod}(\mathbb{Q}[\epsilon_0]/\epsilon_0^2))$ into ∞-category of local systems of $\mathbb{Q}[\epsilon_0]/\epsilon_0^2$-modules over the circle, whose image consists of those local systems of $\mathbb{Q}[\epsilon_0]/\epsilon_0^2$-modules such that the monodromy action on $\pi_*(S^1)$ is ind-unipotent.

In particular, to give an object in $\mathrm{Pic}(R)$ is equivalent to giving an element in $\mathrm{Pic}(\mathbb{Q}[\epsilon_0]/\epsilon_0^2)$ (of which there are only the suspensions of the unit, by Theorem 2.4.4) and an ind-unipotent (monodromy) automorphism, which is necessarily given by multiplication by $1 + q\epsilon$ for $q \in \mathbb{Q}$. We observe that this gives the right group structure to the Picard group because $(1 + q\epsilon)(1 + q'\epsilon) = 1 + (q + q')\epsilon$. □

Proposition 2.4.8 provides a counterexample to [Bal10, Conjecture 74], which states that in a tensor triangulated category generated by the unit with a local spectrum (e.g. with no nontrivial thick subcategories), any element $\mathcal{L}$ in the Picard group has the property that $\mathcal{L}^{\otimes n}$ is a suspension of the unit for suitable $n > 0$. In fact, one can take the (homotopy) category of perfect R-modules for R as in Proposition 2.4.8, which has no nontrivial thick subcategories by [Mat14b, Theorem 1.3].

Remark 2.4.9. Other examples of Picard groups come from the theory of *stable module ∞-categories* of a p-group G over a field k of characteristic p, which from a homotopy-theoretic perspective can be expressed as the module ∞-categories of the Tate construction k^{tG}. The Picard groups of stable module ∞-categories have been studied in the modular representation theory literature (under the name *endotrivial modules*) starting with [Dad78], where it is proved that the Picard group is algebraic (and cyclic) in the case where G is elementary abelian. The classification for a general p-group appears in [CT05].

3. The descent spectral sequence

In this section, we describe a descent spectral sequence for calculating Picard groups. The spectral sequence (studied originally by Gepner and Lawson [GL] in a closely related setting) is based on the observation (Proposition 2.2.3) that the association $\mathcal{C} \mapsto \mathcal{P}\mathrm{ic}(\mathcal{C})$, from symmetric monoidal ∞-categories to $\mathbf{E}_\infty$-spaces, commutes with homotopy limits. We will describe several examples and applications of this in the present section. Explicit computations will be considered in later parts of this paper.

For example, let $\{\mathcal{C}_U\}$ be a sheaf of symmetric monoidal ∞-categories on a site, and let $\Gamma(\mathcal{C})$ denote the global sections (i.e. the homotopy limit) ∞-category. Then we have an equivalence of connective spectra

$$\mathfrak{pic}(\Gamma(\mathcal{C})) \simeq \tau_{\geq 0}\Gamma(\mathfrak{pic}(\mathcal{C}_U)),$$

and one can thus use the descent spectral sequence for a sheaf of spectra to approach the computation of $\mathfrak{pic}(\Gamma(\mathcal{C}))$. We will use this approach, together with a bit of descent theory, to calculate $\mathrm{Pic}(TMF)$. The key idea is that while TMF itself has sufficiently complicated homotopy groups that results such as Theorem 2.4.6 cannot apply, the ∞-category of TMF-modules is built up as an inverse limit of module categories over $\mathbf{E}_\infty$-rings with better behaved homotopy groups.

3.1. Refinements.

Let X be a Deligne-Mumford stack equipped with a flat map $X \to M_{FG}$ to the moduli stack of formal groups. We will use the terminology of [MM13].

Definition 3.1.1. An *even periodic refinement* of X is a sheaf $\mathcal{O}^{\mathrm{top}}$ of $\mathbf{E}_\infty$-rings on the affine, étale site of X, such that for any étale map

$$\mathrm{Spec}R \to X,$$

the multiplicative homology theory associated to the $\mathbf{E}_\infty$-ring $\mathcal{O}^{\mathrm{top}}(\mathrm{Spec}R)$ is functorially identified with the (weakly) even-periodic Landweber exact theory[2] associated to the formal group classified by $\mathrm{Spec}R \to X \to M_{FG}$. We will denote the refinement of the ordinary stack X by $\mathfrak{X}$.

A very useful construction from the refinement $\mathfrak{X}$ is the $\mathbf{E}_\infty$-ring of "global sections" $\Gamma(X, \mathcal{O}^{\mathrm{top}})$, which is the homotopy limit of the $\mathcal{O}^{\mathrm{top}}(\mathrm{Spec}R)$ as $\mathrm{Spec}R \to X$ ranges over the affine, étale site of X.

Example 3.1.2. When X is the moduli stack M_{ell} of elliptic curves, with the natural map $M_{ell} \to M_{FG}$ that assigns to an elliptic curve its formal group, Goerss-Hopkins obstruction theory can be used to prove the existence of an even periodic refinement $\mathfrak{M}_{ell}$; the global sections of $\mathfrak{M}_{ell}$ are defined to be the $\mathbf{E}_\infty$-ring TMF of *topological modular forms*; for a survey, see [Goe10]. There is a similar picture for the compactified moduli stack $\overline{M}_{ell}$, whose global sections are denoted Tmf.

Definition 3.1.3. Given the refinement $\mathfrak{X}$, one has a natural symmetric monoidal stable ∞-category $\mathrm{QCoh}(\mathfrak{X})$ of *quasi-coherent sheaves* on $\mathfrak{X}$, given as a homotopy limit of the (stable, symmetric monoidal) ∞-categories $\mathrm{Mod}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec}R))$ for each étale map $\mathrm{Spec}R \to X$.

[2]See [Lur10, Lecture 18] for an exposition of the theory of weakly even-periodic theories.

There is an adjunction

$$\mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})) \rightleftarrows \mathrm{QCoh}(\mathfrak{X}), \tag{3.1}$$

where the left adjoint "tensors up" and the right adjoint takes global sections.

Our main goal in this paper is to investigate the left hand side; however, the right hand side is sometimes easier to work with, since even periodic, Landweber-exact spectra have convenient properties. Therefore, the following result will be helpful.

Theorem 3.1.4 ([MM13, Theorem 4.1])**.** *Suppose X is noetherian and separated, and $X \to M_{FG}$ is quasi-affine. Then the adjunction (3.1) is an equivalence of symmetric monoidal ∞-categories.*

For example, since the map $M_{ell} \to M_{FG}$ is affine, it follows that $\mathrm{Mod}(TMF)$ is equivalent to $\mathrm{QCoh}(\mathfrak{M}_{ell})$. This was originally proved by Meier, away from the prime 2, in [Mei12]. Theorem 3.1.4 implies the analog for Tmf and the derived *compactified* moduli stack, as well [MM13, Theorem 7.2].

Suppose $X \to M_{FG}$ is quasi-affine. In particular, it follows that there is a *sheaf* of symmetric monoidal ∞-categories on the affine, étale site of X, given by

$$(\mathrm{Spec} R \to X) \to \mathrm{Mod}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec} R)),$$

whose global sections are given by $\mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}))$. This diagram of ∞-categories is a sheaf in view of the descent theory of [Lur11b, Theorem 6.1], but [MM13, Theorem 4.1] gives the global sections. We are now in the situation of the introduction to this section. In particular, we obtain a descent spectral sequence for $\mathfrak{pic}(\Gamma(X, \mathcal{O}^{\mathrm{top}}))$, and we turn to studying it in detail.

3.2. **The Gepner-Lawson spectral sequence.** Keep the notation of the previous subsection: X is a Deligne-Mumford stack equipped with a quasi-affine flat map $X \to M_{FG}$, and $(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$ is an even periodic refinement.

Our goal in this subsection is to prove:

Theorem 3.2.1. *Suppose that X is a regular and connected Deligne-Mumford stack with a quasi-affine flat map $X \to M_{FG}$, and suppose $\mathfrak{X}$ is an even periodic refinement of X. There is a spectral sequence*

$$E_2^{s,t} = \begin{cases} \mathbb{Z}/2 & t = s = 0, \\ H^s(X, \mathcal{O}_X^\times) & t - s = 1 - s, \\ H^s(X, \omega^{(t-1)/2}) & t \geq 3 \text{ odd}, \\ 0 & \text{otherwise}, \end{cases} \tag{3.2}$$

whose abutment is $\pi_{t-s}\Gamma(X, \mathfrak{pic}(\mathcal{O}^{\mathrm{top}}))$. In particular, for $t-s \geq 0$, the abutment is $\pi_{t-s}\mathfrak{pic}(\Gamma(X, \mathcal{O}^{\mathrm{top}}))$.

The differentials run $d_r : E_r^{s,t} \to E^{s+r,t+r-1}$.

The analogous spectral sequence for a faithful Galois extension has been studied in work of Gepner and Lawson [GL], and our approach is closely based on theirs.

Proof. In this situation, as we saw in the previous subsection, we get an equivalence of symmetric monoidal ∞-groupoids,

$$\mathcal{P}\mathrm{ic}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})) \simeq \mathrm{holim}_{\mathrm{Spec} R \to X} \mathcal{P}\mathrm{ic}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec} R)),$$

where $\mathrm{Spec} R \to X$ ranges over the affine étale maps. Equivalently, we have an equivalence of connective spectra

$$\mathfrak{pic}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})) \simeq \tau_{\geq 0}\left(\mathrm{holim}_{\mathrm{Spec} R \to X} \mathfrak{pic}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec} R))\right).$$

Let us study the descent spectral sequence associated to this. We need to understand the homotopy group *sheaves* of the sheaf of connective spectra $(\mathrm{Spec} R \to X) \mapsto \mathfrak{pic}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec} R))$ (i.e.

the sheafification of the homotopy group presheaves $(\mathrm{Spec} R \to X) \mapsto \pi_i \mathfrak{pic}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec} R)))$. First, we know that

$$\pi_1 \mathfrak{pic}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec} R)) \simeq R^{\times},$$

and, for $i \geq 2$, we have

$$\pi_i \left(\mathfrak{pic}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec} R) \right) \simeq \pi_{i-1} \mathcal{O}^{\mathrm{top}}(\mathrm{Spec} R) = \begin{cases} \omega^{(i-1)/2} & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

It remains to determine the homotopy group sheaf π_0. If X is a regular Deligne-Mumford stack, so that each ring R that enters is regular, then we can do this using Theorem 2.4.6. In fact, it follows if R is a local ring, then $\pi_0 \mathfrak{pic}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec} R)) \simeq \mathbb{Z}/2$. Thus, up to suitably suspending once, invertible sheaves are locally trivial. Using the descent spectral sequence for a sheaf of spectra, we get that the above descent spectral sequence for $\Gamma(\mathfrak{X}, \mathfrak{pic}(\mathcal{O}^{\mathrm{top}}))$ is almost entirely the same as the descent spectral sequence for $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$ in the sense that the cohomology groups that appear for $t \geq 3$, i.e. $H^s(X, \omega^{(t-1)/2})$, are the same as those that appear in the descent spectral sequence for $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$. However, the terms for $t = 1$ are the étale cohomology of $\mathbb{G}_m$ on X. In particular, we obtain the term

$$H^1(X, \mathcal{O}_X^{\times}) \simeq \mathrm{Pic}(X),$$

which is the Picard group of the underlying ordinary stack. □

Remark 3.2.2. One may think of the spectral sequence as arising from a totalization, or rather as a filtered colimit of totalizations. Choose an étale hypercover $\mathfrak{A}$ given by $U_\bullet \to X$ by affine schemes $\{U_n\}$. For any $\mathbf{E}_\infty$-ring A, denote by $\mathcal{P}\mathrm{ic}^{\mathbb{Z}}(A)$ the symmetric monoidal subcategory of $\mathcal{P}\mathrm{ic}(A)$ spanned by those A-modules such that, after restricting to each connected component of $\mathrm{Spec} \pi_0 A$, become equivalent to a suspension of A. Denote by $\mathfrak{pic}^{\mathbb{Z}}(A)$ the associated connective spectrum. Then we form the totalization

$$\mathrm{Tot}(\mathfrak{pic}^{\mathbb{Z}}(\mathcal{O}^{\mathrm{top}}(U_\bullet))),$$

whose associated infinite loop space $\Omega^\infty \mathrm{Tot}(\mathfrak{pic}^{\mathbb{Z}}(\mathcal{O}^{\mathrm{top}}(U_\bullet)))$ is, by descent theory, the symmetric monoidal ∞-groupoid of $\mathcal{P}\mathrm{ic}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}))$ spanned by those invertible modules which become (up to a suspension) trivial after pullback along $U_0 \to X$. In particular, the filtered colimit of these totalizations is the spectrum we are after. The descent spectral sequence of Theorem 3.2.1 is the filtered colimit of these Tot spectral sequences.

3.3. **Galois descent.** We next describe the setting of the spectral sequence that was originally considered in [GL]. Let $A \to B$ be a faithful G-Galois extension of $\mathbf{E}_\infty$-ring spectra in the sense of [Rog08]. In particular, G acts on B in the ∞-category of $\mathbf{E}_\infty$-A-algebras and $A \to B^{hG}$ is an equivalence. Then $A \to B$ is an analog of a G-Galois étale cover in the sense of ordinary commutative algebra or algebraic geometry. As in ordinary algebraic geometry, there is a good theory of *Galois descent* along $A \to B$, as has been observed by several authors, for instance [GL, Mei12].

Theorem 3.3.1 (Galois descent)**.** *Let $A \to B$ be a faithful G-Galois extension of $\mathbf{E}_\infty$-rings. Then there is a natural equivalence of symmetric monoidal ∞-categories* $\mathrm{Mod}(A) \simeq \mathrm{Mod}(B)^{hG}$.

The "strength" of the descent is in fact very good. As shown in [Mat14a, Theorem 3.36], any faithful Galois extension $A \to B$ satisfies a form of descent up to nilpotence: the thick tensor-ideal that B generates in $\mathrm{Mod}(A)$ is equal to all of $\mathrm{Mod}(A)$. This imposes strong restrictions on the descent spectral sequences that can arise.

Applying the Picard functor, we get an equivalence of spaces

$$\mathcal{P}\mathrm{ic}(A) \simeq \mathcal{P}\mathrm{ic}(B)^{hG}, \tag{3.3}$$

or an equivalence of *connective* spectra

$$\mathfrak{pic}(A) \simeq \tau_{\geq 0}\mathfrak{pic}(B)^{hG}. \tag{3.4}$$

Remark 3.3.2. The spectrum $\Sigma\mathfrak{gl}_1 B$ is equivalent to $\tau_{\geq 1}(\mathfrak{pic}B)$; consider the induced map of G-homotopy fixed point spectral sequences. All the differentials involving the $t-s=0$ line will be the same for $\mathfrak{pic}B$ and $\Sigma\mathfrak{gl}_1 B$. Hence, we obtain a short exact sequence

$$0 \to \pi_0(\Sigma\mathfrak{gl}_1 B)^{hG} \to \pi_0(\mathfrak{pic}B)^{hG} \to E_\infty^{0,0} \to 0,$$

where $E_\infty^{0,0}$ is the kernel of all the differentials supported on $H^0(G, \pi_0\mathfrak{pic}B)$. This short exact sequence exhibits $\pi_0(\Sigma\mathfrak{gl}_1 B)^{hG}$ as the *relative Picard group* of $A \to B$, which consists of invertible A-modules which after smashing with B become isomorphic to B itself.

Our main interest in Galois theory, for the purpose of this paper, comes from the observation, due to Rognes, that there are numerous examples of G-Galois extensions of $\mathbf{E}_\infty$-rings $A \to B$ where the homotopy groups of B are significantly simpler than that of A. In particular, one hopes to understand the homotopy groups of $\mathfrak{pic}(B)$, and then use (3.3) and (3.4) together with an analysis of the associated homotopy fixed-point spectral sequence

$$H^s(G, \pi_t\mathfrak{pic}(B)) \Rightarrow \pi_{t-s}(\mathfrak{pic}B)^{hG}, \tag{3.5}$$

whose abutment for $t = s$ is the Picard group $\mathrm{Pic}(A)$.

Example 3.3.3 ([Rog08, Proposition 5.3.1])**.** The map $KO \to KU$ and the C_2-action on KU arising from complex conjugation exhibit KU as a C_2-Galois extension of KO.

Example 3.3.3 is fundamental and motivational to us: the study of KO-modules, which is a priori difficult because of the complicated structure of the ring $\pi_*(KO)$, can be approached via Galois descent together with the (much easier) study of KU-modules. In particular, we obtain

$$\mathfrak{pic}(KO) \simeq \tau_{\geq 0}\mathfrak{pic}(KU)^{hC_2},$$

and one can hope to use the homotopy fixed-point spectral sequence (HFPSS) to calculate $\mathfrak{pic}(KO)$. This approach is due to Gepner-Lawson [GL],[3] and we shall give a version of it below in Section 7.1 (albeit using a different method of deducing differentials).

Other examples of Galois extensions come from the theory of topological modular forms with *level structure.*

Example 3.3.4. Let $n \in \mathbb{N}$, and let $TMF(n)$ denote the periodic version of TMF for elliptic curves over $\mathbb{Z}[1/n]$-algebras with a *full level n structure.* Then, by [MM13, Theorem 7.6], $TMF[1/n] \to TMF(n)$ is a faithful $GL_2(\mathbb{Z}/n)$-Galois extension. The advantage is that, if $n \geq 3$, the moduli stack of elliptic curves with level n structure is actually a regular affine scheme (by [KM85, Corollary 2.7.2], elliptic curves with full level $n \geq 3$ structure have no nontrivial automorphisms). In particular, $TMF(n)$ is even periodic with regular π_0, and one can compute its Picard group purely algebraically by Theorem 2.4.6. One can then hope to use $GL_2(\mathbb{Z}/n)$-descent to get at the Picard group of $TMF[1/n]$. We will take this approach below.

3.4. The E_n-local sphere.

In addition, descent theory can be used to give a spectral sequence for $\mathfrak{pic}(L_n S^0)$. This is related to work of Kamiya-Shimomura [KS07] and the upper bounds that they obtain on $\mathrm{Pic}(L_n S^0)$.

Consider the cobar construction on $L_n S^0 \to E_n$, i.e. the cosimplicial $\mathbf{E}_\infty$-ring

$$E_n \rightrightarrows E_n \wedge E_n \overset{\to}{\rightrightarrows} \dots,$$

whose homotopy limit is $L_n S^0$. It is a consequence of the Hopkins-Ravenel smash product theorem that this cosimplicial diagram has "effective descent."

[3] The original calculation of the Picard group of KO, by related techniques, is unpublished work of Mike Hopkins.

Proposition 3.4.1. *The natural functor*

$$\mathrm{Mod}(L_nS^0) \to \mathrm{Tot}\left(\mathrm{Mod}(E_n^{\wedge(\bullet+1)})\right),$$

is an equivalence of symmetric monoidal ∞-categories.

Proof. According to the Hopkins-Ravenel smash product theorem, the map of $\mathbf{E}_\infty$-rings $L_nS^0 \to E_n$ has the property that the thick tensor-ideal that E_n generates in $\mathrm{Mod}(L_nS^0)$ is all of $\mathrm{Mod}(L_nS^0)$. According to [Mat14a, Proposition 3.21], this implies the desired descent statement (the condition is there called "admitting descent"). The argument is a straightforward application of the Barr-Beck-Lurie monadicity theorem [Lur12, §6.2]. □

In particular, we find that

$$\mathfrak{pic}(L_nS^0) \simeq \tau_{\geq 0}\mathrm{Tot}\mathfrak{pic}(E_n^{\wedge(\bullet+1)}).$$

Let us try to understand the associated spectral sequence.

The higher homotopy groups $\pi_i, i \geq 2$ of $\mathfrak{pic}(E_n^{\wedge(\bullet+1)})$ are determined in terms of those of $E_n^{\wedge(\bullet+1)}$. Once again, it remains to determine π_0. Now E_n is an even periodic $\mathbf{E}_\infty$-ring whose π_0 is regular local, so $\mathrm{Pic}(E_n) \simeq \pi_0\mathfrak{pic}(E_n) \simeq \mathbb{Z}/2$ by Theorem 2.4.6. The iterated smash products $E_n^{\wedge m}$ are also even periodic, so their Picard group contains at least a $\mathbb{Z}/2$. We do not need to know their exact Picard groups, however, to run the spectral sequence, as only the $\mathbb{Z}/2$ component is relevant for the spectral sequence (as it is all that comes from $\pi_0\mathfrak{pic}(E_n)$).

Next, we need to determine the algebraic Picard group. After taking π_0, the simplicial scheme

$$\dots \substack{\to\\\to\\\to} \mathrm{Spec}\pi_0(E_n \wedge E_n) \rightrightarrows \mathrm{Spec}\pi_0 E_n,$$

is a presentation of the moduli stack $M_{FG}^{\leq n}$ of formal groups (over $\mathbb{Z}_{(p)}$-algebras) of height at most n.

Proposition 3.4.2. $\mathrm{Pic}(M_{FG}^{\leq n}) \simeq \mathbb{Z}$*, generated by ω.*

Proof. We use the presentation of M_{FG} (localized at p) via the simplicial stack

$$\dots \substack{\to\\\to\\\to} (\mathrm{Spec}(MU \wedge MU)_*)/\mathbb{G}_m \rightrightarrows (\mathrm{Spec}MU_*)/\mathbb{G}_m. \tag{3.6}$$

Since the Picard group of a polynomial ring over $\mathbb{Z}_{(p)}$ is trivial, and each smash power of MU has a polynomial ring for π_*, the Picard group of each of the terms in the simplicial stack *without* the $\mathbb{G}_m$-quotient is trivial, and the group of units is $\mathbb{Z}_{(p)}^\times$, constant across the simplicial object. In other words, the *Picard groupoid* of each $\mathrm{Spec}(MU^{\wedge(s+1)})_*$ is $B\mathbb{Z}_{(p)}^\times$. When we add the $\mathbb{G}_m$-quotient, we get $\mathbb{Z} \times B\mathbb{Z}_{(p)}^\times$ for the Picard groupoid of each term in the simplicial stack because of the possibility of twisting by a character of $\mathbb{G}_m$: this twisting corresponds to the powers of ω. By descent theory, this shows that $\mathrm{Pic}(M_{FG}) \simeq \mathbb{Z}$, generated by ω. More precisely, the Picard groupoid of M_{FG} is the totalization of the Picard groupoids of $\mathrm{Spec}(MU^{\wedge(s+1)})_*/\mathbb{G}_m$, and each of these is $\mathbb{Z} \times B\mathbb{Z}_{(p)}^\times$: that is, the cosimplicial diagram of Picard groupoids is constant and the totalization is $\mathbb{Z} \times B\mathbb{Z}_{(p)}^\times$ again.

When we replace M_{FG} by $M_{FG}^{\leq n}$, we can replace the above presentation by excising from each term the closed substack cut out by $(p, v_1, \dots, v_n)$. This does not affect the Picard *groupoid* since the codimension of the substack removed is at least 2 (i.e. neither the Picard group nor the group of units is affected). That is, when we modify each term in (3.6) to form the associated presentation of $M_{FG}^{\leq n}$, the Picard groupoid is unchanged. It follows by faithfully flat descent that the inclusion $M_{FG}^{\leq n} \to M_{FG}$ induces an isomorphism on Picard groups (or groupoids) and that the Picard group is generated by ω. □

We conclude the following result.

Theorem 3.4.3. *There is a spectral sequence*

$$E_2^{s,t} = \begin{cases} \mathbb{Z}/2 & t = s = 0, \\ H^s(M_{FG}^{\leq n}, \mathcal{O}_{M_{FG}}^\times) & t - s = 1 - s, \\ H^s(M_{FG}^{\leq n}, \omega^{(t-1)/2}) & t \geq 3 \text{ odd}, \\ 0 & \text{otherwise}, \end{cases}$$

which converges for $t - s \geq 0$ to $\pi_{t-s}\mathfrak{pic}(L_n S^0)$. The relevant occurrences of the second case are $H^0(M_{FG}^{\leq n}, \mathcal{O}_{M_{FG}}^\times) \simeq \mathbb{Z}_{(p)}^\times$ and $H^1(M_{FG}^{\leq n}, \mathcal{O}_{M_{FG}}^\times) \simeq \mathbb{Z}$.

Note in particular that the E_2-term is determined entirely in terms of the Adams-Novikov spectral sequence for the E_n-local sphere. As we will see in Section 5, many of the differentials are also determined by the ANSS.

4. First examples

In this section, we will give several examples where descent theory gives a quick calculation of the Picard group. In these examples, we will not need to analyze differentials in the descent spectral sequence (3.5). The main examples of interest, where there will be a number of differentials to determine, will be treated in the last part of this paper.

4.1. The faithfully flat case.

We begin with the simplest case. Suppose $R \to R'$ is a morphism of $\mathbf{E}_\infty$-rings which is faithfully flat. In this case, we know from [Lur11b, Theorem 6.1] and Theorem 2.3.2 that the tensor-forgetful adjunction $\mathrm{Mod}(R) \rightleftarrows \mathrm{Mod}(R')$ is comonadic and we get a descent spectral sequence for the Picard group of R, as

$$\mathfrak{pic}(R) \simeq \tau_{\geq 0}\mathrm{Tot}\, \mathfrak{pic}(R'^{\otimes(\bullet+1)}).$$

This spectral sequence, however, gives essentially no new information that is not algebraic in nature. That is, the entire E_2-term $E_2^{s,t}$ for $t > 1$ vanishes, as it can be identified with the E_2-term for the cobar resolution $R'^{\otimes(\bullet+1)}$ of R, and this cobar resolution has a degenerate spectral sequence with non-zero terms only for $s = 0$ at E_2. For example, an element in $\mathrm{Pic}(R)$ is algebraic if *and only if* its image in $\mathrm{Pic}(R')$ is algebraic, by faithful flatness.

Thus, faithfully flat descent will be mostly irrelevant to us as a tool of computing the non-algebraic parts of Picard groups. In the examples of interest, we want $\pi_*(R')$ to be significantly simpler homologically than $\pi_*(R)$, so that we will be able to conclude (using results such as Theorem 2.4.6) that the Picard group of R' is entirely algebraic. But if $\pi_*(R')$ is faithfully flat over $\pi_*(R)$, it cannot be much simpler homologically. (Recall for example that *regularity* descends under faithfully flat extensions of noetherian rings.)

4.2. Cochain $\mathbf{E}_\infty$-rings and local systems.

In this subsection, we give another example of a family of $\mathbf{E}_\infty$-ring spectra whose Picard groups can be determined, or at least bounded.

Let X be a space and R an $\mathbf{E}_\infty$-ring. Let $R^X = C^*(X; R)$ be the $\mathbf{E}_\infty$-ring of R-valued cochains on X.

Definition 4.2.1. Let $\mathrm{Loc}_X(\mathrm{Mod}(R)) = \mathrm{Fun}(X, \mathrm{Mod}(R))$ denote the ∞-category of *local systems* of R-module spectra on X.

Then we have a fully faithful imbedding of symmetric monoidal ∞-categories

$$\mathrm{Mod}^\omega(R^X) \subset \mathrm{Loc}_X(\mathrm{Mod}(R)),$$

which sends R^X to the constant local system at R and is determined by that. As discussed in [Mat14a, §7], this imbedding is often useful for relating invariants of R^X to those of R. In particular, since any invertible R^X-module is perfect, we have a fully faithful functor of ∞-groupoids

$$\mathcal{P}\mathrm{ic}(R^X) \to \mathcal{P}\mathrm{ic}(\mathrm{Loc}_X(\mathrm{Mod}(R))) = \mathrm{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{Mod}(R))),$$

where the last identification follows because $\mathcal{P}\mathrm{ic}$ commutes with homotopy limits (Proposition 2.2.3). Thus, we get the following useful *upper bound* for the Picard group of R^X.

Proposition 4.2.2. *If R is an $\mathbf{E}_\infty$-ring and X is any space, then $\mathrm{Pic}(R^X)$ is a subgroup of $\pi_0(\mathfrak{pic}(R)^X)$.*

Without loss of generality, we will assume that X is connected. Note that we have a cofiber sequence

$$\Sigma \mathfrak{gl}_1(R) \to \mathfrak{pic}(R) \to H(\mathrm{Pic}(R)),$$

where $H(\mathrm{Pic}(R))$ denotes the Eilenberg-MacLane spectrum associated to the group $\mathrm{Pic}(R)$. If we take the long exact sequence after taking maps from X, we get an exact sequence

$$0 \to \pi_{-1}(\mathfrak{gl}_1(R)^X) \to \pi_0(\mathfrak{pic}(R)^X) \to \mathrm{Pic}(R). \tag{4.1}$$

Our object of interest, $\mathrm{Pic}(R^X)$, is a subobject of the middle term, by the above proposition.

Let us unwind the exact sequence further. First, the composite map $\mathrm{Pic}(R^X) \to \pi_0(\mathfrak{pic}(R)^X) \to \mathrm{Pic}(R)$ comes from the map of $\mathbf{E}_\infty$-rings $R^X \to R$ given by choosing a basepoint of X. In particular, it is *split surjective* as it has a section given by $R \to R^X$ (so (4.1) is a split exact sequence). Next, observe that, using the truncation map $\mathfrak{gl}_1(R) \to HR_0^\times$, we have a map $\pi_{-1}(\mathfrak{gl}_1(R)^X) \to \pi_{-1}((HR_0^\times)^X) = \mathrm{Hom}(\pi_1(X), R_0^\times)$. We can understand this map in terms of $\mathrm{Pic}(R^X)$. Very explicitly, suppose given an invertible R^X-module M with associated local system $\mathcal{L} \in \mathrm{Loc}_X(\mathrm{Mod}(R))$. Then if the image of M in $\mathrm{Pic}(R)$ is trivial, we conclude that $\mathcal{L}_x \simeq R$ for any basepoint $x \in X$. An element in $\pi_1(X, x)$ induces a monodromy automorphism of $\mathcal{L}_x$ and thus defines an element of $R_0^\times$. This defines a map in $\mathrm{Hom}(\pi_1(X, x), R_0^\times)$. Let $\mathrm{Pic}^0(R^X)$ be the kernel of $\mathrm{Pic}(R^X) \to \mathrm{Pic}(R)$. Then we have just described the map

$$\mathrm{Pic}^0(R^X) \xrightarrow{\phi} \mathrm{Hom}(\pi_1(X, x), R_0^\times), \tag{4.2}$$

that comes from the exact sequence (4.1).

The monodromy action cannot be arbitrary, since this local system is not arbitrary: it is in the image of $\mathrm{Mod}^\omega(R^X)$ and can be therefore built up as a colimit of copies of the unit. As in [Mat14a, §8], it follows that the monodromy action of any element of the fundamental group must be *ind-unipotent.* In particular, fix an element M of $\mathrm{Pic}^0(R^X)$. Given any loop $\gamma \in \pi_1(X, x)$, the associated element $u = u_{\gamma, M} \in R_0^\times$ under the homomorphism $\phi(M) : \mathrm{Pic}^0(R^X) \to \mathrm{Hom}(\pi_1(X, x), R_0^\times)$ of (4.2) must have the property that $u - 1$ is nilpotent.

Hence if R_0 is a *reduced* ring, we deduce from (4.1) the following conclusion.

Corollary 4.2.3. *If R is an $\mathbf{E}_\infty$-ring with $\pi_0(R)$ reduced, and X is any connected space, then we have a split short exact sequence*

$$0 \to A \to \mathrm{Pic}(R^X) \to \mathrm{Pic}(R) \to 0,$$

where $A \subset \pi_{-1}(\mathfrak{gl}_1(R)^X)$ is actually contained in $\pi_{-1}((\tau_{\geq 1}\mathfrak{gl}_1(R))^X) \subset \pi_{-1}((\mathfrak{gl}_1(R))^X)$. In particular, if $\pi_{-1}((\tau_{\geq 1}\mathfrak{gl}_1(R))^X) = 0$, then $\mathrm{Pic}(R) \to \mathrm{Pic}(R^X)$ is an isomorphism.

Again, we note that the map $\pi_{-1}((\tau_{\geq 1}\mathfrak{gl}_1(R))^X) \to \pi_{-1}(\mathfrak{gl}_1(R)^X)$ is injective, by the long exact sequence and the fact that $\pi_0(\mathfrak{gl}_1(R)^X) \to \pi_0((HR_0^\times)^X) \simeq R_0^\times$ is surjective.

As an application, we obtain a calculation of the Picard group of a nonconnective $\mathbf{E}_\infty$-ring in a setting far from regularity.

Theorem 4.2.4. *Let A be any finite* abelian *group and let E_n be Morava E-theory. Then the Picard group of E_n^{BA} is $\mathbb{Z}/2$, generated by the suspension ΣE_n^{BA}. The same conclusion holds for any finite group G whose p-Sylow subgroup is abelian, where p is the prime of definition for E_n.*

Proof. We induct upon the rank of A. When $A = 0$, then $E_n^{BA} \simeq E_n$ and Theorem 2.4.6 implies that the Picard group is $\mathbb{Z}/2$.

If the rank of A is positive, write $A \simeq \mathbb{Z}/p^m \times A'$ where A' has smaller rank. The inductive hypothesis gives us that the Picard group of $E_n^{BA'}$ is $\mathbb{Z}/2$. Now $E_n^{BA} \simeq (E_n^{BA'})^{B\mathbb{Z}/p^m}$. Moreover, $E_n^{BA'}$ is well-known to be even periodic (though its π_0 is not regular).

We claim now that $\pi_{-1}((\tau_{\geq 1}\mathfrak{gl}_1(E_n^{BA}))^{B\mathbb{Z}/p^m}) = 0$. To see this, we observe that the homotopy groups of $\tau_{\geq 1}\mathfrak{gl}_1(E_n^{BA'})$ are concentrated in even degrees and are all given by torsion-free p-local abelian groups. Therefore, the cohomology groups $H^i(\mathbb{Z}/p^m; \pi_j \tau_{\geq 1}\mathfrak{gl}_1(E_n^{BA}))$ *vanish* if i is odd, since the $\mathbb{Z}/p^m$-action on them is trivial. In the homotopy fixed point spectral sequence for $(\tau_{\geq 1}\mathfrak{gl}_1(E_n^{BA}))^{B\mathbb{Z}/p^m}$ (i.e. the Atiyah-Hirzebruch spectral sequence), there is no room for contributions to π_{-1}. In fact, there is no room for differentials at all, which indicates that any $\lim^1$ terms cannot occur either. Now Corollary 4.2.3 shows that the map $E_n^{BA'} \to E_n^{BA}$ induces an equivalence on Picard groups, which completes the inductive step.

For the last claim, fix any finite group G with an abelian p-Sylow subgroup $A \subset G$. For any connected space X, denote as before $\mathrm{Pic}^0(R^X)$ the kernel of $\mathrm{Pic}(R^X) \to \mathrm{Pic}(R)$. We have a commutative square

$$\begin{array}{ccc} \mathrm{Pic}^0(E_n^{BG}) & \longrightarrow & \mathrm{Pic}^0(E_n^{BA}) \\ \downarrow & & \downarrow \\ \pi_{-1}(\tau_{\geq 1}\mathfrak{gl}_1(E_n)^{BG}) & \longrightarrow & \pi_{-1}(\tau_{\geq 1}\mathfrak{gl}_1(E_n)^{BA}) \end{array}$$

The bottom horizontal map is injective since $\tau_{\geq 1}\mathfrak{gl}_1(E_n)$ is p-local and BG is p-locally a wedge summand of BA. It follows that $\mathrm{Pic}^0(E_n^{BG}) \to \mathrm{Pic}^0(E_n^{BA})$ is injective, and since the latter is zero, the former must be as well. □

Note that the spectrum E_1 is p-complete complex K-theory.

Proposition 4.2.5. *Let G be any finite group. Then the Picard group of E_1^{BG} is finite.*

Proof. In fact, $\pi_{-1}\left(\tau_{\geq 1}\mathfrak{gl}_1(E_1)^{BG}\right)$ is finite. We know that $\tau_{\geq 3}\mathfrak{gl}_1(E_1) \simeq \Sigma^4 ku_p^\wedge$ by a theorem of Adams-Priddy [AP76]. Moreover, $(ku_p^\wedge)^*(BG)$ is finite in each odd dimension, by comparing with $E_1^*(BG)$ which vanishes in odd dimensions. It follows now from Corollary 4.2.3 that the desired Picard group has to be finite. □

Question 4.2.6. Let G be any finite group. Can the Picard group of E_1^{BG} be any larger than $\mathbb{Z}/2$? What about the higher Morava E-theories?

4.3. **Coconnective rational $\mathbf{E}_\infty$-rings.** We can also determine the Picard groups of coconnective rational $\mathbf{E}_\infty$-ring spectra. A rational $\mathbf{E}_\infty$-ring R is said to be *coconnective* if

(1) $\pi_0(R)$ is a field (of characteristic zero), and
(2) $\pi_i(R) = 0$ for $i > 0$.

Theorem D. If R is a coconnective rational $\mathbf{E}_\infty$-ring, then the Picard group $\mathrm{Pic}(R) \simeq \mathbb{Z}$, generated by ΣR.

Proof. Let $k = \pi_0(R)$. We use [Lur11c, Proposition 4.3.3] to conclude that $R \simeq \mathrm{Tot}(A^\bullet)$, where $A^\bullet$ is a cosimplicial $\mathbf{E}_\infty$-k-algebra with each A^i of the form $k \oplus V[-1]$, where V is a discrete k-vector space; the $\mathbf{E}_\infty$-structure given is the "square-zero" one.

We thus begin with the case of $R = k \oplus V[-1]$: we will show that $\mathrm{Pic}(R) \simeq \mathbb{Z}$ in this case. Since Pic commutes with filtered colimits, we may assume that V is a finite-dimensional vector space. In this case,

$$R \simeq k^{S^1 \vee \dots \vee S^1},$$

where the number of copies of S^1 in the wedge summand is equal to the dimension $n = \dim_k V$; by [Lur11c, Proposition 4.3.1], any rational $\mathbf{E}_\infty$-ring with these homotopy groups is equivalent to $k \oplus V[-1]$. But we can now use Corollary 4.2.3 to see that the Picard group of $k^{S^1 \vee \dots \vee S^1}$ is $\mathbb{Z}$, generated by the suspension, because $\tau_{\geq 1}\mathfrak{gl}_1(k) = 0$.

Now suppose that R is arbitrary. As above, we have an equivalence $R \simeq \mathrm{Tot}(A^\bullet)$ where each A^i is a coconnective $\mathbf{E}_\infty$-ring of the form $k \oplus V[-1]$ for V a discrete k-vector space. We have seen above that $\mathrm{Pic}(A^i) \simeq \mathbb{Z}$. We know, moreover, that we have a fully faithful imbedding of symmetric monoidal ∞-categories

$$\mathrm{Mod}^\omega(R) \subset \mathrm{Tot}(\mathrm{Mod}(A^\bullet)),$$

which implies that we have a fully faithful functor of ∞-groupoids

$$\mathcal{P}\mathrm{ic}(R) \to \mathrm{Tot}(\mathcal{P}\mathrm{ic}(A^\bullet)).$$

But each $\mathcal{P}\mathrm{ic}(A^i)$, as an ∞-groupoid, has homotopy groups given by

$$\pi_j \mathcal{P}\mathrm{ic}(A^i) \simeq \begin{cases} \mathbb{Z} & j = 0 \\ k^\times & j = 1 \end{cases},$$

and in particular, in the cosimplicial diagram $\mathcal{P}\mathrm{ic}(A^\bullet)$, all the maps are *equivalences*. This is a helpful consequence of coconnectivity. Thus, we find that $\mathrm{Tot}\mathcal{P}\mathrm{ic}(A^\bullet)$ maps by equivalences to each $\mathcal{P}\mathrm{ic}(A^i)$, and we get an upper bound of $\mathbb{Z}$ for $\mathcal{P}\mathrm{ic}(R)$. This upper bound is realized by the suspension ΣR (which hits the generator of $\mathbb{Z} \simeq \pi_0 \mathrm{Tot}(\mathcal{P}\mathrm{ic}(A^\bullet))$). □

Remark 4.3.1. If $k = \mathbb{Q}$, then a large class of coconnective $\mathbf{E}_\infty$-rings with $\pi_0 \simeq \mathbb{Q}$ (e.g. those with reasonable finiteness hypotheses and vanishing π_{-1}) arise as cochains on a simply connected space, by Quillen-Sullivan's rational homotopy theory. The comparison with local systems can be carried out directly here to prove D for these $\mathbf{E}_\infty$-rings.

4.4. **Quasi-affine cases.** We now consider a case where the descent spectral sequence enables us to *produce* nontrivial elements in the Picard group. Let A be a weakly even-periodic $\mathbf{E}_\infty$-ring with $\pi_0(A)$ regular noetherian, and write $\omega = \pi_2(A)$. Then A leads to a sheaf of $\mathbf{E}_\infty$-rings on the affine, étale site of $\mathrm{Spec}\pi_0 A$. That is, for every étale $\pi_0 A$-algebra A_0', there is (functorially) associated [Lur12, §8.5] $\mathbf{E}_\infty$-ring A' with $\pi_0 A' \simeq A_0'$ and A' flat over A. We will denote this sheaf by $\mathcal{O}^{\mathrm{top}}$.

Now let $a_1, \dots, a_n \in \pi_0(A)$ be a regular sequence, for $n \geq 2$. We consider the complement U in $\mathrm{Spec}\pi_0(A)$ of the closed subscheme $V(a_1, \dots, a_n)$ and the sections $\overline{A} = \Gamma(U, \mathcal{O}^{\mathrm{top}})$. $\overline{A}$ is an $\mathbf{E}_\infty$-A-algebra and is a type of localization of A, albeit not (directly) an arithmetic one.[4] Note that $\mathrm{Pic}(A)$ is algebraic by Theorem 2.4.6, but the situation for $\overline{A}$ is more complicated.

The homotopy groups $\pi_*(\overline{A})$ are given by the abutment of a descent spectral sequence

$$H^s(U, \omega^{\otimes t}) \implies \pi_{2t-s}(\overline{A}). \tag{4.3}$$

We can, first, determine the zero-line. We have

$$H^0(U, \omega^{\otimes t}) = H^0(\mathrm{Spec}\pi_0 A, \omega^{\otimes t}),$$

because $\mathrm{Spec}\pi_0 A$ is regular and $U \subset \mathrm{Spec}\pi_0 A$ is obtained by removing a codimension ≥ 2 subscheme.

Proposition 4.4.1. *The only other nonzero term in the descent spectral sequence* (4.3) *occurs for* $s = n - 1$. *The descent spectral sequence degenerates.*

[4] A piece of forthcoming work of Bhatt and Halpern-Leinster identifies the universal property of $\overline{A}$.

Proof. Cover the scheme U by the n open affine subsets $U_i = \mathrm{Spec}\pi_0(A) \setminus V(a_i)$, for $1 \leq i \leq n$. Given any quasi-coherent sheaf $\mathcal{F}$ on U, it follows that the coherent cohomology $H^*(U, \mathcal{F})$ is that of the Cech complex (which starts in degree zero)

$$\bigoplus_{i=1}^{n} \mathcal{F}(U_i) \to \bigoplus_{i<j} \mathcal{F}(U_i \cap U_j) \to \cdots \to \mathcal{F}(U_1 \cap \cdots \cap U_n).$$

Let $R = \pi_0(A)$, and suppose $\mathcal{F}$ is the restriction to $U \subset \mathrm{Spec}R$ of the quasi-coherent sheaf $\widetilde{M}$ on $\mathrm{Spec}R$ for an R-module M. Then the final term is the cokernel of the map

$$\bigoplus_{i=1}^{n} M[(a_1 \dots \hat{a}_i \dots a_n)^{-1}] \to M[(a_1 \dots a_n)^{-1}],$$

where the hat denotes omission. If M is flat, the complex is exact away from degrees 0 and $n-1$ as the sequence $a_1, \dots, a_n$ is regular, using a Koszul complex argument (see for instance [ILL$^+$07]), and the zeroth cohomology is given by M itself.

Now, in view of the map $A \to \overline{A}$, clearly everything in the zero-line of the E_2-page of the spectral sequence survives, so the spectral sequence must degenerate. □

We now study the Picard group of $\overline{A}$: as above, $\pi_*(\overline{A})$ is not regular but instead has a great deal of square-zero material. Let $\mathfrak{U} = (U, \mathcal{O}^{\mathrm{top}}|_U)$ denote the "derived scheme" consisting of the topological space $U \subset \mathrm{Spec}\pi_0 A$, but equipped with the sheaf $\mathcal{O}^{\mathrm{top}}$ of $\mathbf{E}_\infty$-rings. $\overline{A}$ arises as the global sections of the structure sheaf $\mathcal{O}^{\mathrm{top}}$ over the derived scheme $\mathfrak{U}$.

Since U is quasi-affine as an (ordinary!) scheme, it follows by [MM13, Corollary 3.24][5] that the global sections functor is the right adjoint of an inverse equivalence

$$\mathrm{Mod}(\overline{A}) \rightleftarrows \mathrm{QCoh}(\mathfrak{U}),$$

of symmetric monoidal ∞-categories. In particular, the Picard group $\mathrm{Pic}(\overline{A})$ can be computed as $\mathrm{Pic}(\mathrm{QCoh}(\mathfrak{U}))$.

As before, we have a descent spectral sequence (3.2) converging to $\pi_{t-s}\mathfrak{pic}(\overline{A})$. But from (3.2), we know that almost all of the terms at E_2 are identified with the descent spectral sequence for $\pi_*(\overline{A})$. In addition, we know that $H^1(U, \mathcal{O}_U^\times) \simeq \mathrm{Pic}(\pi_0(A))$, as $\pi_0(A)$ is regular and the complement of U has codimension ≥ 2. These classes must be permanent cycles as they are realized in $\mathrm{Pic}(\overline{A})$: in fact, they are realized in $\mathrm{Pic}(A)$ itself. Thus, the descent spectral sequence for $\mathfrak{pic}$ degenerates as well, and we get:

Theorem 4.4.2. *Let $\overline{A} = \Gamma(U, \mathcal{O}^{\mathrm{top}})$ as above. Then we have a natural isomorphism*

$$\mathrm{Pic}(\overline{A}) \simeq \mathbb{Z}/2 \times \mathrm{Pic}(\pi_0 A) \times \pi_{-1}(\overline{A}).$$

Observe, moreover, that

$$\pi_{-1}(\overline{A}) = \begin{cases} \mathrm{coker}\left(\bigoplus_{i=1}^{n} \omega^{n/2-1}[(a_1 \dots \hat{a}_i \dots a_n)^{-1}] \to \omega^{n/2-1}[(a_1 \dots a_n)^{-1}]\right) & n \geq 4 \text{ even} \\ 0 & n \text{ odd.} \end{cases} \tag{4.4}$$

Example 4.4.3. Let A be a Landweber-exact, even periodic $\mathbf{E}_\infty$-ring with $\pi_0 A$ regular noetherian; for instance, A could be Morava E-theory E_n. In this case, we take $a_1, \dots, a_k = p, v_1, \dots, v_{k-1}$, so that $\overline{A} \simeq L_k A$. This gives C as a special case of Theorem 4.4.2.

[5]This result is originally due to Jacob Lurie.

Part II. Computational tools

5. The comparison tool in the stable range

This is a technical section in which we develop a tool that will enable us to compare many of the differentials in a Picard spectral sequence for Galois or étale descent with the analogous differentials in the corresponding descent spectral sequence before taking the Picard functor (i.e. for the $\mathbf{E}_\infty$-rings themselves). For example, in the Galois descent setting, we are given a G-Galois extension $A \to B$, and we know the descent, i.e. homotopy fixed point, spectral sequence for $A \simeq B^{hG}$. The tool we develop in this section will allow us to deduce many differentials in the homotopy fixed point spectral sequence for $(\mathfrak{pic}B)^{hG}$.

For a spectrum or a pointed space X, and integers a, b, we denote by $\tau_{\geq a}X$, $\tau_{\leq b}X$, and $\tau_{[a,b]}$ the truncations of X with homotopy groups in the designated range. Our main observation is that if R is any $\mathbf{E}_\infty$-ring, then for any $n \geq 2$, there is a natural equivalence of spectra

$$\tau_{[n,2n-1]}R \simeq \tau_{[n,2n-1]}\mathfrak{gl}_1(R).$$

This equivalence is natural at the level of ∞-categories, and enables us to identify a large number of differentials in descent spectral sequences for $\mathfrak{gl}_1$ and therefore also for $\mathfrak{pic}$. This observation, however, fails if we increase the range by 1, and an identification of the relevant discrepancy (as observed in such spectral sequences) will be the subject of the following section and the formula (6.1).

5.1. Truncated spaces and spectra.

Throughout, $n \geq 2$.

Definition 5.1.1. Let $\mathrm{Sp}_{[n,2n-1]} \subset \mathrm{Sp}$ denote the ∞-category of spectra with homotopy groups concentrated in degrees $[n, 2n-1]$. Let $\mathcal{S}_*$ denote the ∞-category of pointed spaces, and let $\mathcal{S}_{*,[a,b]} \subset \mathcal{S}_*$ denote the subcategory spanned by those pointed spaces whose homotopy groups are concentrated in the interval $[a, b]$.

Theorem 5.1.2. *The functor $\Omega^\infty : \mathrm{Sp}_{[n,2n-1]} \to \mathcal{S}_*$ is fully faithful. The functor $\Omega^\infty : \mathrm{Sp}_{[n,2n-2]} \to \mathcal{S}_{*,[n,2n-2]}$ is an equivalence of ∞-categories.*

Proof. Let $X, Y \in \mathrm{Sp}_{[n,2n-1]}$. We want to show that the natural map

$$\mathrm{Hom}_{\mathrm{Sp}}(X, Y) \to \mathrm{Hom}_{\mathcal{S}_*}(\Omega^\infty X, \Omega^\infty Y) \tag{5.1}$$

is a homotopy equivalence. By adjointness, we can identify this with the map

$$\mathrm{Hom}_{\mathrm{Sp}}(X, Y) \to \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty\Omega^\infty X, Y)$$

that arises from the counit map $\Sigma^\infty\Omega^\infty X \to X$. Observe that we have a natural equivalence $\mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty\Omega^\infty X, Y) \simeq \mathrm{Hom}_{\mathrm{Sp}}(\tau_{\leq 2n-1}\Sigma^\infty\Omega^\infty X, Y)$ because Y is $(2n-1)$-truncated. In particular, to prove Theorem 5.1.2, it will suffice to show that the natural map of spectra

$$\tau_{\leq 2n-1}\Sigma^\infty\Omega^\infty X \to X \simeq \tau_{\leq 2n-1}X,$$

is an equivalence, for any $X \in \mathrm{Sp}_{[n,2n-1]}$. Equivalently, we need to show that for any such spectrum X, the map

$$\pi_k(\Sigma^\infty\Omega^\infty X) \to \pi_k(X) \tag{5.2}$$

is an isomorphism for $k \leq 2n-1$. But we have maps of *spaces*

$$\Omega^\infty X \to \Omega^\infty\Sigma^\infty\Omega^\infty X \to \Omega^\infty X,$$

where the composite is the identity. The first map is the unit $Y \to \Omega^\infty\Sigma^\infty Y$ applicable for any $Y \in \mathcal{S}_*$, and the second map is Ω^∞ applied to the counit. By the Freudenthal suspension theorem, the first map induces an isomorphism on homotopy groups $\pi_k, k \leq 2n-1$, and therefore the second map does as well. This proves the claim that (5.2) is an isomorphism and the first part of the theorem.

The functor $\Omega^\infty : \mathrm{Sp}_{[n,2n-1]} \to \mathcal{S}_{*,[n,2n-1]}$ is not essentially surjective, because spaces with homotopy groups concentrated in degrees $[n, 2n-1]$ can still have *Whitehead products*, and spaces with nontrivial Whitehead products can never be in the image of Ω^∞. However, we claimed in the statement of the theorem that the functor $\Omega^\infty : \mathrm{Sp}_{[n,2n-2]} \to \mathcal{S}_{*,[n,2n-2]}$ is an equivalence of ∞-categories. To show this, it suffices to show that the functor is essentially surjective.

Given a pointed space X with homotopy groups in the desired range, we suppose inductively (on k) that $\tau_{\leq k}X$ is in the image of Ω^∞. If $k \geq 2n-2$, then we are done. Otherwise, we have a pullback square

$$\begin{array}{ccc} \tau_{\leq k+1}X & \longrightarrow & * \\ \downarrow & & \downarrow \\ \tau_{\leq k}X & \longrightarrow & K(\pi_{k+1}X, k+2). \end{array}$$

Observe that the three pointed spaces $\tau_{\leq k}X, K(\pi_{k+1}X, k+2)$, and $*$ are all in the image of Ω^∞ (the first by the inductive hypothesis), and $K(\pi_{k+1}X, k+2) \in \mathcal{S}_{*,[n,2n-1]}$. Moreover, the *maps* in the diagram are in the image of Ω^∞ by the previous part of the result. Therefore, the object $\tau_{\leq k+1}X$ is in the image of Ω^∞, as Ω^∞ preserves homotopy fiber squares. ☐

Given an integer k, we could precompose the functor of Theorem 5.1.2 with the equivalence $\Omega^k : \mathrm{Sp}_{[n+k,2n+k-1]} \to \mathrm{Sp}_{[n,2n-1]}$, and obtain the following:

Corollary 5.1.3. *For any integer k, the functor $\Omega^{\infty+k} : \mathrm{Sp}_{[n+k,2n+k-1]} \to \mathcal{S}_*$ is fully faithful.*

5.2. Comparisons for $\mathbf{E}_\infty$-rings.

Our basic example for all this comes from the spectrum $\mathfrak{gl}_1(R)$ associated to an $\mathbf{E}_\infty$-ring R, and the comparison between the two. This comparison is the main obstacle in understanding the descent spectral sequence for the Picard group: it is generally easier to understand descent spectral sequences for the $\mathbf{E}_\infty$-rings themselves (e.g. for TMF).

Given an $\mathbf{E}_\infty$-ring R, the *spectra* $R, \mathfrak{gl}_1(R)$ are generally very different.

Example 5.2.1 (T. Lawson [hl])**.** Consider the commutative, differential graded algebra $\mathbb{F}_2[x]/x^3$ where $|x| = 1$ and $dx = 0$ (so $d \equiv 0$). Let R be the associated $\mathbf{E}_\infty$-ring under $\mathbb{F}_2$. Then $\mathfrak{gl}_1(R)$ has homotopy groups in dimensions $1, 2$ given by $\mathbb{F}_2$; however, they are connected by multiplication by η. In particular, $\mathfrak{gl}_1(R)$ is not an $\mathbb{F}_2$-module spectrum.

More generally, let R be the $\mathbf{E}_\infty$-ring associated to the commutative, differential graded algebra $\mathbb{F}_2[x]/x^3$ where $|x| = n$, $dx = 0$. R can also be constructed by applying the Postnikov section $\tau_{\leq 2n}$ to the free $\mathbf{E}_\infty$-$\mathbb{F}_2$-algebra on a class in degree n. Then $\pi_n(\mathfrak{gl}_1(R)) \simeq \pi_{2n}(\mathfrak{gl}_1(R)) \simeq \mathbb{F}_2$ and all the other homotopy groups of $\mathfrak{gl}_1(R)$ vanish. Therefore, $\mathfrak{gl}_1(R)$ is the fiber of a k-invariant map $H\mathbb{F}_2[n] \to H\mathbb{F}_2[2n+1]$. In this case, we can identify the k-invariant map and thus identify $\mathfrak{gl}_1(R)$.

Proposition 5.2.2. *Given R as above, the k-invariant of $\mathfrak{gl}_1(R)$ is given by the map Sq^{n+1} : $H\mathbb{F}_2[n] \to H\mathbb{F}_2[2n+1]$.*

Proof. We first argue, following Lawson, that $\mathfrak{gl}_1(R)$ cannot be the spectrum $H\mathbb{F}_2[n] \vee H\mathbb{F}_2[2n]$. In fact, in this case, the map of spectra $H\mathbb{F}_2[n] \to \mathfrak{gl}_1(R)$ would, by adjointness [ABG+08] lead to a map of $\mathbf{E}_\infty$-rings

$$\Sigma^\infty_+ K(\mathbb{F}_2, n) \to R,$$

carrying the class in $\pi_n K(\mathbb{F}_2, n)$ to the nonzero class in $\pi_n R$. Smashing with $H\mathbb{F}_2$, we would get a map of $\mathbf{E}_\infty$-$H\mathbb{F}_2$-algebras

$$H\mathbb{F}_2 \wedge \Sigma^\infty_+ K(\mathbb{F}_2, n) \to R$$

with the same property. Now $\pi_n(H\mathbb{F}_2 \wedge \Sigma^\infty_+ K(\mathbb{F}_2, n)) \simeq \mathbb{F}_2$, with the nontrivial class coming from $\pi_n(K(\mathbb{F}_2, n))$. However, this class squares to zero by [CLM76, Lemma 6.1, Ch. 1] while the nonzero class in $\pi_n R$ does not square to zero. This is a contradiction and proves that such a map cannot exist. Consequently, the k-invariant map for $\mathfrak{gl}_1(R)$ must be nontrivial.

On the other hand, we know $\Omega^\infty \mathfrak{gl}_1(R) \simeq K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, 2n)$ because $\Omega^\infty \mathfrak{gl}_1(R)$ is the connected component at 1 of $\Omega^\infty R$. In particular, the k-invariant $H\mathbb{F}_2[n] \to H\mathbb{F}_2[2n+1]$ defines upon applying Ω^∞ the trivial cohomology class in $H^{2n+1}(K(\mathbb{F}_2, n); \mathbb{F}_2)$.

So, for the k-invariant of $\mathfrak{gl}_1(R)$, we need a nonzero element ϕ of degree $n+1$ in the (mod 2) Steenrod algebra such that, if $\iota_n \in H^n(K(\mathbb{F}_2, n); n)$ is the tautological class, then $\phi\iota_n = 0$. By the calculation of the cohomology of Eilenberg-MacLane spaces (see [MT68] for a textbook reference), the only possibility is Sq^{n+1}. □

Nonetheless, we will show that right below the range of the previous example, the spectra $\mathfrak{gl}_1(R)$ and R can be identified.

Corollary 5.2.3. *Let $n \geq 2$ and let R be any $\mathbf{E}_\infty$-ring. Then there is an equivalence of spectra, functorial in R,*

$$\tau_{[n,2n-1]}\mathfrak{gl}_1(R) \simeq \tau_{[n,2n-1]}R.$$

Similarly, there is an equivalence of spectra, functorial in R,

$$\tau_{[n+1,2n]}\mathfrak{pic}(R) \simeq \Sigma\tau_{[n,2n-1]}R.$$

Proof. For any $\mathbf{E}_\infty$-ring R, the space $\Omega^\infty \mathfrak{gl}_1(R) = GL_1(R)$ is a union of those components of $\Omega^\infty R$ that correspond to units in $\pi_0 R$. In particular, $\Omega^\infty \tau_{\geq 1}\mathfrak{gl}_1(R)$ is *canonically* identified with $\Omega^\infty \tau_{\geq 1} R$ in $\mathcal{S}_*$. Applying Theorem 5.1.2, we now get a canonical identification as desired in the corollary. The second half of Corollary 5.2.3 follows from the first, as $\tau_{\geq 0}\Omega\mathfrak{pic}(R) \simeq \mathfrak{gl}_1(R)$ as spectra. □

Take now a faithful G-Galois extension $A \to B$ of $\mathbf{E}_\infty$-rings, and consider the HFPSS (3.5) for $\mathfrak{pic}B$. We want to understand $\pi_0(\mathfrak{pic}(B)^{hG})$, or equivalently $\pi_{-1}(\Omega\mathfrak{pic}(B)^{hG})$, and we can do this by understanding the HFPSS for the G-action on $\Omega\mathfrak{pic}(B)$. Observe first that $\pi_t\Omega\mathfrak{pic}(B) \simeq \pi_t B$ functorially for $t \geq 1$: in fact, $\Omega^\infty(\Omega\mathfrak{pic}(B)) \simeq GL_1(B)$. In other words, the spectrum $\Omega\mathfrak{pic}(B)$ equipped with the G-action has the property that, after applying Ω^∞, it is identified with a union of connected components of $\Omega^\infty B$ (with the G-action on B).

As a result, we have a map of spaces with G-action

$$\Omega^\infty(\Omega\mathfrak{pic}(B)) \to \Omega^\infty B,$$

which identifies the former with a union of connected components of the latter. As a result, we can identify the respective HFPSS for the spaces $\Omega^\infty(\Omega\mathfrak{pic}(B))$, $\Omega^\infty B$ for $t > 0$, both at E_2 and differentials (including the "fringed" ones).

In particular, shifting by one again, most of the differentials in the HFPSS for $\mathfrak{pic}(B)$ are determined by the HFPSS for B. More precisely, any differential out of $E_r^{s,t}$ for $t - s > 0$, $s > 0$, depends only on the G-space $\Omega\mathcal{P}\mathrm{ic}(B)$, so the equivalence of $\Omega\mathcal{P}\mathrm{ic}(B)$ with a union of connected components of $\Omega^\infty B$ implies that the differential *can be identified* with the analogous differential in the HFPSS for B.

However, to understand $\pi_0(\mathfrak{pic}(B)^{hG}) \simeq \pi_0(\mathcal{P}\mathrm{ic}(B)^{hG}) \simeq \mathrm{Pic}(A)$, we need to determine differentials out of $E_r^{s,t}$ with $t = s$. These differentials cannot be determined by $\Omega\mathcal{P}\mathrm{ic}(B)$, as a space with a G-action. Our strategy to determine these differentials is to use the equivalence of spectra with G-action

$$\tau_{[n+1,2n]}\mathfrak{pic}(B) \simeq \Sigma\tau_{[n,2n-1]}B,$$

which is a special case of Corollary 5.2.3.

Assume that $r \leq t-1$. In this case, any differential $d_r : E_*^{t,t} \to E_*^{t+r,t+r-1}$ in the HFPSS for $\mathfrak{pic}(B)$ is determined by the G-action on $\tau_{[t,t+r-1]}\mathfrak{pic}(B)$. Since we have an equivalence $\tau_{[t,t+r-1]}\mathfrak{pic}(B) \simeq \Sigma\tau_{[t-1,t+r-2]}B$, compatible with the G-actions, we can identify the differentials.

Denote the differentials in the homotopy fixed point spectral sequence

$$H^s(G, \pi_t \mathfrak{pic} B) \Rightarrow \pi_{t-s}(\mathfrak{pic} B)^{hG}$$

by $d_r^{s,t}(\mathfrak{pic}B)$, and similarly $d_r^{s,t}(B)$ for those in the HFPSS for B. The upshot of this discussion is the following.

Comparison Tool 5.2.4. *Let $A \to B$ be a G-Galois extension of $\mathbf{E}_\infty$ rings. Whenever $2 \leq r \leq t-1$, we have an equality of differentials $d_r^{t,t}(\mathfrak{pic}B) = d_r^{t,t-1}(B)$.*

Of course, we also have an identification of differentials out of (s,t) if $t-s>0, s>0$.

Remark 5.2.5. Our original approach to the Comparison Tool 5.2.4 was somewhat more complicated than the above and has been described in [MS14]. Namely, our strategy was to identify the HFPSS with a Bousfield-Kan spectral sequence for a certain cosimplicial space $X^\bullet$ built from $\mathcal{P}\mathrm{ic}(B)$ with its G-action, and argue that these differentials only depended on the fiber of $\mathrm{Tot}_{t+r}(X^\bullet) \to \mathrm{Tot}_{t-1}(X^\bullet)$ (as well as the other fibers in between). In the appropriate range, these fibers depend only on $\Omega X^\bullet$ as a cosimplicial space. However, $\Omega X^\bullet$ can be (almost) identified with the analogous cosimplicial space for the G-action on $\Omega^{\infty-1}(\tau_{\geq 0}B)$ because $\Omega\mathcal{P}\mathrm{ic}(B)$ is a union of components $\Omega^\infty B$. This forces the differentials to correspond to one another.

For the same reasons, we would have an analogous comparison results for the spectral sequence as Theorem 3.2.1. Again, any differential in the descent spectral sequence for $\mathfrak{pic}(\Gamma(X, \mathcal{O}^{\mathrm{top}}))$ that only depends on the *diagram* $\tau_{[n+1,2n]}\mathfrak{pic}(\mathcal{O}^{\mathrm{top}})$ can be identified with the corresponding differential in the descent spectral sequence for $\Gamma(X, \mathcal{O}^{\mathrm{top}})$, thanks to the equivalence of *diagrams* of spectra $\tau_{[n+1,2n]}\mathfrak{pic}(\mathcal{O}^{\mathrm{top}}) \simeq \Sigma\tau_{[n,2n-1]}\mathcal{O}^{\mathrm{top}}$.

Remark 5.2.6. The equivalence $\tau_{[n,2n-1]}R \simeq \tau_{[n,2n-1]}\mathfrak{gl}_1(R)$ resembles the following observation in commutative algebra. Let A be an ordinary commutative ring and let $I \subset A$ be a square-zero ideal. Then $1 + I \subset A^\times$ and there is an isomorphism of groups

$$I \simeq 1 + I \subset A^\times, \quad x \mapsto 1 + x.$$

This correspondence is a very degenerate version of the exponential and logarithm.

Suppose p is a prime number and $(p-1)!$ is invertible in A. Then if $J \subset A$ is an ideal with $J^p = 0$, we have $1 + J \subset A^\times$ and a natural isomorphism of groups

$$J \simeq 1 + J, \quad x \mapsto 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{p-1}}{(p-1)!},$$

given by a p-truncated exponential. Motivated by this, we conjecture:

Conjecture 5.2.7. *Let R be an $\mathbf{E}_\infty$-ring with $(p-1)!$ invertible. Then, for any n, there is a* functorial *equivalence of spectra $\tau_{[n,pn-1]}R \simeq \tau_{[n,pn-1]}\mathfrak{gl}_1(R)$.*

In the case of *rational* $\mathbf{E}_\infty$-ring spectra R, we should obtain a functorial equivalence of spectra $\tau_{\geq 1}\mathfrak{gl}_1(R) \simeq \tau_{\geq 1}R$. At the level of *cohomology theories*, this is described in §2.5 of [Rez06], but the construction there does not give naturality at the ∞-categorical level.

5.3. **A general result on Galois descent.** As a quick application of the preceding ideas, we can prove a general result about Galois descent for Picard groups.

Theorem E. Let $A \to B$ be a faithful G-Galois extension of $\mathbf{E}_\infty$-rings. Then the relative Picard group of B/A is $|G|$-power torsion of finite exponent.

Proof. We know that the relative Picard group of $A \to B$ is given by $\pi_{-1}(\mathfrak{gl}_1(B)^{hG})$ (compare Remark 3.3.2). There is a HFPSS that converges to the homotopy groups, which begins with the group cohomology (with coefficients in G) of $\pi_*(\mathfrak{gl}_1(B))$. Every contributing term is $|G|$-power torsion: in fact, every term is a $H^i(G; \cdot)$ for $i > 0$ and is thus killed by $|G|$. However, in view of the potential infiniteness of the filtration, as well as the possibilities of nontrivial extensions, this alone does not force $\pi_{-1}(\mathfrak{gl}_1(B)^{hG})$ to be $|G|$-power torsion.

Our strategy is to compare the HFPSS for $\pi_{-1}(\mathfrak{gl}_1(B)^{hG})$ with that of $\pi_{-1}(B^{hG})$. The map $A \to B$ admits descent in the sense of [Mat14a, Definition 3.17]. In particular, by [Mat14a, Corollary 4.4], the descent spectral sequence for $A \to B$ (equivalently, the HFPSS) has a *horizontal vanishing line* at a finite stage. It follows that, above a certain filtration, everything in the HFPSS for $\pi_*(A) \simeq \pi_*(B^{hG})$ is killed by a d_k for k bounded.

In view of our Comparison Tool 5.2.4, it follows that any class in the relative Picard group has bounded filtration (though possibly the bound is weaker than the analog in $\pi_{-1}(B)$). Since every term in the spectral sequence is killed by $|G|$, the theorem follows. □

6. The first unstable differential

6.1. **Context.** Let $R^\bullet$ be a cosimplicial $\mathbf{E}_\infty$-ring, and consider the Bousfield-Kan spectral sequences (BKSS) $\{E_r^{s,t}\}, \left\{\overline{E}_r^{s,t}\right\}$ for the two cosimplicial objects $\mathfrak{gl}_1(R^\bullet)$ and $R^\bullet$, converging to π_{t-s} of the respective totalizations in Sp.

For $t - s \geq 0$, the spectral sequences and the differentials are mostly identified with one another, as the space $\Omega^\infty \mathfrak{gl}_1(R)$ is a union of connected components of $\Omega^\infty R$. But for $t - s = -1$, we get differentials

$$d_r : E_r^{t+1,t} \to E_r^{t+r+1,t+r-1}, \quad \overline{d}_r : E_r^{t+1,t} \to E_r^{t+r+1,t+r-1}.$$

These depend on more than the spaces $\Omega^\infty R, \Omega^\infty \mathfrak{gl}_1(R)$: they require the one-fold deloopings. As we saw in Corollary 5.2.3, for any $n \geq 2$, in the range $[n, 2n-1]$, the cosimplicial *spectra* $\tau_{[n,2n-1]} R^\bullet, \tau_{[n,2n-1]} \mathfrak{gl}_1(R^\bullet)$ are identified, and as a result, for $r \leq t$, the groups in question are (canonically) identified and $d_r = \overline{d}_r$.

But in general, $d_{t+1} \neq \overline{d}_{t+1}$. Since all the previous differentials entering or leaving this spot between the two spectral sequences were identified, the groups in question are identified. We let the correspondence $E_{t+1}^{t+1,t} \simeq \overline{E}_{t+1}^{t+1,t}$ be given as $x \mapsto \overline{x}$. Similarly, we have a correspondence $E_{t+1}^{2t+2,2t+1} \simeq \overline{E}_{t+1}^{2t+2,2t+1}$.

In this subsection, we will give a universal formula for the first differential out of the stable range. We will need this for the 2-primary Picard group of TMF.

Theorem 6.1.1. *We have the formula*

$$\overline{d}_{t+1}(\overline{x}) = \overline{d_{t+1}(x) + x^2}, \quad x \in E_{t+1}^{t+1,t}. \tag{6.1}$$

Remark 6.1.2. The above formula actually makes $\overline{d}_{t+1}$ into a linear operator. This follows from the graded-commutativity of the BKSS for $R^\bullet$. Note in particular that the difference between $\overline{d}_{t+1}$ and d_{t+1} is annihilated by two.

Remark 6.1.3. If Conjecture 5.2.7 is true, then we suspect that there is a similar universal formula for the first differential if $(p-1)!$ is invertible.

6.2. **The universal example.** The proof of (6.1) follows a standard technique in algebraic topology: we reduce to a "universal" case and show that (6.1) is essentially the only possibility. We want to consider the universal case of a cosimplicial $\mathbf{E}_\infty$-ring $R^\bullet$ with a class in $E_{t+1}^{t+1,t}$. This class represents an element in $\pi_{-1}\mathrm{Tot}_{2t+1}(R^\bullet)$ trivialized in $\mathrm{Tot}_t(R^\bullet)$; the differential d_{t+1} represents the obstruction to lifting to Tot_{2t+2}. So, we need to make the analysis of differentials in the cosimplicial $\mathbf{E}_\infty$-ring which corepresents the functor $R^\bullet \mapsto \Omega^\infty\left(\Sigma^{-1}\mathrm{fib}\left(\mathrm{Tot}_{2t+1}(R^\bullet) \to \mathrm{Tot}_t(R^\bullet)\right)\right)$.

The relevant cosimplicial $\mathbf{E}_\infty$-ring $X^\bullet$ can be constructed as follows.

Definition 6.2.1. Let Lan denote the operation of left Kan extension, and let $\mathrm{Lan}_{\Delta^{\leq t}\to\Delta}(*)$ denote the left Kan extension of the constant functor $\Delta^{\leq t} \to \mathcal{S}$ at a point to Δ. Similarly, define $\mathrm{Lan}_{\Delta^{\leq 2t+1}\to\Delta}(*)$. Consider the homotopy pushout

$$\begin{array}{ccc} \mathrm{Lan}_{\Delta^{\leq t}\to\Delta}(*)_+ & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathrm{Lan}_{\Delta^{\leq 2t+1}\to\Delta}(*)_+ & \longrightarrow & \mathscr{F}^\bullet, \end{array} \tag{6.2}$$

where $\mathscr{F}^\bullet : \Delta \to \mathcal{S}_*$ is a functor to the ∞-category $\mathcal{S}_*$ of *pointed* spaces.

Consider $\Sigma^\infty \mathscr{F}^\bullet : \Delta \to \mathrm{Sp}$ and the functor

$$\mathscr{X}^\bullet = \mathrm{Free}_{\mathrm{CAlg}}(\Sigma^{\infty-1}\mathscr{F}^\bullet) : \Delta \to \mathrm{CAlg},$$

into the ∞-category CAlg of $\mathbf{E}_\infty$-rings, obtained by applying the free algebra functor everywhere to $\Sigma^{\infty-1}\mathscr{F}^\bullet$. Then $\mathscr{X}^\bullet$, by construction, corepresents the functor we are interested in. In particular, it suffices to prove (6.1) for this particular functor. As we will see in the next paragraph, $\mathscr{F}^\bullet$ takes values in *connective* spectra and therefore so does $\mathscr{X}^\bullet$. Since we are only interested in differentials in a particular range, we may (by naturality) only consider the Postnikov section $\tau_{\leq 2t}\mathscr{X}^\bullet$. We get the following basic step.

Proposition 6.2.2. *In order to prove Theorem 6.1.1, it suffices to prove it for the $\tau_{\leq 2t}\mathscr{X}^\bullet$ (and the tautological class).*

In fact, we have a reasonable handle on what the functor $\tau_{\leq 2t}\mathscr{X}^\bullet$ looks like and can *entirely* determine the BKSS. To see this, we recall the construction of $\mathscr{F}$; compare also the discussion in [MS14]. The functor

$$\mathrm{Lan}_{\Delta^{\leq t}\to\Delta}(*) : \Delta \to \mathcal{S},$$

sends any finite nonempty totally ordered set T to the nerve of the category $\Delta^{\leq t}_{/T}$ of all order-preserving morphisms $\{S \to T\}$ where:

(1) S is a finite, nonempty totally ordered set, and
(2) $|S| \leq t+1$.

Proposition 6.2.3. $\mathrm{Lan}_{\Delta_{\leq t}\to\Delta}(*)$ *is naturally equivalent to the functor which sends $T \in \Delta$ to the nerve of the* poset $P_{\leq t+1}(T)$ *of nonempty subsets of T of cardinality $\leq t+1$.*

Proof. In fact, for any T, there is a natural map $P_{\leq t+1}(T) \to \Delta^{\leq t}_{/T}$, which is a homotopy equivalence as it is right adjoint to the functor $\Delta^{\leq t}_{/T} \to P_{\leq t+1}(T)$ which sends $S \to T$ to $\mathrm{image}(S \to T) \subset T$. $\square$

As in [MS14], the nerve of $P_{\leq t+1}(T)$, for any choice of T, is (pointwise) homotopy equivalent to a wedge of t-spheres, and contractible if $|T| \leq t+1$. We get from (6.2):

Proposition 6.2.4. *The functor $\mathscr{F}^\bullet : \Delta \to \mathcal{S}_*$ constructed above has the following properties:*

(1) *For any T, $\mathscr{F}(T)$ is always a wedge of copies of S^{t+1} and S^{2t+1}.*
(2) *Restricted to $\Delta^{\leq t}$, $\mathscr{F}^\bullet$ is contractible. Restricted to $\Delta^{\leq 2t}$, $\mathscr{F}^\bullet$ is pointwise a wedge of copies of S^{t+1}.*

6.3. **Some technical lemmas.** Our first goal is to understand the BKSS for $\Sigma^{\infty-1}\mathscr{F}^\bullet$. Observe that pointwise, this cosimplicial spectrum is a wedge of copies of S^t and S^{2t}. In order to do this, we need to understand the cosimplicial abelian group $\pi_*(\Sigma^{\infty-1}\mathscr{F}^\bullet)$. We will prove the following:

Proposition 6.3.1. *The cohomology $H^s(\pi_*(\Sigma^{\infty-1}\mathscr{F}^\bullet))$ is given by*

$$H^s(\pi_*(\Sigma^{\infty-1}\mathscr{F}^\bullet)) \simeq \begin{cases} \pi_*(S^t) & s = t+1 \\ \pi_*(S^{2t}) & s = 2(t+1). \end{cases} \tag{6.3}$$

In the spectral sequence, the differential d_{t+1} is an isomorphism.

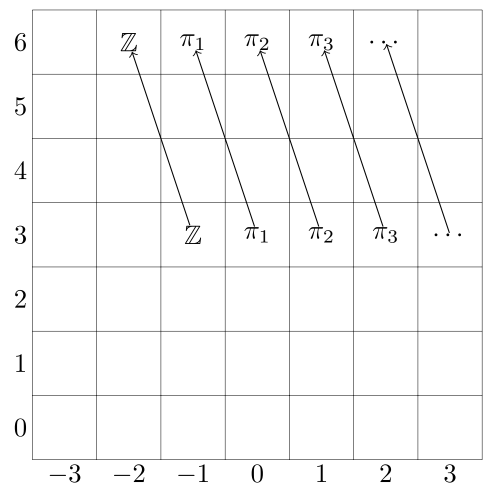

Figure 1. Bousfield-Kan spectral sequence for $\Sigma^{\infty-1}\mathscr{F}^\bullet$, with $t = 2$. π_k denotes $\pi_k S^0$

The spectral sequence is depicted in Figure 1. The proof of Proposition 6.3.1 will take work and will be spread over two subsections. In the present subsection, our main result is that the totalization of $\Sigma^{\infty-1}\mathscr{F}^\bullet$ (and related cosimplicial spectra) is contractible, and we will deduce the differentials from that. The approach to this is not computational and relies instead on ideas involving the ∞-categorical Dold-Kan correspondence of Lurie.

We recall from [Lur09, 1.2.8.4] the *cone* construction, which associates to a simplicial set K, the *cone* $K^{\triangleleft}$. If K is an ∞-category, $K^{\triangleleft}$ is as well, and is obtained by adding a new initial object to K.

Lemma 6.3.2. *Let K be a simplicial set and $\mathcal{D}$ an ∞-category with colimits. Let $F : K^{\triangleleft} \to \mathcal{D}$ be a functor with the property that F carries the cone point to an initial object of $\mathcal{D}$. Then the natural map*

$$\varinjlim_{K} F|_K \to \varinjlim_{K^{\triangleleft}} F$$

is an equivalence in $\mathcal{D}$.

Proof. If K is the empty simplicial set, then the assertion is obvious, since the colimit over the empty set is given by the initial object of $\mathcal{D}$. If $K = \Delta^n$ for some $n \geq 0$, so that $K^{\triangleleft} = \Delta^{n+1}$, then the colimit of F is simply F evaluated at the terminal vertex, and the assertion is again evident.

Suppose we have a pushout diagram of simplicial sets

$$\begin{array}{ccc} K_0 & \longrightarrow & K_2 \\ \downarrow & & \downarrow \\ K_1 & \longrightarrow & K, \end{array}$$

and a functor $F : K \to \mathcal{C}$. Assume moreover that one of the maps $K_0 \to K_1, K_0 \to K_2$ is an inclusion. Then we have a pushout (and homotopy pushout) diagram of simplicial sets

$$\begin{array}{ccc} K_0^{\triangleleft} & \longrightarrow & K_2^{\triangleleft} \\ \downarrow & & \downarrow \\ K_1^{\triangleleft} & \longrightarrow & K^{\triangleleft}, \end{array}$$

because the functor $L \mapsto L^{\triangleleft}$ from simplicial sets to *pointed* simplicial sets preserves colimits [Lur09, Remark 1.2.8.2]. By [Lur09, Proposition 4.4.2.2], given any functor $F : K^{\triangleleft} \to \mathcal{D}$, the following diagrams in $\mathcal{D}$ are pushouts:

$$\begin{array}{ccc} \varinjlim_{K_0} F|_{K_0} & \longrightarrow & \varinjlim_{K_2} F|_{K_2} \\ \downarrow & & \downarrow \\ \varinjlim_{K_1} F|_{K_1} & \longrightarrow & \varinjlim_{K} F|_{K}, \end{array} \qquad \begin{array}{ccc} \varinjlim_{K_0^{\triangleleft}} F|_{K_0^{\triangleleft}} & \longrightarrow & \varinjlim_{K_2^{\triangleleft}} F|_{K_2^{\triangleleft}} \\ \downarrow & & \downarrow \\ \varinjlim_{K_1^{\triangleleft}} F|_{K_1^{\triangleleft}} & \longrightarrow & \varinjlim_{K^{\triangleleft}} F. \end{array}$$

It thus follows that if the statement of the lemma is true for K_0, K_1, and K_2, then it must be true for K, since we have a natural transformation between the above two squares.

As a result, consider the collection $\mathfrak{A}$ of simplicial sets for which the above lemma is true. Then:

(1) $\mathfrak{A}$ contains $\emptyset$ and all the standard simplices $\Delta^n, n \geq 0$.
(2) $\mathfrak{A}$ is closed under pushouts of inclusions.
(3) $\mathfrak{A}$ is closed under filtered colimits. This follows using a similar argument as above, using instead the results of [Lur09, §4.2.3] and in particular [Lur09, Remark 4.2.3.9].

It now follows that $\mathfrak{A}$ is the collection of all simplicial sets. Indeed, the third condition allows us to reduce to showing that $\mathfrak{A}$ contains every *finite* simplicial set. Then, using induction on the dimension and a cell decomposition of the simplicial set, we can conclude using the first two items. □

Lemma 6.3.3. *Let $\mathcal{C}, \mathcal{D}$ be ∞-categories and assume that $\mathcal{D}$ has colimits. Let $F : \mathcal{C}^{\triangleleft} \to \mathcal{D}$ be a functor such that F carries the cone point to an initial object of $\mathcal{D}$. Let $\mathcal{C}' \subset \mathcal{C}$ be a full subcategory. Then the following are equivalent:*

(1) *$F|_{\mathcal{C}}$ is a left Kan extension of its restriction to $\mathcal{C}'$.*
(2) *F is a left Kan extension of its restriction to ${\mathcal{C}'}^{\triangleleft}$.*

Proof. Suppose the first condition satisfied. Then if $c \in \mathcal{C}$ is arbitrary, the natural map

$$\varinjlim_{c' \to c \in \mathcal{C}'_{/c}} F(c') \to F(c)$$

is an equivalence. Now, we have an equivalence of ∞-categories $(\mathcal{C}'_{/c})^{\triangleleft} \simeq ({\mathcal{C}'}^{\triangleleft})_{/c}$, because $\triangleleft$ adds a new initial object. Therefore, if $c \in \mathcal{C}$ is arbitrary, we also get that the natural map

$$\varinjlim_{c' \to c \in ({\mathcal{C}'}^{\triangleleft})_{/c}} F(c') \simeq \varinjlim_{c' \to c \in (\mathcal{C}'_{/c})^{\triangleleft}} F(c') \to F(c)$$

is an equivalence, thanks to Lemma 6.3.2. At the cone point, the left Kan extension condition is automatic. Thus, it follows that F is a left Kan extension of $F|_{\mathcal{C}'^{\triangleleft}}$. The converse is proved in the same way. $\square$

Proposition 6.3.4. *Let $\mathcal{C}$ be a stable ∞-category and let $F : \Delta^{\leq n} \to \mathcal{C}$ be any functor. Suppose F is a left Kan extension of its restriction to $\Delta^{\leq n-1}$. Then $\varprojlim_{\Delta^{\leq n}} F$ is contractible.*

Proof. Observe that the cone $(\Delta^{\leq n})^{\triangleleft}$ is given by the category $\Delta_+^{\leq n}$ of the finite totally ordered sets $\{[i]\}_{-1 \leq i \leq n}$ since $[-1]$ is an initial object of this category. Consider the functor $\widetilde{F} : \Delta_+^{\leq n} \simeq (\Delta^{\leq n})^{\triangleleft} \to \mathcal{C}$ extending F that sends the cone point to the initial object (one can always make such an extension). In order to show that $\varprojlim_{\Delta^{\leq n}} F$ is contractible, it suffices to show that $\widetilde{F}$ is a right Kan extension of $F = \widetilde{F}|_{\Delta^{\leq n}}$.

Now, we recall a basic result of Lurie [Lur12, Lemma 1.2.4.19] (which we use for the opposite category), a piece of the ∞-categorical version of the Dold-Kan correspondence: given any functor $G : \Delta_+^{\leq n} \to \mathcal{C}$, G is a right Kan extension of $G|_{\Delta^{\leq n}}$ if and only if G is a *left* Kan extension of $G|_{\Delta_+^{\leq n-1}}$. In our case, it follows that to show that $\widetilde{F}$ is a right Kan extension of F (as we would like to see), it suffices to show that $\widetilde{F}$ is a *left* Kan extension of $\widetilde{F}|_{\Delta_+^{\leq n-1}}$. But by Lemma 6.3.3, this follows from the fact that $\widetilde{F}|_{\Delta^{\leq n}} = F$ is a left Kan extension of $\widetilde{F}|_{\Delta^{\leq n-1}} = F|_{\Delta^{\leq n-1}}$. $\square$

6.4. **The BKSS for $\mathscr{F}$.** The goal of this subsection is to complete the proof of Proposition 6.3.1. To begin with, we analyze the BKSS for the functor $\Sigma_+^\infty \mathrm{Lan}_{\Delta^{\leq t} \to \Delta}(*) : \Delta \to \mathrm{Sp}$.

Proposition 6.4.1. *The BKSS for the cosimplicial spectrum $\Sigma_+^\infty \mathrm{Lan}_{\Delta^{\leq t} \to \Delta}(*)$ satisfies*

$$E_2^{s,*} = H^s(\pi_*(\Sigma_+^\infty \mathrm{Lan}_{\Delta^{\leq t} \to \Delta}(*))) = \begin{cases} \pi_*(S^0) & s = 0 \\ \pi_*(S^t) & s = t+1. \end{cases} \tag{6.4}$$

The differential d_{t+1} is an isomorphism. (The result for $t = 2$ is displayed in Figure 2.)

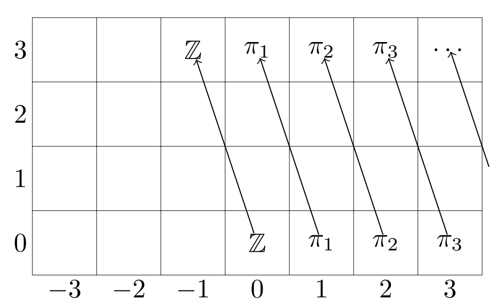

FIGURE 2. Bousfield-Kan spectral sequence for $\Sigma_+^\infty \mathrm{Lan}_{\Delta^{\leq t} \to \Delta}$, with $t = 2$.

Proof. Observe that $\mathrm{Lan}_{\Delta^{\leq t} \to \Delta}(*)$ is, pointwise, a wedge of t-spheres, so to compute the desired cohomology $H^s(\pi_*(\Sigma_+^\infty \mathrm{Lan}_{\Delta^{\leq t} \to \Delta}(*)))$, it suffices to do this for π_t. (The disjoint basepoint contributes the $\pi_*(S^0)$ for $s = 0$ in cohomology.) In other words, we may consider the cosimplicial $H\mathbb{Z}$-module $M^\bullet = H\mathbb{Z} \wedge \Sigma_+^\infty \mathrm{Lan}_{\Delta^{\leq t} \to \Delta}(*)$. Now we know that, for each n, $\pi_*(M^n)$ is concentrated in degrees 0 and t, and that $\pi_0(M^\bullet)$ is the constant cosimplicial abelian group $\mathbb{Z}$. Moreover, by Proposition 6.3.4, $\mathrm{Tot}(M^\bullet)$ is contractible. A look at the spectral sequence for $\mathrm{Tot}(M^\bullet)$ shows that $H^s(\pi_t M^\bullet)$ must be concentrated in degree $s = t+1$ and must be a $\mathbb{Z}$ there. The claim about differentials also follows from contractibility of the totalization. $\square$

Proof of Proposition 6.3.1. The definition (6.2) of $\mathscr{F}^\bullet$, and Proposition 6.4.1 gives the E_2-page of the spectral sequence, when one uses the long exact sequence in homotopy groups. The differentials are forced, again, by Proposition 6.3.4 which implies that $\mathrm{Tot}(\Sigma^{\infty-1}\mathscr{F}^\bullet)$ is contractible. □

6.5. **Completion of the proof.** Now we need to consider the cosimplicial $\mathbf{E}_\infty$-ring

$$\mathscr{Y}^\bullet \overset{\mathrm{def}}{=} \tau_{\le 2t}\mathscr{X}^\bullet \simeq \tau_{\le 2t}\mathrm{Free}_{\mathrm{CAlg}}(\Sigma^{\infty-1}\mathscr{F}^\bullet).$$

In this subsection, we will determine the relevant piece of the BKSS for $\mathscr{Y}$ and then complete the proof. We have that

$$\mathscr{Y}^\bullet \simeq \tau_{\le 2t}S^0 \vee \tau_{\le 2t}\Sigma^{\infty-1}\mathscr{F}^\bullet \vee \tau_{\le 2t}\left(\Sigma^{\infty-1}\mathscr{F}^\bullet\right)^{\wedge 2}_{h\Sigma_2}.$$

In particular, the cohomology $H^s(\pi_*(\mathscr{Y}^\bullet))$ picks up a copy of $\pi_*(S^0)$ for $s=0$ (which is mostly irrelevant). In Proposition 6.3.1, we determined the BKSS for $\mathscr{F}$; in bidegrees $(t+1,t)$ and $(2t+2,2t)$, this picks up copies of $\mathbb{Z}$ such that the first one hits the second one with a d_{t+1}. We will prove:

Proposition 6.5.1. *$E_2^{2t+2,2t} \simeq \mathbb{Z}\oplus\mathbb{Z}/2$ in the BKSS for $\mathscr{Y}$. The $\mathbb{Z}/2$ is generated by the square of the class in bidegree $(t+1,t)$.*

Proof. We will use the notation and results of Appendix C. Let $A^\bullet$ be the cosimplicial abelian group $\pi_t\Sigma^{\infty-1}\mathscr{F}^\bullet$. As we have seen (Proposition 6.3.1), $H^{t+1}(A^\bullet)\simeq\mathbb{Z}$ and the other cohomology of $A^\bullet$ vanishes. Now, using the notation of Definition C.1,

$$\pi_{2t}(\Sigma^{\infty-1}\mathscr{F})^{\wedge 2}_{h\Sigma_2} = \begin{cases}\mathrm{Sym}_2 A^\bullet & t \text{ even}\\ \widetilde{\mathrm{Sym}}_2 A^\bullet & t \text{ odd.}\end{cases}$$

By Proposition C.5, we find that the $E_2^{2t+2,2t}$ term of $(\Sigma^{\infty-1}\mathscr{F})^{\wedge 2}_{h\Sigma_2}$ is as claimed. □

We are now ready to complete the proof and determine the differential in the $\mathfrak{gl}_1$ spectral sequence. Using the notation of the beginning of this section, it follows that $E_{t+1}^{t+1,t}\simeq\mathbb{Z}$ and $E_{t+1}^{2t+1,2t}\simeq\mathbb{Z}\oplus\mathbb{Z}/2$, and similarly for $\overline{E}$. The d_{t+1} carries the $\mathbb{Z}$ into the other $\mathbb{Z}$. By naturality of the spectral sequence, it follows that there must exist a universal formula

$$\overline{d}_{t+1}(\overline{x}) = \overline{a d_{t+1}(x) + \epsilon x^2}, \quad a\in\mathbb{Z},\ \epsilon\in\{0,1\}. \tag{6.5}$$

The main claim is that $a=\epsilon=1$. Our first goal is to compute a.

Lemma 6.5.2. *We have an equivalence of ∞-categories between the ∞-category $\mathrm{Fun}^L(\mathrm{Sp}_{\ge 0},\mathrm{Sp}_{\ge 0})$ of cocontinuous functors $\mathrm{Sp}_{\ge 0}\to\mathrm{Sp}_{\ge 0}$ and $\mathrm{Sp}_{\ge 0}$ given by evaluating at the sphere. The inverse equivalence sends a connective spectrum Y to the functor $X\mapsto X\otimes Y$.*

Proof. It suffices to show that evaluation at the sphere induces an equivalence of ∞-categories $\mathrm{Fun}^L(\mathrm{Sp}_{\ge 0},\mathrm{Sp})\simeq\mathrm{Sp}$ (with inverse given as above). But the ∞-category Sp is the *stabilization* [Lur12, §1.4] of $\mathrm{Sp}_{\ge 0}$, so that we have an equivalence (by [Lur12, Corollary 1.4.4.5]) $\mathrm{Fun}^L(\mathrm{Sp},\mathrm{Sp})\simeq\mathrm{Fun}^L(\mathrm{Sp}_{\ge 0},\mathrm{Sp})$ given by restriction. But we know that $\mathrm{Fun}^L(\mathrm{Sp},\mathrm{Sp})\simeq\mathrm{Sp}$ by evaluation at the sphere spectrum, with inverse given by the smash product (see [Lur12, §4.8.2]). □

We need the following fact about $\mathfrak{gl}_1$.

Proposition 6.5.3. *Let X be a connective spectrum, and let $S^0\vee X$ be the square-zero $\mathbf{E}_\infty$-ring. Then there is a natural equivalence of spectra,*

$$\mathfrak{gl}_1(S^0\vee X)\simeq\mathfrak{gl}_1(S^0)\vee X.$$

Proof. Given the connective spectrum X, we can use the composite $S^0 \to S^0 \vee X \to S^0$, in which the second map sends X to 0, to get a natural splitting

$$\mathfrak{gl}_1(S^0 \vee X) \simeq \mathfrak{gl}_1(S^0) \vee F(X),$$

where $F : \mathrm{Sp}_{\geq 0} \to \mathrm{Sp}_{\geq 0}$ is a certain functor that we want to claim is naturally isomorphic to the identity. First, observe that F commutes with colimits. Namely, F commutes with filtered colimits (as one can check on homotopy groups), F takes $*$ to $*$, and given a pushout square

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_3 & \longrightarrow & X_4, \end{array} \tag{6.6}$$

in $\mathrm{Sp}_{\geq 0}$, the analogous diagram

$$\begin{array}{ccc} F(X_1) & \longrightarrow & F(X_2) \\ \downarrow & & \downarrow \\ F(X_3) & \longrightarrow & F(X_4) \end{array} \tag{6.7}$$

is a pushout square in $\mathrm{Sp}_{\geq 0}$. This in turn follows by considering long exact sequences in homotopy groups. More precisely, given the pushout square (6.6), the diagram of $\mathbf{E}_\infty$-rings

$$\begin{array}{ccc} S^0 \vee X_1 & \longrightarrow & S^0 \vee X_2 \\ \downarrow & & \downarrow \\ S^0 \vee X_3 & \longrightarrow & S^0 \vee X_3, \end{array}$$

is a homotopy *pullback* in $\mathbf{E}_\infty$-rings, so that applying $\mathfrak{gl}_1$ (which is a *right adjoint*) leads to a pullback square

$$\begin{array}{ccc} \mathfrak{gl}_1(S^0 \vee X_1) & \longrightarrow & \mathfrak{gl}_1(S^0 \vee X_2) \\ \downarrow & & \downarrow \\ \mathfrak{gl}_1(S^0 \vee X_3) & \longrightarrow & \mathfrak{gl}_1(S^0 \vee X_4), \end{array}$$

and in particular, (6.7) is homotopy cartesian too in $\mathrm{Sp}_{\geq 0}$. Therefore, it is homotopy cocartesian as well if we can show that the map

$$\pi_0(\mathfrak{gl}_1(S^0 \vee X_3)) \oplus \pi_0(\mathfrak{gl}_1(S^0 \vee X_2)) \to \pi_0(\mathfrak{gl}_1(S^0 \vee X_4))$$

is surjective. This, however, follows from the analogous fact that $\pi_0(X_3) \oplus \pi_0(X_2) \to \pi_0(X_4)$ is surjective as (6.6) is a pushout.

Therefore, as both F commutes with colimits, F is necessarily of the form $X \mapsto X \otimes Y$ for some $Y \in \mathrm{Sp}_{\geq 0}$, by Lemma 6.5.2. For $X = H\mathbb{Z}$, we find $F(X) = H\mathbb{Z}$, so that $H\mathbb{Z} \otimes Y$ is concentrated in degree zero and is isomorphic to $H\mathbb{Z}$. This forces $Y \simeq S^0$ and proves the claim. □

Proof of Theorem 6.1.1. Proposition 6.5.3 implies that in the universal formula (6.5), the constant $a = 1$. In fact, we know that if $X^\bullet$ is any cosimplicial spectrum, then the cosimplicial spectra $\mathfrak{gl}_1(S^0 \vee X^\bullet)$ and $\mathfrak{gl}_1(S^0) \vee X^\bullet$ are identified. In particular, the differentials in the spectral sequence for $\mathfrak{gl}_1(S^0 \vee X^\bullet)$ and in the spectral sequence for $S^0 \vee X^\bullet$ are identified, forcing $a = 1$.

It remains to show that $\epsilon = 1$. For this, we need an example where the two differentials do *not* agree. This will be a generalization of Example 5.2.1. Consider the $\mathbf{E}_\infty$-ring of Proposition 5.2.2, with $n = t$, so that, in particular, $\mathfrak{gl}_1(R)$ has homotopy groups in dimensions t and $2t$ only. Proposition 5.2.2 shows that the k-invariant is *nontrivial*.

Consider the space $X = K(\mathbb{F}_2, t+1)$, and consider the Atiyah-Hirzebruch spectral sequences for the homotopy groups of $\mathfrak{gl}_1(R)^X$ and R^X (these can be identified with BKSS's by choosing simplicial resolutions of X by points). The latter clearly degenerates, but the former does not. In fact, the Atiyah-Hirzebruch spectral sequence for $BGL_1(R)^{K(\mathbb{F}_2,t+1)}$ fails to degenerate precisely because $BGL_1(R)$ has a nontrivial k-invariant *in spaces.* That is, we can produce a map of spaces $K(\mathbb{F}_2, t+1) \to B\Omega^\infty R$ inducing an isomorphism on π_{t+1}, but we cannot do this with $B\Omega^\infty R$ replaced by $BGL_1(R)$.

This completes the proof of Theorem 6.1.1. □

Part III. Computations

7. Picard groups of real K-theory and its variants

Before we embark on the lengthy computations for the Picard groups of the various versions of topological modular forms, let us work out in detail the case of real K-theory, as well as the Tate K-theory spectrum $KO((q))$. In particular, these examples will illustrate our methodology without being computationally cumbersome.

7.1. Real K theory.

In this subsection, we compute the Picard group of KO using C_2-Galois descent from the C_2-Galois extension $KO \to KU$ and the Comparison Tool 5.2.4 (but not the universal formula of Theorem 6.1.1).

We begin with the basic case of *complex* K-theory.

Example 7.1.1 (Complex K-theory)**.** The complex K theory spectrum has a very simple ring of homotopy groups $KU_* = \mathbb{Z}[u^{\pm 1}]$ with u in degree 2. In particular, KU is even periodic with a regular noetherian π_0, so its Picard group is algebraic by Theorem 2.4.6. The inner workings of Theorem 2.4.6 would use that the only (homogeneous) maximal ideals of KU_* are generated by prime numbers p; for each p, there is a corresponding residue field spectrum, namely mod-p K-theory, also known as an extension of the Morava K-theory of height one at the given prime. As the Picard group of $KU_0 = \mathbb{Z}$ is trivial, and $\mathrm{Pic}(KU_*) \simeq \mathbb{Z}/2$, any invertible KU-module is equivalent to either KU or ΣKU.

To compute $\mathrm{Pic}(KO)$, we start with this knowledge that thanks to Example 7.1.1, $\pi_0\mathfrak{pic}(KU) = \mathrm{Pic}(KU)$ is $\mathbb{Z}/2$. We have the spectral sequence from (3.5)

$$H^*(C_2, \pi_*\mathfrak{pic}(KU)) \Rightarrow \pi_*\mathfrak{pic}(KU)^{hC_2} \simeq \pi_*\mathfrak{pic}(KO) \quad \text{for } * \geq 0,$$

which will allow us to compute $\pi_0(\mathfrak{pic}(KU))^{hC_2} \simeq \mathrm{Pic}KO$. We note that $\pi_1\mathfrak{pic}(KU) \simeq (KU_0)^\times = \mathbb{Z}/2$, and

$$H^*(C_2, \mathbb{Z}/2) = \mathbb{Z}/2[x],$$

where x is in cohomological degree 2. The higher homotopy groups of $\mathfrak{pic}(KU)$ coincide (as C_2-modules) with those of KU, suitably shifted by one.

Recall, moreover, that the E_2-page of the HFPSS for $\pi_*(KO)$ is given by the bigraded ring

$$E_2^{*,*} = \mathbb{Z}[u^2, u^{-2}, h_1]/(2h_1^2), \quad |u^2| = (4,0),\ h_1 = (1,2),$$

where u^2 is the square of the Bott class in $\pi_*(KU) \simeq \mathbb{Z}[u^{\pm 1}]$, and h_1 detects in homotopy the Hopf map η. The class h_1 is in bidegree $(s,t) = (1,2)$, so it is drawn using Adams indexing in the $(1,1)$ place. The differentials are determined by $d_3(u^2) = h_1^3$ and the spectral sequence collapses at E_4. For convenience, we reproduce a picture in Figure 3; the interested reader can find the detailed computation of this spectral sequence in [HS14, Sec.5].

Therefore, the E_2-page of the spectral sequence for $\mathfrak{pic}(KU)^{hC_2}$ is as in Figure 4. To deduce differentials, we use our Comparison Tool 5.2.4: in the homotopy fixed point spectral sequence for KU, there are only d_3-differentials. By the Comparison Tool 5.2.4, we conclude that we can

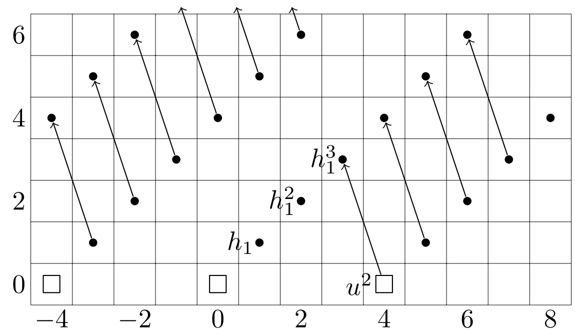

FIGURE 3. Homotopy fixed point spectral sequence for $\pi_*(KO) \simeq \pi_*(KU^{hC_2})$
• denotes $\mathbb{Z}/2$ and □ denotes $\mathbb{Z}$

"import" those differentials to the HFPSS for $\mathfrak{pic}(KU)$ when they involve terms with $t \geq 4$. In particular, we see that the differentials drawn in Figure 4 are non-zero; moreover, everything that is above the drawn range and in the $s = t$ column either supports or is the target of a non-zero differential. Note that we are not claiming that there are no other differentials, but these suffice for our purposes.

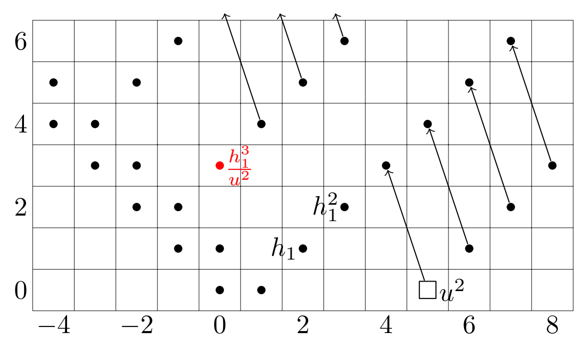

FIGURE 4. Homotopy fixed point spectral sequence for $\mathfrak{pic}(KU)^{hC_2}$

We deduce from this that $\pi_0\mathfrak{pic}(KU)^{hC_2} = \mathrm{Pic}(KO)$ has cardinality at most eight. On the other hand, the fact that KO is 8-periodic gives us a lower bound $\mathbb{Z}/8$ on PicKO. Thus we get:

Theorem 7.1.2 (Hopkins; Gepner-Lawson [GL])**.** *Pic*(KO) *is precisely* $\mathbb{Z}/8$*, generated by* ΣKO*.*

Theorem 7.1.2 was proved originally by Hopkins (unpublished) using related techniques. The approach via descent theory is due to Gepner-Lawson in [GL]. Their identification of the differentials in the spectral sequence is, however, different from ours: they use an explicit knowledge of the structure of $\mathfrak{gl}_1(KU)$ with its C_2-action (which one does not have for TMF).

Remark 7.1.3. In view of Remark 3.3.2, we conclude that the relative Picard group of the C_2-extension $KO \to KU$ is $\pi_{-1}(\mathfrak{gl}_1 KU)^{hC_2} \simeq \mathbb{Z}/4$.

Remark 7.1.4. In the usual descent spectral sequence for KO, the class $\frac{h_1^3}{u^2}$ in red supports a d_3. By Theorem 6.1.1 and the multiplicative structure of the usual SS, $\frac{h_1^3}{u^2}$ does *not* support a d_3 in the descent SS for Pic. We saw that above by counting: if $\frac{h_1^3}{u^2}$ did not survive, the Picard group of

KO would be too small. For 2-local TMF, simple counting arguments will not suffice and we will actually need to use Theorem 6.1.1 as well.

Remark 7.1.5. We can also deduce from the spectral sequence that the cardinality of the relative Brauer group for KO/KU, which is isomorphic to $\pi_{-1}(\mathfrak{pic}KU)^{hC_2}$, is most eight. However, we do not know how to construct necessarily non-trivial elements of this Brauer group in order to deduce a lower bound as in the Picard group case.

7.2. $KO[q], KO[[q]]$ **and** $KO((q))$**.** We now include a variant of the above example where one adds a polynomial (resp. power series, Laurent series) generator, where we will also be able to confirm the answer using a different argument. This example can be useful for comparison with TMF using topological q-expansion maps. We begin by introducing the relevant $\mathbf{E}_\infty$-rings. This subsection will not be used in the sequel and may be safely skipped by the reader.

Definition 7.2.1. We write for $S^0[x]$ the suspension spectrum $\Sigma^\infty_+\mathbb{Z}_{\geq 0}$. Since $\mathbb{Z}_{\geq 0}$ is an $\mathbf{E}_\infty$-monoid in spaces (in fact, a strictly commutative monoid), $S^0[x]$ naturally acquires the structure of an $\mathbf{E}_\infty$-ring. Given an $\mathbf{E}_\infty$-ring R, we will write $R[x] = R \wedge S^0[x]$.

We can also derive several other variants:

(1) We will let $R[[x]]$ denote the x-adic completion of $R[x]$, so its homotopy groups look like a power series ring over $\pi_*(R)$.
(2) We will let $R[x^{\pm 1}]$ denote the localization $R[x][1/x]$, so its homotopy groups are given by Laurent polynomials in $\pi_*(R)$.
(3) We will let $R((x)) = R[[x]][1/x]$, so that its homotopy groups look like formal Laurent series over $\pi_*(R)$.

On the one hand, $\pi_*(R[x]) \simeq \pi_*(R)[x]$ is a polynomial ring over $\pi_*(R)$ on a generator in degree zero. On the other hand, as an $\mathbf{E}_\infty$-algebra under R, the universal property of $R[x]$ is significantly more complicated than that of the "free" $\mathbf{E}_\infty$-R-algebra on a generator (which is often denoted $R\{x\}$). A map $R[x] \to R'$, for an $\mathbf{E}_\infty$-R-algebra R', is equivalent to an $\mathbf{E}_\infty$-map

$$\mathbb{Z}_{\geq 0} \to \Omega^\infty R',$$

where $\Omega^\infty R'$ is regarded as an $\mathbf{E}_\infty$-space under *multiplication.* In general, given a class in $\pi_0(R')$, there is no reason to expect an $\mathbf{E}_\infty$-map $R[x] \to R'$ carrying x to it, since $\mathbb{Z}_{\geq 0}$ as an $\mathbf{E}_\infty$-monoid is quite complicated. Such classes are called "strictly commutative."

Example 7.2.2. There is a map $R[x] \to R$ sending $x \to 1$. This comes from the map of $\mathbf{E}_\infty$-spaces $\mathbb{Z}_{\geq 0} \to * \to \Omega^\infty S^0$ where $*$ maps to the unit in $\Omega^\infty S^0$.

Example 7.2.3. There is a map $R[x] \to R$ sending $x \to 0$. To obtain this, we start with the $\mathbf{E}_\infty$-monoid M with elements $\{1, e\}$ with $e^2 = e$. There is a morphism of $\mathbf{E}_\infty$-monoids (in fact, of strictly commutative monoids)

$$\mathbb{Z}_{\geq 0} \to M,$$

sending $n \in \mathbb{Z}_{\geq 0}$ to 1 for $n = 0$ and e for $n > 0$, giving a map of $\mathbf{E}_\infty$-R-algebras

$$R[x] \to R \wedge \Sigma^\infty_+ M.$$

But $\pi_*(R \wedge \Sigma^\infty_+ M) \simeq \pi_*(R)[e]/(e^2 - e)$, so $R \wedge \Sigma^\infty_+ M$ is *étale* as an $\mathbf{E}_\infty$-R-algebra. Therefore, we get by the étale obstruction theory (see for instance [Lur12, §8.5]) a canonical map $R \wedge \Sigma^\infty_+ M \to R$ carrying $e \mapsto 0$. Composing the maps $R[x] \to R \wedge \Sigma^\infty_+ M \to R$ gives the map we want. Note also that this map factors over the completion to give a map $R[[x]] \to R$ sending $x \to 0$.

The map $R[x] \to R$ given in Example 7.2.3 has the property that it exhibits the $R[x]$-module R as the cofiber $R[x]/x$. It follows in particular that if R' is any $\mathbf{E}_\infty$-R-algebra and $x' \in \pi_0(R')$ is a strictly commutative element, then we can give the cofiber $R'/x' \simeq R' \otimes_{R[x]} R$ the structure of an $\mathbf{E}_\infty$-R'-algebra.

Remark 7.2.4. Consider the sphere spectrum S^0. Then no cofiber S^0/n for $n \notin \{\pm 1, 0\}$ can admit the structure of an $\mathbf{E}_\infty$-ring by, for example, [MNN14, Remark 4.3].[6] It follows that the only element of $\pi_0(S^0) \simeq \mathbb{Z}$, besides 0 and 1, that can potentially be strictly commutative is -1. Now, -1 is not strictly commutative in the $K(1)$-local sphere $L_{K(1)}S^0$ at the prime 2 because of the operator θ of [Hop]: we have $\theta(-1) = \frac{(-1)^2-(-1)}{2} = 1 \neq 0$, while power operations such as θ annihilate strictly commutative elements. Therefore, it cannot be strictly commutative in S^0. (One could have applied a similar argument with power operations to every other integer, too.) However, we observe that -1 is strictly commutative in $S^0[1/2]$: the obstruction is entirely 2-primary (Proposition 7.2.7 below).

Remark 7.2.5. Spaces of strictly commutative elements in $K(n)$-local $\mathbf{E}_\infty$-rings have been recently studied in unpublished work of Hopkins-Lurie.

Example 7.2.6. Let $a, b \in \pi_0(R)$ be strictly commutative elements for R an $\mathbf{E}_\infty$-ring. Then ab is also strictly commutative. If a is a unit, then a^{-1} is strictly commutative. This follows because there is a natural addition on $\mathbf{E}_\infty$-maps $\mathbb{Z}_{\geq 0} \to \Omega^\infty R$.

Proposition 7.2.7. *Let R be an $\mathbf{E}_\infty$-ring with n invertible. Then any $u \in \pi_0(R)$ with $u^n = 1$ (i.e., an nth root of unity) is strictly commutative.*

Proof. We consider the map of $\mathbf{E}_\infty$-monoids $\mathbb{Z}_{\geq 0} \to \mathbb{Z}/n\mathbb{Z}$ and the induced map of $\mathbf{E}_\infty$-ring spectra

$$R[x] \to R \wedge \Sigma^\infty_+ \mathbb{Z}/n\mathbb{Z}. \tag{7.1}$$

Since $\frac{1}{n} \in \pi_0(R)$, $R \wedge \Sigma^\infty_+ \mathbb{Z}/n\mathbb{Z}$ is étale over R and the homotopy groups are given by $\pi_*(R)[x]/(x^n - 1)$. We can thus produce a map of $\mathbf{E}_\infty$-rings $R \wedge \Sigma^\infty_+(\mathbb{Z}/n\mathbb{Z}) \to R$ sending $1 \in \mathbb{Z}/n\mathbb{Z}$ to u by étaleness. Composing with (7.1) gives us the strictly commutative structure on u. □

Using these ideas, we will be able to give a direct computation of the Picard group of the $\mathbf{E}_\infty$-ring $KO[[q]]$. (We have renamed the power series variable to "q" in accordance with "q-expansions.")

Proposition 7.2.8. *The map* $\mathrm{Pic} KO \to \mathrm{Pic} KO[[q]]$ *is an isomorphism, where q is in degree zero.*

Proof. Suppose M is an invertible $KO[[q]]$-module such that $M/qM \simeq M \otimes_{KO[[q]]} KO$ is equivalent to KO. We will show that then M is equivalent to $KO[[q]]$ using Bocksteins. Specifically, consider the generating class in $\pi_0(M/qM) \simeq \mathbb{Z}$; we will lift this to a class in $\pi_0(M)$. It follows that the induced map $KO[[q]] \to M$ becomes an equivalence after tensoring with $KO \simeq KO[[q]]/q$; since M is q-adically complete, it will follow that $KO[[q]] \simeq M$.

By induction on k, suppose that:

(1) $\pi_{-1}(M/q^k M) = 0$.
(2) $\pi_0(M/q^k M) \to \pi_0(M/qM)$ is a surjection.

These conditions are clearly satisfied for $k = 1$. If these conditions are satisfied for k, then the cofiber sequence of $KO[[q]]$-modules

$$M/q^k M \to M/q^{k+1} M \to M/qM$$

shows that they are satisfied for $k+1$. In the limit, we find that there is a map $KO[[q]] \to M$ which lifts the generator of $\pi_0(M/qM)$, which proves the claim. □

Proposition 7.2.8 can also be proved using Galois descent, but unlike for KO, we need to use Theorem 6.1.1.

[6] It is an unpublished result of Hopkins that no Moore spectrum can even admit the structure of an $\mathbf{E}_1$-algebra.

Second proof of Proposition 7.2.8. The faithful C_2-Galois extension $KO \to KU$ induces upon base-change a faithful C_2-Galois extension $KO[[q]] \to KU[[q]]$. The Picard group of $KU[[q]]$, again by Theorem 2.4.6, is $\mathbb{Z}/2$ generated by the suspension. Consider now the descent spectral sequence for $\mathfrak{pic}$, which is a modification of the descent spectral sequence for KO in Figure 4. One difference is that every term with $t \geq 2$ is replaced by its tensor product over $\mathbb{Z}$ with $\mathbb{Z}[[q]]$; the other is that the $t = 1$ line now contains the C_2-cohomology of the units in $\pi_0 KU[[q]]$, which is a bigger module than $(\pi_0 KU)^\times = \mathbb{Z}/2$. Namely, these units are $\mathbb{Z}/2 \oplus q\mathbb{Z}[[q]]$, with trivial C_2-action. The resulting E_2-page is displayed in Figure 5.

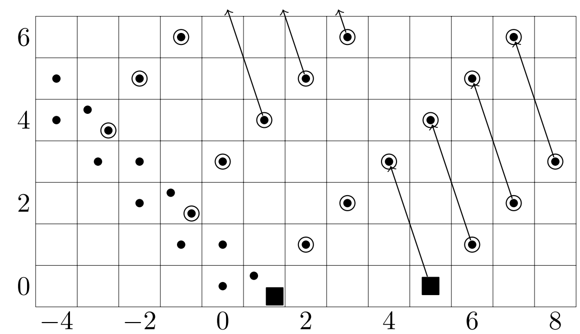

FIGURE 5. Homotopy fixed point spectral sequence for $\mathfrak{pic}(KU[[q]])^{hC_2}$
• denotes $\mathbb{Z}/2$, ⊙ denotes $\mathbb{Z}/2[[q]]$, and ■ denotes $\mathbb{Z}[[q]]$

Since the d_3 is the only differential in the ordinary HFPSS for $\pi_*(KO[[q]])$, as before, it follows that the only contributions to $\mathrm{Pic}(KO[[q]])$ can come from the $\mathbb{Z}/2$ with $t = s = 0$ (the suspension), the $\mathbb{Z}/2$ with $(s,t) = (1,1)$ (i.e., the algebraic Picard group), and the $\mathbb{Z}/2[[q]]$ in bi-degree $(s,t) = (3,3)$.

But here, $E_2^{3,3} = \mathbb{Z}/2[[q]]\frac{h_1^3}{u^2}$ is infinite, so unlike previously, we do not get the automatic upper bound of eight on $|\mathrm{Pic}(KO[[q]])|$. On the other hand, we can use Theorem 6.1.1 to determine the d_3 supported here. Note that in the HFPSS for $(KU[[q]])^{hC_2}$, we have

$$d_3\left(f(q)\frac{h_1^3}{u^2}\right) = f(q)\frac{h_1^6}{u^4}, \quad f(q) \in \mathbb{Z}/2[[q]].$$

Therefore, in view of (6.1), in the HFPSS for $\mathfrak{pic}(KU[[q]])^{hC_2}$, we have

$$d_3\left(f(q)\frac{h_1^3}{u^2}\right) = \left(f(q) + f(q)^2\right)\frac{h_1^6}{u^4}.$$

(Note that a crucial point here is that in the HFPSS for KO, $d_3\left(\frac{h_1^3}{u^2}\right)$ equals its square.) It follows from this that in the HFPSS, the kernel of d_3 on $E_2^{3,3}$ is $\mathbb{Z}/2$ generated by $1\frac{h_1^3}{u^2}$: the equation $f(q) + f(q)^2 = 0$ has only the solutions $f(q) \equiv 0, 1$. Therefore, we do get an upper bound of eight on the cardinality of $\mathrm{Pic}(KO[[q]])$ after all, as nothing else in $E_2^{3,3}$ lives to E_4. □

Corollary 7.2.9. *The maps $KO \to KO[q]$, $KO \to KO((q))$ induce isomorphisms on Picard groups.*

Proof. This result is not a corollary of Proposition 7.2.8 but rather of its second proof. In fact, the same argument shows that d_3 has a $\mathbb{Z}/2$ as kernel on the relevant term $E_2^{3,3}$, which gives an upper bound of cardinality eight on the Picard group of $KO[q]$ or $KO((q))$ as before. □

Remark 7.2.10. Corollary 7.2.9 cannot be proved using the Bockstein spectral sequence argument used in the first proof of Proposition 7.2.8. However, a knowledge of the Picard group of $KO[[q]]$ can be used to describe enough of the C_2-descent spectral sequence to make it possible to prove Corollary 7.2.9 without the explicit formula (6.1). We leave this to the reader.

8. Picard groups of topological modular forms

In the rest of the paper we proceed to use descent to compute the Picard groups of various versions of topological modular forms. We will analyze the following descent-theoretic situations:

- The Galois extension $TMF[1/2] \to TMF(2)$, with structure group $GL_2(\mathbb{Z}/2)$, also known as the symmetric group on three letters.
- The Galois extension $TMF[1/3] \to TMF(3)$, with structure group $GL_2(\mathbb{Z}/3)$; this is a group of order 48, which is a nontrivial extension of the binary tetrahedral group and C_2.
- Étale descent from the (derived) moduli stack of elliptic curves or its compactification.

In each of these cases, we will start with the knowledge of the original descent spectral sequence, computing the homotopy groups of the global sections or homotopy fixed point spectrum. This information plus some additional computation of the differing cohomology groups will provide the data for the E_2-page of the descent spectral sequence for the Picard spectrum. The additional computations are somewhat lengthy, hence we are including them separately in the Appendix. Our computation in Appendix B is inspired by an analogous $K(2)$-local version which Hans-Werner Henn has generously shared with us.

8.1. The Picard group of $TMF[1/2]$.

When 2 is inverted, the moduli stack of elliptic curves M_{ell} has a $GL_2(\mathbb{Z}/2)$-Galois cover by $M_{ell}(2)$, the moduli stack of elliptic curves with full level 2 structure. This remains the case for the derived versions of these stacks, and on global sections gives a Galois extension $TMF[1/2] \to TMF(2)$. The extension is useful for the purposes of descent as the homotopy groups of $TMF(2)$ are cohomologically very simple.

To be precise, we have that

$$TMF(2)_* = \mathbb{Z}[1/2][\lambda_1^{\pm 1}, \lambda_2^{\pm 1}][(\lambda_1 - \lambda_2)^{-1}],$$

where the (topological) degree of each λ_i is four. To see this, one can use the presentation of the moduli stack $M_{ell}(2)$ from [Sto12, Sec.7]: there it is computed that $\overline{M}_{ell}(2)$ is equivalent to (the stacky) $\mathrm{Proj}\mathbb{Z}[1/2][\lambda_1, \lambda_2]$, and that the locus classifying smooth curves, i.e., $M_{ell}(2)$, is given by the non-vanishing of $\lambda_1^2\lambda_2^2(\lambda_1 - \lambda_2)^2$. More precisely, $M_{ell}(2)$, as a stack, is the $\mathbb{G}_m$-quotient of the ring $\mathbb{Z}[1/2][\lambda_1, \lambda_2, (\lambda_1^2\lambda_2^2(\lambda_1 - \lambda_2))^{-1}]$, where the $\mathbb{G}_m$-action is as follows: a unit u acts as $\lambda_i \mapsto u^2\lambda_i$ for $i = 1, 2$, so that it is an open substack of a *weighted projective stack*.

In particular, $TMF(2)_*$ has a unit in degree 4, and is zero in degrees not divisible by 4. It will be helpful to write $TMF(2)_*$ differently, so as to reflect this periodicity more explicitly; for example, we have that $TMF(2)_* = TMF(2)_0[\lambda_2^{\pm 1}]$, and

$$TMF(2)_0 = \mathbb{Z}[1/2][s^{\pm 1}, (s-1)^{-1}], \tag{8.1}$$

where $s = \frac{\lambda_1}{\lambda_2}$. Therefore, Corollary 2.4.7 applies to give the following conclusion.

Lemma 8.1.1. *The Picard group of* $TMF(2)$ *is* $\mathbb{Z}/4$*, generated by the suspension* $\Sigma TMF(2)$*.*

Remark 8.1.2. The proof of Corollary 2.4.7 relies on the construction of residue field spectra; let us specify what they are in the case at hand. The maximal ideals in $TMF(2)_0$ are $\mathfrak{m} = (p, s-a)$, where p is an odd prime and $a \not\equiv 0, 1$ modulo p. For each of these ideals, we have a residue field spectrum which is (an extension of) mod p Morava K-theory at height one or two. By [Sil86, V.4.1], height two occurs precisely when

$$\sum_{i=0}^{(p-1)/2} \binom{(p-1)/2}{i} a^i$$

is zero modulo p.

Next we use descent from $TMF(2)$ to $TMF[1/2]$ to obtain the following result.

Theorem 8.1.3. *The Picard group of $TMF[1/2]$ is $\mathbb{Z}/72$, generated by the suspension $\Sigma TMF[1/2]$. In particular, this Picard group is algebraic.*

Proof. We use the homotopy fixed point spectral sequence (3.5)

$$H^s(GL_2(\mathbb{Z}/2), \pi_t \mathfrak{pic}(TMF(2))) \Rightarrow \pi_{t-s}\mathfrak{pic}(TMF(2))^{hGL_2(\mathbb{Z}/2)}. \tag{8.2}$$

To begin with, note that the homotopy groups $\pi_t\mathfrak{pic}(TMF(2))$ for $t \geq 2$ are isomorphic to $\pi_{t-1}TMF(2)$ as $GL_2(\mathbb{Z}/2)$-modules. This tells us that the $t \geq 2$ part of the E_2-page of the HFPSS (8.2) for $\mathfrak{pic}(TMF(2))$ is a shifted version of the corresponding part for $TMF(2)$.

The latter is immediately obtained from the analogous computation for $Tmf(2)$ in [Sto12] (depicted in Figure 2 of loc.cit.), as we now describe. Recall that $TMF(2) \simeq Tmf(2)[\Delta^{-1}]$; the non-negative homotopy groups $\pi_{\geq 0}Tmf(2)$ are the graded polynomial ring $\Lambda = \mathbb{Z}[1/2][\lambda_1, \lambda_2]$ [Prop.8.1,loc.cit.], and the class $\Delta \in \pi_{24}Tmf(2)$ is

$$\Delta = 16\lambda_1^2\lambda_2^2(\lambda_2 - \lambda_1)^2$$

by [Prop.10.3,loc.cit.]. Now, by [Prop.10.8,loc.cit.] we have that

$$H^*\big(GL_2(\mathbb{Z}/2), \pi_*TMF(2)\big) = H^*\big(GL_2(\mathbb{Z}/2), \Lambda\big)[\Delta^{-1}].$$

In particular, the invariants $H^0\big(GL_2(\mathbb{Z}/2), \Lambda\big)[\Delta^{-1}]$ are the ring of modular forms

$$\mathbb{Z}[1/2][c_4, c_6, \Delta^{\pm 1}]/(12^3\Delta - c_4^3 + c_6^2).$$

The higher cohomology $H^{>0}\big(GL_2(\mathbb{Z}/2), \Lambda\big)$ is computed in [Sec.10.1, loc.cit.], and in particular is killed by c_4 and c_6. Consequently,

$$H^{>0}\big(GL_2(\mathbb{Z}/2), \pi_{\geq 0}TMF(2)\big) = H^{>0}\big(GL_2(\mathbb{Z}/2), \Lambda\big) = H^{>0}(GL_2(\mathbb{Z}/2), \pi_{\geq 0}Tmf(2)).$$

Let us recall (the names of) certain interesting classes in these cohomology groups:

(1) There is a in $H^1(GL_2(\mathbb{Z}/2), \pi_4TMF(2)) = \mathbb{Z}/3$, hence in $H^1\big(GL_2(\mathbb{Z}/2), \pi_5\mathfrak{pic}(TMF(2))\big)$ (so, a is in bidegree $(s,t) = (1,5)$ in the Picard HFPSS, and depicted in position $(s, t-s) = (1,4)$ using the Adams convention). In homotopy, this element detects the Greek element α_1 in the Hurewicz image in $TMF[1/2]$.
(2) There is b in $H^2\big(GL_2(\mathbb{Z}/2), \pi_{13}\mathfrak{pic}(TMF(2))\big) = \mathbb{Z}/3$ (b is in bidegree $(2,13)$ or position $(2,11)$); in homotopy it detects β_1.

Then, $H^{>0}\big(GL_2(\mathbb{Z}/2), TMF(2)_*\big)$ is precisely the ideal of $\mathbb{Z}/3[a,b][\Delta^{\pm 1}]/(a^2)$ of positive cohomological degree. For example

$$H^5\big(GL_2(\mathbb{Z}/2), \pi_5\mathfrak{pic}(TMF(2))\big) = H^5\big(GL_2(\mathbb{Z}/2), \pi_4TMF(2)\big) = \mathbb{Z}/3,$$

generated by $ab^2\Delta^{-1}$. We see this class depicted red below in Figure 6.

Next, we turn to the information which is new for the Picard HFPSS, i.e., the group cohomology of π_0 and π_1 of the spectrum $\mathfrak{pic}(TMF(2))$. By Lemma 8.1.1, we know that the zeroth homotopy group is $\mathbb{Z}/4$, and since it is generated by the suspension $\Sigma TMF(2)$, the action of $GL_2(\mathbb{Z}/2)$ on this $\mathbb{Z}/4$ is trivial. Even though for our purposes only the invariants $H^0\big(GL_2(\mathbb{Z}/2), \pi_0\mathfrak{pic}(TMF(2))\big)$ are necessary, we can in fact compute all the cohomology groups. This is done in Lemma A.1 of Appendix A.

The last piece of data needed for the determination of the E_2-page of the Picard HFPSS is the group cohomology with coefficients in $\pi_1\mathfrak{pic}(TMF(2)) = (\pi_0 TMF(2))^\times$. This is done in Proposition A.2. The range $s \leq 15$ and $-6 \leq t-s \leq 7$ of spectral sequence is depicted in Figure 6. Note that in this range, the $t-s=0$ column has three non-zero entries: there is a $\mathbb{Z}/4$ for $s=0$, $\mathbb{Z}/6$ for $s=1$, and $\mathbb{Z}/3$ for $s=5$.

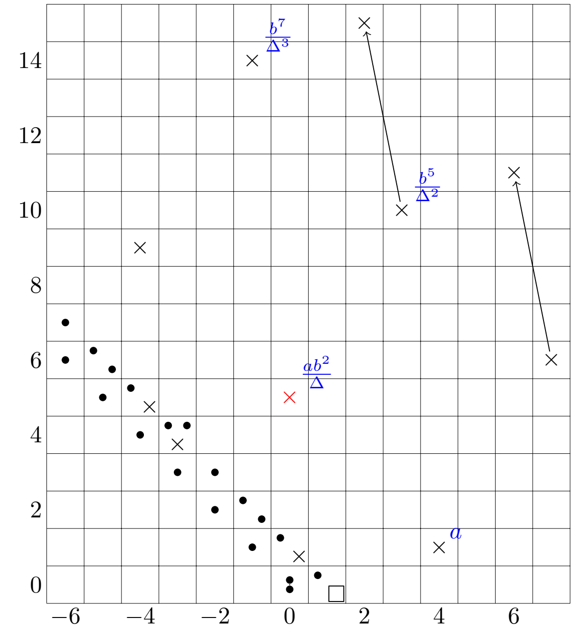

Figure 6. Homotopy fixed point spectral sequence for $(\mathfrak{pic}TMF(2))^{hGL_2(\mathbb{Z}/2)}$
($\square$ denotes $\mathbb{Z}$, $\bullet$ denotes $\mathbb{Z}/2$, and $\times$ denotes $\mathbb{Z}/3$)

Now we are ready to study the differentials in the HFPSS for $\mathfrak{pic}(TMF(2))^{hGL_2(\mathbb{Z}/2)}$. Comparison with the HFPSS for the $GL_2(\mathbb{Z}/2)$-action on $TMF(2)$ gives a number of differentials, using our Comparison Tool 5.2.4. To distinguish between the differentials in the two spectral sequences, let us denote by d_r^o those in the HFPSS of $TMF(2)$. The superscript o stands for "original."

Recall that in the HFPSS for $TMF(2)$, there are non-zero d_5^o and d_9^o differentials, which are obtained, for example, by a comparison with the HFPSS for $Tmf(2)$ which is fully determined in [Sto12]. In particular, in the HFPSS for $TMF(2)$, the first differential is $d_5^o(\Delta) = ab^2$, and the rest of the d_5^o's are determined by multiplicativity. In particular, we have

$$d_5^o\left(\frac{b^5}{\Delta^2}\right) = \frac{ab^7}{\Delta^3} \qquad d_5^o\left(\frac{b^3}{\Delta}\right) = -\frac{ab^5}{\Delta^2}. \tag{8.3}$$

Next (and last) is d_9^o; we have that $d_9^o(a\Delta^2) = b^5$. Consequently, we also have

$$d_9^o\left(\frac{ab^2}{\Delta}\right) = \frac{b^7}{\Delta^3}. \tag{8.4}$$

Let us now see which of these differentials also occur in the HFPSS for $\mathfrak{pic}TMF(2)$; according to Comparison Tool 5.2.4, the d_5-differentials are imported in the range $t > 5$, and the d_9-differentials in the $t > 9$ range. In particular, the differentials in (8.3) are the same in the Picard HFPSS; these are the two differentials drawn in Figure 6. Moreover, everything in the zero column and above the depicted region, i.e., such that $s = t > 16$, either supports a differential or is killed by one which originates in the $t > 9$ range. Hence, everything above the depicted region is killed in the spectral sequence and nothing survives to the E_∞-page.

Note, however, that we cannot (and should not attempt to) import the differential (8.4); this would be a d_9-differential with $t = 5$, so it does not satisfy the hypothesis of Comparison Tool 5.2.4.

Let us analyze the potentially remaining contributions to $\pi_0\mathfrak{pic}(TMF(2))^{GL_2(\mathbb{Z}/2)}$; regardless of what the rest of the differentials could possibly be, we have:

- a group of order at most 4 (and dividing 4) in position $(0, 0)$,
- a group of order at most 6 (and dividing 6) in position $(0, 1)$, and
- a group of order at most 3 (and dividing 3) in position $(0, 5)$.

Therefore $\mathrm{Pic}TMF[1/2] = \pi_0\mathfrak{pic}(TMF(2))^{GL_2(\mathbb{Z}/2)}$ has order at most $4 \times 6 \times 3 = 72$, and dividing 72. This is an upper bound. But we also have a well-known lower bound: the suspension $\Sigma TMF[1/2]$ generates a nontrivial element of $\mathrm{Pic}(TMF[1/2])$ of order 72 because $TMF[1/2]$ is 72-periodic. Thus we have proven the result. □

Remark 8.1.4. Our computations give an independent proof of the result of Fulton-Olsson [FO10] that the Picard group of the classical moduli stack of elliptic curves M_{ell} over $\mathbb{Z}[1/2]$ is $\mathbb{Z}/12$. (Fulton-Olsson carry out the analysis over any base, though.) This is a toy analog of the above analysis.

In fact, the Picard *groupoid* of the moduli stack $M_{ell}[1/2]$ is the homotopy fixed points of the $GL_2(\mathbb{Z}/2)$-action on the Picard groupoid of $M_{ell}(2)$. Now the Picard *group* of $M_{ell}(2)$ is $\mathbb{Z}/2$, as $M_{ell}(2)$ is an open subset in a weighted projective stack over a UFD, so that quasi-coherent sheaves on $M_{ell}(2)$ correspond simply to graded modules over $\mathbb{Z}[1/2, \lambda_1, \lambda_2, (\lambda_1^2\lambda_2^2(\lambda_1 - \lambda_2))^{-1}]$ and the only nontrivial invertible object is the shift by one of the unit.

In the HFPSS for computing $\mathrm{Pic}(M_{ell}[1/2])$, we see by the above computation of

$$H^1\left(GL_2(\mathbb{Z}/2); \Gamma(M_{ell}(2), \mathcal{O}^\times)\right)$$

that one gets a contribution of order 6. Thus, $|\mathrm{Pic}(M_{ell}[1/2])| \le 12$, but we know that ω has order twelve, so we are done.

8.2. **The Picard group of** $TMF[1/3]$**.** This section will be similar to Section 8.1, but with more complicated computations as is to be expected from 2-torsion. In this case we will use the $GL_2(\mathbb{Z}/3)$-Galois extension $TMF[1/3] \to TMF(3)$, coming from the Galois cover $M_{ell}(3) \to M_{ell}[1/3]$ of the moduli stack of elliptic curves with 3 inverted by the moduli stack of elliptic curves equipped with a full level 3-structure.

From [Sto14, 4.2], we can immediately compute the homotopy groups of $TMF(3)$: the moduli stack $M_{ell}(3)$ is affine, and is given as the locus of non-vanishing of

$$\Delta = 3^{-5}\zeta(1-\zeta)\gamma_1^3\gamma_2^3(\gamma_1 + \zeta\gamma_2)^3(\gamma_2 - \zeta\gamma_1)^3$$

in the compact moduli stack $\overline{M}_{ell}(3) = \mathrm{Proj}\mathbb{Z}[1/3, \zeta][\gamma_1, \gamma_2]$. Here γ_i are variables in (topological) degree 2, and ζ is a primitive third root of unity, whose appearance is due to the fact that the Weil pairing on the 3-torsion points of an elliptic curve equips $\overline{M}_{ell}(3)$ with a map to $\mathrm{Spec}\mathbb{Z}[1/3, \zeta]$. Hence the descent spectral sequence computing $TMF(3)_*$ collapses to give

$$TMF(3)_* = \mathbb{Z}[1/3, \zeta][\gamma_1^{\pm 1}, \gamma_2^{\pm 1}][(\gamma_1 + \zeta\gamma_2)^{-1}, (\gamma_2 - \zeta\gamma_1)^{-1}].$$

Written differently, we have that $TMF(3)_* = TMF(3)_0[\gamma_2^{\pm 1}]$, and

$$TMF(3)_0 = \mathbb{Z}[1/3, \zeta][t^{\pm 1}, (1 - \zeta t)^{-1}, (1 + \zeta^2 t)^{-1}], \tag{8.5}$$

for $t = \frac{\gamma_1}{\gamma_2}$. In particular $TMF(3)_0$ is regular noetherian, and $TMF(3)$ is even periodic. Thus, Theorem 2.4.6 (together with the fact that the ring $\mathbb{Z}[\zeta, t]$ and hence any of its localizations has unique factorization) implies the following conclusion.

Lemma 8.2.1. *The Picard group* $\mathrm{Pic}(TMF(3))$ *is* $\mathbb{Z}/2$ *generated by* $\Sigma TMF(3)$*.*

Naturally, we will use this lemma as an input in computing the HFPSS for the associated Picard spectra.

Theorem 8.2.2. *The Picard group of* $TMF[1/3]$ *is* $\mathbb{Z}/192$*, generated by the suspension* $\Sigma TMF[1/3]$*. In particular, this Picard group is algebraic.*

Proof. As is to be expected, we use the HFPSS (3.5)

$$H^s(GL_2(\mathbb{Z}/3), \pi_t\mathfrak{pic}(TMF(3))) \Rightarrow \pi_{t-s}\mathfrak{pic}(TMF(3))^{hGL_2(\mathbb{Z}/3)}. \tag{8.6}$$

The homotopy groups $\pi_t(\mathfrak{pic}TMF(3))$ for $t \geq 2$ are isomorphic to $\pi_{t-1}TMF(3)$ as $GL_2(\mathbb{Z}/3)$-modules, hence the $t \geq 2$ part of the E_2-page of the HFPSS for $\mathfrak{pic}TMF(3)$ is same as the corresponding part in the HFPSS for $TMF(3)$. We will use the fact that $TMF(3) \simeq Tmf(3)[\Delta^{-1}]$ to identify this part of the spectral sequence for $TMF(3)$ and therefore for $\mathfrak{pic}(TMF(3))$.

The E_2-page of the HFPSS commuting the homotopy groups of $Tmf\hat{_2}$ as $(Tmf(3)\hat{_2})^{hGL_2(\mathbb{Z}/3)}$ is computed in [Sto14], and depicted in Figure 9 of loc.cit. Since we are working with 3 inverted, and 2 and 3 are the only primes dividing the order of $GL_2(\mathbb{Z}/3)$, we conclude that

$$H^{>0}(GL_2(\mathbb{Z}/3), \pi_*Tmf(3)) = H^{>0}(GL_2(\mathbb{Z}/3), \pi_*Tmf(3)\hat{_2}).$$

The invariants $H^0(GL_2(\mathbb{Z}/3), \pi_{>0}Tmf(3))$ are the ring of modular forms $\mathbb{Z}[1/3][c_4, c_6, \Delta]/(12^3\Delta - c_4^3 + c_6^2)$.

Let Γ denote the graded ring $\mathbb{Z}[1/3, \zeta][\gamma_1, \gamma_2]$. As in the case of level 2-structures, we have that

$$H^*(GL_2(\mathbb{Z}/3), \pi_*TMF(3)) = H^*(GL_2(\mathbb{Z}/3), \Gamma)[\Delta^{-1}].$$

In the group cohomology of Γ, computed and depicted in Figure 7 of [Sto14], there are a number of interesting torsion classes, including:

(1) h_1 in bidegree $(s,t) = (1,2)$, depicted in position $(s, t-s) = (1,1)$, which detects (the Hurewicz image of) the Hopf map η in homotopy.
(2) h_2 in position $(1,3)$, which detects (the Hurewicz image of) the Hopf map ν.
(3) d in position $(2,14)$, which detects in homotopy the class known as κ.
(4) g in position $(4,20)$, which detects in homotopy the class $\bar{\kappa}$.
(5) c in position $(2,8)$, which detects in homotopy the class ϵ.

The homotopy elements detected by these classes satisfy some relations; for example,

$$\eta^3 = 4\nu, \quad \kappa\nu^2 = 4\bar{\kappa}.$$

Let us also name one of the less famous elements in the descent spectral sequence for $tmf_{(2)}$, which also appear in the HFPSS for $TMF[1/3]$. Namely, there is a $\mathbb{Z}/2$ in position $(1,5)$; we will the generating class by the generic name x (in [Bau08] it bears the name $a_1^2h_1$).

All torsion classes with the exception of (powers of) h_1 are annihilated by c_4 and c_6. In the Picard spectral sequence, all of these classes appear shifted by one to the right; we have labeled some such classes in Figure 9. A "zoomed in" portion of the Picard spectral sequence is depicted in Figure 8. There, and in all of the related spectral sequences, lines of slope 1 denote h_1-multiplication, and lines of slope 1/3 denote h_2-multiplication.

A "zoomed out" portion of the Picard HFPSS (8.6) is depicted in Figure 7; the elements that are to the right of the $t = 2$ line are, of course, a shift of the corresponding elements in the spectral sequence for $TMF[1/3]$. However, to avoid cluttering the picture, a family of classes is not shown, with the exception of the elements depicted in green, namely $h_1^3\frac{c_4c_6}{\Delta}$ and $h_1^6\frac{c_4^2}{\Delta}$, as well as the tower supported on 1, which do belong to this family. The family consists precisely of the h_1-power multiples of non-torsion classes. In the "zoomed in" Figure 9 this family is also not shown.

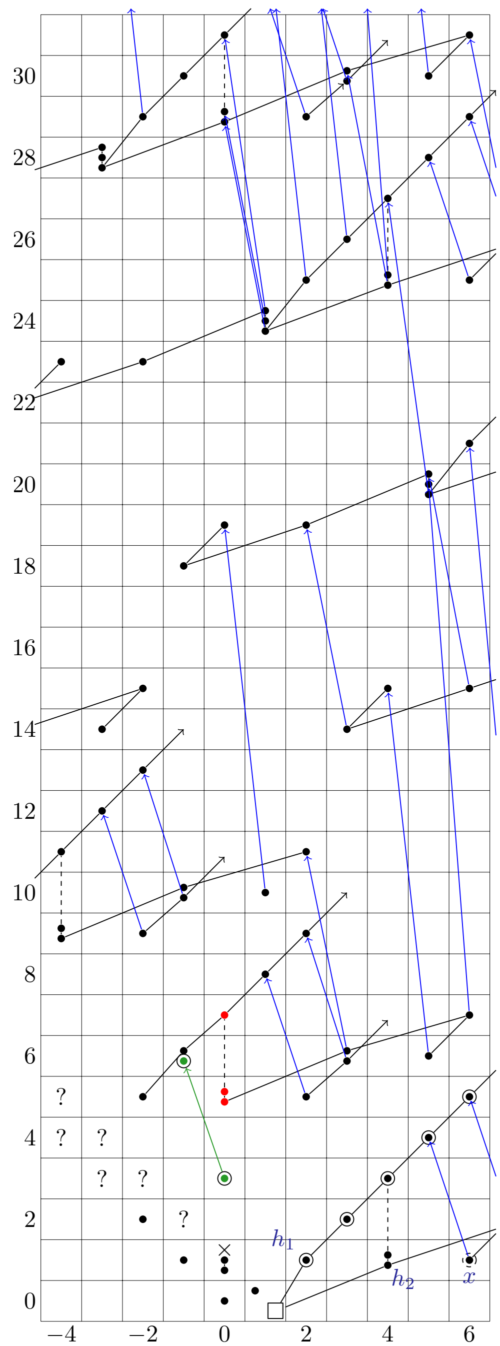

FIGURE 7. Homotopy fixed point spectral sequence for $\mathfrak{pic}(TMF(3))^{hGL_2(\mathbb{Z}/3)}$ ($\square$ denotes $\mathbb{Z}$, $\bullet$ denotes $\mathbb{Z}/2$, $\odot$ denotes $\mathbb{Z}/2[j]$, and $\times$ denotes $\mathbb{Z}/3$)

More specifically, the non-torsion subring of the E_2-page of the $TMF[1/3]$-spectral sequence is precisely the part in cohomological degree 0 and consists of the ring of modular forms $MF_*[1/3] = \mathbb{Z}[1/3][c_4, c_6, \Delta^{\pm 1}]/(12^3\Delta - c_4^3 + c_6^2)$. On the E_2-page, these support infinite h_1-multiples, i.e., $MF_*[1/3][h_1]$ is a subring of the E_2-page. Note that in degree zero, $MF_*[1/3] = \mathbb{Z}[j]$, where

$j = \frac{c_4^3}{\Delta}$ is the classical j-invariant. What we have omitted drawing in Figure 7 and 9 is all of the elements coming from this subring, with the exception of the mentioned classes. For comparison, these elements are drawn in the smaller-range Figure 8.

Remark 8.2.3. These two classes, which we have depicted in green, do not appear in the spectral sequence for $Tmf[1/3]$, since they involve a negative power of Δ. Another difference between the Tmf and TMF situation is that in the E_2-page of the latter, there are infinite groups, isomorphic to $\mathbb{Z}/2[j]$ and generated by h_1, h_1^2, h_1^3, etc., in positions $(1,1)$, $(2,2)$, $(3,3)$, etc. Moreover, the element x in position $(1,5)$ also generates an infinite $\mathbb{Z}/2[j]$, as do all of its h_1-multiples.

Note that in the range that we are considering (namely, $t > 1$), the HFPSS for the $GL_2(\mathbb{Z}/3)$-action on $Tmf(3)$ coincides with the descent spectral for $Tmf[1/3]$ as the sections of $\mathcal{O}^{top}$ over $\overline{\mathfrak{M}}_{ell}[1/3]$, and the differentials in the latter have been fully determined in Johan Konter's master thesis [Kon]. Of course, these differentials really come from the connective tmf, whose descent spectral sequence is fully computed in [Bau08]. In these spectral sequences, d_3^o is the first non-trivial differential, followed by $d_5^o, d_7^o, d_9^o \dots d_{23}^o$. In particular, we have the following differentials [Bau08, Sec.8]:

$$\begin{aligned} d_3^o(c_6) &= c_4 h_1^3 & d_3^o(x) &= h_1^4 \\ d_5^o(\Delta) &= g h_2 & d_7^o(4\Delta) &= g h_1^3 \\ d_9^o(\Delta^2 h_1) &= g^2 c & d_{11}^o(d\Delta^2) &= g^3 h_1, \end{aligned} \tag{8.7}$$

and a number of others.

Let us see now which of these differentials we can import using our Comparison Tool 5.2.4. In the $TMF[1/3]$ spectral sequence, we have that $d_3^o(h_1^3 \frac{c_4 c_6}{\Delta}) = h_1^6 \frac{c_4^2}{\Delta}$; in the Picard SS, the element corresponding to $h_1^3 \frac{c_4 c_6}{\Delta}$ has $t = 3$, thus we *cannot* import this differential. We deal with this class later, i.e., in the next paragraph. However, all the other classes which are on the $s = t$ column and are h_1-power multiples of non-torsion classes, i.e., members of the family which we have not drawn in Figure 7, are well within the $t > 3$ range, so that we can indeed conclude by Comparison Tool 5.2.4 that they either support a differential or are killed by one. For example, the h_1-multiple of the differential just discussed does happen, i.e., in the Picard SS we have $d_3(h_1^4 \frac{c_4 c_6}{\Delta}) = h_1^7 \frac{c_4^2}{\Delta}$. In particular, we need not worry about these omitted classes any more.

Now we turn to the question of whether any differentials are supported on the $(s, t-s) = (3,0)$ position in the HFPSS for $\mathfrak{pic}(TMF(3))^{hGL_2(\mathbb{Z}/3)}$. For this purpose we use the universal formula (6.1) of Theorem 6.1.1, just as we did in the second proof of Proposition 7.2.8. We have that $E_2^{3,3}$ of the Picard spectrum HFPSS is $\mathbb{Z}/2[j]$ generated by $h_1^3 \frac{c_4 c_6}{\Delta}$; the corresponding element in the original HFPSS has

$$d_3^o \left(h_1^3 \frac{c_4 c_6}{\Delta} \right) = h_1^6 \frac{c_4^2}{\Delta}.$$

Now we have that

$$\left(h_1^3 \frac{c_4 c_6}{\Delta} \right)^2 = h_1^6 \frac{c_4^2 c_6^2}{\Delta^2} = (j - 12^3) h_1^6 \frac{c_4^2}{\Delta} = j h_1^6 \frac{c_4^2}{\Delta},$$

using the fact that $12^3 \Delta = c_4^3 - c_6^2$ and that by definition, $j = \frac{c_4^3}{\Delta}$. Therefore, we conclude by (6.1) that in the Picard HFPSS, the differential $d_3 : E_2^{3,3} \to E_2^{6,5}$ is given by

$$d_3 \left(f(j) h_1^3 \frac{c_4 c_6}{\Delta} \right) = (f(j) + j f(j)^2) h_1^6 \frac{c_4^2}{\Delta},$$

where $\left(f(j) h_1^3 \frac{c_4 c_6}{\Delta} \right)$ is an arbitrary element of $E_2^{3,3}$. However, $(f(j) + j f(j)^2)$ in $\mathbb{Z}/2[j]$ is zero only if $f(j)$ is zero, hence this d_3 is injective and has trivial kernel. (Note this is an interesting difference between this situation and the one in Proposition 7.2.8.) Consequently, $E_4^{3,3}$ is zero.

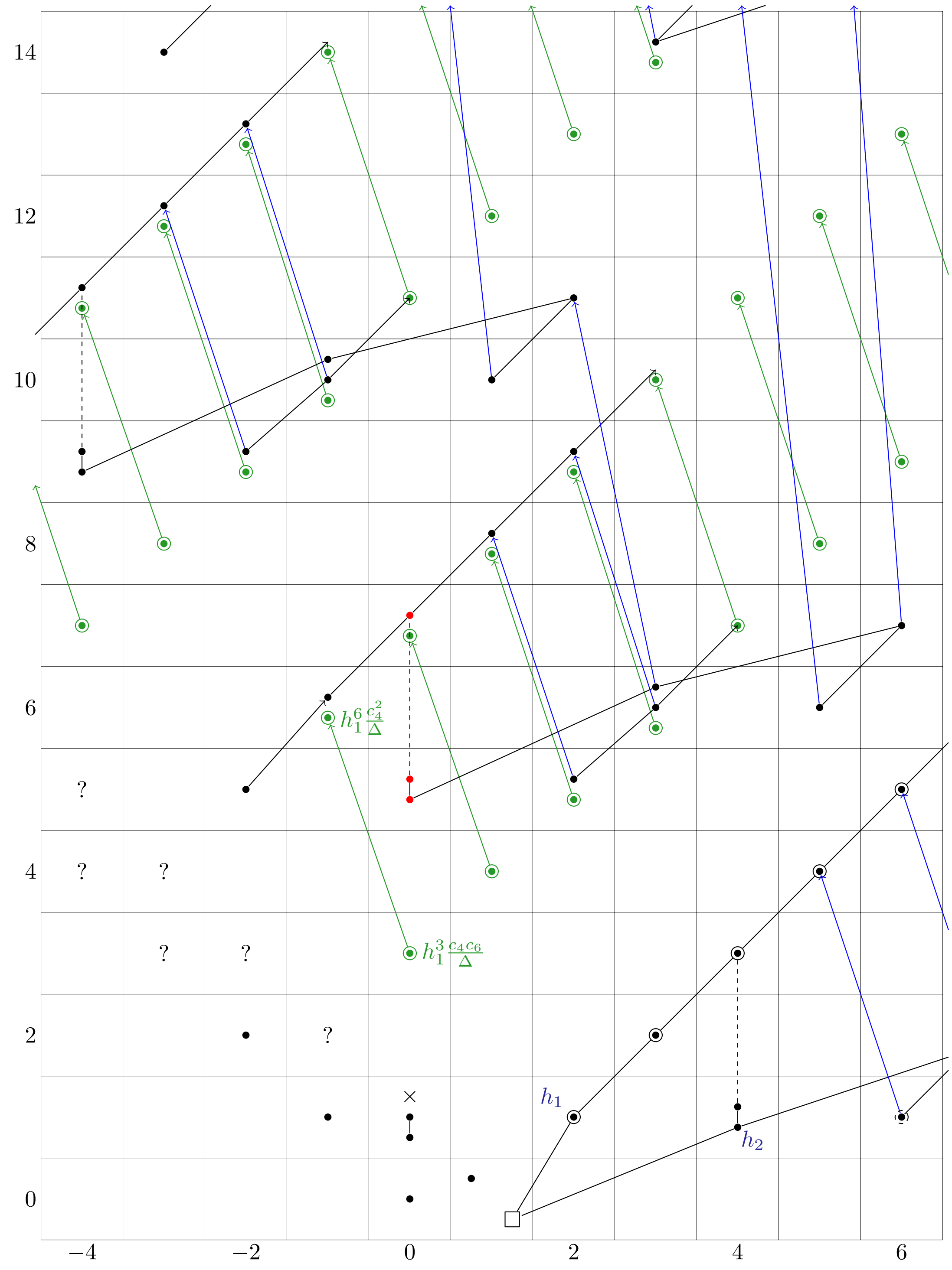

FIGURE 8. Homotopy fixed point spectral sequence for $\mathfrak{pic}(TMF(3))^{hGL_2(\mathbb{Z}/3)}$ ($\square$ denotes $\mathbb{Z}$, $\bullet$ denotes $\mathbb{Z}/2$, $\odot$ denotes $\mathbb{Z}/2[j]$, and $\times$ denotes $\mathbb{Z}/3$)

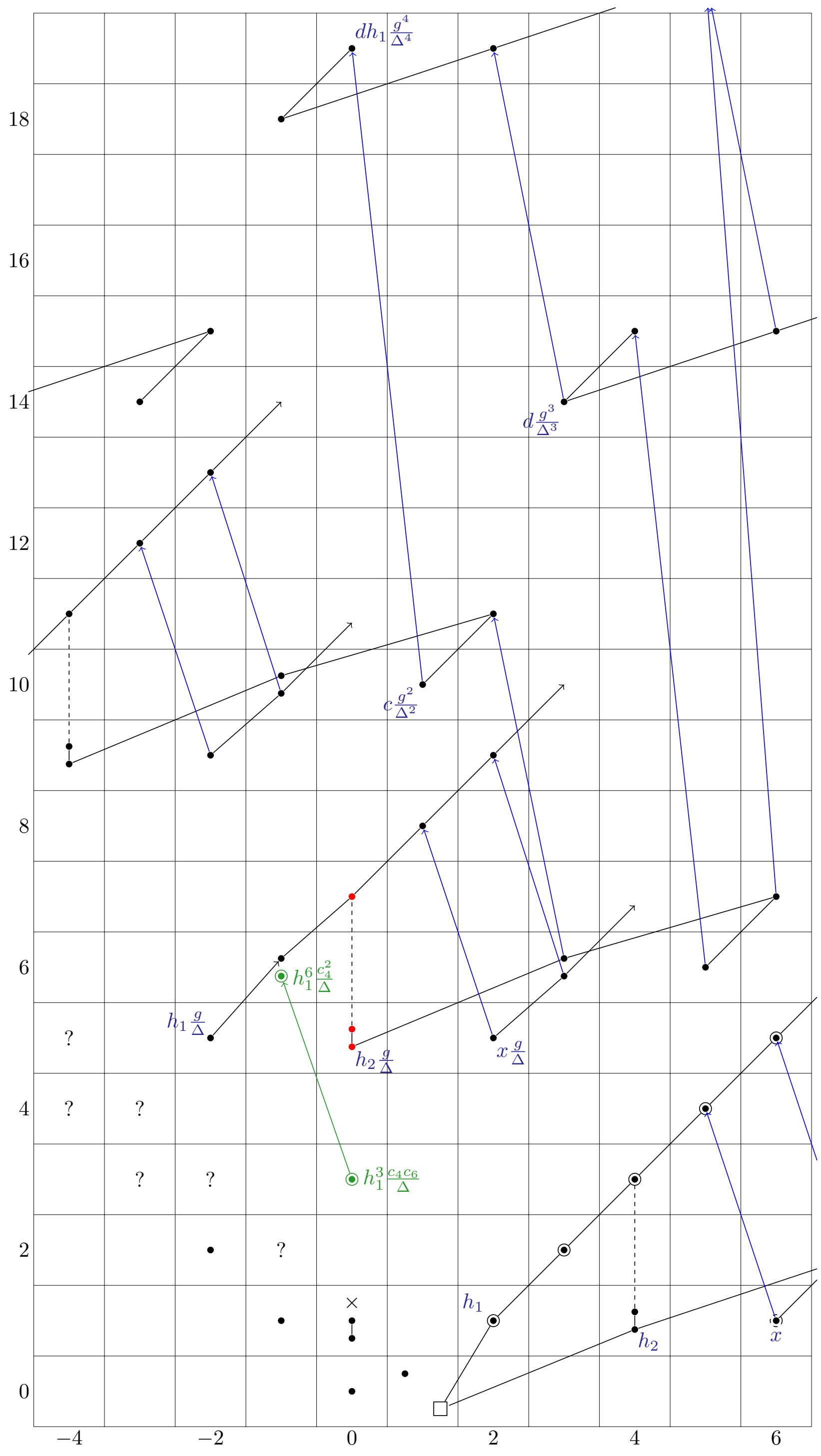

FIGURE 9. Homotopy fixed point spectral sequence for $\mathfrak{pic}(TMF(3))^{hGL_2(\mathbb{Z}/3)}$ ($\square$ denotes $\mathbb{Z}$, $\bullet$ denotes $\mathbb{Z}/2$, $\odot$ denotes $\mathbb{Z}/2[j]$, and $\times$ denotes $\mathbb{Z}/3$)

Further use of Comparison Tool 5.2.4 determines the all the differentials we have drawn in blue in Figures 7 to 9. Note that of the classes in the $s = t$ column, i.e., the one which contributes to the Picard group of $TMF[1/3]$, everything with $s \geq 8$ is killed. However, $h_2 \frac{g}{\Delta}$, generating a $\mathbb{Z}/4$ in $s = 5$, and $h_1^3 \frac{g}{\Delta}$ generating a $\mathbb{Z}/2$ in $s = 7$, remain. In the original spectral sequence, the first one of these supported a d_5^o and a d_{13}^o, and the second supported a d_{25}^o.

Next we need to determine the rest of the spectral sequence, i.e., the part which involves π_0 and π_1 of the Picard spectrum of $TMF(3)$. Computations for this are deferred to the Appendix; the results that we care about are, according to Lemma B.2 and Proposition B.5, $H^*(GL_2(\mathbb{Z}/3), \mathbb{Z}/2) = \mathbb{Z}/2$ for each $0 \leq * \leq 2$, and $H^*(GL_2(\mathbb{Z}/3), \pi_0\mathfrak{pic}(TMF(3)))$ is a group of order 12.

At this point we are ready to make conclusions about the Picard group of $TMF[1/3]$: in the $t = s$ vertical line of the HFPSS, i.e., the one that abuts to $\pi_0\mathfrak{pic}TMF[1/3] = \mathrm{Pic}TMF[1/3]$, nothing above the $s = 7$ line survives the spectral sequence. The following might survive:

- at most a group of order 2 in position $(0, 0)$,
- at most a group of order 12 in $(1, 0)$,
- at most a group of order 4 in $(5, 0)$, and
- at most a group of order 2 in $(7, 0)$.

The upshot is that we get an upper bound of $2 \times 12 \times 4 \times 2 = 192$ on the order of the Picard group. But $TMF[1/3]$ is 192-periodic, so this upper bound must also be a lower bound. In conclusion,

$$\mathrm{Pic}(TMF[1/3]) = \mathbb{Z}/192$$

as claimed, generated by $\Sigma TMF[1/3]$. □

Remark 8.2.4. As in Remark 8.1.4, we can use some of our computations to reprove Fulton-Olsson's [FO10] result that the moduli stack of elliptic curves $M_{ell}[1/3]$ also has a Picard group $\mathbb{Z}/12$. Namely, we start with the knowledge that $\mathrm{Pic}(M_{ell}(3))$ is trivial, as $M_{ell}(3)$ is the prime spectrum of a UFD. Then, we consider the Picard HFPSS for the algebraic stack $M_{ell}[1/3]$, which must collapse due to sparsity. The only contribution towards the Picard group is

$$H^1\left(GL_2(\mathbb{Z}/3), \Gamma(M_{ell}(3), \mathcal{O}^\times)\right)$$

which we saw by Proposition B.5 has order 12. But ω has order 12, hence $\mathrm{Pic}M_{ell}[1/3]$ is cyclic of order 12.

8.3. **Calculation of** $\mathrm{Pic}(TMF)$**.** In this section we will compute the Picard group of the integral period version of topological modular forms TMF. The result, as stated in the introduction, is:

Theorem A. The Picard group of integral TMF is $\mathbb{Z}/576$, generated by ΣTMF.

Proof. There is no nontrivial Galois extension of the integral TMF by [Mat14a, Theorem 10.1], but we can use étale descent, as TMF is obtained as the global sections of the sheaf $\mathcal{O}^{top}$ of even-periodic E_∞-rings on the moduli stack of elliptic curves. Namely, we can use Theorem 3.2.1 because the map $M_{ell} \to M_{FG}$ is known to be affine. The spectral sequence is

$$H^s(M_{ell}, \pi_t\mathfrak{pic}\mathcal{O}^{top}) \Rightarrow \pi_{t-s}\Gamma(\mathfrak{pic}\mathcal{O}^{top}),$$

and we are interested in π_0. Using Theorem 3.2.1, the E_2-page of this spectral sequence is given by

$$E_2^{s,t} = \begin{cases} \mathbb{Z}/2 & t = s = 0 \\ H^s(M_{ell}, \mathcal{O}_{M_{ell}}^\times) & t = 1 \\ H^s(M_{ell}, \omega^{(t-1)/2}) & t \geq 3, \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Over a field k of characteristic $\neq 2, 3$, Mumford [Mum65] showed that

$$H^1((M_{ell})_k, \mathcal{O}_{M_{ell}}^\times) \simeq \mathbb{Z}/12,$$

i.e., the Picard group of the moduli stack is $\mathbb{Z}/12$, generated by the line bundle ω that assigns to an elliptic curve the dual of its Lie algebra. This result is also true over $\mathbb{Z}$ by the work of Fulton-Olsson [FO10]. However, using descent we can reprove that result. Namely, in Remarks 8.1.4 and 8.2.4 we saw that the Picard groups of both $M_{ell}[1/2]$ and $M_{ell}[1/3]$ are $\mathbb{Z}/12$, both generated by ω. Cover the integral stack M_{ell} by these two; their intersection is $M_{ell}[1/6]$, which is the weighted projective stack $\mathrm{Proj}\mathbb{Z}[1/6][c_4, c_6]$ (with c_4 and c_6 in degrees[7] 4 and 6 respectively), and which therefore has Picard group $\mathbb{Z}/12$ also generated by ω. The descent spectral sequence for $\mathfrak{pic}$ associated to this cover gives the result.

Because $M_{ell}[1/6]$ has no higher cohomology, the groups $H^s(M_{ell}, \omega^{(t-1)/2})$, when $s > 0$, are given as the direct sum of the corresponding cohomology groups of $M_{ell}[1/2]$ and $M_{ell}[1/3]$. These groups, in turn, are isomorphic to

$$H^s(GL_2(\mathbb{Z}/p), \pi_{t-1}TMF(p)) = H^s(GL_2(\mathbb{Z}/p), H^0(M_{ell}(p), \omega^{(t-1)/2})),$$

where p is 2 or 3, as the map $M_{ell}(p) \to M_{ell}[1/p]$ is Galois, and $M_{ell}(p)$ has no higher cohomology. We computed these groups in the previous examples.

The machinery of Section 5 now allows us to compare this Picard descent spectral sequence to the one which computes the homotopy groups of TMF. From Corollary 5.2.3 and an analogue of Comparison Tool 5.2.4, we conclude that the differentials involving 3-torsion classes wipe out everything above the $s = 5$ line, and those involving 2-torsion classes wipe out everything above the $s = 7$ line. These differentials are identical to what happens in the homotopy fixed point spectral sequences in the previous two examples. We conclude that the following are the only groups that can survive:

- at most a group of order 2 in $(t - s, s) = (0, 0)$,
- at most a group of order 12 in $(0, 1)$,
- at most a group of order 2 in $(0, 3)$,
- at most a group of order 12 in $(0, 5)$, and
- at most a group of order 2 in $(0, 7)$.

This gives us an upper bound $2^7 3^2$ on the cardinality of π_0, which is twice the periodicity of TMF. The spectral sequence is depicted in Figure 10. □

8.4. **Calculation of** $\mathrm{Pic}(Tmf)$**.** We will now prove the following result stated in the introduction:

Theorem B. The Picard group of Tmf is $\mathbb{Z} \oplus \mathbb{Z}/24$, generated by ΣTmf and a certain 24-torsion invertible module.

Note that while $Tmf[1/n]$, for $n = 2, 3$, can be described as the homotopy fixed point spectrum $Tmf(n)^{hGL_2(\mathbb{Z}/n)}$ just as in the periodic case, the extension $Tmf[1/n] \to Tmf(n)$ is *not* Galois, and therefore we cannot use Galois descent to compute the Picard group. However, we can use Theorem 3.2.1 for the compactified moduli stack $\overline{M}_{ell}$.

First, we need a lemma.

Lemma 8.4.1. *Let $\mathcal{L}$ be the line bundle on $\overline{M}_{ell}$ obtained by gluing the trivial line bundles on $M_{ell} = \overline{M}_{ell}[\Delta^{-1}]$ and $\overline{M}_{ell}[c_4^{-1}]$ via the clutching function j. Then $\mathcal{L} \simeq \omega^{-12}$.*

Proof. To give a section of $\mathcal{L} \otimes \omega^{12}$ over $\overline{M}_{ell}$ is equivalent to giving sections $s_1 \in \Gamma(M_{ell}, \omega^{12})$ and $s_2 \in \Gamma(\overline{M}_{ell}[c_4^{-1}], \omega^{12})$ such that $(js_1)|_{M_{ell}[c_4^{-1}]} = (s_2)|_{M_{ell}[c_4^{-1}]}$. We take $s_1 = \Delta$ and $s_2 = c_4^3$, and we get a nowhere vanishing section of $\mathcal{L} \otimes \omega^{12}$. □

[7] These are the algebraic degrees, which get doubled in topology

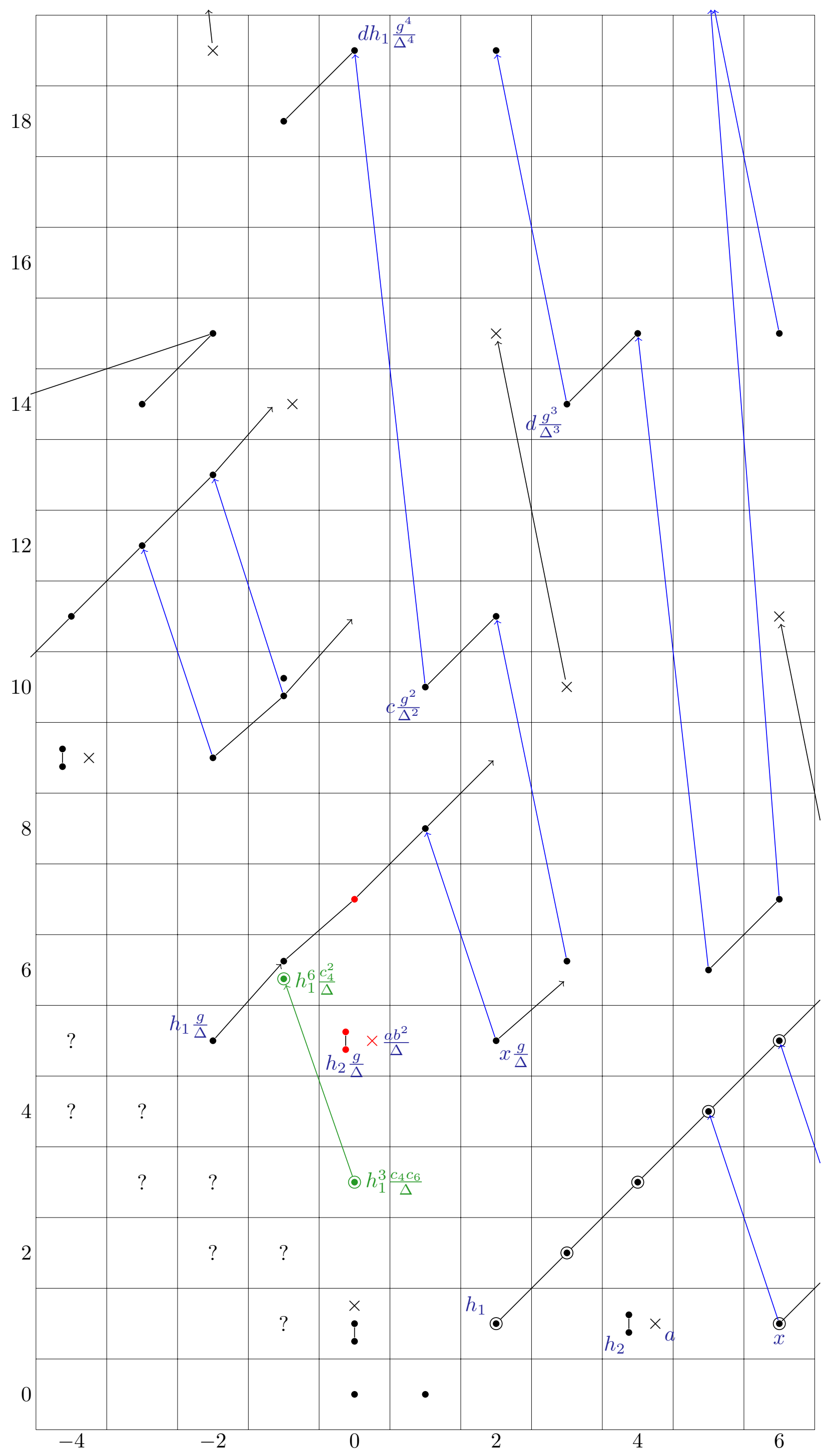

FIGURE 10. Descent spectral sequence for $\Gamma(\mathfrak{pic}\mathcal{O}^{top})$ on $\mathfrak{M}_{ell}$
($\square$ denotes $\mathbb{Z}$, $\bullet$ denotes $\mathbb{Z}/2$, and $\times$ denotes $\mathbb{Z}/3$)

Proof of Theorem B. The relevant part of the (Picard) descent spectral sequence is similar as for TMF, with the following exceptions: the algebraic part $H^1(\overline{M}_{ell}, \mathcal{O}^\times)$ is now $\mathbb{Z}$, by Fulton-Olsson [FO10], and all the torsion groups are now finite, i.e., there are no $\mathbb{Z}/2[j]$'s appearing. In particular, $E_2^{3,3}$ is zero, and we have

- at most a group of order 2 in $(t-s, s) = (0, 0)$,
- a sub-quotient of $\mathbb{Z}$ in $(0, 1)$,
- at most a group of order 12 in $(0, 5)$, and
- at most a group of order 2 in $(0, 7)$,

as potential contributions to the E_∞^0-page. The depiction is in Figure 11.

Note that the $\mathbb{Z}/2$ in $(0, 0)$, which corresponds to a single suspension of the even-periodic spectra that Tmf is built from, is represented by ΣTmf in the Picard group of Tmf. Similarly, the element $1 \in \mathbb{Z} = E_2^{0,1} = \mathrm{Pic}\overline{M}_{ell}$ corresponds to the line bundle ω, which topologically is represented by $\Sigma^2 Tmf$. Thus these groups survive to the E_∞-page and are related by an extension. The rest of the E_∞-filtration now tells us that $\mathrm{Pic}Tmf$ sits in an extension

$$0 \to A \to \mathrm{Pic}Tmf \to \mathbb{Z} \to 0,$$

where A is a finite group of order at most 24.

We claim that $A = \mathbb{Z}/24$ and therefore $\mathrm{Pic}(Tmf) = \mathbb{Z} \oplus \mathbb{Z}/24$. To see this, we will construct a line bundle $\mathcal{I}$ such that $\mathcal{I}^{\otimes 24} \simeq \mathcal{O}^{top}$, but no lower power of $\mathcal{I}$ is equivalent to $\mathcal{O}^{top}$.

Construction 8.4.2. Consider the cover of $\overline{M}_{ell}$ by $\overline{M}_{ell}[\Delta^{-1}] = M_{ell}$ and $\overline{M}_{ell}[c_4^{-1}]$ which fit in the pushout diagram

$$\begin{array}{ccc} \overline{M}_{ell}[\Delta^{-1}, c_4^{-1}] & \longrightarrow & \overline{M}_{ell}[\Delta^{-1}] \\ \downarrow & & \downarrow \\ \overline{M}_{ell}[c_4^{-1}] & \longrightarrow & \overline{M}_{ell}. \end{array}$$

Let $\mathcal{J}$ be the line bundle on the *derived* moduli stack $\overline{\mathfrak{M}}_{ell} = (\overline{M}_{ell}, \mathcal{O}^{\mathrm{top}})$ obtain by gluing $\mathcal{O}^{top}$ on $\overline{M}_{ell}[\Delta^{-1}]$ and $\mathcal{O}^{top}$ on $\overline{M}_{ell}[c_4^{-1}]$ using the clutching function $j = \frac{c_4^3}{\Delta}$ on $\overline{M}_{ell}[\Delta^{-1}, c_4^{-1}]$.

We claim that $\mathcal{J}$ is not a suspension of $\mathcal{O}^{top}$, and that $\mathcal{I} = \Sigma^{24}\mathcal{J}$ is an element of the Picard group of order 24.

To see this, we note that $\pi_0\mathcal{J}$ is $\omega^{\otimes -12}$, so if $\mathcal{J}$ is a suspension of $\mathcal{O}^{top}$, it ought to be $\Sigma^{-24}\mathcal{O}^{top}$. However, $\Sigma^{-24}\mathcal{O}^{top}$ restricted to $\overline{M}_{ell}[\Delta^{-1}]$ is $\Sigma^{-24}\mathcal{O}^{top}$, whereas $\mathcal{J}$ restricts to $\mathcal{O}^{top}$ on $\overline{M}_{ell}[\Delta^{-1}]$. As we know from Section 8.3, these are not isomorphic on $\overline{M}_{ell}[\Delta^{-1}]$; nonetheless, they do become isomorphic after taking 24-th tensor powers. Consequently, $\mathcal{J}^{\otimes 24} \simeq (\Sigma^{24}\mathcal{O}^{top})^{\otimes 24} \simeq \Sigma^{-576}\mathcal{O}^{top}$. Note that this analysis also shows that no lower power of $\mathcal{J}$ is a suspension of $\mathcal{O}$. Therefore $\mathcal{I} = \Sigma^{24}\mathcal{J}$ defines an element of order 24 of the Picard group of $\overline{\mathfrak{M}}_{ell}$ and thus of Tmf. □

The same analysis shows that $\mathrm{Pic}Tmf_{(2)} = \mathbb{Z} \oplus \mathbb{Z}/8$ and $\mathrm{Pic}Tmf_{(3)} = \mathbb{Z} \oplus \mathbb{Z}/3$, the torsion being generated by the respective localizations of $\mathcal{I}$. Moreover, $\mathrm{Pic}Tmf_{(p)}$, for $p > 3$ is $\mathbb{Z}$.

8.5. **Relation to the E_2-local Picard group.** Notice that $\mathcal{I}$ is the only "exotic" element in all of our examples involving the various forms of topological modular forms. Let us see how it relates to the exotic piece of the Picard group of the category of E_2-local spectra, i.e., modules over the E_2-local sphere spectrum. The exotic phenomena only occur at $p = 2$ and $p = 3$, but since only the 3-primary E_2-local Picard group is known, let us concentrate on that case for the remainder of this section.

In [GHMR12], the authors compute κ_2, the exotic part of the Picard group of the category of 3-primary $K(2)$-local spectra; they show $\kappa_2 = \mathbb{Z}/3 \times \mathbb{Z}/3$.

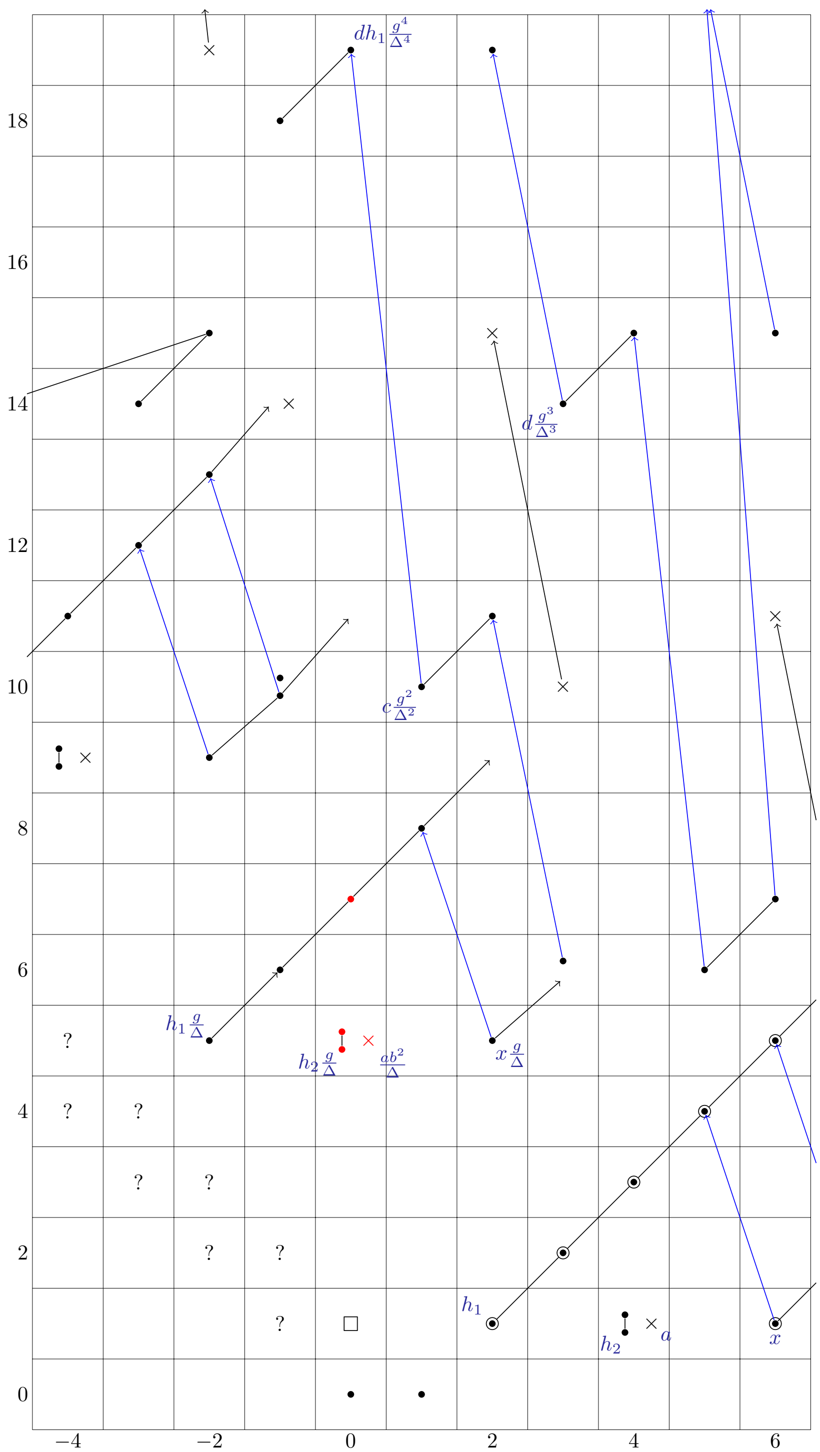

FIGURE 11. Descent spectral sequence for $\Gamma(\mathfrak{pic}\mathcal{O}^{top})$ on $\overline{\mathfrak{M}}_{ell}$ ($\square$ denotes $\mathbb{Z}$, $\bullet$ denotes $\mathbb{Z}/2$, and $\times$ denotes $\mathbb{Z}/3$)

In addition, they look at the localization map from the E_2-local category to the $K(2)$-local category, and show that it induces an isomorphism $\kappa_{\mathcal{L}_2} \to \kappa_2$, where $\kappa_{\mathcal{L}_2}$ denotes the exotic E_2-local Picard group.

Consider now the commutative diagram

$$\begin{array}{ccc} \kappa_{\mathcal{L}_2} & \longrightarrow & \kappa_2 \\ \downarrow t & & \downarrow t_{K(2)} \\ \mathrm{Pic} Tmf_{(3)} & \longrightarrow & \mathrm{Pic} Tmf_{K(2)}, \end{array}$$

in which the horizontal maps are given by $K(2)$-localization, and the vertical maps are given by smashing with Tmf and $Tmf_{K(2)}$, respectively. In Thm.5.5 of loc.cit, the authors show that there is an element P of κ_2 such that $L_{K(2)}(P \wedge Tmf_{K(2)}) \simeq \Sigma^{48} Tmf_{K(2)}$, i.e., $t_{K(2)} P = 48 \in \mathbb{Z}/72 \subseteq \mathrm{Pic} Tmf_{K(2)}$. Under the top horizontal isomorphism, this P lifts to an element $\widetilde{P}$ of $\kappa_{\mathcal{L}_2}$, such that $t(\widetilde{P})$ has order three in $\mathrm{Pic} Tmf$ and such that the $K(2)$-localization of $t(\widetilde{P})$ is $L_{K(2)}(\Sigma^{48} Tmf)$. Therefore, $t(\widetilde{P})$ must be twice the class of $\mathcal{I}$. In other words, the exotic element $\widetilde{P}$ of $\kappa_{\mathcal{L}_2}$ is detected as an exotic element of $\mathrm{Pic} Tmf_{(3)}$.

The other $\mathbb{Z}/3$ in κ_2, i.e., κ_2 modulo the subgroup generated by P, is generated by a spectrum Q such that $t_{K(2)} Q = 0$. This Q lifts to $\widetilde{Q} \in \kappa_{\mathcal{L}_2}$, still of order 3, which must map under t to an element of order 3 in $\mathrm{Pic} Tmf$ which is in the kernel of the bottom localization map. But there are no non-trivial elements of finite order in this kernel, hence $\widetilde{Q}$ is not detected in $\mathrm{Pic} Tmf_{(3)}$.

Perhaps at the prime 2 as well there is an element of the exotic E_2-local Picard group which is detected in the torsion of $\mathrm{Pic} Tmf_{(2)}$.

Appendices

Appendix A. Group cohomology computations for $TMF(2)$

In this section of the appendix we compute the group cohomology for the $GL_2(\mathbb{Z}/2)$-action on $\pi_0 \mathfrak{pic}(TMF(2)) = \mathbb{Z}/4$ (with trivial action), and on $\pi_1 \mathfrak{pic}(TMF(2)) = TMF(2)_0^\times$ with the natural action. The group $GL_2(\mathbb{Z}/2)$ is the symmetric group on three letters, so it has a (unique) normal subgroup of order 3, which we denote C_3, with quotient C_2. We can therefore use the associated Lyndon-Hochschild-Serre spectral sequence (LHSSS)

$$H^p(C_2, H^q(C_3, M)) \Rightarrow H^{p+q}(GL_2(\mathbb{Z}/2), M) \tag{A.1}$$

for $GL_2(\mathbb{Z}/2)$-modules M.

Let us first deal with the easier case.

Lemma A.1. *The group cohomology for the $GL_2(\mathbb{Z}/2)$-action on the trivial module $\mathbb{Z}/4$ is*

$$H^*(GL_2(\mathbb{Z}/2), \pi_0 \mathfrak{pic}(TMF(2))) = \begin{cases} \mathbb{Z}/4, & * = 0 \\ \mathbb{Z}/2, & * > 0. \end{cases}$$

Proof. Since 3 is invertible in $\mathbb{Z}/4$, we have that $H^*(C_3, \mathbb{Z}/4) = \mathbb{Z}/4$ concentrated in degree zero, and with trivial action by $C_2 = GL_2(\mathbb{Z}/2)/C_3$. Hence the LHSSS (A.1) collapses, giving

$$H^s(GL_2(\mathbb{Z}/2), \mathbb{Z}/4) = H^s(C_2, \mathbb{Z}/4),$$

which is $\mathbb{Z}/4$ for $s = 0$ and $\mathbb{Z}/2$ otherwise. □

Next we will compute the group cohomology for the action of $GL_2(\mathbb{Z}/2)$ on $\pi_1 \mathfrak{pic}(TMF(2))$ which is the multiplicative group of units in $\pi_0 TMF(2)$. First of all, we should explicitly describe the module that we are working with.

Let σ and τ be the generators of $GL_2(\mathbb{Z}/2)$ of order 3 and 2 respectively as chosen in [Sto12, Lem.7.3]; of course, σ generates the normal subgroup C_3. For brevity, let M denote the module of units in $TMF(2)_0$; by (8.1) is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}^{\oplus 3}$, where $\mathbb{Z}/2$ is multiplicatively generated by -1, and the $\mathbb{Z}$'s are multiplicatively generated by $2, s$ and $(s-1)$. The action is determined by [Lem.7.3,loc.cit.], where it is shown that the chosen generators σ and τ act as

$$\sigma : s \mapsto \frac{s-1}{s} \qquad\qquad \tau : s \mapsto \frac{1}{s}.$$

For example, a generic element m in M is transformed by σ as

$$\begin{aligned} m = (-1)^{\epsilon} 2^k s^a (s-1)^b \mapsto \sigma m &= (-1)^{\epsilon+b} 2^k s^{-a-b} (s-1)^a \\ &\mapsto \sigma^2 m (-1)^{\epsilon+a+b} 2^k s^b (s-1)^{-a-b}, \end{aligned}$$

which, written additively gives

$$\sigma : (\epsilon, k, a, b) \mapsto (\epsilon + b, k, -a-b, a) \mapsto (\epsilon + a + b, k, b, -a-b).$$

To use the LHSSS (A.1), the first step is to compute $H^*(C_3, M)$. From our formula for the σ action, we see that

$$\begin{aligned} (1-\sigma)(\epsilon, k, a, b) &= (b, 0, 2a+b, b-a) \\ N_\sigma(\epsilon, k, a, b) &= (1+\sigma+\sigma^2)(\epsilon, k, a, b) = (\epsilon + a, 3k, 0, 0); \end{aligned}$$

thus we conclude that for $q \geq 1$

$$\begin{aligned} H^0(C_3, M) = M^{C^3} &= \ker(1-\sigma) = \mathbb{Z}/2 \oplus \mathbb{Z} = \{\pm 2^k\} \\ H^{2q}(C_3, M) &= \ker(1-\sigma)/\operatorname{im}(N_\sigma) = \mathbb{Z}/3 = 2^{\mathbb{Z}}/2^{3\mathbb{Z}} \\ H^{2q-1}(C_3, M) &= \ker N_\sigma / \operatorname{im}(1-\sigma) = \mathbb{Z}/3 = (-s)^{\mathbb{Z}}/(-s)^{3\mathbb{Z}}. \end{aligned}$$

In the odd cohomology groups $\ker N_\sigma / \operatorname{im}(1-\sigma)$, $(s-1)$ is congruent to $(-s)^2$.

Further, τ acts trivially on $H^s(C_3, M)$ when $s \equiv 0, 1$ modulo 4, and nontrivially when $s \equiv 2, 3$ modulo 4. This gives the E_2-page of the LHSSS, which must collapse and give the following computation:

$$H^s(GL_2(\mathbb{Z}/2), M) = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}, & \text{for } s = 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/3, & \text{for } s \equiv 1(4) \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & \text{for } s \equiv 2(4) \\ \mathbb{Z}/2, & \text{for } s \equiv 3(4) \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3, & \text{for } s \equiv 0(4), s > 0. \end{cases} \tag{A.2}$$

We have thus proven the following result.

Proposition A.2. *The group cohomology for the $GL_2(\mathbb{Z}/2)$-action on $\pi_0\mathfrak{pic}(TMF(2)) = (TMF(2)_0)^\times$ is as in (A.2). In particular, we have that $H^1(GL_2(\mathbb{Z}/2), TMF(2)_0^\times) = \mathbb{Z}/6$.*

Appendix B. Group cohomology computations for $TMF(3)$

This section of the appendix is devoted to computing the group cohomology for $GL_2(\mathbb{Z}/3)$ acting on $\pi_0\mathfrak{pic}(TMF(3)) = \mathbb{Z}/2$, and $\pi_1\mathfrak{pic}(TMF(3)_0)^\times$. The group $GL_2(\mathbb{Z}/3)$ has order 48 and has the binary tetrahedral group as normal subgroup, in the guise of $SL_2(\mathbb{Z}/3)$. We have found it difficult to compute the higher cohomology groups of $(TMF(3)_0)^\times$, but since we are only using $H^1(GL_2(\mathbb{Z}/3), (TMF(3)_0)^\times)$ in Section 8.2, we will concentrate on computing this group only.

From (8.1), we see that $TMF(3)_0^\times \subset TMF(3)_0$ is $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}^{\oplus 4}$ multiplicatively generated by $-1, \zeta, 3, t, (1-\zeta t)$, and $(1+\zeta^2 t)$. The $GL_2(\mathbb{Z}/3)$-module structure is determined in [Sto14, 4.3]; to describe it, let x, y, z be the elements of $GL_2(\mathbb{Z}/3)$ chosen in loc.cit. Explicitly,

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Then x and y generate a quaternion group Q_8, and x, y, z generate $SL_2(\mathbb{Z}/3)$. Let σ be the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. These generate the whole group, and their action on the element $t = \frac{\gamma_1}{\gamma_2}$ is as determined in loc.cit. to be

$$x(t) = -\frac{1}{t} \qquad y(t) = \zeta^2 \frac{1-\zeta t}{1+\zeta^2 t} \qquad z(t) = \zeta^2 \frac{t}{1-\zeta t} \qquad \sigma(t) = \frac{1}{t}.$$

The rest is is determined by the fact that everything fixes $\mathbb{Z}[1/3] \subset TMF(3)_0$, a matrix A in $GL_2(\mathbb{Z}/3)$ takes ζ to $\zeta^{\det A}$, and the action respects the ring structure.

Consequently, we see that the submodule $\mathbb{Z}$ of $TMF(3)_0^\times$ multiplicatively generated by 3 has trivial action and splits off. Let M be its complement in $TMF(3)_0^\times$; then for any integer s we have

$$H^s(GL_2(\mathbb{Z}/3), TMF(3)_0^\times) = H^s(GL_2(\mathbb{Z}/3), \mathbb{Z}) \oplus H^s(GL_2(\mathbb{Z}/3), M). \tag{B.1}$$

Lemma B.1. *The first cohomology group $H^1(GL_2(\mathbb{Z}/3), \mathbb{Z})$ of the trivial module $\mathbb{Z}$ is zero.*

Proof. We show this by a couple of applications of the LHSSS. Recall from [CE99, XII.7] that $H^s(Q_8, \mathbb{Z})$ is $\mathbb{Z}$ for $s = 0$, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ for $s = 4p + 2$, $\mathbb{Z}/8$ for $s = 4p$, and zero otherwise. In particular, H^1 is zero. Now look at the LHSSS for the extension

$$1 \to Q_8 \to SL_2(\mathbb{Z}/3) \to C_3 \to 1;$$

since $H^1(C_3, \mathbb{Z}) = 0$, we get that $H^1(SL_2(\mathbb{Z}/3), \mathbb{Z})$ is also trivial. Next we apply the LHSSS for the extension

$$1 \to SL_2(\mathbb{Z}/3) \to GL_2(\mathbb{Z}/3) \to C_2 \to 1;$$

again $H^1(C_2, \mathbb{Z}) = 0$, proving the result. ☐

While we are working with trivial modules, let us also compute the cohomology of $\mathbb{Z}/2$.

Lemma B.2. *The cohomology $H^s(GL_2(\mathbb{Z}/3), \mathbb{Z}/2)$ is $\mathbb{Z}/2$ for $0 \le s \le 2$.*

Proof. The method of proof is same as for the previous lemma; however, now we start with the fact that

$$H^s(Q_8, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{for } s \equiv 0, 3(4) \\ \mathbb{Z}/ \oplus \mathbb{Z}/2, & \text{for } s \equiv 1, 2(4). \end{cases}$$

See for example [CE99, XII.7]. The modules $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ in degrees congruent to 1 or 2 modulo 4 have non-trivial C_3-action (without fixed points), where C_3 is the quotient $SL_2(\mathbb{Z}/3)/Q_8$. Therefore, the LHSSS gives that $H^s(SL_2(\mathbb{Z}/3), \mathbb{Z}/2)$ is non-zero for $s \equiv 0, 3$ modulo 4, in which case it is $\mathbb{Z}/2$, and is zero otherwise. Hence in the last LHSSS iteration

$$H^p(C_2, H^q(SL_2(\mathbb{Z}/3, \mathbb{Z}/2))) \Rightarrow H^{p+q}(GL_2(\mathbb{Z}/3), \mathbb{Z}/2),$$

for $s = p + q \le 2$, only $q = 0$ can contribute, and contributes precisely a $Z/2$. Due to scarcity, no differentials can possibly happen in this initial part of the spectral sequence. ☐

Going back to the case of interest, i.e. $TMF(2)_0^\times$, we conclude by Lemma B.1 that

$$H^1(GL_2(\mathbb{Z}/3), TMF(3)_0^\times) = H^1(GL_2(\mathbb{Z}/3), M), \tag{B.2}$$

and we now compute the latter cohomology group. The module M is isomorphic to $\mathbb{Z}/2\oplus\mathbb{Z}/3\oplus\mathbb{Z}^{\oplus 3}$ as an abelian group, and the action is as follows. If an element of M is written as $m=(\epsilon,\alpha,a,b,c)$, then the group generators x,y,z,σ act as

$$\begin{aligned}x : m \mapsto&(\epsilon+a+c,\alpha+b-c,-a-b-c,c,b)\\ y : m \mapsto&(\epsilon+b+c,\alpha-a-c,b,a,-a-b-c)\\ z : m \mapsto&(\epsilon,\alpha-a,a,-a-b-c,b)\\ \sigma : m \mapsto&(\epsilon+b,-\alpha-b+c,-a-b-c,b,c).\end{aligned}\tag{B.3}$$

There is a (non-split) short exact sequence

$$0\to\mathbb{Z}/2\to M\to\bar{M}\to 0;$$

we will show below in Proposition B.3 that $H^1(GL_2(\mathbb{Z}/3),\bar{M})=\mathbb{Z}/12$, and by Lemma B.2 we know the first few cohomology groups of $\mathbb{Z}/2$. Moreover, one easily sees that the invariant group $H^0(GL_2(\mathbb{Z}/3),\bar{M})$ is trivial. The long exact sequence in cohomology associated to this exact sequence of modules is, therefore,

$$0\to\mathbb{Z}/2\to H^1(GL_2(\mathbb{Z}/3),M)\to\mathbb{Z}/12\xrightarrow{\partial_1}\mathbb{Z}/2;\tag{B.4}$$

in Proposition B.5, we will compute the connecting homomorphism $\partial_1:\mathbb{Z}/12\to\mathbb{Z}/2$.

Proposition B.3. *The cohomology group $H^1(GL_2(\mathbb{Z}/3),\bar{M})$ is cyclic of order* 12.

Proof. There is a short exact sequence

$$0\to\bar{M}\to\widetilde{M}\to\mathbb{Z}\to 0$$

of $GL_2(\mathbb{Z}/3)$-modules, in which $\widetilde{M}$ can be described as the multiplicative group of units in

$$\big(TMF(3)_0[(1-t)^{-1}]\big)/2.$$

Note that $\bar{M}$ has a similar description, as the group of units in $\big(TMF(3)_0\big)/2$. As an abelian group, $\bar{M}$ is isomorphic to $\mathbb{Z}/3\oplus\mathbb{Z}^{\oplus 3}$, multiplicatively generated (in $\big(TMF(3)_0\big)/2$) by $\zeta,t,(1+\zeta t)$, and $(1+\zeta^2 t)$. The module $\widetilde{M}$ is isomorphic as an abelian group to $\mathbb{Z}/3\oplus\mathbb{Z}^{\oplus 4}$, where in addition to the generators for $\bar{M}$, the last $\mathbb{Z}$ is generated by $(1+t)$. The map $\widetilde{M}\to\mathbb{Z}$ is simply given by projection onto this last coordinate.

The action of $GL_2(\mathbb{Z}/3)$ is the one induced from the action on $\big(TMF(3)_0[(1-t)^{-1}]\big)/2$; explicitly, if $\widetilde{m}=(\alpha,a,b,c,d)\in\widetilde{M}$, the group elements x,y,z,σ act as

$$\begin{aligned}x : \widetilde{m} \mapsto&(\alpha+b-c,-a-b-c-d,c,b,d)\\ y : \widetilde{m} \mapsto&(\alpha-a-c+d,b,a,-a-b-c-d,d)\\ z : \widetilde{m} \mapsto&(\alpha-a,a,-a-b-c-d,b,d)\\ \sigma : \widetilde{m} \mapsto&(-\alpha-b+c,-a-b-c-d,b,c,d).\end{aligned}\tag{B.5}$$

We will show below in Proposition B.4 that $H^1(GL_2(\mathbb{Z}/3),\widetilde{M})=0$, implying that we have an exact sequence

$$0\to H^0(GL_2(\mathbb{Z}/3),\widetilde{M})\to\mathbb{Z}\xrightarrow{\partial_0}H^1(GL_2(\mathbb{Z}/3),\bar{M})\to 0.\tag{B.6}$$

Let us determine the invariants in $\widetilde{M}$; an element $\widetilde{m}=(\alpha,a,b,c,d)$ is invariant if and only if

$$a=b=c=-a-b-c-d,$$

and

$$\alpha\equiv\alpha+b-c\equiv\alpha-a-c+d\equiv\alpha-a\equiv-\alpha-b+c\mod(3).$$

The first set of these equalities implies that $d = -4a$, and the second implies that a must be divisible by 3, and that $\alpha = 0$. Therefore, the invariants in $\widetilde{M}$ are $\mathbb{Z}$, generated by $(0, -3, -3, -3, 12)$. This also proves that the map

$$\mathbb{Z} = H^0(GL_2(\mathbb{Z}/3), \widetilde{M}) \to H^0(GL_2(\mathbb{Z}/3), \mathbb{Z}) = \mathbb{Z},$$

induced by projection onto the last coordinate, is multiplication by 12. Hence, (B.6) identifies the cohomology group $H^1(GL_2(\mathbb{Z}/3), \bar{M})$ with $\mathbb{Z}/12$. □

Proposition B.4. *For the module $\widetilde{M}$ defined in the proof of Proposition B.3 above, we have*

$$H^1(GL_2(\mathbb{Z}/3), \widetilde{M}) = 0.$$

Proof. In this proof we first consider $GL_2(\mathbb{Z}/3)$ via a different extension from Lemma B.1 and B.2 above. Namely, we look at the center $Z = C_2$ of $GL_2(\mathbb{Z}/3)$ generated by $x^2 = y^2$, and hence

$$1 \to Z \to GL_2(\mathbb{Z}/3) \to PGL_2(\mathbb{Z}/3) \to 1.$$

Note that Z acts trivially on $\widetilde{M}$, and therefore, as $\widetilde{M}$ has no 2-torsion, $H^1(Z, \widetilde{M})$ is zero. Consequently, from the LHSSS associated to this extension

$$H^p(PGL_2(\mathbb{Z}/3), H^q(Z, \widetilde{M})) \Rightarrow H^{p+q}(GL_2(\mathbb{Z}/3), \widetilde{M}),$$

we see that $H^1(GL_2(\mathbb{Z}/3), \widetilde{M}) = H^1(PGL_2(\mathbb{Z}/3), \widetilde{M})$. It is this latter group that we will compute by a couple of more applications of the LHSSS.

The (reductions modulo the center Z of the) elements x, y generate a normal subgroup of $PGL_2(\mathbb{Z}/3)$ isomorphic to $C_2 \times C_2$, and along with z, the three elements generate a subgroup of order 12 that we will therefore call G_{12}. This group G_{12} is also normal in $GL_2(\mathbb{Z}/3)$.

First, let us compute the $C_2 \times C_2$ cohomology; to do so, use the LHSSS for the extension

$$1 \to C_2^x \to C_2 \times C_2 \to C_2^y \to 1,$$

where the subgroup is the C_2 generated by x and the quotient by y. From (B.5), we see that

$$H^0(C_2^x, \widetilde{M}) = \widetilde{M}^x = \ker(1 - x) = \mathbb{Z}/3 \oplus \mathbb{Z} \oplus \mathbb{Z}$$

generated by $(1, 0, 0, 0, 0), (0, 1, 0, 0, -2)$, and $(0, 0, 1, 1, -2)$. To compute H^1, note that $\ker(1 + x)$ is $\mathbb{Z} \oplus \mathbb{Z}$ generated by $(0, 1, 0, 0, 0)$ and $(-1, 0, 1, -1, 0)$, by (B.5). Both of these are in the image of $(1 - x)$; the first is $(1 - x)(0, 0, 0, 0, 1)$, and the second is $(1 - x)(0, 0, 0, -1, 1)$, implying that $H^1(C_2^x, \widetilde{M}) = 0$. Therefore, for $s = 0$ and $s = 1$, we have that

$$H^s(C_2 \times C_2, \widetilde{M}) = H^s(C_2^y, \widetilde{M}^x).$$

Denote an element of $\widetilde{M}^x$ as (α, a, b), in the basis given in the previous paragraph. This element corresponds to $(\alpha, a, b, b, -2a - 2b) \in \widetilde{M}$. Then, by (B.5), we have that

$$y(\alpha, a, b) = (\alpha, b, a).$$

By computing the kernel of $(1 - y)$ on $\widetilde{M}^x$ we get that

$$H^0(C_2 \times C_2, \widetilde{M}) = \widetilde{M}^{x,y} = \mathbb{Z}/3 \oplus \mathbb{Z},$$

generated by the elements $(1, 0, 0)$ and $(0, 1, 1)$ of $\widetilde{M}^x$. The kernel of the norm $(1 + y)$ on $\widetilde{M}^x$ is $\mathbb{Z}$, but it is contained in the image of $(1 - x)$, hence $H^1(C_2 \times C_2, \widetilde{M}) = 0$.

Now we turn to computing the G_{12} cohomology, for which by the above we have that if $s = 0$ or $s = 1$,

$$H^s(G_{12}, \widetilde{M}) = H^s(C_3, \widetilde{M}^{x,y}),$$

where the C_3 is generated by z, or more precisely, the image of z under a suitable quotient. Similarly as above, since $\widetilde{M}^{x,y} = \mathbb{Z}/3 \oplus \mathbb{Z}$, denote an element of it by (α, a). Then (α, a) corresponds to the element $(\alpha, a, a, a, -4a) \in \widetilde{M}$, so by (B.5), z acts on it by

$$z(\alpha, a) = (\alpha - a, a).$$

The kernel of $(1 - z)$ on $\widetilde{M}^{x,y}$ therefore is

$$\mathbb{Z}/3 \oplus \mathbb{Z} = H^0(G_{12}, \widetilde{M}) = \widetilde{M}^{x,y,z},$$

consisting of elements of the form $(\alpha, 3a)$, whereas the kernel of the norm $(1 + z + z^2)$ is $(\mathbb{Z}/3)$ generated by $(1, 0)$. However, $\ker(1 + z + z^2) = \operatorname{im}(1 - z)$, hence $H^1(G_{12}, \widetilde{M}) = 0$.

By the above discussion, we get that for $s = 0, 1$,

$$H^s(PGL_2(\mathbb{Z}/3), \widetilde{M}) = H^s(C_2^\sigma, \widetilde{M}^{x,y,z}).$$

To compute what this is, note that a generic element of $\widetilde{M}^{x,y,z}$ is $(\alpha, 3a)$, corresponding to $(\alpha, 3a, 3a, 3a, -12a) \in \widetilde{M}$, and by (B.5),

$$\sigma(\alpha, 3a) = (-\alpha, 3a).$$

Again we have that $\ker(1 + \sigma) = \operatorname{im}(1 - \sigma)$, hence $H^1(C_2^\sigma, \widetilde{M}^{x,y,z})$ is trivial, implying the claimed result. ☐

Proposition B.5. *The connecting homomorphism* $\partial_1 : H^1(GL_2(\mathbb{Z}/3), \bar{M}) \xrightarrow{\partial_1} H^2(GL_2(\mathbb{Z}/3), \mathbb{Z}/2)$ *induced by the short exact sequence in (B.4) above is the surjection* $\mathbb{Z}/12 \to \mathbb{Z}/2$. *Therefore, we have an exact sequence*

$$0 \to \mathbb{Z}/2 \to H^1(GL_2(\mathbb{Z}/3), M) \to \mathbb{Z}/6 \to 0.$$

The proof will consist of explicitly computing this connecting homomorphism, by its very definition. Therefore, let us first recall the construction of connecting homomorphisms in group cohomology. Let G be a finite group, and let A be a G-module. Then the group cohomology $H^*(G, A)$ can be computed as the cohomology of the cochain complex $C^\bullet(G, A)$, where $C^n(A) = \operatorname{Map}(G^{\times n}, A)$ is the abelian group of *set* maps from the n-fold direct product $G^{\times n}$ to A, and the differential $d_A^n : C^n(A) \to C^{n+1}(A)$ takes an n-cochain φ to the cochain $d_A^n \varphi$ defined by

$$(d_A^n \varphi)(g_0, g_1, \ldots, g_n) = g_0 \varphi(g_1, \ldots, g_n) + \sum_{i=1}^{n} (-1)^i \varphi(g_0 \ldots, g_{i-1} g_i, \ldots) + (-1)^{n+1} \varphi(g_0, \ldots, g_{n-1}).$$

(See, for example, [Wei94, 6.5].) For example $C^0(A) = A$, and $d_A^0(a)(g) = ga - a$. Let $Z^n(A)$ and $B^n(A)$ denote the groups of n-cocycles (i.e. the kernel of d_A^n) and n-coboundaries (i.e. the image of d_A^{n-1}), respectively.

Now suppose we have a short exact sequence

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$$

of G-modules. It is easily seen that then for each $n \geq 0$,

$$0 \to C^n(A) \to C^n(B) \to C^n(C) \to 0$$

is also exact. Then, an application of the Snake lemma to the map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^n(A) & \longrightarrow & C^n(B) & \longrightarrow & C^n(C) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_A^n} & & \downarrow{\scriptstyle d_B^n} & & \downarrow{\scriptstyle d_C^n} & & \\ 0 & \longrightarrow & C^{n+1}(A) & \longrightarrow & C^{n+1}(B) & \longrightarrow & C^{n+1}(C) & \longrightarrow & 0 \end{array}$$

is what gives the connecting homomorphism $\partial_n : H^n(G, C) \to H^{n+1}(G, A)$. Explicitly, let $\bar{\varphi}$ be an element of $H^n(G, C) = Z^n(C)/B^n(C)$, and let $\varphi \in Z^n(C) \subseteq C^n(C)$ be any representative of $\bar{\varphi}$.

(The connecting map ∂_n does not depend on this choice.) Since the map $\pi : B \to C$ is surjective, there exists a lift $\widetilde{\varphi} : G^n \to B$ such that $\varphi = \pi \circ \widetilde{\varphi}$. This $\widetilde{\varphi}$ need not be a cocycle, but $\psi = d_B^n \widetilde{\varphi}$ surely is. However, since φ is a cocycle, $\pi \circ \psi$ is identically zero, hence there is a (unique) lift $\widetilde{\psi} : G^{n+1} \to A$ of ψ, i.e. such that $\psi = \iota \circ \widetilde{\psi}$. This $\widetilde{\psi}$ is also a cocycle, by the injectivity of ι, and $\partial_n(\bar{\varphi})$ is defined to be the class of $\widetilde{\psi}$ in $H^{n+1}(G, A)$. In particular, $\partial_n(\bar{\varphi})$ is zero if and only if $\widetilde{\psi}$ is a coboundary.

Proof of Proposition B.5. To start, let us understand the cocycle representatives of

$$H^1(GL_2(\mathbb{Z}/3), \bar{M}) \cong \mathbb{Z}/12.$$

In other words, for we are looking for set maps $f : G \to \bar{M}$, which are cocycles, i.e such that $f(gh) = gf(h) + f(g)$, and which are not coboundaries. We can do this by computing the boundary homomorphism from (B.6)

$$\partial_0 : H^0(GL_2(\mathbb{Z}/3), \mathbb{Z}) = \mathbb{Z} \to H^1(GL_2(\mathbb{Z}/3), \bar{M}),$$

which we saw is surjective. It came from the exact sequence

$$0 \to \bar{M} \to \widetilde{M} \to \mathbb{Z} \to 0.$$

Indeed, let $d \in \mathbb{Z}$ be an arbitrary element, i.e. a zero cocyle; then, by the procedure described above, we lift d arbitrarily to $\widetilde{M}$, for example as $\widetilde{d} = (0, 0, 0, 0, d) \in \widetilde{M} = C^0(\widetilde{M})$. Next we take $\psi_d = d^0_{\widetilde{M}} : G \to \widetilde{M}$; by definition, it is the set function

$$\psi_d(g) = g\widetilde{d} - \widetilde{d}.$$

Since $\psi_d(g)$ maps to zero in $\mathbb{Z}$ for every $g \in GL_2(\mathbb{Z}/3)$, it is in image of $\bar{M}$. Call the resulting function $\widetilde{\psi}_d : G \to \bar{M}$. Then $\widetilde{\psi}_d$ represents the image of d in $H^1(GL_2(\mathbb{Z}/3), \bar{M})$ under ∂_0. In particular, unless d is a multiple of 12, there is no $m \in \bar{M}$ such that $\widetilde{\psi}_d$ is of the form $\widetilde{\psi}_d(g) = gm - m$.

Using (B.5), we can explicitly compute the values of $\widetilde{\psi}_d$ in $\bar{M}$. For example,

$$\text{(B.7)} \qquad \begin{array}{lll} \widetilde{\psi}_d(x) = (0, -d, 0, 0) & \widetilde{\psi}_d(z) = (0, 0, -d, 0) & \widetilde{\psi}_d(z^2) = (0, 0, 0, -d) \\ \widetilde{\psi}_d(y) = (d, 0, 0, -d) & \widetilde{\psi}_d(\sigma) = (0, -d, 0, 0). & \end{array}$$

Next, we compute the connecting homomorphism

$$\partial_1 : H^1(GL_2(\mathbb{Z}/3), \bar{M}) \to H^2(GL_2(\mathbb{Z}/3), \mathbb{Z}/2)$$

of the function $\widetilde{\psi}_d$ by the same procedure. To do so, we take an arbitrary lift of it to a set function $\rho_d : G \to M$; for example, choose ρ_d to be the function which is always zero in the first coordinate, i.e. the coordinate which corresponds to $\mathbb{Z}/2 \subset M$. (Recall, $\bar{M} = M/(\mathbb{Z}/2)$.) Next we apply the differential d^1_M to this ρ_d; we have that $d^1_M \rho_d : G \times G \to M$ is given by

$$(d^1_M \rho_d)(g, h) = g\rho_d(h) - \rho_d(gh) + \rho_d(g).$$

For all $g, h \in G$, $d^1_M \rho_d$ maps to zero in $\bar{M}$, hence is in the image of $\mathbb{Z}/2 \to M$. Call the resulting function $\xi_d : G \times G \to \mathbb{Z}/2$. This ξ_d represents the image of $\widetilde{\psi}_d$ under ∂_1.

Notice that we can simplify the formula for $\xi_d(g, h)$ because we chose $\rho_d(g) = (0, \widetilde{\psi}_d)$. Namely, we have

$$(d^1_M \rho_d)(g, h) = g\rho_d(h) - \rho_d(gh) + \rho_d(g) = g(0, \widetilde{\psi}_d(h)) - (0, \widetilde{\psi}_d(gh)) + (0, \widetilde{\psi}_d(g));$$

from where it follows that $\xi_d(g, h)$ equals the first (i.e. the $\mathbb{Z}/2$-) coordinate of the element $g(0, \widetilde{\psi}_d(h)) \in M$. Let us use this, along with results from (B.7) (and the action described in

(B.3)), to compute some values of ξ_d:

$$
\begin{array}{lllll}
\xi_d(z,-)=0 & & & & \\
\xi_d(x,x)=d & \xi_d(x,y)=d & \xi_d(x,z)=0 & \xi_d(x,\sigma)=d & \xi_d(\sigma z,\sigma)=d \\
\xi_d(y,x)=0 & \xi_d(y,y)=d & \xi_d(y,z)=d & \xi_d(y,\sigma)=0 & \xi_d(xy,z^2)=0 \\
\xi_d(\sigma,x)=0 & \xi_d(\sigma,y)=0 & \xi_d(\sigma,z)=d & \xi_d(\sigma,\sigma)=0 & \xi_d(xy,\sigma)=d.
\end{array}
\tag{B.8}
$$

The question is whether $\xi=\xi_1$ is a coboundary, i.e. whether there exists a set function $\lambda : G \to \mathbb{Z}/2$ such that

$$\xi(g,h)=g\lambda(h)-\lambda(gh)+\lambda(g).$$

Since the action on $\mathbb{Z}/2$ must be trivial, and $-1=1$ in $\mathbb{Z}/2$, this condition is the same as

$$\xi(g,h)=\lambda(g)+\lambda(h)+\lambda(gh).$$

We will show that such λ cannot exist, therefore showing that ξ is not a coboundary, i.e. that it must represent the non-trivial cocycle in $H^2(GL_2(\mathbb{Z}/3),\mathbb{Z}/2)=\mathbb{Z}/2$, meaning that ∂_1 is surjective.

Assume the contrary, i.e. assume that such λ exists. Then we would have for all $g\in GL_2(\mathbb{Z}/3)$ that $\lambda(g^2)=\xi(g,g)$. In particular, we get (using (B.8)) that $\lambda(z^2)=0=\lambda(z)$, that $\lambda(1)=\lambda(\sigma^2)=0$, and $\lambda(x^2)=\lambda(y^2)=1$. Next, we have

$$\xi(x,z)=0=\lambda(x)+\lambda(z)+\lambda(xz)$$
$$\xi(z,y^3)=0=\lambda(z)+\lambda(y^3)+\lambda(zy^3);$$

the relation $xz=zy^3$ therefore implies that (after adding these two equalities)

$$\lambda(x)+\lambda(y^3)=0.$$

On the other hand, we also have

$$\xi(y,y^2)=0=1+\lambda(y)+\lambda(y^3),$$

so we conclude that

$$\lambda(x)+\lambda(y)=1.$$

However,

$$\xi(x,y)=1=\lambda(x)+\lambda(y)+\lambda(xy)$$

then implies that $\lambda(xy)=0$.

Now we will use the relation $\sigma z\sigma z=xy$ in $GL_2(\mathbb{Z}/3)$. On the one hand, we have

$$\xi(\sigma z,\sigma z)=\lambda(\sigma z\sigma z)=\lambda(xy)=0,$$

but on the other, we can directly compute $\xi(\sigma z,\sigma z)$ to be 1 (see the next paragraph for details). This is a contradiction, hence such λ does not exist, hence ξ is not a coboundary, and our connecting map must be a surjection as claimed.

Here is how to compute $\xi(\sigma z,\sigma z)$. First of all we compute $\widetilde{\psi}_1(\sigma z)$; by construction this is obtained by looking at the element $\widetilde{1}=(0,0,0,0,1)\in\widetilde{M}$, and acting on it by σz. We have by (B.5) that

$$\sigma z(\widetilde{1})=\sigma(0,0,-1,0,1)=(1,0,-1,0,1),$$

hence $\widetilde{\psi}_1(\sigma z)=(1,0,-1,0)\in\bar{M}$. Next, by the argument above (B.8), we take the element $(0,1,0,-1,0)\in M$, act on it by σz to get (by (B.3))

$$\sigma z(0,1,0,-1,0)=\sigma(0,1,0,1,-1)=(1,0,0,1,-1),$$

and the value of $\xi(\sigma z,\sigma z)$ is the first coordinate in this result, i.e. 1. □

Appendix C. Derived functors of the symmetric square

The purpose of this appendix is to prove the necessary auxiliary results on symmetric squares of cosimplicial abelian groups. The main lemma (Lemma C.3 below) is by no means new; it is stated (without proof) as a special case of a more general result in [Goe90, Ch. II, Prop. 3.7], but for the convenience of the reader, we include proofs here.

Definition C.1. Let A be an abelian group. We write $\mathrm{Sym}^2(A) = (A \otimes A)^{C_2}$ be the C_2-invariants in the tensor square $A \otimes A$ and we let $\mathrm{Sym}_2(A) = (A \otimes A)_{C_2}$ be the coinvariants. We also let $\widetilde{\mathrm{Sym}}_2(A)$ denote the C_2-coinvariants in $(A \otimes A) \otimes \mathbb{Z}_\epsilon$ where the first factor is given the permutation action and $\mathbb{Z}_\epsilon$ is the sign representation.

We will need some tools for comparing these different functors. Let V be any finite-dimensional $\mathbb{F}_2$-vector space. Then, we have a *norm map*

$$\mathrm{Sym}_2(V) \to \mathrm{Sym}^2(V),$$

from coinvariants to invariants. The composite

$$V \otimes V \to \mathrm{Sym}_2(V) \simeq (V \otimes V)_{C_2} \xrightarrow{N} (V \otimes V)^{C_2}$$

sends $v \otimes w \mapsto v \otimes w + w \otimes v$.

Proposition C.2. *For any $\mathbb{F}_2$-vector space V, there is a natural exact sequence*

$$0 \to V \to \mathrm{Sym}_2(V) \xrightarrow{N} \mathrm{Sym}^2(V) \to V \to 0, \tag{C.1}$$

where the first map is the "Frobenius" that sends $v \in V$ to the image of $v \otimes v \in V \otimes V$ in $\mathrm{Sym}_2(V)$.

The identification of the cokernel of the norm map with V is similar: it sends $v \in V$ to the image of $v \otimes v \in \mathrm{Sym}^2(V)$ in the cokernel.

Proof. It is easy to see that we have a natural sequence $V \to \mathrm{Sym}_2(V) \to \mathrm{Sym}^2(V)$, and a natural map from V to the *cokernel* of the norm map given by $v \mapsto v \otimes v$. In other words, we get maps $\phi_1 : V \to \ker(N), \phi_2 : V \to \mathrm{coker}(N)$.

It remains to verify that ϕ_1, ϕ_2 are isomorphisms. Without loss of generality, we may assume that $\dim_{\mathbb{F}_2} V < \infty$, since all terms involved commute with filtered colimits in V. Let $\{e_1, \dots, e_n\}$ be a basis for V. Then $\mathrm{Sym}_2(V)$ has basis vectors $\{e_i e_j\}_{i \le j}$ and $\mathrm{Sym}^2(V)$ has basis vectors given by $\{e_i \otimes e_j + e_j \otimes e_i\}_{i<j} \cup \{e_i \otimes e_i\}$. Under the norm map, the subspace of $\mathrm{Sym}_2(V)$ spanned by the $\{e_i e_j\}_{i<j}$ is mapped isomorphically to the subspace of $\mathrm{Sym}^2(V)$ spanned by $\{e_i \otimes e_j + e_j \otimes e_i\}_{i<j}$, while the subspace spanned by the $\left\{e_i^2\right\}$ is annihilated. The cokernel of the norm map has a basis given by the images of the $\{e_i \otimes e_i\}$, and so the map from V to it is an isomorphism. Similarly, we see that ϕ_1 is an isomorphism. □

Our basic lemma is the following.

Lemma C.3. *Let $r \geq 3$, and let $A^\bullet$ be a cosimplicial abelian group such that $H^i(A^\bullet) = \mathbb{Z}$ if $i = r$ and 0 otherwise. Then we have*

$$H^{2r}(\mathrm{Sym}_2(A^\bullet \otimes_{\mathbb{Z}} \mathbb{F}_2)) \simeq \mathbb{F}_2.$$

generated by the square. Moreover, $H^i(\mathrm{Sym}_2(A^\bullet \otimes_{\mathbb{Z}} \mathbb{F}_2)) = 0$ for $i > 2r$.

If one works modulo 2 (and it is easy to see that 2 is the relevant prime here), a more general result, giving the computation of the homotopy groups of the symmetric square (stated in terms of the dual simplicial $\mathbb{F}_2$-vector space) is given in [Goe90, Prop. 3.7, Ch II].

Proof. We will reduce Lemma C.3 to a known statement about the homology of the symmetric squares of spheres. Indeed, let $\overline{A^\bullet} = A^\bullet \otimes_{\mathbb{Z}} \mathbb{F}_2$. It suffices now to compute the *homology* of the *simplicial* $\mathbb{F}_2$-vector space

$$\mathrm{Sym}^2(\overline{A}_\bullet^\vee) \simeq \mathrm{Sym}_2(\overline{A}^\bullet)^\vee,$$

and show that $H_{2r}(\mathrm{Sym}^2(\overline{A}_\bullet^\vee)) \simeq \mathbb{F}_2$. So, in general, our problem is equivalent to the following: given a simplicial $\mathbb{F}_2$-vector space $B_\bullet$ with $H_r(B_\bullet) \simeq \mathbb{F}_2$ and $H_s(B_\bullet) = 0$ for $s \neq r$, we need to show that

$$H_{2r}(\mathrm{Sym}^2 B_\bullet) \simeq \mathbb{F}_2, \quad H_i(\mathrm{Sym}^2 B_\bullet) = 0 \text{ for } i > 2r. \tag{C.2}$$

In other words, we need to understand the nonabelian derived functors of the functor Sym^2 on vector spaces.

Unfortunately, (C.2) is phrased in terms of the functor Sym^2 rather than the more classically studied Sym_2. We can get around this using Proposition C.2. We consider the exact sequence of *simplicial* $\mathbb{F}_2$-vector spaces from (C.1),

$$0 \to B_\bullet \to \mathrm{Sym}_2(B_\bullet) \xrightarrow{N} \mathrm{Sym}^2(B_\bullet) \to B_\bullet \to 0,$$

and the induced exact sequence in chain complexes obtained by applying the Dold-Kan correspondence. But $B_\bullet$ has homology only in degree r. It follows that the norm map $\mathrm{Sym}_2(B_\bullet) \to \mathrm{Sym}^2(B_\bullet)$ induces an isomorphism on homology in degrees $2r$ and above. Therefore, it suffices to show that $H_{2r}(\mathrm{Sym}_2(B_\bullet)) \simeq \mathbb{F}_2$ and $H_i(\mathrm{Sym}_2(B_\bullet)) = 0$ for $i > 2r$.

For this, in turn, the choice of simplicial $\mathbb{F}_2$-vector space $B_\bullet$ is entirely irrelevant. In particular, if we choose a pointed simplicial set $X_\bullet$ modeling S^r, we may take $B_\bullet \simeq \mathbb{F}_2[X_\bullet]/\mathbb{F}_2[*]$. In this case, it follows that $\mathrm{Sym}_2(B_\bullet) \simeq \mathrm{Sym}_2(\mathbb{F}_2[X_\bullet]/\mathbb{F}_2[*])$, and we have a natural isomorphism

$$\mathrm{Sym}_2(B_\bullet) \simeq \mathrm{Sym}_2(\mathbb{F}_2[X_\bullet])/\mathbb{F}_2[X_\bullet], \tag{C.3}$$

because for any $\mathbb{F}_2$-vector space V with a one-dimensional subspace $\mathbb{F}_2\iota \subset V$, we have a natural isomorphism

$$\mathrm{Sym}_2(V/\mathbb{F}_2) \simeq \mathrm{Sym}_2(V)/V,$$

where $V \subset \mathrm{Sym}_2(V)$ via $v \mapsto v.\iota$.

Since $\mathbb{F}_2[X_\bullet]$ has homology concentrated in degrees $0, r$, it follows that we have a natural isomorphism

$$H_i(\mathrm{Sym}_2(\mathbb{F}_2[X_\bullet])) \simeq H_i(\mathrm{Sym}_2(B_\bullet)), \quad i \geq 2r.$$

But the simplicial vector space $\mathrm{Sym}_2(\mathbb{F}_2[X_\bullet])$ is the simplicial $\mathbb{F}_2$-homology of the simplicial set $\mathrm{Sym}_2(X_\bullet) \simeq (X_\bullet \times X_\bullet)_{C_2}$. This simplicial set, via geometric realization, is a model for the topological *symmetric square* $(S^r \times S^r)_{C_2}$; this follows as geometric realization for finite simplicial sets commutes with finite products and colimits. Our assertion is now, finally, reduced to the following Lemma. □

Lemma C.4. *Let S^r be the r-sphere. Then $H_{2r}(\mathrm{Sym}_2(S^r); \mathbb{F}_2) \simeq \mathbb{F}_2$.*

Proof. This is a special case of the calculation [Nak58] of the homology of the symmetric powers of spheres. □

Proposition C.5. *Let $t \geq 2$ and let $A^\bullet$ be a cosimplicial abelian group with $H^*(A^\bullet)$ concentrated in degree $* = t+1$ and $H^{t+1}(A^\bullet) = \mathbb{Z}$ generated by ι. Then:*

(1) *If t is even, $H^{2t+2}(\mathrm{Sym}_2 A^\bullet) \simeq \mathbb{Z}/2$, generated by ι^2.*
(2) *If t is odd, then $H^{2t+2}(\widetilde{\mathrm{Sym}_2} A^\bullet) \simeq \mathbb{Z}/2$, generated by ι^2.*

Proof. Consider first the case t even. In this case, we have maps of cosimplicial abelian groups

$$\mathrm{Sym}_2 A^\bullet \to A^\bullet \otimes A^\bullet \to \mathrm{Sym}_2 A^\bullet$$

where the first map is the norm map and the second map is projection. The composite is multiplication by two. Note that $H^{2t+2}(A^\bullet \otimes A^\bullet) \simeq \mathbb{Z}$, but since t is even, the C_2-action is the sign representation, so that the map $H^*(\mathrm{Sym}_2 A^\bullet) \to H^*(A^\bullet \otimes A^\bullet)$ must be the zero map as it lands in the C_2-invariants on cohomology. In particular, the cohomology of $\mathrm{Sym}_2(A^\bullet)$ is all annihilated by 2. By the universal coefficient theorem, it suffices to show that $H^{2t+2}(\mathrm{Sym}_2 A^\bullet \otimes_{\mathbb{Z}} \mathbb{Z}/2) \simeq \mathbb{F}_2$ and $H^k(\mathrm{Sym}_2 A^\bullet \otimes_{\mathbb{Z}} \mathbb{Z}/2) = 0$ for $k > 2t+2$, which we showed in Lemma C.3. Moreover, Lemma C.3 gives us our claim that ι^2 generates the cohomology.

Now suppose t is odd. Again, using the norm maps

$$\widetilde{\mathrm{Sym}}_2 A^\bullet \to A^\bullet \otimes A^\bullet \otimes \epsilon \to \widetilde{\mathrm{Sym}}_2 A^\bullet,$$

we find that the cohomology of $\widetilde{\mathrm{Sym}}_2 A^\bullet$ is annihilated by two. We note that $\widetilde{\mathrm{Sym}}_2 A^\bullet \otimes_{\mathbb{Z}} \mathbb{F}_2 \simeq \mathrm{Sym}_2 A^\bullet \otimes_{\mathbb{Z}} \mathbb{F}_2$ at the level of cosimplicial abelian groups. If we take the derived tensor product, we obtain in addition a copy of $A^\bullet \otimes_{\mathbb{Z}} \mathbb{F}_2$ (i.e., the 2-torsion in $\widetilde{\mathrm{Sym}}_2 A^\bullet$) in π_1 that does not contribute in the relevant dimension, so we may ignore it. In this case, we know that $H^k(\widetilde{\mathrm{Sym}}_2 A^\bullet \otimes_{\mathbb{Z}} \mathbb{F}_2) \simeq \mathbb{F}_2$ for $k = 2t+2$ and 0 for $k > 2t+2$, so that we can apply the universal coefficient theorem as in the previous case. □

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