Existence and Uniqueness Theorems for Fuzzy Differential Equations

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Abstract

Fuzzy differential equations are important tools to deal with dynamic systems in fuzzy environments. However, it is difficult to find the solutions to all fuzzy differential equations. In this paper, methods to solve general linear fuzzy differential equations and reducible fuzzy differential equations are given. Moreover, some other solvable fuzzy differential equations are discussed.

Keywords: Fuzzy variable; Liu process; Fuzzy differential equation

1 Introduction

Fuzziness is a basic type of uncertainty in real world. To describe a set without definite boundary, fuzzy set was initiated by Zadeh [16] in 1965, whose membership function indicates the degree of an element belonging to it. In order to measure a fuzzy event, a self-duality credibility measure was introduced by Liu and Liu [7] in 2002. Since an axiomatic foundation for credibility theory was constructed by Liu [8] and a sufficient and necessary condition is given by Li and Liu [6], credibility theory has been a perfect mathematical system. A survey of credibility theory can be found in Liu [9], and interested reader may consult the book [10].

In order to describe dynamic fuzzy phenomena, a fuzzy process was proposed by Liu [11]. Based on this fuzzy process, a fuzzy integral and a fuzzy differential formula were introduced by Liu [11]. For the importance and usefulness, they were named as Liu process, Liu integral and Liu formula, which are the counterparts of Brownian motion, Ito integral and Ito formula. Some researchers concerning Liu process have be done. You [15] extended Liu process, Liu integral and Liu formula to the case of multi-dimensional and higher-order Liu formula was given. Complex Liu process was studied by Qin [13]. Moreover, Liu process is Lipschitz continuous and has finite variation, which were proved in Dai[1]. Some researches on the application of Liu process were made. Credibilistic stopping problems was considered in Peng [12], an option pricing formula on asset modeled as a geometric Liu process was obtained by Qin and Li [14], a new model of credibilistic option pricing was derived by Gao [3], the problem of fuzzy portfolio selection was studied by Zhu [17].

For a deterministic differential equation, the easiest way to introduce fuzziness is to assume that the initial value is a fuzzy variable. Such an equation was first called fuzzy differential equation by Kandel and Byatt. Its solution does not require fuzzy calculus; we can use classical method to solve them. Such fuzzy differential equations can be considered as deterministic differential equations with a perturbed initial condition. Many researches have been made on this type of fuzzy differential equations, such as Kaleva [4], Kaleva [5] and Ding, Ma and Kandel [2] and etc. In 2008, by introducing the fuzziness in the differential equation via an additional fuzzy noise term, a new fuzzy differential equation was defined by Liu [11] as

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t. \tag{1.1}$$

Here C_t is a standard Liu process, and f and g are some given functions. Such a fuzzy differential equation is a type of differential equations driven by Liu process. Its solution is a fuzzy process. The fuzziness of X_t results, on the one hand, from the initial condition, and on the other hand, from the noise generated by Liu process. It is just the fuzzy counterpart of stochastic differential equation. In this paper, we will study the fuzzy differential equation proposed by Liu [11].

In Section 2 of this paper, some concepts and results of Liu process will be given as a preliminaries. Then in Section 3, the solutions of some special fuzzy differential equations will be discussed. a brief summary is given in Section 4.

2 Preliminaries

A fuzzy process $X(t, \theta)$ is defined as a dual function of t and θ such that $X(t^*, \theta)$ is a fuzzy variable for each t^*, where t is time and θ is a point in credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. For simplicity, we use the symbol X_t to replace $X(t, \theta)$ in the following section.

Definition 2.1 *(Liu [11]) A fuzzy process C_t is said to be a Liu process if*
(i) $C_0 = 0$,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{t+s} - C_s$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$ whose membership function is

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}, \ -\infty < x < +\infty.$$

The Liu process is said to be standard if $e = 0$ and $\sigma = 1$.

Definition 2.2 *(Liu [11]) Let X_t be a fuzzy process and let C_t be a standard Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\triangle = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then the Liu integral of X_t with respect to C_t is defined as follows,

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\triangle \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limitation exists almost surely and is a fuzzy variable.

In order to simplify the calculation of Liu integral, Liu formula was introduced by Liu [11], which corresponds to Ito formula.

Differential equations are used to describe the evolution of a dynamic systems. In what follows we will discuss the solutions to fuzzy differential equations (FDEs) proposed by Liu [11] .

3 Solutions to Some Special FDEs

It follows from the definition of FDE and solution to FDE that a solution is some function of the given Liu process C_t. When $g(t, X_t) = 0$, FDE (1.1) becomes an ordinary differential equation (ODE).

3.1 General Linear FDEs

Linear FDEs form a class of FDEs that can be solved explicitly. Consider general linear FDEs in one dimension

$$\mathrm{d}X_t = (\alpha_t + \beta_t X_t)\mathrm{d}t + (\gamma_t + \delta_t X_t)\mathrm{d}C_t \tag{3.1}$$

where $\alpha_t, \beta_t, \gamma_t, \delta_t$ are given fuzzy processes, and are continuous functions of t.

Definition 3.1 *The linear FDE is called linear equations with additive noise if δ_t in FDE* (3.1).

The equation gained its name since the process X_t is not directly involved in Liu integral.

Definition 3.2 *The linear FDE is called homogenous if $\alpha_t = 0$ and $\gamma_t = 0$ in FDE* (3.1).

Example 3.1 (Liu [11]) Let C_t be a standard Liu process. Then the FDE

$$\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t \tag{3.2}$$

has a solution $X_t = \exp(at + bC_t)$, where a, b are two constants. It is just a geometric Liu process.

The FDE in Example 3.1 is a homogenous FDE.

Example 3.2 (Liu [11]) Let C_t be a standard Liu process. Then the FDE

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t \tag{3.3}$$

has a solution $X_t = at + bC_t$, where a, b are two constants. It is just a Liu process.

The FDE in Example 3.2 is an autonomous FDE.

Example 3.3 Let C_t be a standard Liu process. Then $X_t = C_t/(1 + t)$ is a solution of

$$\mathrm{d}X_t = -\frac{X_t}{1+t}\mathrm{d}t + \frac{1}{1+t}\mathrm{d}C_t. \tag{3.4}$$

In fact, it follows from Liu formula that

$$\mathrm{d}X_t = -\frac{C_t}{(1+t)^2}\mathrm{d}t + \frac{1}{1+t}\mathrm{d}C_t$$

which is equivalent to FDE (3.4).

Example 3.4 (Liu process on the unit circle, Liu [11]) Let C_t be a standard Liu process, and $\boldsymbol{h}(t,x) = e^{ix} = (\cos x, \sin x)$ for $x \in R$. Define $\boldsymbol{X}_t = g(t, C_t) = e^{iC_t} = (\cos C_t, \sin C_t)$. It follows from multi-dimensional Liu formula that

$$\begin{cases} \mathrm{d}X_{1t} = -\sin C_t \mathrm{d}C_t \\ \mathrm{d}X_{2t} = \cos C_t \mathrm{d}C_t. \end{cases}$$

Then the fuzzy process $\boldsymbol{X}_t = (X_{1t}, X_{2t})$, which we will call Liu process on the unit circle, is the solution of the FDEs

$$\begin{cases} \mathrm{d}X_{1t} = -X_{2t} \mathrm{d}C_t \\ \mathrm{d}X_{2t} = X_{1t} \mathrm{d}C_t. \end{cases} \tag{3.5}$$

Or, in matrix notation,

$$\mathrm{d}\boldsymbol{X}_t = K\boldsymbol{X}_t \mathrm{d}C_t, \quad \text{where } K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Example 3.5 (Liu process on the ellipse) A fuzzy process $\boldsymbol{X}_t = (X_{1t}, X_{2t}) = (a\cos C_t, b\sin C_t)$ is called Liu process on the ellipse if C_t is a standard Liu process, and a and b are two nonnegative constants. It follows from multi-dimensional Liu formula that

$$\begin{cases} \mathrm{d}X_{1t} = -a\sin C_t \mathrm{d}C_t \\ \mathrm{d}X_{2t} = b\cos C_t \mathrm{d}C_t. \end{cases}$$

Then the fuzzy process (X_{1t}, X_{2t}) is the solution of the FDEs

$$\begin{cases} \mathrm{d}X_{1t} = -\dfrac{a}{b} X_{2t} \mathrm{d}C_t \\ \mathrm{d}X_{2t} = \dfrac{b}{a} X_{1t} \mathrm{d}C_t. \end{cases} \tag{3.6}$$

Or, in matrix notation,

$$\mathrm{d}\boldsymbol{Y}_t = K\boldsymbol{Y}_t \mathrm{d}C_t, \quad \text{where } K = \begin{pmatrix} 0 & -a/b \\ b/a & 0 \end{pmatrix}.$$

Example 3.6 Let $(C_{1t}, C_{2t}, \cdots, C_{nt})$ be an n-dimensional standard Liu process, and X_t a fuzzy process defined by

$$\mathrm{d}X_t = u_t X_t \mathrm{d}t + X_t \sum_{k=1}^{n} \alpha_k \mathrm{d}C_{kt}, \tag{3.7}$$

where u_t is a absolutely integrable fuzzy process, α_k and r are constants for all k. We immediately have

$$\ln\left(\frac{X_t}{X_0}\right) = \int_0^t u_s \mathrm{d}s + \int_0^t \sum_{k=1}^{n} \alpha_k \mathrm{d}C_{ks}$$

which means

$$X_t = X_0 \exp\left(\int_0^t u_s \mathrm{d}s + \int_0^t \sum_{k=1}^{n} \alpha_k \mathrm{d}C_{kt}\right).$$

Example 3.7 Let C_t be a standard Liu process. Suppose a fuzzy process X_t is defined by

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma \mathrm{d}C_t \tag{3.8}$$

where μ and σ are constants. Multiplying both sides of (3.8) by the integrating factor $\exp(-\mu t)$, we have

$$\exp(-\mu t)\mathrm{d}X_t - \mu\exp(-\mu t)X_t\mathrm{d}t = \sigma\exp(-\mu t)\mathrm{d}C_t$$

i.e.

$$\mathrm{d}\left(\exp(-\mu t)X_t\right) = \sigma\exp(-\mu t)\mathrm{d}C_t.$$

It follows from integration by parts that

$$\exp(-\mu t)X_t = X_0 + \sigma\exp(-\mu t)C_t + \sigma\mu\int_0^t \exp(-\mu s)C_s\mathrm{d}s$$

i.e.

$$X_t = \exp(\mu t)\left(X_0 + \sigma\mu\int_0^t \exp(-\mu s)C_s\mathrm{d}s\right) + \sigma C_t.$$

Example 3.8 Let C_t be a standard Liu process, and u_t, v_t fuzzy processes. Suppose fuzzy process X_t is defined by

$$\mathrm{d}X_t = u_t\mathrm{d}t + v_tX_t\mathrm{d}C_t. \tag{3.9}$$

Multiplying both sides of (3.1.9) by the integrating factor $\exp(-v_tC_t)$, we have

$$\exp(-v_tC_t)\mathrm{d}X_t - v_tX_t\exp(-v_tC_t)\mathrm{d}C_t = u_t\exp(-v_tC_t)\mathrm{d}t$$

i.e.

$$\mathrm{d}\left(\exp(-v_tC_t)X_t\right) = u_t\exp(-v_tC_t)\mathrm{d}t.$$

Thus

$$X_t\exp(-v_tC_t) = X_0 + \int_0^t u_s\exp(-v_sC_s)\mathrm{d}s,$$

i.e.

$$X_t = \exp(v_tC_t)\left(X_0 + \int_0^t u_s\exp(-v_sC_s)\mathrm{d}s\right).$$

Next, we will discuss the solution to a general linear FDE.

Solutions to General Linear FDE

Firstly, if PDE (3.1) degenerates to

$$\mathrm{d}X_t = \beta_tX_t\mathrm{d}t + \delta_tX_t\mathrm{d}C_t, \tag{3.10}$$

where β_t, δ_t are given fuzzy processes, and are continuous functions of t. Then

$$\ln\frac{X_t}{X_0} = \int_0^t \beta_s\mathrm{d}s + \int_0^t \delta_s\mathrm{d}C_s$$

i.e.

$$X_t = X_0\exp\left(\int_0^t \beta_s\mathrm{d}s + \int_0^t \delta_s\mathrm{d}C_s\right).$$

To find the solution of (3.1), let

$$X_t = U_tV_t \tag{3.11}$$

where $\mathrm{d}U_t = \beta_t U_t \mathrm{d}t + \delta_t U_t \mathrm{d}C_t$, and $\mathrm{d}V_t = a_t \mathrm{d}t + b_t \mathrm{d}C_t$.

Set $U_0 = 1$ and $V_0 = X_0$. Taking the differentials of both sides of (3.11), we have

$$\mathrm{d}X_t = U_t \mathrm{d}V_t + V_t \mathrm{d}U_t$$

by using multi-dimensional Liu formula, i.e.

$$\mathrm{d}X_t = (U_t a_t + V_t \beta_t U_t)\mathrm{d}t + (U_t b_t + V_t \delta_t U_t)\mathrm{d}C_t. \tag{3.12}$$

Comparing (3.12) with (3.1), we can choose coefficients a_t and b_t such that $X_t = U_t V_t$. The desired coefficients satisfy equations

$$U_t a_t = \alpha_t, \text{ and } U_t b_t = \gamma_t.$$

It follows from the solution of (3.10) that

$$U_t = \exp\left(\int_0^t \beta_s \mathrm{d}s + \int_0^t \delta_s \mathrm{d}C_s\right).$$

Then X_t is

$$X_t = U_t\left(X_0 + \int_0^t \frac{\alpha_s}{U_s}\mathrm{d}s + \int_0^t \frac{\gamma_s}{U_s}\mathrm{d}C_s\right).$$

Example 3.9 Let C_t be a standard Liu process. Suppose a fuzzy process X_t is defined by

$$\mathrm{d}X_t = \frac{b - X_t}{1-t}\mathrm{d}t + \mathrm{d}C_t, \quad 0 \le t < 1, X_0 = a. \tag{3.13}$$

Then in a same way above, we can deduce that

$$X_t = a(1-t) + bt + (1-t)\int_0^t \frac{\mathrm{d}C_s}{1-s}$$

is the solution of FDE (3.1.13).

3.2 Reducible FDE

For certain FDE, the solution can be found by performing a substitution (change of variables) which reduces the given FDE to a linear equation.

Definition 3.3 *A FDE*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

is called reducible if there exists a substitution (change of variables) $Y_t = U(t, X_t)$ *such that* (3.2.1) *deduces to a linear FDE*

$$\mathrm{d}Y_t = (\alpha_t + \beta_t Y_t)\mathrm{d}t + (\gamma_t + \delta_t Y_t)\mathrm{d}C_t,$$

where $\alpha_t, \beta_t, \gamma_t, \delta_t$ *are chosen as fuzzy processes satisfying the conditions*

$$\begin{cases} \alpha_t + \beta_t U = \dfrac{\partial U}{\partial t} + \dfrac{\partial U}{\partial x} f \\ \gamma_t + \delta_t U = \dfrac{\partial U}{\partial x} g. \end{cases}$$

Example 3.10 Let C_t be a standard Liu process. Suppose a fuzzy process X_t is defined by

$$\mathrm{d}X_t = rX_t(K - X_t)\mathrm{d}t + \beta X_t \mathrm{d}C_t,\ X_0 = x > 0 \tag{3.15}$$

where K is a positive constant and $r, \beta \in R$.

Let $Y_t = (t+1)/X_t$. Then $Y_0 = x^{-1}$ and FDE (3.15) reduces to

$$\mathrm{d}Y_t = \left(r(t+1) + \left(\frac{1}{t+1} - rk\right) Y_t\right) \mathrm{d}t - \beta Y_t \mathrm{d}C_t.$$

Thus,

$$Y_t = (t+1)\exp(-rKt - \beta C_t)\left(Y_0 + \int_0^t \frac{r}{\exp(-rKs - \beta C_s)}\mathrm{d}s\right),$$

i.e.

$$X_t = \frac{\exp(rKt + \beta C_t)}{x^{-1} + r\int_0^t(\exp(rKs + \beta C_s))\mathrm{d}s}.$$

Example 3.11 Let C_t be a standard Liu process. Suppose a fuzzy process X_t is defined by

$$\mathrm{d}X_t = k(a - \ln X_t)X_t \mathrm{d}t + bX_t \mathrm{d}C_t,\ X_0 = x > 0 \tag{3.16}$$

where k, a, b are positive constants. Letting $Y_t = \ln X_t$, FDE (3.2.3) is transformed as

$$\mathrm{d}Y_t = k(a - Y_t)\mathrm{d}t + b\mathrm{d}C_t.$$

Thus

$$Y_t = e^{-kt}\ln x + a + be^{-kt}\int_0^t e^{ks}\mathrm{d}C_s,$$

i.e.

$$X_t = \exp\left(e^{-kt}\ln x + a + be^{-kt}\int_0^t e^{ks}\mathrm{d}C_s\right).$$

3.3 Other solvable FDEs

In this section, some other solvable FDEs are discussed.

Type I:

$$\mathrm{d}X_t = u_t \mathrm{d}t + f(C_t)\mathrm{d}C_t, \tag{3.17}$$

where C_t is a standard Liu process, u_t a fuzzy process, and $f(x)$ is a absolutely integrable and continuous function. It is obtained easily that

$$X_t = X_0 + \int_0^t u_s \mathrm{d}s + \int_0^t f(C_s)\mathrm{d}C_s$$

i.e.

$$X_t = X_0 + \int_0^t u_s \mathrm{d}s + \int_0^{C_t} f(s)\mathrm{d}s.$$

Type II:

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t)X_t \mathrm{d}C_t, \quad X_0 = x \tag{3.18}$$

where $f : R \times R \to R$ and $g : R \to R$ are given continuous functions.

To obtain the solution of (3.18), we define an integrating factor

$$U_t = U_t(\theta) = \exp\left(-\int_0^t g(s)\mathrm{d}C_s\right).$$

In a similar method used in Example 3.1.8, FDE (3.18) is equivalent to

$$\mathrm{d}(U_t X_t) = U_t f(t, X_t)\mathrm{d}t.$$

Letting $Y_t(\theta) = U_t(\theta)X_t(\theta)$, FDE (3.18) gets the form

$$\mathrm{d}Y_t(\theta) = U_t(\theta)f(t, U_t^{-1}(\theta)Y_t(\theta))\mathrm{d}t, \quad Y_0 = x. \tag{3.19}$$

Note that (3.19) is a deterministic differential equation for each $\theta \in \Theta$. Then the solution to FDE (3.18) can be found.

Example 3.3.1 Let C_t be a standard Liu process, u_t a fuzzy process. Suppose fuzzy process X_t is defined by

$$\mathrm{d}X_t = u_t X_t \mathrm{d}t + f(C_t)X_t \mathrm{d}C_t, \tag{3.20}$$

where $f(x)$ is a absolutely integrable and continuous function. Then

$$\ln\frac{X_t}{X_0} = \int_0^t u\mathrm{d}t + \int_0^t f(C_s)\mathrm{d}C_s$$

i.e.

$$X_t = X_0 \exp\left(\int_0^t u\mathrm{d}t + \int_0^{C_t} f(s)\mathrm{d}s\right).$$

Example 3.3.2 Let C_t be a standard Liu process. Suppose fuzzy process X_t is defined by

$$\mathrm{d}X_t = \frac{1}{X_t}\mathrm{d}t + aX_t\mathrm{d}C_t, \quad X_0 = x > 0 \tag{3.21}$$

where a is a constant. Letting $U_t = \exp(-aC_t)$ and $Y_t = U_t X_t$, we get $X_t = \exp(aC_t)Y_t, Y_0 = x$ and

$$\mathrm{d}Y_t = \frac{\exp(-2aC_t)}{Y_t}\mathrm{d}t.$$

Then

$$X_t = \pm\exp(aC_t)\left(x^2 + 2\int_0^t \exp(-2aC_s)\mathrm{d}s\right)^{\frac{1}{2}}.$$

Example 3.3.3 Let C_t be a standard Liu process Suppose fuzzy process X_t is defined by

$$\mathrm{d}X_t = X_t^b\mathrm{d}t + aX_t\mathrm{d}C_t, \quad X_0 = x > 0 \tag{3.22}$$

where a and b are constants, $b \le 1$. Letting $U_t = \exp(-aC_t)$ and $Y_t = U_t X_t$, we get $X_t = \exp(aC_t)Y_t, Y_0 = x$ and

$$\mathrm{d}Y_t = \frac{\exp(-2aC_t)}{Y_t}\mathrm{d}t.$$

Next, we will discuss the solution of (3.3.5) in three cases.

Case 1: If $b = 0$, then

$$X_t = \exp(aC_t)\left(x + \int_0^t \exp(-aC_s)\mathrm{d}s\right).$$

Case 2: If $b = 1$, then

$$X_t = \pm x\exp(t + aC_t).$$

Case 3: Otherwise, if b is a negative odd number, the solution of (3.3.5) is

$$X_t = \pm\exp(aC_t)\left(x^{1-b} + (1-b)\int_0^t \exp(a(b-1)C_s)\mathrm{d}s\right)^{\frac{1}{1-b}}.$$

Otherwise,

$$X_t = \exp(aC_t)\left(x^{1-b} + (1-b)\int_0^t \exp(a(b-1)C_s)\mathrm{d}s\right)^{\frac{1}{1-b}}.$$

4 Conclusions

This paper was mainly to discuss the solutions of some special fuzzy differential equations based on Liu process. The methods were provided to solve general linear FDEs and reducible FDE.

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