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Brief paper

Local stability analysis using simulations and sum-of-squares programming☆

Ufuk Topcu [a], Andrew Packard [a,*], Peter Seiler [b]

[a] *Department of Mechanical Engineering, University of California, Berkeley, 94720-1740, USA*

[b] *Honeywell Technology Center, Minneapolis, MN, 55418, USA*

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ABSTRACT

The problem of computing bounds on the region-of-attraction for systems with polynomial vector fields is considered. Invariant subsets of the region-of-attraction are characterized as sublevel sets of Lyapunov functions. Finite-dimensional polynomial parametrizations for Lyapunov functions are used. A methodology utilizing information from simulations to generate Lyapunov function candidates satisfying necessary conditions for bilinear constraints is proposed. The suitability of Lyapunov function candidates is assessed solving linear sum-of-squares optimization problems. Qualified candidates are used to compute invariant subsets of the region-of-attraction and to initialize various bilinear search strategies for further optimization. We illustrate the method on small examples from the literature and several control oriented systems.

1. Introduction

The region-of-attraction (ROA) of a locally asymptotically stable equilibrium point is an invariant set such that all trajectories emanating from points in this set converge to the equilibrium point. Computing the exact ROA for nonlinear dynamics is very hard if not impossible. Therefore, researchers have focused on determining invariant subsets of the ROA. Among all other methods, those based on Lyapunov functions are dominant in the literature (Chesi, Garulli, Tesi, & Vicino, 2005; Chiang & Thorp, 1989; Davison & Kurak, 1971; Genesio, Tartaglia, & Vicino, 1985; Hachicho & Tibken, 2002; Papachristodoulou, 2005; Tan & Packard, 2006; Tibken, 2000; Tibken & Fan, 2006; Vannelli & Vidyasagar, 1985). These methods compute a Lyapunov function as a local stability certificate and sublevel sets of this Lyapunov function, in which the function decreases along the flow, provide invariant subsets of the ROA.

Using sum-of-squares (SOS) relaxations for polynomial nonnegativity (Parrilo, 2003), it is possible to search for polynomial Lyapunov functions for systems with polynomial and/or rational dynamics using semidefinite programming (Hachicho & Tibken, 2002; Papachristodoulou, 2005; Tan & Packard, 2006). Reliable and efficient solvers for linear semidefinite programs (SDPs) are available (Sturm, 1999). However, the SOS relaxation for the problem of computing invariant subsets of the ROA leads to bilinear matrix inequality (BMI) constraints. BMIs are nonconvex and bilinear SDPs, those with BMI constraints, are known to be NP-hard in general (Toker & Ozbay, 1995). Consequently, the state-of-the-art of the solvers for bilinear SDPs is far behind that for the linear ones. Recently PENBMI, a solver for bilinear SDPs, was introduced (Kočvara & Stingl, 2005) and subsequently used for computing invariant subsets of the ROA (Tan & Packard, 2006; Tibken & Fan, 2006). It is a local optimizer and its behavior (speed of convergence, quality of the local optimal point, etc.) depends on the point from which the optimization starts.

By contrast, simulating a nonlinear system of moderate size, except those governed by stiff differential equations, is computationally efficient. Therefore, extensive simulation is a tool used in real applications. Although the information from simulations is inconclusive, i.e., cannot be used to find provably invariant subsets of the ROA, it provides insight into the system behavior. For example, if, using Lyapunov arguments, a function certifies that a set $\mathcal{P}$ is in the ROA, then that function must be positive and decreasing on any solution trajectory initiating in $\mathcal{P}$. Using a finite number of points on finitely many convergent trajectories and a linear parametrization of the Lyapunov function V, those constraints become affine, and the feasible polytope (in V-coefficient space) is a convex outer bound on the set of coefficients of valid Lyapunov functions. It is intuitive that drawing samples from this set to seed the bilinear SDP solvers may

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* Corresponding author. Tel.: +1 510 642 6152; fax: +1 510 642 6163.
E-mail addresses: utopcu@jagger.me.berkeley.edu (U. Topcu), pack@jagger.me.berkeley.edu (A. Packard), peter.j.seiler@honeywell.com (P. Seiler).

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improve the performance of the solvers. In fact, if there are a large number of simulation trajectories, samples from the set often are suitable Lyapunov functions (without further optimization) themselves. Effectively, we are relaxing the bilinear problem (using a very specific system theoretic interpretation of the problem) to a linear problem, and the true feasible set is a subset of the linear problem's feasible set. Information from simulations is also used in Prokhorov and Feldkamp (1999) and Serpen (2005) for computing *approximate* Lyapunov functions.

Notation: For $x \in \mathcal{R}^n$, $x \succeq 0$ means that $x_k \geq 0$ for $k = 1, \ldots, n$. For $Q = Q^T \in \mathcal{R}^{n\times n}$, $Q \succeq 0$ $(Q \succ 0)$ means that $x^TQx \geq 0$ (≤ 0) for all $x \in \mathcal{R}^n$. $\mathbb{R}[x]$ represents the set of polynomials in x with real coefficients. The subset $\Sigma[x] := \{\pi \in \mathbb{R}[x] : \pi = \pi_1^2 + \pi_2^2 + \cdots + \pi_m^2, \pi_1, \ldots, \pi_m \in \mathbb{R}[x]\}$ of $\mathbb{R}[x]$ is the set of SOS polynomials. For $\pi \in \mathbb{R}[x]$, $\partial(\pi)$ denotes the degree of π. $\mathcal{C}^1$ denotes the space of continuously differentiable functions. We use the term "semidefinite programming" to mean optimization problems with affine objective function and *general* (not necessarily affine) matrix (semi)definiteness constraints. $\lhd$

2. Characterization of invariant subsets of the ROA and bilinear SOS problem

Consider the autonomous nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the state vector and $f : \mathcal{R}^n \to \mathcal{R}^n$ is such that $f(0) = 0$, i.e., the origin is an equilibrium point of (1), and f is locally Lipschitz. Let $\phi(\xi, t)$ denote the solution to (1) at time t with the initial condition $x(0) = \xi$. If the origin is asymptotically stable but not globally attractive, one often wants to know which trajectories converge to the origin as time approaches ∞. The region-of-attraction R_0 of the origin for the system (1) is $R_0 := \{\xi \in \mathcal{R}^n : \lim_{t\to\infty} \phi(\xi, t) = 0\}$. A modification of a similar result in Vidyasagar (1993) provides a characterization of invariant subsets of the ROA. For $\eta > 0$ and a function $V : \mathcal{R}^n \to \mathcal{R}$, define the η-sublevel set $\Omega_{V,\eta}$ of V as $\Omega_{V,\gamma} := \{x \in \mathcal{R}^n : V(x) \leq \eta\}$.

Lemma 1. *Let $\gamma \in \mathcal{R}$ be positive. If there exists a $\mathcal{C}^1$ function $V : \mathcal{R}^n \to \mathcal{R}$ such that*

$$\Omega_{V,\gamma} \text{ is bounded, and} \tag{2}$$

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \text{for all } x \in \mathcal{R}^n \tag{3}$$

$$\Omega_{V,\gamma} \setminus \{0\} \subset \left\{x \in \mathcal{R}^n : \nabla V(x) f(x) < 0\right\}, \tag{4}$$

then for all $\xi \in \Omega_{V,\gamma}$, the solution of (1) exists, satisfies $\phi(\xi, t) \in \Omega_{V,\gamma}$ for all $t \geq 0$, and $\lim_{t\to\infty} \phi(\xi, t) = 0$, i.e., $\Omega_{V,\gamma}$ is an invariant subset of R_0. $\lhd$

In order to enlarge the computed invariant subset of the ROA, we define a variable sized region $\mathcal{P}_\beta := \{x \in \mathcal{R}^n : p(x) \leq \beta\}$, where $p \in \mathbb{R}[x]$ is a fixed positive definite convex polynomial, and maximize β while imposing the constraint $\mathcal{P}_\beta \subseteq \Omega_{V,\gamma}$ along with the constraints (2)–(4). This can be written as

$$\beta^*(\mathcal{V}) := \max_{\beta>0, V\in\mathcal{V}} \beta \quad \text{subject to (2)–(4)}, \qquad \mathcal{P}_\beta \subseteq \Omega_{V,\gamma}. \tag{5}$$

Here $\mathcal{V}$ denotes the set of candidate Lyapunov functions over which the maximum is defined, for example all $\mathcal{C}^1$ functions. Lemma 1 and the associated optimization problem in (5) provide a characterization of the invariant subsets of the ROA in terms of the sublevel sets of Lyapunov functions.

The problem in (5) is an infinite-dimensional problem. In order to make it amenable to numerical optimization (specifically SOS optimization), we restrict $\mathcal{V}$ to be all polynomials of some fixed degree. We use the well-known sufficient condition: for any $\pi \in \mathbb{R}[x]$, if $\pi \in \Sigma[x]$, then π is positive semidefinite (Parrilo, 2003). Using simple generalizations of the S-procedure (Lemmas 2 and 3), we obtain sufficient conditions for set containment constraints. Specifically, let l_1 and l_2 be positive definite polynomials (typically $\epsilon x^T x$ for some small real number ϵ). Then, since l_1 is radially unbounded, the constraint

$$V - l_1 \in \Sigma[x] \tag{6}$$

and $V(0) = 0$ are sufficient conditions for (2) and (3). By Lemma 2, if $s_1 \in \Sigma[x]$, then

$$-[(\beta - p)s_1 + (V - \gamma)] \in \Sigma[x] \tag{7}$$

implies the set containment $\mathcal{P}_\beta \subseteq \Omega_{V,\gamma}$, and by Lemma 3, if $s_2, s_3 \in \Sigma[x]$, then

$$-[(\gamma - V)s_2 + \nabla V f s_3 + l_2] \in \Sigma[x] \tag{8}$$

is a sufficient condition for (4). Using these sufficient conditions, a lower bound on $\beta^*(\mathcal{V})$ can be defined as

$$\beta_B^*(\mathcal{V}, \mathcal{S}) := \max_{V\in\mathcal{V}, \beta, s_i\in\mathcal{S}_i} \beta \quad \text{subject to (6)–(8)}, \\ V(0) = 0, s_i \in \Sigma[x], \beta > 0. \tag{9}$$

Here, the sets $\mathcal{V}$ and $\mathcal{S}_i$ are prescribed finite-dimensional subspaces of polynomials. Although β_B^* depends on these subspaces, it will not always be explicitly notated. Note that since conditions (6)–(8) are only sufficient conditions, $\beta_B^*(\mathcal{V}, \mathcal{S}) \leq \beta^*(\mathcal{V}) \leq \beta^*(\mathcal{C}^1)$. The optimization problem in (9) is bilinear because of the product terms βs_1 in (7) and $V s_2$ and $\nabla V f s_3$ in (8). However, the problem has more structure than a general BMI problem. If V is fixed, the problem becomes affine in $S = \{s_1, s_2, s_3\}$ and vice versa. In Section 3, we will construct a convex outer bound on the set of feasible V and sample from this outer bound set to obtain candidate V's, and then solve (9) for S, holding V fixed.

3. Relaxation of the bilinear SOS problem using simulation data

The usefulness of simulation in understanding the ROA for a given system is undeniable. Faced with the task of performing a stability analysis (e.g. "for a given p, is $\mathcal{P}_\beta$ contained in the ROA?"), a pragmatic, fruitful and wise approach begins with a linearized analysis and at least a modest number of simulation runs. Certainly, just one divergent trajectory starting in $\mathcal{P}_\beta$ certifies that $\mathcal{P}_\beta \not\subset R_0$. Conversely, a large collection of only convergent trajectories hints at the likelihood that indeed $\mathcal{P}_\beta \subset R_0$. Suppose this latter condition is true, let $\mathbf{C}$ be the set of N_{conv} trajectories $\mathbf{c}$ converging to the origin with initial conditions in $\mathcal{P}_\beta$. In the course of simulation runs, divergent trajectories $\mathbf{d}$ whose initial conditions are not in $\mathcal{P}_\beta$ may also get discovered, so let the set of $\mathbf{d}$'s be denoted by $\mathbf{D}$ and N_{div} be the number of elements of $\mathbf{D}$. Although $\mathbf{C}$ and $\mathbf{D}$ depend on β and the manner in which $\mathcal{P}_\beta$ is sampled, this is not explicitly notated.

With β and γ fixed, the set of Lyapunov functions which certify that $\mathcal{P}_\beta \subset R_0$, using conditions (6)–(8), is simply $\{V \in \mathbb{R}[x] : \text{(6)–(8) hold for some } s_i \in \Sigma[x]\}$. Of course, this set could be empty, but it must be contained in the convex set $\{V \in \mathbb{R}[x] : \text{(10) holds}\}$, where

$$\nabla V(\mathbf{c}(t)) f(\mathbf{c}(t)) < 0, \\ l_1(\mathbf{c}(t)) \leq V(\mathbf{c}(t)), \quad \text{and} \quad V(\mathbf{c}(0)) \leq \gamma, \\ \gamma + \delta \leq V(\mathbf{d}(t)), \tag{10}$$

for all $\mathbf{c} \in \mathbf{C}$, $\mathbf{d} \in \mathbf{D}$, and $t \geq 0$, where δ is a fixed (small) positive constant. Informally, these conditions simply say that any V which verifies that $\mathcal{P}_\beta \subset R_0$ using conditions (6)–(8) must, on the trajectories starting in $\mathcal{P}_\beta$, be decreasing and take on values between 0 and γ. Moreover, V must be greater than γ on divergent trajectories. In fact, with the exception of the strengthened lower bound on V (beyond mere positivity), the conditions in (10) are even necessary conditions for any $V \in \mathcal{C}^1$ which verify $\mathcal{P}_\beta \subset R_0$ using conditions (2)–(4).

3.1. Affine relaxation using simulation data

Let $\mathcal{V}$ be linearly parametrized as $\mathcal{V} := \{V \in \mathbb{R}[x] : V(x) = \varphi(x)^{\mathrm{T}}\alpha\}$, where $\alpha \in \mathcal{R}^{n_b}$ and φ is an n_b-dimensional vector of polynomials in x. Given $\varphi(x)$, constraints in (10) can be viewed as constraints on $\alpha \in \mathcal{R}^{n_b}$ yielding the convex set $\{\alpha \in \mathcal{R}^{n_b}$: (10) holds for $V = \varphi(x)^{\mathrm{T}}\alpha\}$. For each $\mathbf{c} \in \mathbf{C}$, $\mathbf{d} \in \mathbf{D}$, let $\mathcal{T}_{\mathbf{c}}$ and $\mathcal{T}_{\mathbf{d}}$ be finite subsets of the interval $[0, \infty)$ including the origin. A polytopic outer bound for this set described by finitely many constraints is $\mathcal{Y}_{sim} := \{\alpha \in \mathcal{R}^{n_b}$: (11) holds$\}$, where

$$
\begin{aligned}
&[\nabla\varphi(\mathbf{c}(\tau_c))f(\mathbf{c}(\tau_c))]^{\mathrm{T}}\alpha < 0, \\
&l_1(\mathbf{c}(\tau_c)) \le \varphi(\mathbf{c}(\tau_c))^{\mathrm{T}}\alpha, \quad \text{and} \quad \varphi(\mathbf{c}(0))^{\mathrm{T}}\alpha \le \gamma, \\
&\varphi(\mathbf{d}(\tau_d))^{\mathrm{T}}\alpha \ge \gamma + \delta
\end{aligned} \tag{11}
$$

for all $\mathbf{c} \in \mathbf{C}$, $\tau_c \in \mathcal{T}_{\mathbf{c}}$, $\mathbf{d} \in \mathbf{D}$, and $\tau_d \in \mathcal{T}_{\mathbf{d}}$. Note that $\varphi(\mathbf{c}(0))^{\mathrm{T}}\alpha \le \gamma$ in (11) provides necessary conditions for $\mathcal{P}_\beta \subseteq \Omega_{V,\gamma}$ since $\mathbf{c}(0) \in \mathcal{P}_\beta$ for all $\mathbf{c} \in \mathbf{C}$. In practice, we replace the strict inequality in (11) by $[\nabla\varphi(\mathbf{c}(\tau_c))f(\mathbf{c}(\tau_c))]^{\mathrm{T}}\alpha \le -l_3(\mathbf{c}(\tau_\mathbf{c}))$, where l_3 is a fixed, positive definite polynomial imposing a bound on the rate of decay of V along the trajectories.

The constraint that ∇Vf be negative on a sublevel set of V implies that ∇Vf is negative on a neighborhood of the origin. While a large number of sample points from the trajectories will approximately enforce this, in some cases (e.g. exponentially stable linearization) it is easy to analytically express as a constraint on the low order terms of the polynomial Lyapunov function. For instance, assume V has a positive definite quadratic part, and that separate eigenvalue analysis has established that the linearization of (1) at the origin, i.e., $\dot{x} = \nabla f(0)x$, is asymptotically stable. Define $\mathcal{L}(P) := (\nabla f(0))^{\mathrm{T}} P + P(\nabla f(0))$, where $P^{\mathrm{T}} = P \succ 0$ is such that $x^{\mathrm{T}}Px$ is the quadratic part of V. Then, if (8) holds, it must be that

$$\mathcal{L}(P) \prec 0. \tag{12}$$

Let $\mathcal{Y}_{lin} := \{\alpha \in \mathcal{R}^{n_b} : P = P^{\mathrm{T}} \succ 0$ and (12) holds$\}$. It is well-known that $\mathcal{Y}_{lin}$ is convex (Boyd & Vandenberghe, 2004). Again, in practice, (12) is replaced by the condition $\mathcal{L}(P) \preceq -\epsilon I$, for some small real number ϵ. Furthermore, define $\mathcal{Y}_{SOS} := \{\alpha \in \mathcal{R}^{n_b}$: (6) holds$\}$. By Parrilo (2003), $\mathcal{Y}_{SOS}$ is convex. Since $\mathcal{Y}_{sim}$, $\mathcal{Y}_{lin}$ and $\mathcal{Y}_{SOS}$ are convex, $\mathcal{Y} := \mathcal{Y}_{sim} \cap \mathcal{Y}_{lin} \cap \mathcal{Y}_{SOS}$ is a convex set in $\mathcal{R}^{n_b}$. Eqs. (11) and (12) constitute a set of necessary conditions for (6)–(8); thus, we have $\mathcal{Y} \supseteq \mathcal{B} := \{\alpha \in \mathcal{R}^{n_b} : \exists s_2, s_3 \in \Sigma[x]$ such that (6)–(8) hold$\}$. Since (8) is not jointly convex in V and the multipliers, $\mathcal{B}$ may not be a convex set and even may not be connected.

A point in $\mathcal{Y}$ can be computed solving an affine (feasibility) SDP with the constraints (6), (11) and (12). An arbitrary point in $\mathcal{Y}$ may or may not be in $\mathcal{B}$. However, if we generate a collection $\mathcal{A} := \{\alpha^{(k)}\}_{k=0}^{N_V-1}$ of N_V points distributed approximately uniformly in $\mathcal{Y}$, it may be that some of the points are in $\mathcal{B}$. To this end, we use the so-called "Hit-and-Run" (*H&R*) random point generation algorithm as described in Tempo, Calafiore, and Dabbene (2005). When applied to generate a sample of $\mathcal{Y}$, each step of *H&R* algorithm requires solving four small affine SDPs.

3.2. Algorithms

Since a feasible value of β is not known a priori, an iterative strategy to simulate and collect convergent and divergent trajectories is necessary. This process when coupled with the *H&R* algorithm constitutes the Lyapunov function candidate generation.

Simulation and Lyapunov function generation (SimLFG) algorithm: Given positive definite convex $p \in \mathbb{R}[x]$, a vector of polynomials $\varphi(x)$ and constants $\beta_{SIM}, N_{conv}, N_V, \beta_{shrink} \in (0, 1)$, and empty sets $\mathbf{C}$ and $\mathbf{D}$, set $\gamma = 1$, $N_{more} = N_{conv}$, $N_{div} = 0$.

(i) Integrate (1) from N_{more} initial conditions in the set $\{x \in \mathcal{R}^n : p(x) = \beta_{SIM}\}$.
(ii) If there is no diverging trajectory, add the trajectories to $\mathbf{C}$ and go to (iii). Otherwise, add the divergent trajectories to $\mathbf{D}$ and the convergent trajectories to $\mathbf{C}$, let N_d denote the number of diverging trajectories found in the last run of (i) and set N_{div} to $N_{div} + N_d$. Set β_{SIM} to the minimum of $\beta_{shrink}\beta_{SIM}$ and the minimum value of p along the diverging trajectories. Set N_{more} to $N_{more} - N_d$, and go to (i).
(iii) At this point $\mathbf{C}$ has N_{conv} elements. For each $i = 1, \ldots, N_{conv}$, let $\overline{\tau}_i$ satisfy $\mathbf{c}_i(\tau) \in \mathcal{P}_{\beta_{SIM}}$ for all $\tau \ge \overline{\tau}_i$. Eliminate times in $\mathcal{T}_i$ that are less than $\overline{\tau}_i$.
(iv) Find a feasible point for (6), (11) and (12). If (6), (11) and (12) are infeasible, set $\beta_{SIM} = \beta_{shrink}\beta_{SIM}$, and go to (iii). Otherwise, go to (v).
(v) Generate N_V Lyapunov function candidates using *H&R* algorithm, and return β_{SIM} and Lyapunov function candidates. ◁

The suitability of a Lyapunov function candidate is assessed by solving two optimization problems. Both problems require bisection and each bisection step involves a linear SOS problem. Alternative linear formulations appear in the Appendix. These do not require bisection, but generally involve higher degree polynomial expressions.

Problem 1. Given $V \in \mathbb{R}[x]$ (from *SimLFG* algorithm) and positive definite $l_2 \in \mathbb{R}[x]$, define

$$
\begin{aligned}
&\gamma_L^* := \max_{\gamma, s_2, s_3} \gamma \quad \text{subject to } s_2, s_3 \in \Sigma[x], \gamma > 0, \\
&-[(\gamma - V)s_2 + \nabla Vfs_3 + l_2] \in \Sigma[x].
\end{aligned} \tag{13}
$$

If Problem 1 is feasible, then $\gamma_L^* > 0$ and define:

Problem 2. Given $V \in \mathbb{R}[x]$, $p \in \mathbb{R}[x]$, and γ_L^*, solve

$$
\begin{aligned}
&\beta_L^* := \max_{\beta, s_1} \beta \quad \text{subject to } s_1 \in \Sigma[x], \beta > 0, \\
&-\left[(\beta - p)s_1 - (V - \gamma_L^*)\right] \in \Sigma[x].
\end{aligned} \tag{14}
$$

Although γ_L^* and β_L^* depend on the allowable degree of s_1, s_2, and s_3, this is not explicitly notated.

Assuming Problem 1 is feasible, it is true that $\mathcal{P}_{\beta_L^*} \setminus \{0\} \subseteq \Omega_{V,\gamma_L^*} \setminus \{0\} \subset \{x \in \mathcal{R}^n : \nabla V(x)f(x) < 0\}$, so V certifies that $\mathcal{P}_{\beta_L^*} \subset R_0$. Solutions to Problems 1 and 2 provide a feasible point for the problem in (9). This feasible point can be further improved by solving the problem in (9) using PENBMI and/or iterative coordinate-wise linear optimization schemes, one of which is given next.

Coordinate-wise optimization (CWOpt) algorithm: Given $V \in \mathbb{R}[x]$, positive definite $l_1, l_2 \in \mathbb{R}[x]$, a constant ε_{iter}, and maximum number of iterations N_{iter}, set $k = 0$.

(i) Solve Problems 1 and 2.
(ii) Given s_1, s_2, s_3, and γ_L^* from step (i), set γ in (7) and (8) to γ_L^*, solve (9) for V and β, and set $\beta_L^* = \beta_B^*$.
(iii) If $k = N_{iter}$ or the increase in β_L^* between successive applications of (ii) is less than ε_{iter}, return V, γ_L^*, and β_L^*. Otherwise, set k to $k + 1$ and go to (i). ◁

The algorithms (*SimLFG*, Problems 1 and 2, and *CWOpt*) yield lower bounds on $\beta^*(\mathcal{C}^1)$, as they produce a Lyapunov function which certifies that a particular value of β satisfies $\mathcal{P}_\beta \subset R_0$. Upper bounds (i.e., values of β that are not certifiable) may also be obtained. More specifically, diverging trajectories found in the course of simulation runs provide upper bounds on $\beta^*(\mathcal{C}^1)$ while inconsistency of the constraints (6), (11) and (12) provide upper bounds on β_B^*. A diverging trajectory with the initial

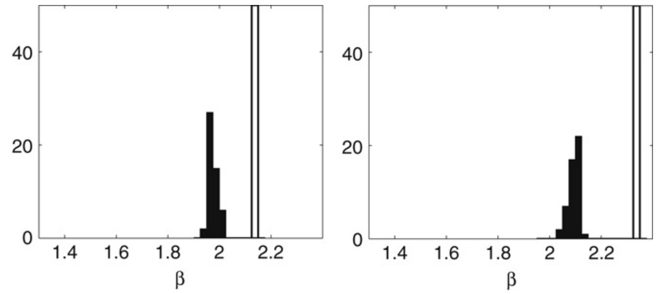

Fig. 1. Histograms of β_L^* (black bars) and β_B^* (white bars) from seeded PENBMI runs for $\partial(V) = 4$ (left), 6 (right).

condition x_0 satisfying $p(x_0) = \beta$ proves that $\mathcal{P}_\beta$ cannot be a subset of the ROA, i.e., $\beta^*(\mathcal{C}^1) < \beta$. Furthermore, restricting Lyapunov function candidates to $\mathcal{V}_\varphi := \{\varphi(x)^\mathrm{T}\alpha : \alpha \in \mathcal{R}^{n_b}\}$ has additional implications. Infeasibility of any of the constraints (6), (11) and (12) for some value of β (recall (11) implicitly depends on β) verifies $\beta_B^*(\mathcal{V}_\varphi, \mathcal{S}) \leq \beta^*(\mathcal{V}_\varphi) < \beta$, regardless of the subspaces constituting $\mathcal{S}$. Moreover, the gap between the value of β proven unachievable and what we actually certify, namely a lower bound to $\beta_B^*(\mathcal{V}_\varphi, \mathcal{S})$, can be used as a measure of suboptimality introduced due to the finiteness of the degree of the multipliers and the fact that the bilinear search and the coordinate-wise linear search are only local optimization schemes. Finally, *H&R*, *SimLFG* and *CWOpt* algorithms become more efficient using parallel computing.

4. Examples

Certifying Lyapunov functions, multipliers and missing parameters for all examples in this paper are available at http://jagger.me.berkeley.edu/~pack/certify. In the examples, $l_i(x) = 10^{-6}x^\mathrm{T}x$ for $i = 1, 2, 3$.

4.1. Van der Pol dynamics

The Van der Pol dynamics $\dot{x}_1 = -x_2, \dot{x}_2 = x_1 + (x_1^2 - 1)x_2$ have a stable equilibrium point at the origin and an unstable limit cycle. The limit cycle is the boundary of the ROA. We applied *SimLFG* algorithm with $p(x) = x^\mathrm{T}x$ and the parameters $N_{conv} = 200$, $\beta_{SIM} = 3.0$ (initial value), $\beta_{shrink} = 0.9$, and $N_V = 50$ for $\partial(V) = 2, 4$, and 6. We found $N_{div} = 21$ diverging trajectories during the simulation runs and feasible solutions for (6), (11) and (12) in step (iv) with $\beta_{SIM} = 1.44, 1.97$, and 2.19 for $\partial(V) = 2, 4$, and 6, respectively. We assessed (computed corresponding values of β_L^* for) the Lyapunov function candidates generated in step (v) solving Problems 1 and 2 and further optimized initializing PENBMI with the solutions of these problems. Fig. 1 shows β_L^* and corresponding β_B^* values for $\partial(V) = 4$ and 6. Practically, every seeded PENBMI run terminated with the same β_B^* value which is the largest known (at least by us) value of β for which (9) is feasible with the prescribed families of Lyapunov functions and multipliers. In addition, we performed 10 unseeded PENBMI runs for $\partial(V) = 4$ and 6. Of these runs 90% and 50%, respectively, terminated successfully (with an optimal value of β equal to that from the seeded PENBMI runs). Moreover, unseeded PENBMI runs took longer computation times than seeded PENBMI runs. For comparison, seeded PENBMI runs took 3–8 and 11–24 s for $\partial(V) = 4$ and 6, respectively, on a desktop PC, whereas they took 50–250 and 1000–2500 s, respectively, for unseeded PENBMI runs. Fig. 2 shows the level sets of the Lyapunov functions corresponding to the value of β_B^*.

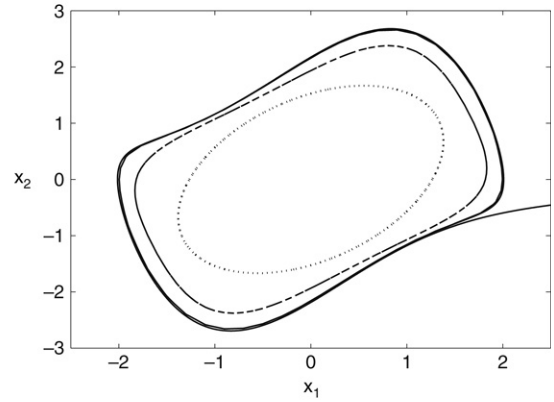

Fig. 2. The invariant subsets of the ROA (dot: $\partial(V) = 2$, dash: $\partial(V) = 4$, and solid: $\partial(V) = 6$ (indistinguishable from the outermost curve for the limit cycle)).

Table 1
Volume ratios for (E_1)–(E_7)

Example	Volume ratio	Example	Volume ratio
(E_1)	16.7/10.2	(E_2)	0.99/0.85
(E_3)	37.2/23.5	(E_4)	1.00/0.28
(E_5)	62.3 /7.3	(E_6)	35.0/15.3
(E_7)	1.44/0.70		

4.2. Examples from the literature

We present results obtained using the method from the previous section for the systems in (15). (E_1)–(E_3) are from Chesi et al. (2005), (E_4) and (E_7) are from Vannelli and Vidyasagar (1985), and (E_5) and (E_6) are from Hauser and Lai (1992) and Hachicho and Tibken (2002), respectively. Since the dynamics in (E_1)–(E_7) have no physical meaning and there is no p given, we applied *SimLFG* algorithm sequentially: Apply *SimLFG* algorithm with $p(x) = x^\mathrm{T}x$ and $N_V = 1$ for $\partial(V) = 2$. Call the quadratic Lyapunov function obtained $\hat{V}$. Set p to $\hat{V}$ and apply *SimLFG* algorithm with this p and $N_V = 1$ for $\partial(V) = 4$. For (E_5)–(E_7), we further applied *CWOpt* algorithm with $N_{iter} = 10$. Table 1 shows the ratio of the volume of the invariant subset of the ROA obtained using this procedure to that reported in the corresponding references. Empirical volumes of sublevel sets of V are computed by randomly sampling a hypercube containing the sublevel set. Values in Table 1 are volumes normalized by π and $4\pi/3$ for 2- and 3-dimensional problems, respectively. For (E_4), (E_6), and (E_7), we also empirically verified that the invariant subsets of the ROA reported in the corresponding references are contained in those computed by this sequential procedure.

$$
\begin{aligned}
&(E_1): \{\dot{x}_1 = x_2, \quad \dot{x}_2 = -2x_1 - 3x_2 + x_1^2x_2. \\
&(E_2): \begin{cases}\dot{x}_1 = x_2, \\ \dot{x}_2 = -2x_1 - x_2 + x_1x_2^2 - x_1^5 + x_1x_2^4 + x_2^5.\end{cases} \\
&(E_3): \begin{cases}\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \\ \dot{x}_3 = -4x_1 - 3x_2 - 3x_3 + x_1^2x_2 + x_1^2x_3.\end{cases} \\
&(E_4): \begin{cases}\dot{x}_1 = -x_2, \quad \dot{x}_2 = -x_3, \\ \dot{x}_3 = -0.915x_1 + (1 - 0.915x_1^2)x_2 - x_3.\end{cases} \\
&(E_5): \begin{cases}\dot{x}_1 = x_2 + 2x_2x_3, \quad \dot{x}_2 = x_3, \\ \dot{x}_3 = -0.5x_1 - 2x_2 - x_3.\end{cases} \\
&(E_6): \{\dot{x}_1 = -x_1 + x_2x_3^2, \quad \dot{x}_2 = -x_2 + x_1x_2, \quad \dot{x}_3 = -x_3. \\
&(E_7): \begin{cases}\dot{x}_1 = -0.42x_1 - 1.05x_2 - 2.3x_1^2 - 0.5x_1x_2 - x_1^3, \\ \dot{x}_2 = 1.98x_1 + x_1x_2.\end{cases}
\end{aligned}
\tag{15}
$$

Table 2
Certified values of β before and after applying *CWOpt* algorithm and from unseeded PENBMI run

	$\partial(V) = 2$	$\partial(V) = 4$
Before iterations	6.56	8.99
After iterations	8.56	14.4
PENBMI (unseeded)	8.60	15.2

4.3. Controlled short period aircraft dynamics

The closed-loop dynamics in (16) have an asymptotically stable equilibrium point at the origin.

$$\dot{x} = \begin{bmatrix} \sum_{i=1}^{5} a_{1i}x_i + \sum_{i=1}^{5} r_{1i}x_ix_2 + r_{16}x_2^3 \\ \sum_{i=1}^{5} a_{2i}x_i + \sum_{i,j=2,5} r_{ij}x_ix_j \\ A_{345}x \end{bmatrix}. \quad (16)$$

Here, $x = [x_1, x_2, x_3, x_4, x_5]^T$ is the state vector, x_1, x_2 and x_5 are pitch rate, angle of attack, and pitch angle, respectively, x_3 and x_4 are the controller states, and $A_{345} \in \mathcal{R}^{3\times 5}$. Before applying our method, we performed excessive simulations and found a diverging trajectory whose initial condition x_0 satisfies $x_0^T x_0 = 16.1$; therefore, initialized β_{SIM} with 16.0. We applied algorithm *SimLFG* with $p(x) = x^T x$, $\beta_{shrink} = 0.85$, $N_{conv} = 4000$, $N_V = 1$ for $\partial(V) = 2$ and 4. We assessed the Lyapunov function candidates solving Problems A.1 and A.2 and further optimized using *CWOpt* algorithm with $N_{iter} = 6$. Certified values of β before and after applying iterations and from unseeded PENBMI runs are shown in Table 2. Unseeded PENBMI runs led to slightly higher values of β. However, this benefit was at the expense of high computational effort. For example, the unseeded PENBMI run took 38 h for $\partial(V) = 4$ whereas our method took 36 min (15 min for the *SimLFG* algorithm and 21 min for the *CWOpt* algorithm). Finally, the dependence that the starting point of *CWOpt* algorithm has on its performance is significant. For example, simply initializing *CWOpt* algorithm with $V(x) = x^T Px + 0.001\sum_{i=1}^{5} x_i^4$, where $P^T = P \succ 0$ satisfies $\mathcal{L}(P) = -I$ yields poor results. After 30 iterations, the *CWOpt* iteration converges, but the resultant Lyapunov function only certifies $\mathcal{P}_{8.5} \subset R_0$.

4.4. Pendubot dynamics

The pendubot is an underactuated two-link pendulum with torque action only on the first link. We designed an LQR controller to balance the two-link pendulum about its upright position. Third order polynomial approximation of the closed-loop dynamics is $\dot{x}_1 = x_2$, $\dot{x}_2 = 782x_1 + 135x_2 + 689x_3 + 90x_4$, $\dot{x}_3 = x_4$ and $\dot{x}_4 = 279x_1x_3^2 - 1425x_1 - 257x_2 + 273x_3^3 - 1249x_3 - 171x_4$. Here, x_1 and x_3 are angular positions of the first link and the second link (relative to the first link). We applied *SimLFG* algorithm sequentially exactly as described in Section 4.2 and *CWOpt* algorithm with 10 iterations and obtained $\beta_L^* = 1.69$. Conversely, we found a diverging trajectory with the initial condition $\bar{x}$ with $p(\bar{x}) = 1.95$ proving that $1.69 \le \beta^*(\mathcal{C}^1) < 1.95$. Fig. 3 shows the $x_2 = 0$ and $x_4 = 0$ slice of the invariant subset of the ROA along with initial conditions (with $x_2 = 0$ and $x_4 = 0$) for some diverging trajectories.

4.5. Closed-loop dynamics with nonlinear observer based controller

For the dynamics $\dot{x}_1 = u$, $\dot{x}_2 = -x_1 + x_1^3/6 - u$ and $y = x_2$, where x_1 and x_2 are the states, u is the control input and y is the output, an observer L with polynomial vector field $\dot{z} = L(y, z)$ with $\partial(L) = 3$ and a control law in the form $u = -145.9z_1 + 12.3z_2$, where z_1 and z_2 are the observer states, were computed in Tan (2006). The application of *SimLFG* algorithm with $\partial(V) = 2$ and p from Tan (2006) and *CWOpt* algorithm with $N_{iter} = 4$ leads to $\beta_L^* = 0.32$. We also applied *CWOpt* algorithm (initialized with the quadratic V found in the first application) with $\partial(V) = 4$ and $N_{iter} = 6$ and obtained $\beta_L^* = 0.52$. Conversely, we found a diverging trajectory with the initial condition $(\bar{x}, \bar{z})$ satisfying $p(\bar{x}, \bar{z}) = 0.54$ proving that $0.52 \le \beta^*(\mathcal{C}^1) < 0.54$.

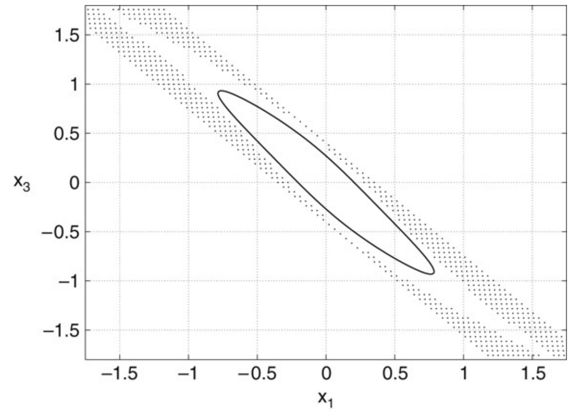

Fig. 3. A slice of the invariant subset of the ROA (solid line) and initial conditions (with $x_2 = 0$ and $x_4 = 0$) for diverging trajectories (dots).

5. Critique and conclusions

5.1. Sampling vs. simulating

A common question we get is "why simulate to get the sample points? — just sample some region, and impose $\nabla V(x)f(x) < 0$ there". There are a few answers to this. Intuitively, even running a few simulations gives insight into the system behavior. Engineers commonly use simulation to assess rough measures of stability robustness and ROA. Moreover, as converse Lyapunov theorems (Vidyasagar, 1993) implicitly define a certifying Lyapunov function in terms of the flow, it makes sense to sample the flow when looking for a Lyapunov function of a specific form. Furthermore, we have the following observation demonstrating that merely sampling some region and imposing $\nabla V(x)f(x) < 0$ there carries misleading information. Consider the Van der Pol dynamics with $p(x) = x^T x$ and let $\mathbf{S}_\beta$ denote a finite sample of $\mathcal{P}_\beta$. It can be shown that the set of quadratic positive definite functions V that satisfy

$$\mathbf{S}_{1.8} \setminus \{0\} \subset \left\{x \in \mathcal{R}^n : \nabla V(x)f(x) < 0\right\} \quad (17)$$

is nonempty. In fact, for $V(x) = 0.32x_1^2 - 0.25x_1x_2 + 0.31x_2^2$, (17) is satisfied (actually for all $x \in \mathcal{P}_{1.8}$, $\nabla V(x)f(x) \le -l_3(x)$). This naively suggests drawing samples from the set of quadratic positive definite functions satisfying (17) in order to try to prove that $\mathcal{P}_{1.8} \subset R_0$. However, simulations reveal a contradicting fact: Using trajectories with initial conditions in $\mathbf{S}_{1.8}$ for $\partial(V) = 2$, i.e., with $\varphi(x) = [x_1^2, x_1x_2, x_2^2]^T$, constraints (6), (11) (with $\gamma = 1$), and (12) turn out to be infeasible. This verifies that *no quadratic Lyapunov function can prove* $\mathcal{P}_{1.8} \subset R_0$ using conditions (6)–(8), with the additional constraint that $\dot{V}(x) \le -10^{-6}x^T x$ on all trajectories starting in $\mathcal{P}_{1.8}$. Recall though, that using quartic Lyapunov functions we know $\beta^*\left(\mathcal{V}_\varphi, \mathcal{S}\right) \ge 2.14$. By these observations, we have the following series of inclusions for the subsets of the positive definite quadratic polynomials

$$\begin{aligned} &\{V : V \text{ certifies } \mathcal{P}_\beta \subset R_0 \text{ using (6)–(8)}\} \\ &\subset \{V : \nabla V(\mathbf{c_s}(\tau))f(\mathbf{c_s}(\tau)) < 0\ \forall\tau, \forall \mathbf{s} \in \mathbf{S}_\beta\} \\ &\subset \{V : \nabla V(\mathbf{s})f(\mathbf{s}) < 0\ \forall \mathbf{s} \in \mathbf{S}_\beta\}, \end{aligned}$$

where $\mathbf{c_s}$ denotes the trajectory with the initial condition $\mathbf{s} \in \mathbf{S}_\beta$. Therefore, merely sampling instead of using simulations leads to a larger outer set from which the samples for V are taken in step (v) of *SimLFG* algorithm and it is less likely to find a function that certifies that $\mathcal{P}_\beta \subset R_0$.

5.2. Conclusions

We proposed a method for computing invariant subsets of the region-of-attraction for asymptotically stable equilibrium points of dynamical systems with polynomial vector fields. We used polynomial Lyapunov functions as local stability certificates whose certain sublevel sets are invariant subsets of the region-of-attraction. Similar to many local analysis problems, this is a nonconvex problem. Furthermore, its sum-of-squares relaxation leads to a bilinear optimization problem. We developed a method utilizing information from simulations for easily generating Lyapunov function candidates. For a given Lyapunov function candidate, checking its feasibility and assessing the size of the associated invariant subset are affine sum-of-squares optimization problems. Solutions to these problems provide invariant subsets of the region-of-attraction directly and/or they can further be used as seeds for local bilinear search schemes or iterative coordinate-wise linear search schemes for improved performance of these schemes. We reported promising results in all these directions.

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Appendix

Lemma 2. *Given $g_0, g_1, \ldots, g_m \in \mathbb{R}[x]$, if there exist $s_1, \ldots, s_m \in \Sigma[x]$ such that $g_0 - \sum_{i=1}^m s_i g_i \in \Sigma[x]$, then $\{x \in \mathcal{R}^n : g_1(x), \ldots, g_m(x) \geq 0\} \subseteq \{x \in \mathcal{R}^n : g_0(x) \geq 0\}$.*

Lemma 3. *Given $g_0, g_1, g_2 \in \mathbb{R}[x]$ such that g_0 is positive definite and $g_0(0) = 0$, if there exist $s_1, s_2 \in \Sigma[x]$ such that $g_1 s_1 + g_2 s_2 - g_0 \in \Sigma[x]$, then $\{x \in \mathcal{R}^n : g_1(x) \leq 0\} \setminus \{0\} \subset \{x \in \mathcal{R}^n : g_2(x) > 0\}$.* ◁

Problems 1 and 2 in Section 3 compute lower bounds on the largest value of γ and β such that, for given V and p, $\Omega_{V,\gamma} \setminus \{0\} \subset \{x \in \mathcal{R}^n : \nabla V(x) f(x) < 0\}$ and $\mathcal{P}_\beta \subset \Omega_{V,\gamma}$. We propose alternative formulations, that do not require line search, to compute similar lower bounds. Labeled γ_a^* and β_a^*, these are generally different than γ_L^* and β_L^*. For $h, g \in \mathbb{R}[x]$ and a positive integer d, define $\mu^o(h, g) := \inf_{x \neq 0} h(x)$ such that $g(x) = 0$, and

$$\mu^*(h, g, d) := \sup_{\mu > 0, r \in \mathbb{R}[x]} \mu \quad \text{subject to}$$

$$(h - \mu)\left(x_1^{2d} + \cdots + x_n^{2d}\right) - gr \in \Sigma[x].$$

Note that $\mu^*(h, g, d) \leq \mu^o(h, g)$.

Lemma 4. *Let $g, h : \mathcal{R}^n \to \mathcal{R}$ be continuous, h be positive definite, $g(0) = 0$, and $g(x) < 0$ for all nonzero $x \in \mathcal{O}$, a neighborhood of the origin. Define $\gamma^o := \mu^o(h, g)$. Then, the connected component of $\{x \in \mathcal{R}^n : h(x) < \gamma^o\}$ containing the origin is a subset of $\{x \in \mathcal{R}^n : g(x) < 0\} \cup \{0\}$.* ◁

Proof. Suppose not and let $\bar{x} \neq 0$ be in the connected component of $\{x \in \mathcal{R}^n : h(x) < \gamma^o\}$ containing the origin but $g(\bar{x}) \geq 0$. Then, there exists a continuous function $\vartheta : [0, 1] \to \mathcal{R}^n$ such that $\vartheta(0) = 0$, $\vartheta(1) = \bar{x}$, and $h(\vartheta(t)) < \gamma^o$ for all $t \in [0, 1]$. Since $g(0) = 0$ and $g(x) < 0$ for all nonzero $x \in \mathcal{O}$, there exists $0 < \epsilon < 1$ such that $g(\vartheta(\epsilon)) < 0$. Since $\bar{x}$ is not in $\{x \in \mathcal{R}^n : g(x) < 0\}$, $g(\vartheta(1)) \geq 0$. Since g and ϑ are continuous, there exists $t^* \in (0, 1]$ such that $g(\vartheta(t^*)) = 0$, which implies $h(\vartheta(t^*)) \geq \gamma^o$. This contradiction leads to $\bar{x} \in \{x \in \mathcal{R}^n : g(x) < 0\}$. □

Corollary 5. *Let $V \in \mathbb{R}[x]$ be a positive definite $\mathcal{C}^1$ function and satisfy (12) and $V(0) = 0$. Then, for all γ such that $0 < \gamma < \mu^o(V, \nabla V f)$, the connected component of $\Omega_{V,\gamma}$ containing the origin is an invariant subset of the ROA.* ◁

Proof. Since the quadratic part of V is a Lyapunov function for the linearized system, there exists a neighborhood $\mathcal{O}$ of the origin such that $\nabla V(x) f(x) < 0$ for all nonzero $x \in \mathcal{O}$. By Lemma 4, the connected component of $\Omega_{V,\gamma}$ containing the origin, a subset of the connected component of $\{x \in \mathcal{R}^n : V(x) < \mu^o(V, \nabla V f)\}$ containing the origin, is contained in $\{x \in \mathcal{R}^n : \nabla V(x) f(x) < 0\} \cup \{0\}$. Corollary 5 follows from regular Lyapunov arguments (Vidyasagar, 1993). □

Corollary 6. *For some positive integer d_1, define $\gamma_a^* := \mu^*(V, \nabla V f, d_1)$. Then, if $\gamma < \gamma_a^*$ for some positive integer d_1, then the connected component of $\Omega_{V,\gamma}$ containing the origin is an invariant subset of the ROA.* ◁

Corollary 7. *Let $0 < \gamma < \gamma_a^*$, d_2 be a positive integer, $V, p \in \mathbb{R}[x]$ be positive definite and p be convex. Define $\beta_a^* := \mu^*(p, V - \gamma, d_2)$. Then for any $\beta < \beta_a^*$, $\mathcal{P}_\beta \subset \Omega_{V,\gamma}$ and $\mathcal{P}_\beta \subset R_0$.* ◁

References

Boyd, S., & Vandenberghe, L. (2004). *Convex optimization*. Cambridge Univ. Press.
Chesi, G., Garulli, A., Tesi, A., & Vicino, A. (2005). LMI-based computation of optimal quadratic Lyapunov functions for odd polynomial systems. *International Journal of Robust Nonlinear Control*, *15*, 35–49.
Chiang, H.-D., & Thorp, J. S. (1989). Stability regions of nonlinear dynamical systems: A constructive methodology. *IEEE Transactions on Automatic Control*, *34*(12), 1229–1241.
Davison, E. J., & Kurak, E. M. (1971). A computational method for determining quadratic Lyap. functions for nonlinear systems. *Automatica*, *7*, 627–636.
Genesio, R., Tartaglia, M., & Vicino, A. (1985). On the estimation of asymptotic stability regions: State of the art and new proposals. *IEEE Transactions on Automatic Control*, *30*(8), 747–755.
Hachicho, O., & Tibken, B. (2002). Estimating domains of attraction of a class of nonlinear dynamical systems with LMI methods based on the theory of moments. In *Proc. CDC* (pp. 3150–3155).
Hauser, J., & Lai, M. C. (1992). Estimating quadratic stability domains by nonsmooth optimization. In *Proc. ACC* (pp. 571–576).
Kočvara, M., & Stingl, M. (2005). *PENBMI user's guide*.
Papachristodoulou, A. (2005). Scalable analysis of nonlinear systems using convex optimization. *Ph.D. dissertation*. Caltech.
Parrilo, P. (2003). Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming Series B*, *96*(2), 293–320.
Prokhorov, D. V., & Feldkamp, L. A. (1999). Application of SVM to Lyapunov function approximation. In *Proc. int. joint conf. on neural networks*.
Serpen, G. (2005). Search for a Lyapunov function through empirical approximation by artificial neural nets: Theoretical framework. In *Proc. int. joint conf. on artificial neural networks*. (pp. 735–740).
Sturm, J. (1999). Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, *11*, 625–653.
Tan, W. (2006). Nonlinear control analysis and synthesis using sum-of-squares programming. *Ph.D. dissertation*. UC, Berkeley.
Tan, W., & Packard, A. (2006). Stability region analysis using sum of squares programming. In *Proc. ACC* (pp. 2297–2302).
Tempo, R., Calafiore, G., & Dabbene, F. (2005). *Randomized algorithms for analysis and control of uncertain systems*. Springer.
Tibken, B. (2000). Estimation of the domain of attraction for polynomial systems via LMIs. In *Proc. CDC* (pp. 3860–3864).

Tibken, B., & Fan, Y. (2006). Computing the domain of attraction for polynomial systems via BMI optimization methods. In *Proc. ACC* (pp. 117–122).
Toker, O., & Ozbay, H. (1995). On the NP-hardness of solving bilinear matrix inequalities and simultaneous stabilization with static output feedback. In *Proc. ACC* (pp. 2525–2526).
Vannelli, A., & Vidyasagar, M. (1985). Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems. *Automatica*, *21*(1), 69–80.
Vidyasagar, M. (1993). *Nonlinear systems analysis* (2nd ed.). Prentice Hall.

Ufuk Topcu received his B.S. in 2003 at Bogazici University and M.S. in 2005 at the University of California, Irvine both in Mechanical Engineering. He is currently working on his Ph.D. in Mechanical Engineering at the University of California, Berkeley. His research interests include quantitative nonlinear system analysis, nonlinear time-scale analysis, and applications of robust and convex optimization.

Andrew Packard joined UC Berkeley Mechanical Engineering in 1990. His technical interests include quantitative nonlinear systems analysis and optimization and data structure issues associated with large-scale collaborative research for predictive modeling of complex physical processes. He is an author of the Robust Control toolbox distributed by Mathworks. The Meyersound X-10 loudspeaker utilizes novel feedback control circuitry developed by his UCB research group.

He is a recipient of the campus Distinguished Teaching Award, the 1995 Eckman Award, the 2005 IEEE Control System Technology Award, and a 2007 IEEE Fellow.

Peter Seiler received B. S. degrees in Mathematics and Mechanical Engineering from the University of Illinois at Urbana-Champaign in 1996. He received a Ph.D. in Mechanical Engineering from the University of California at Berkeley in the fall of 2001. Dr. Seiler is currently a Principal Scientist R&D at Honeywell where he works on the flight control systems, sensor fusion algorithms, and nonlinear/linear robustness analysis tools.