

# Asymptotic Connectivity Properties of Cooperative Wireless Ad Hoc Networks

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**Abstract**—Extensive research has demonstrated the potential improvement in physical layer performance when multiple radios transmit concurrently in the same radio channel. We consider how such cooperation affects the requirements for full connectivity and percolation in large wireless ad hoc networks. Both noncoherent and coherent cooperative transmission are considered. For one-dimensional (1-D) extended networks, in contrast to noncooperative networks, for any path loss exponent less than or equal to one, full connectivity occurs under the noncoherent cooperation model with probability one for any node density. Conversely, there is no full connectivity with probability one when the path loss exponent exceeds one, and the network does not percolate for any node density if the path loss exponent exceeds two. In two-dimensional (2-D) extended networks with noncoherent cooperation, for any path loss exponent less than or equal to two, full connectivity is achieved for any node density. Conversely, there is no full connectivity when the path loss exponent exceeds two, but the cooperative network percolates for node densities above a threshold which is strictly less than that of the noncooperative network. A less conclusive set of results is presented for the coherent case. Hence, even relatively simple noncoherent cooperation improves the connectivity of large ad hoc networks.

**Index Terms**—Connectivity, Percolation, Physical layer cooperation, Ad hoc networks, Wireless networks.

## I. INTRODUCTION

WIRELESS ad hoc networks have been a topic of extreme interest recently. Naturally, connectivity is one of the key issues that requires significant study, since few network services can function properly if the network is disconnected. Due to the wireless medium, any node in a wireless network receives non-zero signal energy from every other node in the network, and thus every node is connected

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in some sense to every other node. However, in practice, to support routing and scheduling algorithms, networks define one-hop neighbors for a node as those to which it is able to establish productive direct communication, where the latter requires some minimal received signal quality for physical layer functions (e.g. synchronization) and achieving some minimal utility, as considered in detail in Section II. Two nodes are then defined as being connected if there exists some sequence of these hops between them.

Although wireless ad hoc networks are finite, asymptotic (in a large number of nodes) analyses have proven useful for understanding the characteristics of large networks and will be considered here. In conventional noncooperative networks, nodes can communicate directly if they are within a distance  $r$  determined by the required received signal-to-noise ratio (SNR), the peak transmission power, and the path loss attenuation function. However, when a set of already-connected nodes transmits simultaneously, cooperation helps meet the received signal requirements, thus allowing a node at a distance greater than  $r$  away to be pulled into the connected component. The asymptotic connectivity properties of such cooperative networks is of our interest. This paper surveys our previous work in the field, while also providing a number of new results. In particular, we follow the model that we originally presented in [1], for which a much stronger set of results is established in [2]. Here, we present a number of new results and proof techniques beyond [2]: the results on full connectivity of Theorems 3.3 and 3.7 are stronger than the analogous results in our work [2] and employ a quite different construction. In addition, the proofs of Theorems 3.4, 3.8, and 4.3 have been modified, often substantially, from those in [2] so as to improve the precision and clarity. Finally, we include more precise statements on percolation, including Theorem 4.1, rather than simply deducing what we can about percolation from our full connectivity results.

In extended networks, as considered here, nodes are distributed across an infinite region according to a Poisson point process with density  $\lambda > 0$ . Connectivity for extended networks is generally defined as the existence of one cluster containing an infinite number of connected nodes, and continuum percolation with the Poisson Boolean model has been the most common approach to study connectivity. For a given  $r$ , when the node density  $\lambda$  exceeds a given threshold  $\lambda_c$ , there will be one infinite cluster almost surely, whereas for node densities less than  $\lambda_c$  there is no infinite cluster with probability one [3]. Previous work has also shown that there is no

TABLE I  
MAJOR RESULTS FOR FULL CONNECTIVITY OF THE NONCOHERENT COOPERATIVE TRANSMISSION MODEL  
( $\alpha$  IS THE PATH LOSS EXPONENT AND  $\lambda$  IS THE NODE DENSITY)

Extended Networks			
1-D		2-D	
$\alpha \leq 1$	Full connectivity $\diamond \forall \lambda > 0$	$\alpha \leq 2$	Full connectivity $\diamond \forall \lambda > 0$
$\alpha > 1$	No full connectivity $\diamond \forall \lambda > 0$	$\alpha > 2$	No full connectivity $\diamond \forall \lambda > 0$

TABLE II  
MAJOR RESULTS FOR PERCOLATION OF THE NONCOHERENT COOPERATIVE TRANSMISSION MODEL  
( $\alpha$  IS THE PATH LOSS EXPONENT AND  $\lambda$  IS THE NODE DENSITY)

Extended Networks			
1-D		2-D	
$\alpha \leq 1$	Percolation $\diamond \forall \lambda > 0$	$\alpha \leq 2$	Percolation $\diamond \forall \lambda > 0$
$1 < \alpha \leq 2$	Open problem		
$\alpha > 2$	No percolation $\diamond \forall \lambda > 0$	$\alpha > 2$	Percolation threshold reduced

percolation in noncooperative one-dimensional networks [4]. In the cooperative setting, we only consider percolation in the noncoherent case, where the symmetry of the links allows us to employ a definition analogous to the noncooperative case. In particular, we will define the noncoherent cooperative network as percolating if a node located at the origin is in an infinite cluster with strictly positive probability. Because of the connectivity improvements afforded by cooperation, we also consider in detail whether an extended network is fully connected, which is defined as *every* node belonging to a single connected component.

This paper focuses on the connectivity of one-dimensional (1-D) and two-dimensional (2-D) cooperative wireless ad hoc networks. 1-D networks are of interest, for example, in modeling networks distributed in river valleys or along transportation corridors, whereas 2-D networks are more appropriate for open regions. For the noncoherent cooperative transmission portion of this work, we adopt the cooperative framework of [1] and develop new analytical approaches that allow us to obtain a complete set of necessary and sufficient conditions for full connectivity with respect to node density  $\lambda$  for a given path loss exponent  $\alpha$  in the channel model. A summary of the results for full connectivity are shown in Table I. The achievability results in Table I include the important  $\alpha = 1$  case in 1-D and  $\alpha = 2$  case in 2-D, and we note that  $\alpha < 2$  occurs in many 2-D communication scenarios where there is a “waveguiding” effect, such as in underwater acoustic communications where the power attenuation in commonly employed transmission bands is mainly dictated by a power law decay with exponent  $\alpha \approx 1.5$  [7]. Our results for the existence of percolation in the network are shown in Table II. Unlike the case of full connectivity, we do not have a complete set of necessary and sufficient conditions when considering percolation. Rather, we describe a number of results and then highlight the open problems.

The positive results in Tables I and II for the noncoherent cooperative transmission case serve as a lower bound to the connectivity gains of cooperative networks, and we discuss briefly in Section II how the negative results can be applied to other cooperative systems, as well. However, to also in-

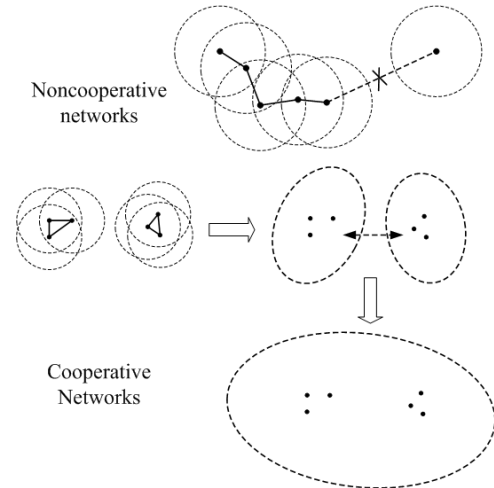


Fig. 1. Connectivity in noncooperative and cooperative networks

investigate an upper bound, we consider a coherent cooperative transmission framework modeled upon perfect distributed beamforming and present a less conclusive collection of results under this model.

The rest of this paper is organized as follows. In Section II, the cooperative transmission models are precisely defined. The full connectivity of extended networks assuming noncoherent cooperative transmission is the core of the paper and is studied in Section III. The percolation of extended networks assuming noncoherent cooperative transmission is studied in Section IV. Section V presents connectivity results for extended networks under the distributed beamforming model. Finally, we conclude in Section VI with comments on the problem considered and future work.

## II. COOPERATION MODELS

Cooperation techniques allow clusters of nodes that have already formed under a noncooperative model to pool their resources together to further connect isolated nodes; thus, the size of each cluster keeps growing until no more nodes can be pulled into any current cluster, as shown in Fig. 1. In

this section, we derive the requirement for one set of already connected nodes to establish a connection to another set of already connected nodes.

Under the conventional  $r$ -radius model [5] employed in the classical connectivity problem [6], each node is able to communicate directly with others (namely, one-hop neighbors) within a distance of  $r$ , which is determined as follows. Noting that wireless transmitters are generally constrained by their peak (rather than average) transmission power, let  $P_t$  be the peak transmission power of a single node and  $\tau$  be the minimum received power required at the receiver for physical layer functionality and a minimal utility under a given channel model and transceiver architecture (e.g. to achieve a minimum signal-to-noise required for synchronization and some minimal data rate). For two nodes to communicate directly, they must be within a distance  $r$  that satisfies:

$$P_t \cdot r^{-\alpha} = \tau \quad (1)$$

There are a number of possible methods for realizing physical layer cooperation. For example, at the high end of performance (and implementation complexity) are techniques such as distributed beamforming [8], which provides coherent voltage summing at the receiver by precisely phasing transmissions. Other techniques include cooperative diversity [9], distributed multiple-input multiple-output (MIMO) [10], etc. Since our strongest conclusion is tied to achievability, we present in detail a simple distributed frequency-shift keyed (FSK) cooperation method that does not require phase coherence at the transmitter and represents a worst case of cooperation that achieves only power summing. However, having the conclusive results of Table I for this model, we then discuss below how these results can be applied to rigorously arrive at analogous lower and upper bounds for the performance of the more conventional technique of cooperative diversity [9]. We also present a distributed beamforming model to find results for an optimistic form of cooperation.

For each model, the key is understanding when a connected set of nodes  $\Omega_A$  can connect to another connected set of nodes  $\Omega_B$ , where the size of either  $\Omega_A$  or  $\Omega_B$  can be unity to correspond to the noncooperative transmit or noncooperative receive case, respectively. Consider first the model for the noncoherent distributed FSK scheme under the worst-case (for connectivity purposes) assumption of a frequency-nonselective quasi-static Rayleigh fading channel between each pair of nodes in the network, where the fading between different pairs is assumed to be independent and identically distributed (IID). From the detailed physical layer description and performance derivation in the Appendix, clusters  $\Omega_A$  and  $\Omega_B$  can be connected if

$$P_t \sum_{j \in \Omega_A} \sum_{k \in \Omega_B} (d_{j,k})^{-\alpha} \geq \tau \quad (2)$$

where  $d_{j,k}$  is the distance between nodes  $j$  and  $k$ , and  $\tau$  is the same as in (1). Equation (2) is symmetric in  $\Omega_A$  and  $\Omega_B$ : (2) is true for  $\Omega_A$  to be able to transmit to  $\Omega_B$  if and only if (2) is true for  $\Omega_B$  to be able to transmit to  $\Omega_A$ .

Throughout Sections III and IV, the sufficient (but not always necessary) condition of (2) is treated as both sufficient and necessary, which is rigorously justified as follows. First,

since it is sufficient, it can clearly be used to establish the positive (achievability) results in Tables I and II (Theorems 3.3, 3.7, and 4.3). Next, from (16) and the discussion that follows it, (2) becomes both sufficient and necessary when  $|\Omega_B| = 1$ . Since the converse (negative) results in Table I (Theorems 3.4 and 3.8) only employ (2) in the  $|\Omega_B| = 1$  case, these results also hold for the exact physical layer model. Finally, since the reduction in required received signal power due to diversity is bounded, converging down to the threshold for an additive white Gaussian noise (AWGN) system in the limit of high diversity, the converse result of Table II (Theorem 4.1), which is true for any fixed threshold, still holds.

Likewise, suppose that we are interested in the performance of networks employing cooperative diversity techniques as described in [9], which would provide a  $|\Omega_A|$  times higher diversity than the model of (2). Clearly, the positive results still apply. Again, since the reduction in the required received signal power due to diversity is bounded, this does not change the negative results in Tables I and II, since these results hold for any fixed threshold.

Finally, we present a model for distributed beamforming [8]. Because we want to use this as an upper bound to all forms of cooperation, we allow the distributed beamforming model to encompass both the distributed FSK scheme, when that is advantageous due to its receiver cooperation, plus a scheme that assumes phase coherence at the transmitter. Furthermore, we will be optimistic (from the perspective of connectivity) and assume an AWGN channel between any pair of nodes. Under these assumptions, a group of nodes  $\Omega_A$  can transmit to a group of nodes  $\Omega_B$  if there exists a node  $i$  in  $\Omega_B$  such that:

$$\sum_{j \in \Omega_A} \sqrt{\frac{P_t}{(d_{i,j})^\alpha}} \geq \tau_{\text{AWGN}} \quad (3)$$

as derived in the Appendix, where  $\tau_{\text{AWGN}}$  is the required received signal power to establish direct communication on an AWGN channel. Unlike the noncoherent cooperative transmission scheme, this model is not symmetric, in that there may exist cluster  $\Omega_A$  that can transmit messages to nodes in cluster  $\Omega_B$ , but for which nodes in cluster  $\Omega_B$  cannot transmit messages to nodes in  $\Omega_A$ .

### III. FULL CONNECTIVITY UNDER THE NONCOHERENT COOPERATIVE TRANSMISSION MODEL

In this section, we establish conditions for full connectivity laws under the noncoherent cooperative transmission model for various path loss exponents  $\alpha$  in 1-D and 2-D extended networks. The results are shown in Table I. As with prior work of ours and others, the results provide conditions on  $\lambda r$  (1-D) or  $\lambda r^2$  (2-D) for connectivity for a given  $\alpha$ ; hence, for clarity, we assume without loss of generality that each node has transmission radius  $r = 1$  and suppress the straightforward modifications for the  $r \neq 1$  case. In fact, for the results of this section, statements are established for any  $\lambda > 0$  in the  $r = 1$  case, which implies that they are also true for any  $\lambda > 0$  for any  $r > 0$ .

### A. One-dimensional Networks

1) *Path Loss Exponent*  $\alpha \leq 1$ : The main idea of the achievability results is to show that, with probability one, there exists some segment of the line that satisfies a particular quite restrictive node distribution property; then, this segment can be shown to be fully connected, and, starting with this segment, other adjacent segments can be captured successively until the entire line is covered. The successive capture requires two technical lemmas, which are presented before the full theorem.

**Lemma 3.1:** For arbitrary  $T > 0$  and integer  $n > 0$ , consider an interval  $[0, (1 + 1/n)T]$  with each of the  $n + 1$  subintervals  $[iT/n, (i + 1)T/n]$  ( $i = 0, 1, \dots, n$ ) containing at least  $\theta T/n$  nodes,  $\theta > 1/[\sum_{i=1}^n 1/(i + 1)]$ , and all of the nodes within  $[0, T)$  are fully connected. Then under the noncoherent cooperative transmission model, the nodes within  $[0, (1 + 1/n)T)$  are fully connected.

*Proof:* It suffices to consider the case where the nodes are situated as follows: groups of  $\theta T/n$  nodes at locations  $iT/n$  ( $i = 0, \dots, n - 1$ ) and a single node at location  $(1 + 1/n)T$ . Thus, given that the nodes in  $[0, T)$  are fully connected, the received power at  $(1 + 1/n)T$  is lower bounded by

$$\frac{\theta T}{n} \sum_{i=0}^{n-1} \left( \frac{(i+2)T}{n} \right)^{-1} = \theta \sum_{i=1}^n \frac{1}{i+1} > 1$$

**Lemma 3.2:** For arbitrary  $T > 0$  and integer  $n > 0$ , consider an interval  $[0, T)$  with  $M$  nodes arranged in two different configurations. In Configuration 1, the nodes are divided into  $n$  equal size groups at locations  $iT/n$  ( $i = 0, \dots, n - 1$ ) and in Configuration 2, they are divided into  $n + 1$  equal size groups at locations  $iT/(n + 1)$  ( $i = 0, \dots, n$ ). Let  $R_k(x)$  denote the received power at any location  $x + T$  ( $x \geq 0$ ) under Configuration  $k$  ( $k = 1, 2$ ) and assume that the nodes within  $[0, T)$  are fully connected. Then  $R_1(x) \leq R_2(x), \forall x \geq 0$ .

*Proof:*

$$\begin{aligned} -R_1(x) &= \frac{-M}{n} \sum_{i=1}^n \frac{1}{x + iT/n} \\ &= \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right) \\ -R_2(x) &= \frac{-M}{n+1} \sum_{i=1}^{n+1} \frac{1}{x + iT/(n+1)} \\ &= \frac{1}{n+1} \sum_{i=1}^{n+1} f\left(\frac{i}{n+1}\right) \end{aligned}$$

where  $f(y) \triangleq -M/(x + Ty)$ . Since  $f(y)$  is strictly increasing and concave for  $y \geq 0$ , Theorem 2 of [16] yields  $-R_1(x) \geq -R_2(x)$ . ■

**Theorem 3.3:** In a 1-D extended network with  $\alpha \leq 1$ , transmission radius  $r = 1$ , and node density  $\lambda > 0$ , full connectivity occurs with probability one.

*Proof:* First, assume  $\alpha = 1$ . Consider  $N$  segments of the line, each of length  $L$ . For any  $\lambda > 0$ , consider arbitrary  $\theta$  such that  $0 < \theta < \lambda$ , and let  $\varepsilon = \lambda - \theta > 0$ . We will generally demonstrate connectivity results when there is a density of  $\theta$  nodes per unit length in a particular sequence

of (large) segments, which we can then show is true with high probability since  $\theta < \lambda$ . Choose the smallest integer  $n$  such that  $\theta > 1/[\sum_{i=1}^n 1/(i + 1)]$ , which is always possible since the sum in the denominator diverges as  $n \rightarrow \infty$ . Divide each segment of length  $L$  into  $n$  subsegments of equal length  $\log \log N / \gamma$ , where  $\gamma = \varepsilon^2 / [2(\theta + \varepsilon)]$ .

We are interested in finding a segment of length  $L$  that has a particular distribution of groups of connected nodes. Let  $C$  be the event that there exists at least one segment of the  $N$  of length  $L$  such that each of its  $n$  subsegments of length  $L/n$  has a set of greater than or equal to  $\theta L/n$  nodes within a single unit length. Now, we show that  $P(C) \rightarrow 1$  as  $N \rightarrow \infty$ . Define  $C_i$  as the event that the  $i^{\text{th}}$  segment of length  $L$  has the desired node distribution. Let  $n_{i,k}$  be the number of nodes in the  $k^{\text{th}}$  subsegment of the  $i^{\text{th}}$  segment. Write:

$$\begin{aligned} P(C_i) &= P\left(C_i \mid \left\{n_{i,k} \geq \frac{\theta L}{n}, k = 1, \dots, n\right\}\right) \\ &P\left(\left\{n_{i,k} \geq \frac{\theta L}{n}, k = 1, \dots, n\right\}\right) \end{aligned} \quad (4)$$

The first term can be lower bounded by requiring each subsegment to have at least  $\theta L/n$  nodes in a specific unit length, recalling the uniformity of distribution of a fixed number of nodes in a given length of a Poisson process, and recognizing that node locations in different subsegments are independent. Then,

$$\begin{aligned} &P\left(C_i \mid \left\{n_{i,k} \geq \frac{\theta L}{n}, k = 1, \dots, n\right\}\right) \\ &\geq P\left(C_i \mid \left\{n_{i,k} = \frac{\theta L}{n}, k = 1, \dots, n\right\}\right) \\ &\geq \left[\left(\frac{1}{L/n}\right)^{\frac{\theta L}{n}}\right]^n \\ &= \left(\frac{n}{L}\right)^{\theta L} \end{aligned}$$

The second term of (4) is lower bounded through a Chernoff bound argument. The number of nodes  $n_{i,k}$  in subsegment  $k$  of length  $L/n$  is a Poisson random variable with parameter  $\mu = \lambda L/n$ . Thus, for any  $\delta \in (0, 1]$ ,

$$P(n_{i,k} < (1 - \delta)\mu) < \exp(-\mu\delta^2/2) \quad (5)$$

which implies (using  $\delta = \varepsilon/\lambda$ ):

$$\begin{aligned} &P\left(\left\{n_{i,k} \geq \frac{\theta L}{n}, k = 1, \dots, n\right\}\right) \\ &= \prod_{k=1}^n \left(1 - P\left(n_{i,k} < \frac{\theta L}{n}\right)\right) \\ &\geq \prod_{k=1}^n \left(1 - \exp\left(-\frac{\lambda L}{n} \frac{\varepsilon^2}{2\lambda^2}\right)\right) \\ &= \left(1 - \frac{1}{\log N}\right)^n \end{aligned}$$

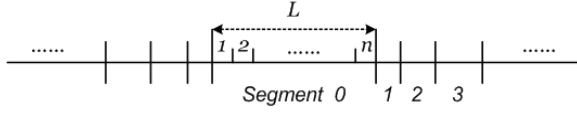


Fig. 2. Adding segments in 1-D extended networks

The above arguments, combined with the continuity of probability, yield:

$$\begin{aligned}
 P\left(\lim_{N \rightarrow \infty} \bigcup_{i=1}^N C_i\right) &= \lim_{N \rightarrow \infty} P\left(\bigcup_{i=1}^N C_i\right) \\
 &= \lim_{N \rightarrow \infty} (1 - (P(\overline{C}_i))^N) \\
 &\geq \lim_{N \rightarrow \infty} \left(1 - \left(1 - \left(\frac{\theta L}{n}\right) \left(1 - \frac{1}{\log N}\right)^n\right)^N\right) \\
 &= 1
 \end{aligned}$$

Per above, we will use such a segment to grow our cooperative cluster. First, for this segment, note that all of the nodes in the  $n$  groups, each with number of nodes  $\theta L/n$  and confined to a unit length, are connected. To see this, observe that the  $\theta L/n$  nodes within a unit length of any subsegment are connected (because  $r = 1$ ). Next, the aggregate received power at a group in one subsegment from a group in an adjacent subsegment is lower bounded by:

$$\left(\frac{\theta L}{n}\right) \left(\frac{\theta L}{n}\right) \frac{1}{2L/n} = \frac{\theta^2 L}{2n} > 1$$

for sufficiently large  $N$ . Since every group can reach the group in the subsegment to either side, all of these groups of nodes are connected. It is also possible to show that any other single node in the segment will also be connected to this emerging cluster by using the exact proof of Lemma 3.1 above, except with the single node at the worst-case location  $T = L$ . Thus, if  $C$  occurs, there exists a segment of length  $L = n \log \log N / \gamma$  that both satisfies the desired node distribution condition and is fully connected.

Now, relabel this originating segment as segment 0. We add to both sides of segment 0 an infinite number of adjacent segments, where the  $k$ -th segment to the right or left of segment 0 has length  $l_k = (1 + 1/n)^{k-1} L/n$ ,  $k = 1, 2, \dots$ , as shown in Fig. 2. Consider  $N - 1$  segments to the right of segment 0. Let  $m_k$  denote the number of nodes in the  $k$ -th segment, which is Poisson distributed with parameter  $\mu_k = E[m_k] = \lambda l_k$ . The application of Chernoff's bound (with  $\delta = \varepsilon/\lambda$ ) yields:

$$\begin{aligned}
 P\left(m_k < \left(1 + \frac{1}{n}\right)^{k-1} \frac{\theta L}{n}\right) \\
 &< \exp\left(-\frac{\varepsilon^2}{2(\theta + \varepsilon)} \left(1 + \frac{1}{n}\right)^{k-1} \frac{L}{n}\right) \\
 &= \exp\left(-\gamma \left(1 + \frac{1}{n}\right)^{k-1} \frac{L}{n}\right)
 \end{aligned}$$

Since the numbers of nodes in the segments are independent Poisson random variables, we have:

$$\begin{aligned}
 P\left(m_k \geq (1 + 1/n)^{k-1} \theta L/n, \forall k\right) \\
 &\geq \prod_{k=1}^{N-1} (1 - \exp(-\gamma (1 + 1/n)^{k-1} L/n))
 \end{aligned} \tag{6}$$

Since  $L = n \log \log N / \gamma$ ,

$$\exp(-\gamma (1 + 1/n)^{k-1} L/n) = 1/(\log N)^{(1+1/n)^{k-1}}$$

and

$$\begin{aligned}
 P\left(m_k \geq (1 + 1/n)^{k-1} \theta L/n, \forall k\right) \\
 &\geq \prod_{k=1}^{N-1} \left(1 - \frac{1}{(\log N)^{(1+1/n)^{k-1}}}\right) \\
 &\xrightarrow{N \rightarrow \infty} 1
 \end{aligned} \tag{7}$$

We now apply Lemmas 3.1 and 3.2 to establish that the network is fully connected with probability one as  $N \rightarrow \infty$ , as follows. Per above, segment 0 is the segment of length  $L$  whose corresponding event  $C_i$ , as defined above, occurs. Now, by (7), we know that, with high probability, segments 1, 2, 3,  $\dots$  have a node density greater than  $\theta$  nodes per unit length. Applying Lemma 3.1, every node in segment 1 is able to join the cluster of segment 0. Furthermore, by Lemma 3.2, when trying to pick up all successive segments, the power  $R_2(x)$  received from the connected length  $(1 + 1/n)L$  is lower bounded by one whose nodes are evenly distributed in  $n$  (rather than  $n+1$ ) equal-length subsegments on  $(1 + 1/n)L$ . Now, Lemma 3.1 reveals that the latter pessimistic configuration is sufficient to join all nodes in the segment of length  $(1 + 1/n)(1 + 1/n)L = L + L/n + (1 + 1/n)L/n$ , thus connecting segment 2. Then, Lemma 3.2 is applied again. Obviously, the repetition of this process in each direction from segment 0 connects the entire line into the cluster. Since  $\lambda$  was arbitrary, the result is established for  $\alpha = 1$ . Finally, a system with  $\alpha < 1$  performs at least as well as one with  $\alpha = 1$ . Therefore the result holds for all  $\alpha \leq 1$ . ■

2) *Path Loss Exponent*  $\alpha > 1$ :

**Theorem 3.4:** Under the noncoherent cooperative transmission model, full connectivity occurs with probability zero in a 1-D extended network for any node density  $\lambda > 0$  when  $\alpha > 1$ .

*Proof:* For any probability of disconnection  $1 - \varepsilon/2$  arbitrarily close to one, we will show that there exists a distance  $d^*$  such that a node  $j$  with no other nodes within distance  $d^*$  of node  $j$  has greater than probability  $1 - \varepsilon/2$  of being disconnected. Observing that such a node exists somewhere on the line with high probability completes the proof.

Consider arbitrary  $\varepsilon > 0$  and consider node  $j$  located at  $x_j$ . Assuming the optimistic case that all other nodes except node  $j$  are in a single connected cluster, the power received at node  $j$  is given by:

$$X = \sum_{\forall x_k, k \neq j} \frac{1}{(d_{k,j})^\alpha}$$

When  $\alpha > 1$ ,  $X$  converges with high probability. The singularity at the origin causes technical difficulties, so, for some small  $\delta$  such that  $0 < \delta < 1$ , define

$$\tilde{X} = \sum_{\forall x_k, |x_k - x_j| > \delta} \frac{1}{(d_{k,j})^\alpha}$$

which has finite mean  $\mu \triangleq E[\tilde{X}]$ . Moreover, define  $\tilde{X}(n)$  for integer  $n$  as

$$\tilde{X}(n) = \sum_{\forall x_k, \delta < |x_k - x_j| < n} \frac{1}{(d_{k,j})^\alpha}.$$

Now, for any sample point for which  $X$  converges,  $\tilde{X}(n)$  is non-negative and non-decreasing as it converges to  $\tilde{X}$ , and, hence, by the monotone convergence theorem of integration,

$$E[\tilde{X}(n)] \rightarrow E[\tilde{X}].$$

Thus, there exists a  $d^*$  such that for  $n \geq d^*$ ,  $E[\tilde{X}] - E[\tilde{X}(n)] < \varepsilon/2$ . Define:

$$Z(n) = \tilde{X} - \tilde{X}(n) = \sum_{x_k \notin (x_j - n, x_j + n)} \frac{1}{(d_{k,j})^\alpha}$$

By the Markov inequality,

$$P(Z(d^*) > 1) \leq E[Z(d^*)] < \frac{\varepsilon}{2}. \quad (8)$$

Now, with probability greater than  $1 - \varepsilon/2$ , we can find a node somewhere on the line who has no neighbors within distance  $d^*$ . By (8), such a node is disconnected with probability greater than  $1 - \varepsilon/2$ . ■

## B. Two-dimensional Networks

1) *Path Loss Exponent  $\alpha \leq 2$* : The main line of thought is analogous to the proof of Theorem 3.3: we show that, with high probability, there exists at least one  $L \times L$  region with a desired node distribution, and then we grow the fully connected component from that region. However, because we have not been able to establish a 2-D analog of Lemma 3.2, the growth from the initial region is constructed differently.

Two technical lemmas precede the proof of the full statement.

*Lemma 3.5*: For arbitrary  $T > 0$  and integer  $n > 0$ , consider a square  $[0, 2T]^2$  with  $4n^2$  equally sized subsquares  $[iT/n, (i+1)T/n] \times [jT/n, (j+1)T/n]$  ( $i, j = 0, 1, \dots, 2n-1$ ), each of which containing at least  $\theta T^2/n^2$  nodes,  $\theta > 1 / \left( \sum_{i=4}^{n+3} 1/i - \sum_{i=4}^{n+3} 3/i^2 \right)$ , and all of the nodes within the square  $[0, T]^2$  are fully connected. Then, under the noncoherent cooperative transmission model, the nodes within  $[0, 2T]^2$  are fully connected.

*Proof*: The doubling takes place in  $n$  steps, resulting in a square of size  $(1+k/n)T \times (1+k/n)T$  at step  $k = 1, 2, \dots, n$ . At step  $k$ , we start with a fully connected square of size  $(1+(k-1)/n)T \times (1+(k-1)/n)T$ , consisting of  $(n+k-1)^2$  small squares of size  $T/n \times T/n$ , each with  $\theta T^2/n^2$  nodes. Assume the worst case that these nodes are located in the lower left corner of each of the small squares and that we are trying to connect a single node at location  $((1+k/n)T, (1+k/n)T)$ . Since, at any stage, there is always an  $n \times n$  set of

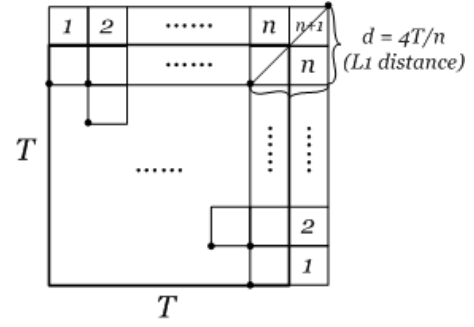


Fig. 3. A lower bound on the received power

squares with identical distances to  $((1+k/n)T, (1+k/n)T)$  as those between the  $n \times n$  set  $[iT/n, (i+1)T/n] \times [jT/n, (j+1)T/n]$  ( $i, j = 0, 1, \dots, n-1$ ) and the point  $((1+1/n)T, (1+1/n)T)$ , it is sufficient to consider the  $k = 1$  case. In the  $k = 1$  case, the received power at  $((1+1/n)T, (1+1/n)T)$  is lower bounded by:

$$\begin{aligned} & \frac{\theta T^2}{n^2} \sum_{i=1}^n \frac{i}{((i+3)T/n)^2} \\ &= \theta \sum_{i=4}^{n+3} \frac{i-3}{i^2} \\ &= \theta \left( \sum_{i=4}^{n+3} \frac{1}{i} - \sum_{i=4}^{n+3} \frac{3}{i^2} \right) \\ &> 1 \end{aligned}$$

The lower bound arises by taking the  $L_1$  distance rather than the Euclidean distance and by only counting power from nodes in the upper right half of the square  $[0, T]^2$ , as shown in Fig. 3. ■

*Lemma 3.6*: For arbitrary  $T > 0$  and integer  $n > 0$ , consider a square  $[0, T]^2$  with  $M$  nodes arranged in two different configurations. In Configuration 1, the nodes are divided into  $n^2$  equal size groups at locations  $(iT/n, jT/n)$  ( $i, j = 0, 1, \dots, n-1$ ), and, in Configuration 2, they are divided into  $4n^2$  equal size groups at locations  $(iT/2n, jT/2n)$  ( $i, j = 0, 1, \dots, 2n-1$ ). Let  $R_k(x)$  denote the received power at any location  $(x+T, x+T)$  ( $x \geq 0$ ) under Configuration  $k$  ( $k = 1, 2$ ) and assume that the nodes within  $[0, T]^2$  are fully connected. Then  $R_1(x) \leq R_2(x), \forall x \geq 0$ .

*Proof*: Consider modifying Configuration 2 to arrive at Configuration 1 in the following manner. Form  $n^2$  sets of four locations each in Configuration 2:

$$A_{i,j} = \left\{ \left( \frac{iT}{n}, \frac{jT}{n} \right), \left( \frac{(2i+1)T}{2n}, \frac{jT}{n} \right), \left( \frac{iT}{n}, \frac{(2j+1)T}{2n} \right), \left( \frac{(2i+1)T}{2n}, \frac{(2j+1)T}{2n} \right) \right\}, \\ i = 0, 1, \dots, n-1, \quad j = 0, 1, \dots, n-1$$

and now, for each set, co-locate the nodes at the other three locations with those at  $(iT/n, jT/n)$ . Clearly, this results in Configuration 1 and reduces the power received at  $(x+T, x+T)$  from each set. ■

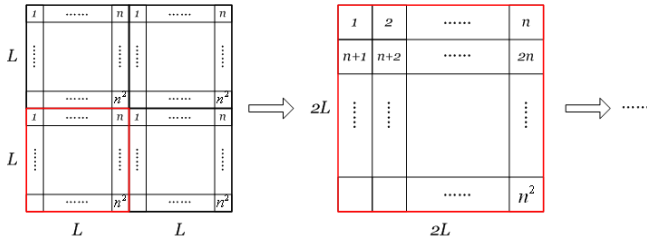


Fig. 4. Adding squares in 2-D extended networks

**Theorem 3.7:** In a 2-D extended network with  $\alpha \leq 2$ , transmission radius  $r = 1$  and node density  $\lambda > 0$ , full connectivity occurs with probability one.

*Proof:* First, assume  $\alpha = 2$ . Consider  $N^2$  squares each of size  $L \times L$  in a 2-D network. For any  $\lambda > 0$ , consider arbitrary  $\theta$  such that  $0 < \theta < \lambda$ , and let  $\varepsilon = \lambda - \theta > 0$ . Define  $n$  as the smallest integer such that  $\theta > 1 / \left[ \sum_{i=4}^{n+3} 1/i - \sum_{i=4}^{n+3} 3/i^2 \right]$ , which we can always do because the denominator diverges as  $n \rightarrow \infty$ . Divide each  $L \times L$  square into  $n^2$  subsquares of size  $L/n \times L/n$ . Let  $L = n\sqrt{\log \log N / \gamma}$ , where  $\gamma = \varepsilon^2 / [2(\theta + \varepsilon)]$ .

Now, we are interested in finding an  $L \times L$  square that satisfies a node distribution property analogous to the 1-D case. Let  $C$  be the event that there exists at least one square of size  $L \times L$  such that each of its  $n^2$  subsquares of size  $L/n \times L/n$  has greater than or equal to  $\theta L^2/n^2$  nodes within a single area of size  $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$ . Then, similarly to the 1-D case, it can be shown that  $P(C) \rightarrow 1$  as  $N \rightarrow \infty$ .

Now, for the  $L \times L$  square satisfying the subsquare distribution requirement, each set of at least  $\theta L^2/n^2$  nodes within a single area of size  $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$  in a given subsquare is connected and can connect to the corresponding set in the subsquares in all four primary directions (up, down, left, right) if

$$\left( \frac{\theta L^2}{n^2} \right) \left( \frac{\theta L^2}{n^2} \right) \left( \frac{n}{\sqrt{5}L} \right)^2 = \frac{\theta^2 L^2}{5n^2} > 1,$$

which is true for sufficiently large  $N$ . Hence, all of these groups of nodes are connected. As in the 1-D case, it is then straightforward to establish that any node in the  $L \times L$  square is also connected to this cluster (In this case, use a simplified version of Lemma 3.5).

Now, we grow in a slightly different manner than in the 1-D case. In particular, at the first stage, we will grow from size  $L \times L$  to size  $2L \times 2L$  by adding  $3n^2$  (small) squares of size  $L/n \times L/n$  to the cluster. Then, we will show that dividing the new  $2L \times 2L$  square into  $n^2$  (rather than  $4n^2$ ) subsquares is pessimistic for further growth, yet is sufficient for that growth to proceed indefinitely, as shown in Fig. 4.

To establish this line of thought, we first demonstrate that, with high probability, all (small) squares at all stages have a density of at least  $\theta$  nodes per unit area. Let  $m_{k,i}$  ( $k = 1, 2, 3, \dots$ ,  $i = 0, 1, \dots, 3n^2 - 1$ ) be the number of nodes in the  $i^{\text{th}}$  (small) square to be added at the  $k^{\text{th}}$  stage, and consider  $N$  stages of growth. Since there are  $3n^2$  squares of size  $(2^{k-1}L/n) \times (2^{k-1}L/n)$  at the  $k^{\text{th}}$  stage, the desired

probability is written as:

$$\begin{aligned} & P(m_{k,i} \geq 2^{2(k-1)}\theta L^2/n^2, \forall k, \forall i) \\ &= \prod_{k=1}^N P(m_{k,i} \geq 2^{2(k-1)}\theta L^2/n^2, \forall i) \\ &= \prod_{k=1}^N \left[ P(m_{k,0} \geq 2^{2(k-1)}\theta L^2/n^2) \right]^{3n^2} \\ &\geq \prod_{k=1}^N \left[ 1 - \exp(-2^{2(k-1)} \log \log N) \right]^{3n^2} \\ &= \prod_{k=1}^N \left[ 1 - \frac{1}{(\log N)^{2^{2(k-1)}}} \right] \xrightarrow{N \rightarrow \infty} 1 \end{aligned}$$

where the inequality follows from the Chernoff bound (see the proof of Theorem 3.3).

Given the high probability event characterized in (9), the successive application of Lemma 3.5 and Lemma 3.6 allows the  $L \times L$  square satisfying the subsquare distribution requirement (a square that results in the occurrence of event  $C$ ) to grow to cover  $[0, \infty) \times [0, \infty)$ .

Finally, start with a  $2L \times 2L$  region, where each  $L \times L$  quarter of the region satisfies the subsquare distribution requirement and grow each of the quarters as above to cover a different quadrant of the 2-D plane, thus connecting all nodes in the 2-D plane for  $\alpha = 2$  and any  $\lambda > 0$ . A system with  $\alpha < 2$  performs at least as well as one with  $\alpha = 2$ . Therefore the result holds for all  $\alpha \leq 2$ . ■

## 2) Path Loss Exponent $\alpha > 2$ :

**Theorem 3.8:** Under the noncoherent cooperative transmission model, full connectivity occurs with probability zero in a 2-D extended network for any node density  $\lambda > 0$  when  $\alpha > 2$ .

*Proof:* The proof follows that of Theorem 2: there exists a  $d^*$  such that a node with no neighbors within 2-D distance  $d^*$  is disconnected with high probability, and such a node exists with high probability. ■

## IV. PERCOLATION UNDER THE NONCOHERENT COOPERATIVE TRANSMISSION MODEL

In this section, we establish conditions for percolation under the noncoherent cooperative transmission model in terms of node density  $\lambda$ , when the path loss exponent  $\alpha$  takes different values in 1-D and 2-D extended networks. The results are summarized in Table II.

We define the cooperatively occupied region to be the region which is reachable by the nodes distributed according to the Poisson process using cooperative communication. We define by  $W^*$  the cooperatively occupied component of the origin, and let  $\theta_r^*(\lambda) = \theta^*(\lambda)$  be the probability that  $W^*$  is an unbounded cooperatively occupied component. This is similar to the percolation function  $\theta(\lambda)$  in the non-cooperative case. As for  $\theta$ ,  $\theta^*(\lambda)$  is an increasing function in  $\lambda$  (the proof of Proposition 2.9 in [3] applies almost identically).

We can thus define  $\lambda_c^*$  to be the critical density, as for the non-cooperative case:

$$\lambda_c^*(r) = \inf\{\lambda \geq 0 : \theta_r^*(\lambda) > 0\} \quad (9)$$

One obvious equality is that  $0 \leq \lambda_c^* \leq \lambda_c$ . This section focuses on discussing whether  $\lambda_c^*$  is trivial (0 or  $\infty$ ) in 1-D and 2-D for some values of  $\alpha$ , and on proving bounds on  $\lambda_c^*$  in the 2-D case.

### A. One-dimensional Networks

Clearly, Theorem 3.3 implies percolation for all  $\lambda$  for any pathloss exponent  $\alpha \leq 1$ . Hence, our interest here is for  $\alpha > 1$ . The following result establishes a negative result for path loss exponents  $\alpha > 2$ .

*Theorem 4.1:* Under the noncoherent cooperative transmission model, the node at the origin in a 1-D network is in a finite cluster with high probability for any node density  $\lambda > 0$  when  $\alpha > 2$ .

*Proof:* The first part of the proof follows that of Theorem 3.4. In particular, we show that there exists a distance  $d^*$  such that a gap in the line of length  $2d^*$  leads to a disconnection of any clusters to its left and right with high probability. Showing that such a gap occurs to the left and right of node at the origin within a fixed length with high probability completes the proof.

Consider any  $\varepsilon > 0$ . For some point  $x_0$  on the line, let:

$$\tilde{X} = \sum_{\forall x_k < x_0, |x_k - x_0| > \delta} \sum_{\forall x_j > x_0, |x_j - x_0| > \delta} \frac{1}{(d_{k,j})^\alpha}$$

which, excepting nodes near  $x_0$  for technical reasons, is the aggregate of the received power of all nodes to the right of  $x_0$  if all of the nodes to the left of  $x_0$  transmitted cooperatively. Note that  $E[\tilde{X}] < \infty$  for  $\alpha > 2$ . Define:

$$\tilde{X}(n) = \sum_{x_0 - n < x_k < x_0 - \delta} \sum_{x_0 + \delta < x_j < x_0 + n} \frac{1}{(d_{k,j})^\alpha}$$

Now,  $\tilde{X}(n)$  converges up to  $\tilde{X}$ ; hence, by the monotone convergence theorem,  $E[\tilde{X}(n)] \rightarrow E[\tilde{X}]$ . Thus, there exists a  $d^*$  such that  $\forall n \geq d^*$ ,  $E[\tilde{X}] - E[\tilde{X}(n)] < \varepsilon/2$ . Define:

$$Z(n) = \tilde{X} - \tilde{X}(n) = \sum_{x_k < x_0 - n} \sum_{x_j > x_0 + n} \frac{1}{(d_{k,j})^\alpha}$$

By the Markov Inequality,

$$P(Z(n) > 1) \leq E[Z(d^*)] < \frac{\varepsilon}{2}$$

for any  $n > d^*$ . Hence, a gap of length  $2d^*$  causes disconnection of portions to the left and right of  $x_0$  with probability  $\geq 1 - \varepsilon/2$ .

Now, consider a node at the origin, and divide the line to its right into segments of length  $2d^*$ . The number of nodes in a segment is Poisson with parameter  $2d^*\lambda$ , and thus each segment has an identical non-zero probability of having zero nodes. Hence, one can choose an  $M^*$  large enough such that the probability of not observing such a segment in  $M^*$  lengths is less than  $\varepsilon/4$ . Hence, the node at the origin has gaps of length  $\geq 2d^*$  to its left and right within the interval  $(-2M^*d^*, 2M^*d^*)$  with probability  $\geq 1 - \varepsilon/2$ . Since the number of nodes within a length of  $4M^*d^*$  is Poisson, it is finite with probability 1, and, with probability  $\geq 1 - \varepsilon$ , the number of nodes to which a node at the origin is connected is upper bounded by the number of nodes in a length  $4M^*d^*$ .

Hence, the node at the origin is in a finite cluster with probability  $\geq 1 - \varepsilon$  for any  $\varepsilon > 0$ . ■

Unfortunately, we are unable to obtain a positive or negative result for  $1 < \alpha \leq 2$ . A path to a positive result would likely demonstrate that connected sets emerge between gaps in the line with sufficient regularity to allow connectivity to “jump” the gaps, while a negative result would require exactly the opposite.

### B. Two-dimensional Networks

Clearly, Theorem 3.3 implies percolation for all  $\lambda$  for any pathloss exponent  $\alpha \leq 2$ . Hence, our main interest here is for  $\alpha > 2$ .

Consider the critical density for percolation to occur in 2-D networks employing noncoherent cooperative transmission with  $\alpha > 2$ . Consider  $r = 1$ , and normalize the power so that two nodes connect if the received power is greater than  $1/r^\alpha = 1$ . In order to consider the effect of cooperation, consider two nodes  $i_0$  and  $i_1$ . Without loss of generality, we can assume that  $i_1$  is at the origin, and that  $i_0$  lies on the  $x$ -axis with a negative  $x$ -coordinate. Let  $d$  be the distance between nodes  $i_0$  and  $i_1$ . Then, nodes  $i_0$  and  $i_1$  can jointly connect to a node  $j$  if

$$\frac{1}{d_{i_0,j}^\alpha} + \frac{1}{d_{i_1,j}^\alpha} \geq 1 \quad (10)$$

Consider a node  $k$  on the positive  $x$ -axis such that  $1/d_{i_0,k}^\alpha + 1/d_{i_1,k}^\alpha = 1$ . The coordinates of node  $k$  are  $(x_k, 0)$ , with  $x_k$  satisfying

$$\frac{1}{x_k^\alpha} + \frac{1}{(x_k + d)^\alpha} = 1 \quad (11)$$

*Lemma 4.2:* If node  $i_0$  and node  $i_1$  are connected, they can cooperatively connect to any node within the circle of center  $x_{i_1}$  and radius  $x_k$ .

*Proof:* For any node  $j$  within the circle of radius  $x_k$  centered at the origin,

$$\frac{1}{d_{i_0,j}^\alpha} + \frac{1}{d_{i_1,j}^\alpha} \geq \frac{1}{x_k^\alpha} + \frac{1}{(x_k + d)^\alpha} = 1$$

due to the fact that  $d_{i_1,j} = |x_j| \leq x_k$  and  $d_{i_0,j} \leq |x_j| + d$  by the triangle inequality. ■

It is easy to check that  $x_k > 1$ . Also,  $x_k$  is a function of  $d$  such that  $x_k$  is monotonously decreasing as  $d \rightarrow \infty$ . Writing  $x_k$  as a function of  $d$ , one can observe that  $x_k(1) > 1$  and  $\lim_{d \rightarrow \infty} x_k(d) = 1$ . This implies that there exists some  $d^* > 1$  such that  $x_k(d^*) = d^*$ . Lemma 4.2 implies that two nodes within distance  $d^*$  can jointly connect to any third node within distance  $d^*$  of either one.

*Theorem 4.3:* In a 2-D extended network with  $\alpha > 2$  and transmission radius  $r = 1$ , percolation occurs with probability one for any  $\lambda \geq \lambda_c^*$ , where  $\lambda_c^*$  satisfies:

$$\lambda_c^* \leq \frac{\lambda_c}{\left(\sqrt[\alpha]{1 + 1/2^\alpha}\right)^2} \quad (12)$$

*Proof:* As in Theorem 3.3, the main idea is to show that, with probability one, there exists an area of space which satisfies a particular node distribution property; then, this area



can be shown to be fully connected, and starting from there, an infinite connected component can be constructed. To identify the connected component, we couple the cooperative system with a non-cooperative system which does percolate. We restrict ourselves to pairwise cooperation, where  $\Omega_a$  contains two nodes, cooperatively communicating with a singleton  $\Omega_b$ .

Consider a Poisson process with density  $\lambda < \lambda_c$  and a Poisson Boolean model with radius  $d^*$  such that  $(d^*)^2\lambda \geq \lambda_c$ . Two nodes are connected if their distance is less than or equal to  $d^*$ . Since the Poisson Boolean model with radius  $d^*$  and density  $\lambda$  is a scaled version of the one with radius 1 and density  $(d^*)^2\lambda$ , and since  $(d^*)^2\lambda \geq \lambda_c$ , the Poisson Boolean model with radius  $d^*$  and density  $\lambda$  percolates with probability one.

This super-criticality of the Poisson Boolean model implies that there exists an infinite connected component almost surely.

Now consider a circle of radius  $1/2$  centered at the origin. With a fixed, positive probability, this circle contains two nodes,  $i_0$  and  $i_1$ . Further,  $i_0$  and  $i_1$  belongs to the infinite connected component of the Poisson Boolean model with a fixed probability, say  $\theta$  (for further discussion of the percolation function  $\theta(\rho, \lambda)$ , we refer the reader to [3]). The event that two nodes exist in  $\mathcal{C}(0, 1/2)$  and belong to the infinite connected component of the Poisson Boolean model has thus a probability  $\theta' > 0$ .

Conditioned on the fact that  $i_1$  belongs to the infinite connected component, we now select any other node  $j$  which also belongs to the infinite connected component. By definition of the connected component, this means that there exists an integer  $N$  and a sequence of nodes  $\{i_k\}$  ( $2 \leq k \leq N$ ) satisfying  $i_N = j$ , and for all  $1 \leq k < N$ , node  $i_k$  and  $i_{k+1}$  are within distance  $d^*$ .

According to Lemma 4.2, if two nodes are located within distance  $d^*$ , they can cooperatively reach any node within distance  $d^*$  of either one. Therefore, since node  $i_2$  is within distance  $d^*$  of node  $i_1$  and  $d_{i_0, i_1} \leq 1 < d^*$ , nodes  $i_0$  and  $i_1$  can jointly connect to node  $i_2$  (indeed, since we do not assume that for  $k \geq 1$ ,  $i_{k+1}$  is distinct from  $i_k$  or  $i_{k-1}$ , this connection might even be trivial). The fact that  $d_{i_0, i_1} < 1$  allows us to initiate the cooperative communication. Again, since node  $i_3$  is within distance  $d^*$  of node  $i_2$  and  $d_{i_1, i_2} \leq d^*$ , nodes  $i_1$  and  $i_2$  can connect to node  $i_3$ . In general, nodes  $i_{k-1}$  and  $i_k$  can connect to node  $i_{k+1}$ ; thus, we have a connected path between  $i_1, i_1, i_2, \dots, i_{N-1}, j$  in the cooperative communications model. This shows that both  $i_0$  and  $i_1$  can communicate to any node in the infinite connected component of the Poisson Boolean model.

Since the event “ $i_0$  and  $i_1$  exist and belong to the infinite connected component of the (non-cooperative) Poisson Boolean model” has probability  $\theta'$  strictly greater than 0, the cooperative model percolates as well for any  $\lambda$  greater than  $\lambda_c/(d^*)^2$ . Clearly,  $1/(d^*)^\alpha + 1/(2d^*)^\alpha = 1$  implies  $d^* = \sqrt[\alpha]{1 + 1/2^\alpha}$ , and this completes the proof. For  $\alpha = 2$ ,  $d^* = \sqrt{5/4}$ , and the cooperation effectively reduces the critical density by at least 20%. ■

For  $\alpha > 2$ , an open problem remains as to whether the critical density is strictly greater than zero, as we have been unable to demonstrate that there exists a (small)  $\lambda > 0$  where

the 2-D network does not percolate.

## V. EXTENDED NETWORKS: DISTRIBUTED BEAMFORMING MODEL

The  $\alpha \leq 1$  and  $\alpha > 2$  cases are straightforward extensions of the noncoherent transmission case, but the regime between  $\alpha = 1$  and  $\alpha = 2$  is interesting. The challenge here is the lack of reciprocity described under (3), without which it is difficult to establish full connectivity.

*Theorem 5.1:* In a 1-D extended network under the coherent cooperative transmission model:

- $\alpha \leq 1$ : full connectivity occurs for any  $\lambda > 0$ ;
- $1 < \alpha < 2$ : there are an infinite number of nodes that can transmit to every other node in the network for  $\lambda > 0$ ;
- $\alpha = 2$ : there are an infinite number of nodes that can transmit to every other node in the network for  $\lambda > 2$ ;
- $\alpha > 2$ : full connectivity never occurs for any  $\lambda > 0$ .

*Proof:* For the  $\alpha \leq 1$  case, the distributed FSK scheme achieves full connectivity for any  $\lambda > 0$ . The  $\alpha > 2$  case follows similar lines to the proof of Theorem 3.4.

For the  $1 < \alpha < 2$  case, consider  $N$  segments of length  $L$ , where  $N$  is a positive integer. For any finite node density  $\lambda > 0$ , choose arbitrary  $\theta$  such that  $0 < \theta < \lambda$  and denote  $\varepsilon = \lambda - \theta$ . Let  $L = \log \log N / \gamma$ , where  $\gamma = \varepsilon^2 / [2(\theta + \varepsilon)]$ . Similar to the proof of Theorem 3.3, we have:

$$P(\exists \text{ fully connected segment of length } L) \xrightarrow{N \rightarrow \infty} 1$$

Suppose we start from such a segment and divide the network into an infinite number of adjacent segments of exponentially growing length in both directions, such as  $\dots 4L, 2L, L, 2L, 4L \dots$ . We number these segments symmetrically starting from sequence number 0, such as  $\dots 2, 1, 0, 1, 2 \dots$ . Let  $n_k$  be the number of nodes in segment  $k$  ( $k = 0, 1, \dots, N-1$ ), where  $k$  is the sequence number and  $L_k = 2^k L$  is the segment length. Using Chernoff's bound,

$$P(n_k \geq 2^k \theta L, \text{ all } k) \xrightarrow{N \rightarrow +\infty} 1$$

Thus, when  $N$  tends to infinity and covers the whole line, each segment  $k$  contains at least  $2^k \theta L$  nodes with probability one. In order to connect the adjacent segment,  $n_k$  must satisfy:

$$\begin{aligned} n_k &= 2^k \theta L \geq [(2^k + 2^{k+1})L]^{\alpha/2} \\ \theta &\geq \frac{3^{\alpha/2}}{(2^k L)^{1-\alpha/2}} \\ &= \frac{3^{\alpha/2} \gamma^{1-\alpha/2}}{(2^k \log \log N)^{1-\alpha/2}} \xrightarrow{N \rightarrow +\infty} 0 \end{aligned}$$

Therefore, starting from the fully connected segment 0, the connected segments will reach neighboring segments in each step, expanding exponentially to cover the whole line.

For the  $\alpha = 2$  case, consider  $N$  segments of length  $L$ . Let  $\lambda = 2 + \varepsilon$ , where  $\varepsilon > 0$ . Based on the proof of Theorem 1 in [2],  $\beta = \varepsilon^2 / [2(2 + \varepsilon)]$  and  $L = 2 \log N / \beta$ , we have

$$P(n_k \geq 2L, \text{ all } k) \xrightarrow{N \rightarrow +\infty} 1$$

For any segment, if there are  $2L$  nodes located within a unit length, they will be able to reach all the other nodes in the

segment. Similarly to the  $1 < \alpha < 2$  case, we can prove that there exists a fully connected segment with probability 1.

$$\begin{aligned} P(\exists \text{ fully connected segment of length } L) &\geq 1 - (1 - (1/L)^{2L})^N \\ &= 1 - \left(1 - \left(\frac{\beta}{2 \log N}\right)^{2 \log N / \beta}\right)^N \\ &\xrightarrow{N \rightarrow +\infty} 1 \end{aligned}$$

Starting from a fully connected segment, the distance  $d$  that nodes in the segment will reach using beamforming can be computed as follows.

$$\left(2L \sqrt{\frac{P_t}{d^\alpha}}\right)^2 = P_t$$

Therefore, we have  $d = 2L$ . The original fully connected segment can reach neighboring segments at both sides. Since each segment contains at least  $2L$  nodes, the expansion will continue to cover the whole line. ■

*Theorem 5.2:* In a 2-D extended network under the coherent cooperative transmission model:

- $\alpha \leq 2$ : full connectivity occurs for any  $\lambda > 0$ ;
- $2 < \alpha < 4$ : there are an infinite number of nodes that can reach every other nodes in the network for any  $\lambda > 0$ ;
- $\alpha = 4$ : there are an infinite number of nodes that can reach every other nodes in the network for  $\lambda > 5$ ;
- $\alpha > 4$ : full connectivity never occurs for any  $\lambda > 0$ .

*Proof:* The proof follows that of Theorem 5.1. ■

## VI. CONCLUSION

In this paper, we have shown that physical layer cooperation is able to significantly improve the connectivity in wireless ad hoc networks. Conclusive results are shown in Table I for the consideration of full connectivity, and the results of Table II have been established when considering percolation. Extensions to coherent cooperative transmission (beamforming) have been considered. For dense networks, it is straightforward to apply these results to establish the conjecture from [1] that  $O(1/N)$  transmission area is sufficient for complete connectivity with probability one when  $\alpha \leq 2$  in the 2-D case [2]. We hasten to note that the results here are based on the assumption that establishing connectivity (rather than maximizing capacity) is the key goal. For example, perhaps an application only desires the ability to get a single message across a wireless network when absolutely necessary, and thus we desire to know at what density this is possible. Because the goal is connectivity rather than capacity, the resulting constructions can require significant capacity overhead in the asymptotic limit.

In this paper we have assumed a path loss attenuation function in the form of a power law, i.e.,  $l(r) = r^{-\alpha}$ , where  $r$  is the distance between the sender and receiver, and  $\alpha$  is the path loss exponent. Note that this attenuation function has a singularity at the origin, and the received power increases without bound as  $r$  decreases. However, in reality, the received power is always finite and bounded from above by the transmission power. To reflect the finite received power, a more realistic

path loss attenuation function is  $l(r) = \min(1, r^{-\alpha})$  [15]. It can be easily verified that the proofs in the paper do not rely on the increasing scaling property of the power law function as  $r$  decreases. On the contrary, we consider the worst case scenario where the tail behavior of the power law for large  $r$  is used. Thus, we have verified that the results still hold under the more realistic bounded path loss attenuation function.

## APPENDIX DERIVATIONS OF COOPERATION MODELS

Before beginning the derivations for the distributed FSK model, consider how the threshold  $\tau$  in (1) is derived. There are many metrics that could be employed to determine whether two nodes are one-hop neighbors, including average bit error probability, bit error outage, average rate, rate outage probability, etc. These metrics all lead to the model of (2), but, so as to make the presentation clear we focus on the average bit error probability rather than trying to capture all of these metrics in a single formulation. Given the quasi-static Rayleigh fading assumption on a given link, the average bit error probability of a coded coherently received binary modulation would be given by [11, pg. 818]

$$P_b = \frac{1}{2} \left[ 1 - \sqrt{\frac{\gamma}{2 + \gamma}} \right] \quad (13)$$

where  $\gamma = G_c P_t T_s r^{-\alpha} / N_0$ ,  $G_c$  is the gain of the coded modulation over a binary orthogonal scheme,  $T_s$  is the symbol interval, and  $N_0/2$  is the (two-sided) power spectral density of the additive white Gaussian noise (AWGN) process at the receiver. From the desired average bit error probability, this equation can then be employed to find the threshold  $\tau$  on the required  $P_t r^{-\alpha}$  for successful communication.

Now, consider a set of connected nodes  $\Omega_A$  attempting to transmit to a connected set of nodes  $\Omega_B$  using distributed FSK. For simplicity, assume the timing asynchronism relative to the symbol interval  $T_s$  is negligible (see [17] if it is not) and that any residual frequency offset between the connected transmitters is insignificant relative to the frequency shift between the signals used to represent a data bit 0 and a data bit 1. Suppressing the pulse shaping, the complex baseband signal during the time interval  $(iT_s, (i+1)T_s)$  from the  $j^{\text{th}}$  transmitter is given by:

$$x_j(t) = \sqrt{2P_t} e^{j(2\pi f_i t + \theta_j)}$$

where  $f_i$  is the frequency shift of the carrier corresponding to the  $i^{\text{th}}$  data bit  $b_i \in \{0, 1\}$ , which is to be transmitted during interval  $(iT_s, (i+1)T_s)$ , and  $\theta_j$  is the phase offset of the  $j^{\text{th}}$  transmitter. Recalling the IID frequency-nonsselective Rayleigh fading model assumed, let  $X_{j,k}$  denote the zero-mean, variance  $(d_{j,k})^{-\alpha}$  complex Gaussian random variable representing the gain between nodes  $j$  and  $k$ , where  $d_{j,k}$  is the distance between node  $j$  and node  $k$ . Then, the received

signal at the  $k^{\text{th}}$  node of the receive cluster is given by:

$$\begin{aligned} y_k(t) &= \sum_{j \in \Omega_A} \sqrt{2P_t} X_{j,k} e^{j(2\pi f_i t + \theta_j)} + n_k(t) \\ &= \left( \sum_{j \in \Omega_A} X_{j,k} e^{j\theta_j} \right) \sqrt{2P_t} e^{j2\pi f_i t} + n_k(t) \\ &= h_k \sqrt{2P_t} e^{j2\pi f_i t} + n_k(t) \end{aligned} \quad (14)$$

where  $h_k$  is complex Gaussian with mean 0 and variance  $\sum_{j \in \Omega_A} (d_{j,k})^{-\alpha}$ , and  $n_k(t)$  is a set of IID zero-mean complex Gaussian random processes each with (two-sided) power spectral density  $N_0/2$ . Note that interference from other simultaneously transmitting clusters, as would be present in many wireless data networks, is not addressed in the model. Rather, we are interested in the minimum requirements when connectivity is the key goal. Besides being of theoretical interest, this would find direct application when one considers a network where interest is in the ability of all (or a large set) of nodes being able to broadcast a high-priority (e.g. alarm) message to all (or a large set) of other nodes.

We will also assume cooperation in the reception by nodes within the receive cluster, and make the common physical layer assumption that each receiving node is able to measure its equivalent complex gain  $h_k$  from the transmitters, which is easily established from pilot symbols within the transmitted signal. The more difficult part of receiver cooperation is that the received symbol samples must be routed to a single node in the cluster for joint processing. This can be done in systems where connectivity is the critical goal, and has even been employed in systems where capacity maximization is the goal [10]. After the receivers estimate  $h_k$ , they correlate their signal with each of the frequencies corresponding to bit 0 and bit 1, weight the results by  $h_k^*$ , and transmit the resulting two scalars to a central decision unit. The processing of these scalars is equivalent to the central decision unit observing the signal:

$$\begin{aligned} y(t) &= \sum_{k \in \Omega_B} (h_k^* y_k(t) + h_k^* n_k(t)) \\ &= \left( \sum_{k \in \Omega_B} |h_k|^2 \right) \sqrt{2P_t} e^{j2\pi f_i t} + \sum_{k \in \Omega_B} h_k^* n_k(t) \end{aligned}$$

which is binary (receiver coherent) orthogonal signaling with  $|\Omega_B|$ -fold diversity with diversity paths of unequal power. Assuming a coding gain of  $G_c$  and recalling the quasi-static IID Rayleigh fading assumption, the average probability of bit error is obtained as:

$$P_b = E \left[ Q \left( \sqrt{\sum_{k \in \Omega_B} |h_k|^2 \frac{G_c P_t T_s}{N_0}} \right) \right] \quad (15)$$

where the expectation is over the random variables  $\{h_k; k = 1, 2, \dots, |\Omega_B|\}$ . An upper bound to this is obtained as:

$$P_b \leq \frac{1}{2} \left[ 1 - \sqrt{\frac{\gamma_c}{2 + \gamma_c}} \right] \quad (16)$$

where  $\gamma_c = G_c P_t T_s \left( \sum_{j \in \Omega_A} \sum_{k \in \Omega_B} (d_{j,k})^{-\alpha} \right) / N_0$ , by assuming that all of the received power is in a single (rather

than  $|\Omega_B|$ ) diversity paths and applying [11, pg. 818]. In other words, the right side assumes that the diversity gain is ignored. However, as noted in the main text, it is important that the inequality in (16) becomes an equality when there is only a single receiving node ( $|\Omega_B| = 1$ ). Now, comparing (13) with (16) yields the sufficiency condition (2) for cluster  $\Omega_A$  to connect to cluster  $\Omega_B$ .

Next, consider the system model for distributed beamforming on an AWGN channel, and recall that we seek a system to serve as an upper bound on the connectivity performance of cooperative schemes. Assuming accurate node location information, frequency synchronization, and phase locking of the transmitters in the cooperating group, the group can set their phase offsets so as to produce positive reinforcement of their signals at some location in space. In other words, the received signal at a node  $k$  at such a position for, say, a transmitted quadrature amplitude modulation (QAM) symbol  $s_i$  would be given by:

$$y(t) = \sum_{j \in \Omega_A} \sqrt{\frac{P_t}{(d_{j,k})^\alpha}} s_i \sqrt{2} \cos(2\pi f_c t + \theta) \quad (17)$$

where the phase synchronization has eliminated the transmitter dependence of the phase offset of the sinusoidal carrier. This then leads to the model of (3).

## REFERENCES

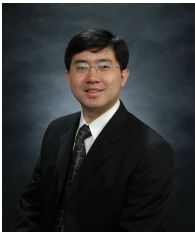
- [1] S. Song, D. L. Goeckel and D. Towsley, "Collaboration improves the connectivity of wireless networks," in *Proc. IEEE Infocom*, 2006.
- [2] L. Wang, B. Liu, D. Goeckel, D. Towsley and C. Westphal, "Connectivity in cooperative wireless ad hoc networks," in *Proc. ACM Mobihoc*, 2008.
- [3] R. Meester and R. Roy, *Continuum Percolation*. Cambridge University Press, 1996.
- [4] P. Hall, "On continuum percolation," *Ann. Prob.*, Vol. 13: 1250-1266, 1985.
- [5] P. Gupta and P. R. Kumar, "The capacity of wireless networks," *IEEE Trans. Inform. Theory*, vol. 46(2), pp. 388-404, Mar. 2000.
- [6] P. Gupta and P. R. Kumar, "Critical power for asymptotic connectivity in wireless networks," *Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W. H. Fleming*, 1998, edited by W. M. McEneaney, G. Yin and Q. Zhang, (Eds.) Birkhauser.
- [7] E. M. Sozer, M. Stojanovic and J. G. Proakis, "Underwater acoustic networks," in *IEEE J. Ocean. Eng.*, vol. 25(1), pp. 72-83, Jan. 2000.
- [8] R. Mudumbai, G. Barriac and U. Madhow, "On the feasibility of distributed beamforming in wireless networks," *IEEE Trans. Wireless Commun.*, vol. 6, pp. 1754-1763, May 2007.
- [9] A. Scaglione, D. Goeckel and J. Laneman, "Cooperative communications in mobile ad-hoc networks: rethinking the link abstraction," *IEEE Signal Processing Mag.: Special Issue on Signal Processing for Ad hoc Communication Networks*, vol. 23, pp. 18-29, Sept. 2006.
- [10] A. Ozgur, O. Leveque and D. Tse, "Hierarchical cooperation achieves optimal capacity scaling in ad hoc networks," *IEEE Trans. Inform. Theory*, vol. 53, pp. 3549-3572, Oct. 2007.
- [11] J. Proakis, *Digital Communications*, 4th ed., New York: McGraw-Hill, 2001.
- [12] L. Wang, B. Liu, D. Goeckel, D. Towsley and C. Westphal, "Connectivity in cooperative wireless ad hoc networks," University of Massachusetts, Amherst, Tech. Rep. UM-CS-2007-015, Mar. 2007.
- [13] O. Dousse, P. Thiran and M. Hasler, "Connectivity in ad hoc and hybrid networks," in *Proc. IEEE Infocom*, 2002.
- [14] S. B. Lowen and M. C. Teich, "Fractal shot noise," *Phys. Rev.*, vol. 63, pp. 1755-1759, Oct. 1989.
- [15] O. Dousse and P. Thiran, "Connectivity vs capacity in dense ad hoc networks," in *Proc. IEEE Infocom*, 2004.
- [16] J. Kuang, "Some extensions and refinements of Minc-Sathre Inequality," *Math. Gaz.*, vol. 83, pp. 123-127, Mar. 1999.
- [17] S. Wei, D. L. Goeckel and M. Valenti, "Asynchronous cooperative diversity," *IEEE Trans. Wireless Commun.*, vol. 5, pp. 1547-1557, Jun. 2006.



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