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Receding horizon H_∞ control of time-delay systems

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Abstract

This paper deals with the disturbance rejection problem for discrete-time linear systems having time-varying state delays and control constraints. The study proposes a novel receding horizon H_∞ control method utilizing a linear matrix inequality based optimization algorithm which is solved in each step of run-time. The proposed controller attenuates disturbances having bounded energies on controlled output and ensures the closed-loop stability and dissipation while meeting the physical control input constraints. The originality of the work lies on the extension of the idea of the well-known H_∞ receding horizon control technique developed for linear discrete-time systems to interval time-delay systems having time-varying delays. The efficiency of the proposed method is illustrated through simulation studies that are carried out on a couple of benchmark problems.

Keywords

Time delay, predictive control, H_∞ optimization

Introduction

Time-delay systems (TDSs) have been a popular and attractive topic in the literature for many years, since the phenomenon can be considered as one of the main reasons for instability and poor performance in many physical and dynamical systems. Due to this, many researchers have paid attention to the stabilization and control of this type of system. The time delay usually affects the system states, control inputs and state derivatives, which leads to different classes of time-delay systems (Hale, 1977; Mahmoud, 2000; Richard, 2003).

The existing results in the literature are often grouped into two classes, due to their dependence on the size of the delay, i.e. *delay-independent methods* and *delay-dependent methods* where the latter is usually less conservative than the former. Therefore, most of the work in literature is devoted to delay-dependent controller design problems (Fridman and Shaked, 2003; Huang and Xiang, 2014; Mahmoud and Ismail, 2005; Moon et al., 2001; Parlakçı and Küçükdemiral, 2011).

On the other hand, the receding horizon control (RHC) method, also known as model predictive control (MPC), is known to be an efficient control method when dealing with linear systems having input–output constraints and disturbances. It is an advanced method of process control that has been in use in the process industries in chemical plants and oil refineries since the 1980s. Carried out for a finite control horizon, this celebrated method is used to solve optimization problems at every step of the run-time by utilizing the current plant measurements, and the current dynamic state of the process, together with the mathematical model of the system. Hence, the varying parameters in the system are all taken into consideration and the controller parameters are updated in every step. During the last four decades, the studies on RHC were mostly focused on ordinary linear time-invariant systems without delays. However, in recent years, some results on RHC for systems having time delays started to appear in the literature, such as Houda et al. (2013), Jeong and Park (2005), Kwon et al. (2003), Shi et al. (2008) and Zhang et al. (2014), in which the proposed techniques are all based on delay-independent schemes. Therefore, they are considered to be highly conservative.

The receding horizon H_∞ control method was first introduced by Chen and Scherer (2006), in which an additional constraint for the optimization problem based on linear matrix inequalities (LMIs) was introduced in order to ensure the dissipativity of the closed-loop system. This method was adapted to a time-delay system by Mei et al. (2009) in which the time delay was assumed to be constant and a delay-independent controller design method was proposed.

Inspired by the lack of delay-dependent receding horizon H_∞ controllers for time-delay systems having time-varying delays in the literature, a delay-dependent receding horizon H_∞ controller design method is proposed in this study, where the time delay is assumed to be time-varying over a known interval. Moreover, the closed-loop stability, dissipativity and

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H_∞ performance of the time-delay system is guaranteed by the proposed scheme. On the other hand, control constraints are also taken into account during the controller design.

The rest of the paper is organized as follows: The control problem is stated in Section "Problem Definition". The main outcomes of the study about receding horizon H_∞ control are presented in detail in Section "Receding Horizon H_∞ Control of Time-delay Systems". In Section "Simulation Studies", the efficiency of the proposed theory and algorithm is demonstrated through a couple of benchmark problems which consist of retarded-type time-delay systems. Finally, the results are discussed and possible future studies are addressed in Section "Conclusion".

Problem definition

Let us consider a class of linear discrete time-delay system of the form

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-d(k)) + B_w w(k) + B_u u(k) \\ z(k) &= Cx(k) + C_d x(k-d(k)) + D_w w(k) + D_u u(k) \\ x(k) &= \phi(k), \quad k \in [-d_{\max}, 0] \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $z(k) \in \mathbb{R}^p$, $\Phi(k) \in \mathbb{R}^n$ and $\omega(k) \in \mathbb{R}^l$ stand for the state vector, control signal, controlled output, the vector of initial conditions, and disturbance, respectively. The control signal applied to the system also fulfils the constraint given as

$$\|u_i(k)\|_\infty \le u_{i,\max}^2, \quad \forall k \ge 0,\ i = 1, \ldots, m \tag{2}$$

where $u_{i,\max}$ is a known magnitude bound on the control effort in the ith control channel. Also, the time-varying delay $d(k)$ satisfies

$$d_{\min} \le d(k) \le d_{\max} \quad \forall k \ge 0 \tag{3}$$

where $d_{\min}$ and $d_{\max}$ are the known lower and upper bound of the delay, respectively.

Here, the *known interval for the time delay* term corresponds to the upper and lower bounds of the time delay. The time-delay interval including the upper and lower bounds is highly related to the dynamics of the system. By using advanced system identification and modelling techniques, it is possible to calculate the lower and upper bounds of the time delay, which leads us to a pre-defined time-delay interval.

Our design goal is to find a feedback control law of the form, $u(k) = K(k)x(k)$ for each step k, which ensures the minimum achievable H_∞ gain for the closed system

$$\begin{aligned} x(k+1) &= A_{\mathrm{cl}} x(k) + A_d x(k-d(k)) + B_w w(k) \\ z(k) &= C_{\mathrm{cl}} x(k) + C_d x(k-d(k)) + D_w w(k) \\ x(k) &= 0, \quad k \in [-d_{\max}, 0] \end{aligned} \tag{4}$$

where $A_{\mathrm{cl}} = A + B_u K(k)$, $C_{\mathrm{cl}} = C + D_u K(k)$ and guarantees the stability of the closed-loop system under the constraints in equations (2) and (3).

Receding horizon H_∞ control of time-delay systems

In order to construct the optimization problem for receding horizon H_∞ control of time-delay systems, we shall present the asymptotic stability condition for the system in equation (4), the bounding constraint for state trajectories and the dissipation condition, respectively. Finally, an algorithm which can be used in real-time control is also proposed.

Stability in receding horizon control

The theorem associated with the H_∞ state feedback control law $u(k) = Kx(k)$ which assures the asymptotic stability of the closed-loop time-delay system given in equation (4) is introduced next.

Theorem 1. *At each sample k of run-time, given positive integers $d_{\min}$, $d_{\max}$ and the positive real number γ, the H_∞ state feedback control law $u(k) = K(k)x(k)$ with $K(k) = L(k)Y^{-1}(k)$ asymptotically stabilizes the system in equation (4) with a disturbance attenuation level of γ, if there exists matrices $Y(k) = Y^{\mathrm{T}}(k)$, $W(k) = W^{\mathrm{T}}(k) > 0$ and $L(k)$ with appropriate dimensions which satisfy*

$$\begin{bmatrix} Y(k) & * & * & * & * & * \\ 0 & W(k) & * & * & * & * \\ 0 & 0 & \gamma^2 I & * & * & * \\ AY(k) + B_u L(k) & A_d W(k) & B_w & Y(k) & * & * \\ CY(k) + D_u L(k) & C_d W(k) & D_w & 0 & I & * \\ Y(k) & 0 & 0 & 0 & 0 & d_{\mathrm{m}}^{-1} W(k) \end{bmatrix} > 0 \tag{5}$$

where $d_{\mathrm{m}} \triangleq d_{\max} - d_{\min} + 1$.

Proof. Let us choose a Lyapunov–Krasovskii functional candidate as

$$V(x(k)) = V_1(k) + V_2(k) + V_3(k) \tag{6}$$

where

$$\begin{aligned} V_1(x(k)) &= x^{\mathrm{T}}(k) P(k) x(k), \\ V_2(x(k)) &= \sum_{s=k-d(k)}^{k-1} x^{\mathrm{T}}(s) Q(k) x(s), \\ V_3(x(k)) &= \sum_{i=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=k+i-1}^{k-1} x^{\mathrm{T}}(s) Q(k) x(s) \end{aligned}$$

Considering the closed-loop system in equation (4), the forward difference of energy functional $V(x(k))$ at step k is defined as

$$\Delta V(k) \triangleq \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) \tag{7}$$

where

$$\begin{aligned}\Delta V_1(k) &= x^{\mathrm{T}}(k+1)P(k)x(k+1) - x^{\mathrm{T}}(k)P(k)x(k)\\ &= x^{\mathrm{T}}(k)(A_{\mathrm{cl}}^{\mathrm{T}}P(k)A_{\mathrm{cl}} - P(k))x(k)\\ &\quad + 2x^{\mathrm{T}}(k)A_{\mathrm{cl}}^{\mathrm{T}}P(k)A_d x(k-d(k))\\ &\quad + 2x^{\mathrm{T}}(k)A_{\mathrm{cl}}^{\mathrm{T}}P(k)B_w w(k)\\ &\quad + x^{\mathrm{T}}(k-d(k))A_d^{\mathrm{T}}P(k)A_d x(k-d(k))\\ &\quad + 2x^{\mathrm{T}}(k-d(k))A_d^{\mathrm{T}}P(k)B_w w(k)\\ &\quad + w^{\mathrm{T}}(k)B_w^{\mathrm{T}}P(k)B_w w(k)\end{aligned} \tag{8}$$

and

$$\begin{aligned}\Delta V_2(k) &= \sum_{s=k-d(k+1)+1}^{k} x^{\mathrm{T}}(s)Q(k)x(s)\\ &\quad - \sum_{s=k-d(k)}^{k-1} x^{\mathrm{T}}(s)Q(k)x(s)\\ &= x^{\mathrm{T}}(k)Q(k)x(k) - x^{\mathrm{T}}(k-d(k))Q(k)x(k-d(k))\\ &\quad + \sum_{s=k-d(k+1)+1}^{k-1} x^{\mathrm{T}}(s)Q(k)x(s)\\ &\quad - \sum_{s=k-d(k)+1}^{k-1} x^{\mathrm{T}}(s)Q(k)x(s)\end{aligned} \tag{9}$$

However, $\Delta V_2(k)$ relies on the exact knowledge of the delay term $d(k+1)$. Therefore, in order to get rid of the term $d(k+1)$ in equation (9), one can use the following bound

$$\begin{aligned}\Delta V_2(k) &\le x^{\mathrm{T}}(k)Q(k)x(k) - x^{\mathrm{T}}(k-d(k))Q(k)x(k-d(k))\\ &\quad + \sum_{s=k-d_{\max}+1}^{k-d_{\min}} x^{\mathrm{T}}(s)Q(k)x(s)\end{aligned} \tag{10}$$

Note that this inequality is obtained with the help of the following identity

$$\begin{aligned}\sum_{s=k-d(k+1)+1}^{k} x^{\mathrm{T}}(s)Q(k)x(s) &= \sum_{s=k-d_{\min}+1}^{k} x^{\mathrm{T}}(s)Q(k)x(s)\\ &\quad + \sum_{s=k-d(k+1)+1}^{k-d_{\min}} x^{\mathrm{T}}(s)Q(k)x(s)\\ &\le \sum_{s=k-d(k)+1}^{k} x^{\mathrm{T}}(s)Q(k)x(s)\\ &\quad + \sum_{s=k-d_{\max}+1}^{k-d_{\min}} x^{\mathrm{T}}(s)Q(k)x(s)\end{aligned} \tag{11}$$

Therefore

$$\begin{aligned}\sum_{s=k-d(k+1)+1}^{k} x^{\mathrm{T}}(s)Q(k)x(s) - \sum_{s=k-d(k)+1}^{k} x^{\mathrm{T}}(s)Q(k)x(s)\\ \le \sum_{s=k-d_{\max}+1}^{s=k-d_{\min}} x^{\mathrm{T}}(s)Q(k)x(s)\end{aligned}$$

and $\sum_{s=k-d_{\max}+1}^{s=k-d_{\min}} x^{\mathrm{T}}(s)Q(k)x(s)$ can be used in equation (9) in order to replace and bound the term

$$\sum_{s=k-d(k+1)+1}^{k} x^{\mathrm{T}}(s)Q(k)x(s) - \sum_{s=k-d(k)+1}^{k} x^{\mathrm{T}}(s)Q(k)x(s)$$

which leads us to equation (10).

Finally

$$\begin{aligned}\Delta V_3 &= \sum_{i=-d_{\max}+2}^{-d_{\min}+1}\left(\sum_{s=k+i}^{k} x^{\mathrm{T}}(s)Q(k)x(s) - \sum_{s=k+i-1}^{k-1} x^{\mathrm{T}}(s)Q(k)x(s)\right)\\ &= \sum_{i=-d_{\max}+2}^{-d_{\min}+1}\left(x^{\mathrm{T}}(k)Q(k)x(k)) - x^{\mathrm{T}}(k+i-1)Q(k)x(k+i-1)\right)\\ &= (d_{\max}-d_{\min})x^{\mathrm{T}}(k)Q(k)x(k)\\ &\quad - \sum_{i=k-d_{\max}+1}^{k-d_{\min}} x^{\mathrm{T}}(s)Q(k)x(s)\end{aligned} \tag{12}$$

is obtained. In view of equations (8), (9) and (12), one can write

$$\begin{aligned}&\Delta V(k) + z^{\mathrm{T}}(k)z(k) - \gamma^2 w^{\mathrm{T}}(k)w(k)\\ &\quad \le \bar{\Delta} V(k) + z^{\mathrm{T}}(k)z(k) - \gamma^2 w^{\mathrm{T}}(k)w(k)\end{aligned} \tag{13}$$

where

$$\begin{aligned}\bar{\Delta} V(k) &= x^{\mathrm{T}}(k)(A_{\mathrm{cl}}^{\mathrm{T}}P(k)A_{\mathrm{cl}} - P(k))x(k)\\ &\quad + 2x^{\mathrm{T}}(k)A_{\mathrm{cl}}^{\mathrm{T}}P(k)A_d x(k-d(k))\\ &\quad + 2x^{\mathrm{T}}(k)A_{\mathrm{cl}}^{\mathrm{T}}P(k)B_w w(k)\\ &\quad + x^{\mathrm{T}}(k-d(k))A_d^{\mathrm{T}}P(k)A_d x(k-d(k))\\ &\quad + 2x^{\mathrm{T}}(k-d(k))A_d^{\mathrm{T}}P(k)B_w w(k)\\ &\quad + w^{\mathrm{T}}(k)B_w^{\mathrm{T}}P(k)B_w w(k)\\ &\quad + x^{\mathrm{T}}(k)Q(k)x(k) - x^{\mathrm{T}}(k-d(k))Q(k)x(k-d(k))\\ &\quad + \sum_{s=k-d_{\max+1}}^{k-d_{\min}} x^{\mathrm{T}}(s)Q(k)x(s)\\ &\quad + (d_{\max}-d_{\min})x^{\mathrm{T}}(k)Q(k)x(k)\\ &\quad - \sum_{s=k-d_{\max}+1}^{k-d_{\min}} x^{\mathrm{T}}(s)Q(k)x(s)\end{aligned} \tag{14}$$

is an upper bound for $\Delta V(k)$. Hence, the control law satisfying

$$\bar{\Delta} V(k) + z^{\mathrm{T}}(k)z(k) - \gamma^2 w^{\mathrm{T}}(k)w(k) < 0 \tag{15}$$

for each step $k \ge 0$ step, asymptotically stabilizes the system in equation (4) for $\omega(k) = 0$ and satisfies the $\|z(k)\|_2^2 \le \gamma^2\|\omega(k)\|_2^2$ performance when $\omega(k) \in l_2[0, \infty)$. Rearranging equation (15), one obtains

$$\begin{bmatrix} x^{\mathrm{T}}(k) \\ x^{\mathrm{T}}(k-d) \\ w^{\mathrm{T}}(k) \end{bmatrix} \begin{bmatrix} A_{\mathrm{cl}}^{\mathrm{T}}P(k)A_{\mathrm{cl}} - P(k) + d_{\mathrm{m}}Q(k) + C_{\mathrm{cl}}^{\mathrm{T}}C_{\mathrm{cl}} & A_{\mathrm{cl}}^{\mathrm{T}}P(k)A_d + C_{\mathrm{cl}}^{\mathrm{T}}C_d & A_{\mathrm{cl}}^{\mathrm{T}}P(k)B_w + C_{\mathrm{cl}}^{\mathrm{T}}D_w \\ A_d^{\mathrm{T}}P(k)A_{\mathrm{cl}} + C_d^{\mathrm{T}}C_{\mathrm{cl}} & A_d^{\mathrm{T}}P(k)A_d - Q(k) + C_d^{\mathrm{T}}C_d & A_d^{\mathrm{T}}P(k)B_w + C_d^{\mathrm{T}}D_w \\ B_w^{\mathrm{T}}P(k)A_{\mathrm{cl}} + D_w^{\mathrm{T}}C_{\mathrm{cl}} & B_w^{\mathrm{T}}P(k)A_d + D_w^{\mathrm{T}}C_d & B_w^{\mathrm{T}}P(k)B_w + D_w^{\mathrm{T}}D_w - \gamma^2 \end{bmatrix} \begin{bmatrix} x(k) \\ x(k-d) \\ w(k) \end{bmatrix} < 0 \quad (16)$$

Then, employing the Schur complement on the multiplier of equation (16) allows us to obtain

$$\begin{bmatrix} d_{\mathrm{m}}Q(k) - P(k) & * & * & * & * \\ 0 & -Q(k) & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * \\ A_{\mathrm{cl}} & A_d & B_w & -P^{-1}(k) & * \\ C_{\mathrm{cl}} & C_d & D_w & 0 & -I \end{bmatrix} < 0 \quad (17)$$

Ultimately, replacing A_{cl} with $A + B_uK$, C_{cl} with $C + D_uK$, utilizing the definitions $Y(k) := P^{-1}(k)$, $W(k) := Q^{-1}(k)$ and $L(k) := K(k)P^{-1}(k)$ in (17) and pre- and post-multiplying equation (17) with diag $\{P^{-1}(k), Q^{-1}(k), I, I, I\}$ leads to

$$\begin{bmatrix} d_{\mathrm{m}}Y^{\mathrm{T}}(k)W^{-1}(k)Y(k) - Y(k) & * & * & * & * \\ 0 & -W(k) & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * \\ AY(k) + B_uL(k) & A_dW(k) & B_w & -Y(k) & * \\ CY(k) + D_uL(k) & C_dW(k) & D_w & 0 & -I \end{bmatrix} < 0$$

where the Schur complement formula is employed for the $d_{\mathrm{m}}Y^{\mathrm{T}}(k)W^{-1}(k)Y(k) - Y(k)$ term, which leads to equation (5). This concludes the proof.

Bounding the state trajectories

Lemma 1. *Let us assume that there exists a solution (γ, Y, W, X) to the stability condition given in equation (5) and the input constraint such as given in Boyd et al. (1994)*

$$\begin{bmatrix} 1/rX & L \\ L^{\mathrm{T}} & Y \end{bmatrix} \geq 0, \quad X_{ii} \leq u_{i,\max}^2 \quad (18)$$

Given the positive constant r, the state trajectory starting from $x(0)$ always lies within the ellipsoid

$$\varepsilon(P, r) := \{x \in \Re^n | V(x(k)) \leq r\} \quad \forall k \geq 0 \quad (19)$$

if there exists matrices $Y(k) = Y^{\mathrm{T}}(k) > 0$ and $W(k) = W^{\mathrm{T}}(k) > 0$ with appropriate dimensions which solve

$$\begin{bmatrix} r & * & * & \cdots & * & \cdots & * & * & * \\ x(k) & Y(k) & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ x(k-1) & 0 & d_{\mathrm{m}}^{-1}W(k) & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x(k-d_{\min}) & 0 & 0 & \cdots & d_{\mathrm{m}}^{-1}W(k) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x(k-d_{\max}+2) & 0 & 0 & \cdots & 0 & \cdots & 3^{-1}W(k) & 0 & 0 \\ x(k-d_{\max}+1) & 0 & 0 & \cdots & 0 & \cdots & 0 & 2^{-1}W(k) & 0 \\ x(k-d_{\max}) & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & W(k) \end{bmatrix} \geq 0 \quad (20)$$

where $d_{\mathrm{m}} \triangleq d_{max} - d_{\min} + 1$.

Proof. Employing the Lyapunov–Krasovskii functional introduced in equation (6), the ellipsoid in equation (19) can be rewritten as

$$\begin{aligned} &r - x^{\mathrm{T}}(k)P(k)x(k) - \sum_{s=k-d(k)}^{k-1} x^{\mathrm{T}}(s)Q(k)x(s) \\ &- \sum_{i=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=k+i-1}^{k-1} x^{\mathrm{T}}(s)Q(k)x(s) \geq 0 \end{aligned} \quad (21)$$

Employing a bounding such as

$$\begin{aligned} \sum_{s=k-d(k)}^{k-1} x^{\mathrm{T}}(s)Q(k)x(s) \leq & \sum_{s=k-d_{\min}}^{k-1} x^{\mathrm{T}}(s)Q(k)x(s) \\ &+ \sum_{s=k-d_{\max}}^{k-d_{\min}-1} x^{\mathrm{T}}(s)Q(k)x(s) \end{aligned} \quad (22)$$

and plugging into equation (21) results in

$$\begin{aligned} &r - x^{\mathrm{T}}(k)P(k)x(k) - \sum_{s=k-d_{\min}}^{k-1} x^{\mathrm{T}}(s)Q(k)x(s) + \sum_{s=k-d_{\max}}^{k-d_{\min}-1} x^{\mathrm{T}}(s)Q(k)x(s) \\ &- \sum_{i=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=k+i-1}^{k-1} x^{\mathrm{T}}(s)Q(k)x(s) \geq 0 \end{aligned} \quad (23)$$

which is a sufficient condition for equation (21) and lets us avoid the use of the unmeasurable term, $d(k)$. Then, the LMI given in equation (20) is readily obtained if the Schur complement formula is employed successively on the inequality in equation (23). □

Dissipation condition

The dissipativity of a system is related to the input–output relationship and can be defined as follows:

Definition 1. *If a system with supply function*

$$s(w(k), z(k)) := \gamma^2 \|w(k)\|^2 - \|z(k)\|^2$$

has a storage function $V(x(k))$ *which satisfies*

$$V(x(k)) + (s(w(k), z(k)) \geq V(x(k+1)) \quad \forall k \geq 0 \tag{24}$$

then the system is called dissipative and equation (24) is called the dissipation inequality.

Theorem 2. *Let us assume that there exists a solution* (γ, Y, W, X) *such that the stability condition, input constraint and state constraint given in equations (5), (18) and (20) are all satisfied. Then, the* H_∞*feedback control law* $u(k) = L(k)Y^{-1}(k)x(k)$ *guarantees the dissipativity of the system in equation (4) with a disturbance attenuation level of* γ *and holds the dissipation inequality*

$$\sum_{i=0}^{k} \left(\|z(i)\|^2 - \gamma^2 \|w(i)\|^2 \right) \leq x^{\mathrm{T}}(0)P(0)x(0) + \sum_{s=-d_0}^{-1} x^{\mathrm{T}}(s)Q(0)x(s) + \sum_{j=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=j-1}^{-1} x^{\mathrm{T}}(s)Q(0)x(s) \tag{25}$$

if there exists positive matrices $Y(k) = Y^{\mathrm{T}}(k) > 0$, $W(k) = W^{\mathrm{T}}(k) > 0$ *and* $L(k)$ *with appropriate dimensions that solve*

$$\begin{bmatrix} \Omega(k) & * & * & \cdots & * & \cdots & * & * \\ x(k) & Y(k) & 0 & \cdots & 0 & \cdots & 0 & 0 \\ x(k-1) & 0 & d_{\mathrm{m}}^{-1}W(k) & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ x(k-d_{\min}) & 0 & 0 & \cdots & d_{\mathrm{m}}^{-1}W(k) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ x(k-d_{\max}+2) & 0 & 0 & \cdots & 0 & \cdots & 2^{-1}W(k) & 0 \\ x(k-d_{\max}+1) & 0 & 0 & \cdots & 0 & \cdots & 0 & W(k) \end{bmatrix} \geq 0 \tag{26}$$

where $d_{\mathrm{m}} \triangleq d_{\max} - d_{\min} + 1$ *and*

$$\Omega(k) \triangleq p(0) + q(0) - p(k-1) - q(k-1) + x^{\mathrm{T}}(k)P(k-1)x(k) + \sum_{s=k-d_{\max}}^{k-1} x^{\mathrm{T}}(s)Q(k-1)x(s) + \sum_{j=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=j+k-1}^{k-1} x^{\mathrm{T}}(s)Q(k-1)x(s)$$

in addition to the dissipation level variables which are given inside Ω *in equation (26) and defined as*

$$p(k) = p(k-1) - x^{\mathrm{T}}(k)[P(k-1) - P(k)]x(k)$$

$$q(k) = q(k-1) - \sum_{s=k-d_{\min}}^{k-1} x^{\mathrm{T}}(s)Q(k-1)x(s) - \sum_{k-d_{\min}}^{k-1} x^{\mathrm{T}}(s)Q(k)x(s) - \sum_{j=-d_{\max+2}}^{-d_{\min}+1} \sum_{s=j+k-1}^{k-1} x^{\mathrm{T}}(s)[Q(k-1) - Q(k)]x(s) \tag{27}$$

Proof. By rearranging the dissipation inequality with respect to $i = 0, 1, \ldots, k$, one can obtain

$$V(i+1) + \|z_i\|^2 \leq \gamma^2 \|w_i\|^2 + V(i) \tag{28}$$

On the other hand, above-mentioned Lyapunov–Krasovskii functional can be rewritten as

$$V(i) = x^{\mathrm{T}}(i)P(i)x(i) + \sum_{s=i-d(i)}^{i-1} x^{\mathrm{T}}(s)Q(i)x(s) + \sum_{j=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=i+j-1}^{i-1} x^{\mathrm{T}}(s)Q(i)x(s) \tag{29}$$

If the equation (29) is used in equation (28) for $i = 0, 1, \ldots, k$ and the negative terms in the inequality are neglected due to the assumptions $P(k) > 0$ and $Q(k) > 0$, one can obtain

$$\sum_{i=0}^{k} \left(\|z(i)\|^2 - \gamma^2 \|w(i)\|^2 \right) \leq x^{\mathrm{T}}(0)P(0)x(0) - \sum_{i=1}^{k} x^{\mathrm{T}}(i)[P(i-1) - P(i)]x(i) + \sum_{s=-d_0}^{-1} x^{\mathrm{T}}(s)Q(0)x(s) - \sum_{i=1}^{k} \sum_{s=i-d_i}^{i-1} x^{\mathrm{T}}(s)[Q(i-1) - Q(i)]x(s) + \sum_{j=-d_{\max+2}}^{-d_{\min}+1} \sum_{s=j-1}^{-1} x^{\mathrm{T}}(s)Q(0)x(s) - \sum_{i=1}^{k} \sum_{j=-d_{\max+2}}^{-d_{\min}+1} \sum_{s=j+i-1}^{i-1} x(s)^{\mathrm{T}}[Q(i-1) - Q(i)]x(s) \tag{30}$$

Hence, in view of

$$\sum_{i=1}^{k} x^{\mathrm{T}}(i)(P(i-1)-P(i))x(i) + \sum_{i=1}^{k}\sum_{s=i-d_i}^{i-1} x^{\mathrm{T}}(s)[Q(i-1)-Q(i)]x(s) + \sum_{i=1}^{k}\sum_{j=-d_{\max+2}}^{-d_{\min}+1}\sum_{s=j+i-1}^{i-1} x^{\mathrm{T}}(s)[Q(i-1)-Q(i)]x(s) \geq 0 \tag{31}$$

the dissipativity of the closed-loop system is assured. However, the inequality in equation (31) depends on the exact knowledge of the delay term $d(i)$. In order to avoid the dependence, we can use a lower bound of the form

$$\sum_{i=1}^{k}\sum_{s=i-d_i}^{i-1} x^{\mathrm{T}}(s)[Q(i-1)-Q(i)]x(s) \geq \sum_{i=1}^{k}\sum_{s=i-d_{\min}}^{i-1} x^{\mathrm{T}}(s)[Q(i-1)-Q(i)]x(s) \tag{32}$$

Therefore, equation (31) can be rearranged as

$$\sum_{i=1}^{k} x^{\mathrm{T}}(i)(P(i-1)-P(i))x(i) + \sum_{i=1}^{k}\sum_{s=i-d_{\min}}^{i-1} x^{\mathrm{T}}(s)[Q(i-1)-Q(i)]x(s) + \sum_{i=1}^{k}\sum_{j=-d_{\max+2}}^{-d_{\min}+1}\sum_{s=j+i-1}^{i-1} x^{\mathrm{T}}(s)[Q(i-1)-Q(i)]x(s) \geq 0 \tag{33}$$

With the help of the definition given in equation (27), we obtain

$$\begin{aligned} p(k-1) &= p(0) - \sum_{i=1}^{k-1} x^{\mathrm{T}}(i)[P(i-1)-P(i)]x(i) \\ q(k-1) &= q(0) - \sum_{i=1}^{k}\sum_{s=i-d_i}^{i-1} x^{\mathrm{T}}(s)[Q(i-1)-Q(i)]x(s) \\ &\quad - \sum_{i=1}^{k}\sum_{j=-d_{\max+2}}^{-d_{\min}+1}\sum_{s=j+i-1}^{i-1} x^{\mathrm{T}}(s)[Q(i-1)-Q(i)]x(s) \end{aligned} \tag{34}$$

If the Schur complement formula is applied to equation (26)

$$p(0) - p(k-1) + x^{\mathrm{T}}(k)[P(k-1)-P(k)]x(k) + q(0) - q(k-1) + \sum_{s=k-d_{\min}}^{k-1} x^{\mathrm{T}}(s)[Q(k-1)-Q(k)]x(s) + \sum_{j=-d_{\max+2}}^{-d_{\min}+1}\sum_{s=j+k-1}^{k-1} x^{\mathrm{T}}(s)[Q(k-1)-Q(k)]x(s) \geq 0 \tag{35}$$

is deduced, and utilizing the terms given in equation (34) in equation (35), we obtain equation (33). Here, in order to point out the equivalence of the Schur complement of equation (26) and the inequality in equation (35), it is necessary to expand the summations in equation (35). With the help of the expansion, it is revealed that each term obtained by expanding the summation terms in equation (35) is exactly same as the quadratic terms which are obtained by applying the Schur complement formula to equation (26). This concludes the proof. □

Theorem 3. *Solving the following optimization problem at each step, allows us to calculate the optimal H_∞controller for the system given in equations (1)–(3).*

$$\min_{L,Y,X,W} \gamma^2 \quad \textit{subject to equations } (5), (18), (20) \textit{ and } (26) \tag{36}$$

Then the feedback gain $K(k) = L(k)Y^{-1}(k)$ is determined by solving the optimization problem given in equation (36) at each step of run-time. The control signal $u(k) = K(k)x(k)$ which is obtained by using the feedback gain $K(k)$ is applied to the system, and the values of the $p(k)$ and $q(k)$ variables are calculated and then used at the next step. Note that the gain of the feedback controller varies at every step, due to the $x(k)$ term that exists in equations (20) and (26). The LMI in equation (26), which involves the $Y(k)$ and $W(k)$ variables and the values of $P(k-1)$ and $p(k-1)$, is neglected at the initial step, $k = 0$. The derivation of the LMI in equation (18) is omitted in this study since it has already been introduced in previous studies such as Boyd et al. (1994) and Chen and Scherer (2006). The derivation of the LMI in equation (5) satisfying the stability condition, the LMI in equation (20) which bounds the state trajectories, and the LMI in equation (26) that guarantees the dissipativity of the closed-loop system are given in Sections 3.1, 3.2 and 3.3, respectively. Next we introduce an algorithm which can be used in a real-time application.

Algorithm

Step 1: Given r_0, $d_{\max}$ and $d_{\min}$, optimal γ is obtained by solving the optimization problem in equation (36) in view of the LMIs in equations (5), (18) and (20). First, a sufficiently high initial value is assigned to γ and the value of γ is updated as $\gamma = \gamma - d\gamma$ as long as the optimization problem in equation (36) has a feasible solution set. Then, the minimum feasible value of γ is assigned as γ_{opt}. Utilizing the feasible solution set to the optimization problem, the LMI variables $P(0)$, $Q(0)$, the feedback controller gain $K(0) = L(0)Y(0)$, dissipation levels $p(0) = x(0)^{\mathrm{T}}P(0)x(0)$ and

$$q(0) = \sum_{s=-d_{\min}}^{-1} x^{\mathrm{T}}(s)Q(0)x(s) + \sum_{s=-d_{\max}+2}^{-d_{\min}+1}\sum_{s=j-1}^{-1} x^{\mathrm{T}}(s)Q(0)x(s)$$

are calculated in order to be used in the next step. Jump to Step 3.

Step 2: Given r_0, $d_{\max}$ and $d_{\min}$, optimal γ is obtained by solving the optimization problem in equation (36) with the LMIs in equations (5), (18), (20) and (26). The value of γ is decreased as $\gamma = \gamma - d\gamma$ as long as the optimization problem in equation (36) can be solved. The value of r is increased in the case where there is no solution for the optimization problem in equation (36) and then γ minimization is re-initiated. $\gamma(k) = \gamma + d\gamma$ is obtained for the first value of γ where there is no solution to the optimization problem in equation (36). Finally, $P(k)$, $Q(k)$, the feedback controller gain $K(k) = L(k)Y(k)$, dissipation levels $p(k)$ and $q(k)$ are calculated by the help of obtained $\gamma(k)$. Then, go to Step 3.

Step 3: Control signal $u(k) = K(k)x(k)$ is applied to the system. Then, go to Step 2.

Simulation studies

In order to demonstrate the application of the proposed algorithm, two numerical examples are introduced in this section. The simulations were carried out under MATLAB using YALMIP with the SEDUMI solver.

Example 1. *Let us consider the following linear time-delay system given in Tang et al. (2008)*

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-d) + B_w w(k) + B_u u(k) \\ y(k) &= Cx(k) \\ x(k) &= \phi(k), \quad k \in [-d_{\max}, 0] \end{aligned} \tag{37}$$

where

$$A = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.9 \end{bmatrix}; A_d = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix}; B_u = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$$
$$B_w = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}; C = [0 \quad 20]; \phi(k) = [0 \quad 0.5]^T$$

Different from the proposed method in this note, the considered system is subject to time-invariant state delay. On the other hand, the disturbance vector $w(k)$ has the dynamics

$$\begin{aligned} v(k+1) &= Gv(k) \\ w(k) &= Dv(k) \end{aligned} \tag{38}$$

where

$$G = \begin{bmatrix} 0.95 & 0.18 \\ -0.18 & 0.9 \end{bmatrix}; D = [1 \quad 0.1]; v(0) = [0 \quad 1]^T$$

Also, the desired reference output $y_{ref}(k)$ is given by the dynamic system governed by

$$\begin{aligned} \tau(k+1) &= F\tau(k) \\ y_{ref}(k) &= H\tau(k) \end{aligned} \tag{39}$$

with

$$F = \begin{bmatrix} 0.9 & 0.1 \\ -0.2 & 0.8 \end{bmatrix}; \ H = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}^T; \tau(0) = [1 \quad 0]^T$$

The controlled output $z(k)$ is defined as the output trajectory error

$$e(k) = y_{ref}(k) - y(k)$$

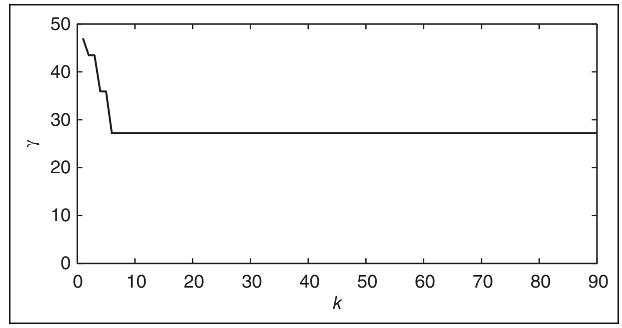

Figure 1. The variation of the disturbance attenuation level of the system.

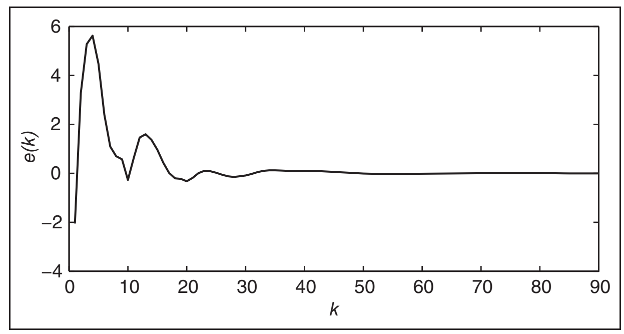

Figure 2. The variation of the controlled output defined to be the error between the desired reference and system output.

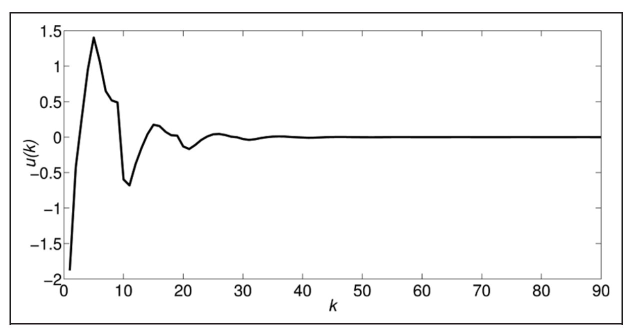

Figure 3. The control signal $u(k)$ applied to the system.

and u_{max} is selected as 2. By choosing $r_0 = 34$ and $d_{max} = d_{min} = 8$, the optimization algorithm for the proposed controller is carried out and γ is calculated as 47.01 at the first step. Then, by solving the optimization problem in equation (36) with the LMIs in equations (5), (18), (20) and (26) recursively for the given example, the controlled output z and control signal u are obtained. Also, the variation of the disturbance attenuation level γ is found to be as illustrated in Figure 1. The controlled output and control signal are illustrated in Figures 2 and 3, respectively. The plots confirm that the proposed receding horizon H_∞ control (RHHC). method has a satisfactory performance for the system given above. Figure 2 shows that $e(k)$ converges to zero very rapidly with the constrained control effort shown in Figure 3. Hence, it can be concluded that the

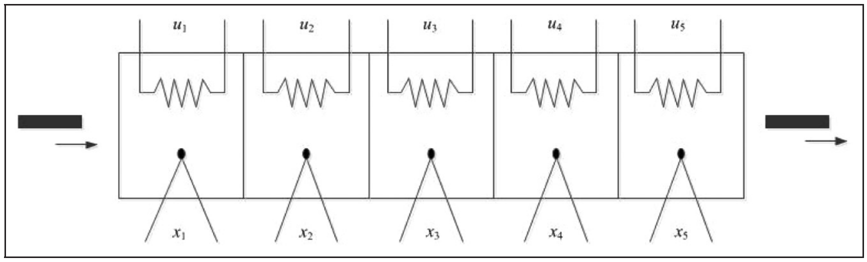

Figure 4. Schematic diagram of the industrial electric heater.

RHHC algorithm is very effective for state-delayed systems having state and input constraints.

Example 2. *An industrial electric heater which is defined with a fifth-order state space model is considered in this example. The model of the heater is given in detail in Chu (1995) and Leite et al. (2009) and involves five heating zones, each equipped with their own thermocouple and electric heater. The temperatures in each zone are state variables* ($x_1,\dots,x_5$) *and the electrical current signals applied to each zone of the heater are control inputs* ($u_1,\dots,u_5$)*, which are shown in Figure 4.*

A state-delayed nominal discrete model for this system was obtained in Chu (1995) in the form

$$x(k+1) = Ax(k) + A_d x(k-d) + B_u u(k) \tag{40}$$

where the system matrices are calculated as

$$A = \begin{bmatrix} 0.97421 & 0.15116 & 0.19667 & -0.05870 & 0.07144 \\ -0.01455 & 0.88914 & 0.26953 & 0.11866 & -0.22047 \\ 0.06376 & 0.12056 & 1.00049 & -0.03491 & -0.02766 \\ -0.05084 & 0.09254 & 0.28774 & 0.82569 & 0.02570 \\ 0.01723 & 0.01939 & 0.29285 & 0.03544 & 0.87111 \end{bmatrix}$$

$$A_d = \begin{bmatrix} -0.01000 & -0.08837 & -0.06989 & 0.18874 & 0.20505 \\ 0.02363 & 0.03384 & 0.05282 & -0.09906 & -0.00191 \\ -0.04468 & -0.00798 & 0.05618 & 0.00157 & 0.03593 \\ -0.04082 & 0.01153 & -0.07116 & 0.16472 & 0.00083 \\ -0.02537 & 0.03878 & -0.04683 & 0.05665 & -0.03130 \end{bmatrix}$$

$$B_u = \begin{bmatrix} 0.53706 & -0.11185 & 0.09978 & 0.04652 & 0.25867 \\ -0.51718 & 0.73519 & 0.57518 & 0.40668 & -0.12472 \\ 0.29469 & 0.31528 & 1.16420 & -0.29922 & 0.23883 \\ -0.20191 & 0.19739 & 0.41686 & 0.66551 & 0.11366 \\ -0.11835 & 0.16287 & 0.20378 & 0.23261 & 0.36525 \end{bmatrix}$$

and the state and control vectors are defined as

$$x = \begin{bmatrix} T_1-\bar{T}_1 & T_2-\bar{T}_2 & T_3-\bar{T}_3 & T_4-\bar{T}_4 & T_5-\bar{T}_5 \end{bmatrix}^{\mathrm{T}}$$

$$u = \begin{bmatrix} u_1-\bar{u}_1 & u_2-\bar{u}_2 & u_3-\bar{u}_3 & u_4-\bar{u}_4 & u_5-\bar{u}_5 \end{bmatrix}^{\mathrm{T}}$$

where T_i *and* u_i ($i = 1,2,3,4,5$) *are the temperature and control electric current in each zone, and* $\bar{T}_i$ *and* $\bar{u}_i$ *are their operating points. In addition to the nominal discrete model given in Chu (1995), Leite et al. (2009) take the presence of disturbance and uncertainties due to changes in the operation points into consideration. Similar to Leite et al. (2009), a performance output* $z(k)$ *which is in the form of equation (1) with*

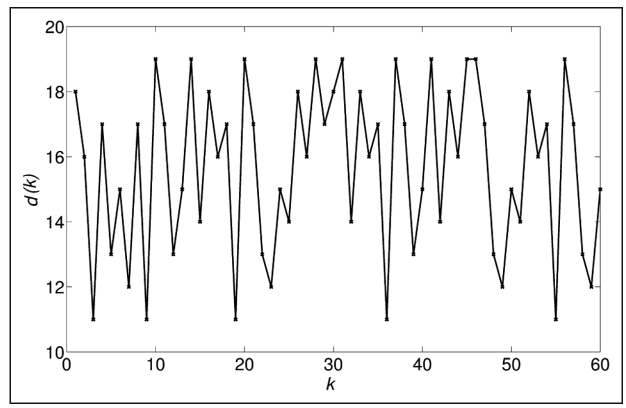

Figure 5. The time-varying delay $d(k)$.

$$B_w = \begin{bmatrix} -0.2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -0.2 \\ 0 & 0 & 1 & 0.2 & 0 \end{bmatrix}^{\mathrm{T}};$$

$$C = \begin{bmatrix} 0.03 & 0.03 & 0 & 0 & 0 \\ 0 & 0 & 0.03 & 0.03 & 0 \\ 0 & 0 & 0 & 0 & 0.03 \end{bmatrix};$$

$$C_d = \begin{bmatrix} 0.3 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.3 \end{bmatrix};$$

$$D_u = \begin{bmatrix} 0 & 0 & 0.01 & 0 & 0.01 \\ 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 & 0 \end{bmatrix};$$

$$D_w = diag\{0.1, 0.2, -0.4\}$$

is employed in this study in order to prove the efficiency of the proposed controller. Different from Chu (1995) and Leite et al. (2009), the time-delay in the states given in Figure 5, is assumed to be time-varying such as $d_{\min} \le d(k) \le d_{\max}$.

On the other hand, the disturbance vector is defined as $w(k) = [w_1(k) \quad w_2(k) \quad w_3(k)]^{\mathrm{T}}$ *where*

$$w_1(k) = \begin{cases} 0.1 & when\ 10 \ge k > 0 \\ 0 & when\ k > 10 \end{cases}$$

$$w_2(k) = \begin{cases} 0.2 & when\ 10 \ge k > 0 \\ 0 & when\ k > 10 \end{cases}$$

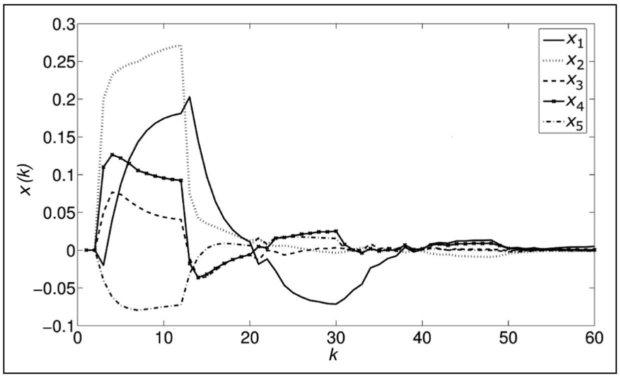

Figure 6. The states $x(k)$ of the system.

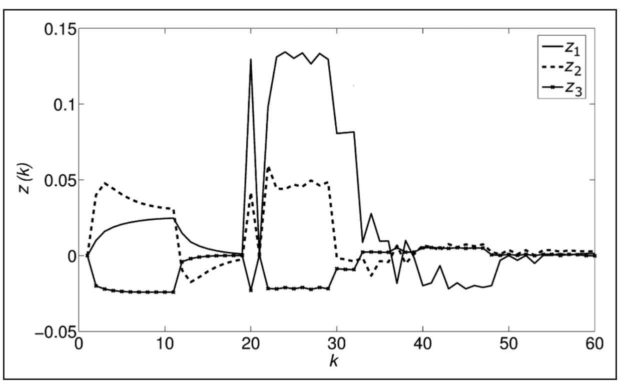

Figure 7. The performance outputs $z(k)$ of the system.

$$w_3(k) = \begin{cases} 0.05 & \text{when } 10 \geq k > 0 \\ 0 & \text{when } k > 10 \end{cases}$$

The control signal $u(k)$ is bounded as $|u_i(k)| \leq 1$, $\forall k \geq 0$, $i = 1,2,\ldots,5$ and the initial conditions are assumed to be all equal to zero. Choosing $r_0 = 1$, $\gamma(k) = 1.8708$ is obtained for $d_{\min} = 10$ and $d_{\max} = 20$. The simulation results of the states $x(k)$, performance outputs $z(k)$ and control inputs $u(k)$ are given in Figures 6–8, respectively.

Similar to the previous example, the plots of $x(k)$, $z(k)$ and $u(k)$ confirm that the proposed RHHC method has a satisfactory performance for the industrial electric heater system. Different from the previous example, the proposed method is employed for the control of a system with time-varying delay. Figures 6 and 7 clearly demonstrate the efficiency of the proposed RHHC algorithm when the system is subject to disturbance $w(k)$ which is given above. Moreover, the results illustrate that the proposed method results in satisfactory performance for the systems involving time-varying delay. Also note that the control effort applied to the system is constrained, as illustrated in Figure 8.

Conclusion

A systematic approach to obtaining a delay-dependent receding horizon H_∞ controller design method for discrete-time state-delayed systems has been addressed in this study. The main contributions of this note to the literature can be stated as follows: First, different from the existing ones, the proposed method is based on delay-dependent LMI conditions and is therefore less conservative than its counterparts. Secondly, different from the existing literature, this note considers discrete-time state-delayed systems having time-varying and interval delays, which is much more realistic than constant-delayed systems. Moreover, thanks to the iterative structure of the proposed method, the controller can adopt to varying parameters such as disturbance and control input constraints and effectively control time-delay systems in the presence of constrained inputs and disturbances, especially when the system is subject to a sufficiently large disturbance. The efficiency of the proposed approach has been validated on a couple of numerical benchmark examples borrowed from literature. The authors believe that the proposed method can be expanded to time-delay systems having parametric uncertainties as a future study, which are highly related to the robustness of the method.

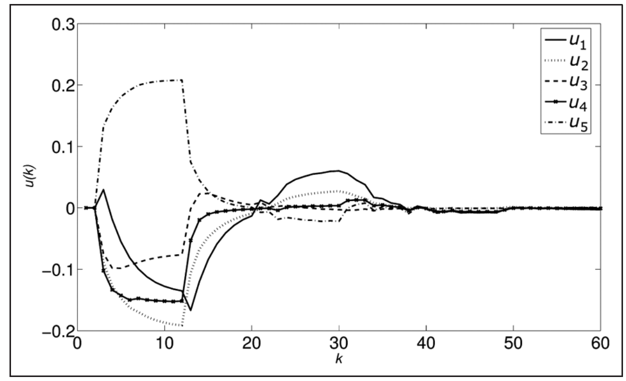

Figure 8. The control signal $u(k)$ applied to the system.

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