A Set Coverage Problem

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Abstract

This paper shows that with $B = \{1, 2, \ldots, n\}$, the smallest k such that

$$(B \times B) - \{(j, j) \mid j \in B\} = \bigcup_{i=1}^{k} (C_i \times D_i)$$

is $s(n)$, where $s(n)$ is the smallest integer k such that $n \leqslant \binom{k}{\lfloor \frac{k}{2} \rfloor}$. This provides a simple set-based formulation and a new proof of a result for boolean ranks [2] and biclique covering of bipartite graphs [1, 5], making these intricate results more accessible.

Key words: Combinatorial problems; Boolean rank; Bipartite covering; Sperner's Theorem

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1. Introduction

The boolean rank *br* of a binary matrix $M_{m,n}$ is the least k such that $M_{m,n} = S_{m,k} T_{k,n}$, where matrix product is carried out in the boolean algebra. Boolean rank is an important topic for role-based access control [4, 8] and communication complexity [3].

There are two equivalent problems related to boolean rank. One is the additive version: the boolean rank of a binary matrix M is the same as the minimal number k of rank-1 boolean matrices M_i such that $M = \Sigma_{i=1}^{k} M_i$. The second is the (edge) covering problem for bipartite graphs: given a bipartite graph G, find the minimum number of bicliques (complete bipartite subgraphs) covering all the edges in G. The translation between these two versions is straightforward: a boolean matrix $M_{m,n}$ corresponds to a bipartite graph, where rows and columns form two disjoint sets of nodes without any edges between nodes of the same set. Each rank-1 sub-matrix of $M_{m,n}$ is a biclique.

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The fact that the decision problem for boolean rank is NP-complete [6] makes the determination of boolean ranks of even seemingly simple binary matrices non-trivival. De Caen, Gregory and Pullman [2] proved that $br(\bar{I}_n) = s(n)$, where $\bar{I}_n$ is the complement of the identity matrix I_n, and $s(n)$ is the smallest integer k such that $n \leqslant \binom{k}{\lfloor \frac{k}{2} \rfloor}$. The same result was recently rediscovered in [1, 5] (see previous paragraph), with a new proof, for biclique covering of the bipartite graph $K^-_{n,n}$, where $K^-_{n,n}$ is the complete bipartite graph $K_{n,n}$ with a perfect matching removed.

In this paper, we give a simple set-based formulation and a new proof of the result $br(\bar{I}_n) = s(n)$: with $B = \{1, 2, \ldots, n\}$, the smallest k such that

$$(B \times B) - \{(j, j) \mid j \in B\} = \bigcup_{i=1}^{k} (C_i \times D_i)$$

is $s(n)$. This simple proof can be easily reinterpreted in either the boolean rank setting or the bipartite graph coverage setting.

2. Main Result

Let $B = \{1, 2, \ldots, n\}$ and $1_B = \{(j, j) \mid j \in B\}$. Suppose $B \times B - 1_B = \bigcup_{j=1}^{k} (C_j \times D_j)$. Then $C_j \cap D_j = \emptyset$ for each $1 \leq j \leq k$. In fact, as long as this disjointness property is enforced, we can make each "block" $C_j \times D_j$ as large as possible and have $B \times B - 1_B = \bigcup_{j=1}^{k} (C_j \times C_j^c)$, where C_j^c is the complement of C_j in B.

Theorem 1. *Let $B = \{1, 2, \ldots, n\}$. If $B \times B - 1_B = \bigcup_{i=1}^{k} (C_i \times D_i)$, then $k \geq s(n)$.*

Proof. Suppose $B \times B - 1_B = \bigcup_{i=1}^{k} (C_i \times D_i)$. For $j = 1, 2, \ldots, n$, let $A_j = \{i \mid j \in C_i\}$. It is clear that $A_j \neq \emptyset$ for any $j \in \{1, 2, \ldots, n\}$. To show that $A_1, A_2, \ldots, A_n$ form an antichain on the set $\{1, 2, \ldots, k\}$, it suffices to prove that there is no pair of sets A_{j_1} and A_{j_2} such that $A_{j_1} \subseteq A_{j_2}$ when $j_1 \neq j_2$.

Suppose to the contrary that $A_{j_1} \subseteq A_{j_2}$ for some $j_1 \neq j_2$, then $j_1 \in C_i$ implies $j_2 \in C_i$. Since $(j_1, j_2) \in \bigcup_{i=1}^{k} (C_i \times D_i)$, there exists $m \in \{1, 2, \ldots, k\}$ such that $(j_1, j_2) \in C_m \times D_m$, i.e., $j_1 \in C_m$ and $j_2 \in D_m$. By the remark earlier, we have $j_2 \in D_m \subseteq C_m^c$, that is,

$j_2 \notin C_m$. This contradicts $j_1 \in C_i$ implies $j_2 \in C_i$. Therefore, $\{A_1, A_2, \ldots, A_n\}$ is an antichain on $\{1, 2, \ldots, k\}$.

By Sperner's Theorem [7], we have $n \leqslant \binom{k}{\lfloor \frac{k}{2} \rfloor}$. This completes the proof. $\square$

Our next lemma provides the key for an antichain-based "block" design.

Lemma 1. *Let $A = \{1, 2, \ldots, k\}$ and $\{A_1, A_2, \ldots, A_n\}$ be an antichain on A. Let $C_i := \{j \mid i \in A_j\}$ for $i \in A$. Then for any $j \in \{1, 2, \ldots, n\}$ we have*

$$\bigcap_{i \in A_j} C_i = \{j\}.$$

Proof. Since $j \in C_i$ for any $i \in A_j$, we have $j \in \bigcap_{i \in A_j} C_i$. Therefore, $\{j\} \subseteq \bigcap_{i \in A_j} C_i$. Suppose $t \in \bigcap_{i \in A_j} C_i$. Then $t \in C_i$ for each $i \in A_j$, that is, $i \in A_t$ for each $i \in A_j$, from which it follows that $A_j \subseteq A_t$. By the assumption that $\{A_1, A_2, \ldots, A_n\}$ is an antichain, we have $j = t$. Hence $\bigcap_{i \in A_j} C_i = \{j\}$. $\square$

Theorem 2. *Let $B = \{1, 2, \ldots, n\}$. There exist $C_i \subseteq B$, for $1 \leq i \leq s(n)$, such that*

$$B \times B - 1_B = \bigcup_{i=1}^{s(n)} (C_i \times C_i^c).$$

Proof. Consider $A = \{1, 2, \ldots, s(n)\}$. Let $\{A_1, A_2, \ldots, A_n\}$ be a size-n antichain on A. Such an antichain exists. For example, there are $\binom{s(n)}{\lfloor \frac{s(n)}{2} \rfloor}$ ($\geqslant n$) different subsets of A with cardinality $\lfloor \frac{s(n)}{2} \rfloor$, and one can take all the size-$\lfloor \frac{s(n)}{2} \rfloor$ subsets of A to form such an antichain.

For $i = 1, 2, \ldots, k$, define $C_i := \{j \mid i \in A_j\}$. Now we prove that $B \times B - 1_B = \bigcup_{i=1}^{s(n)} (C_i \times C_i^c)$. It is clear that $\bigcup_{i=1}^{s(n)} (C_i \times C_i^c) \subseteq B \times B - 1_B$. For any $(s, t) \in B \times B - 1_B$, $s \in C_i$ for all $i \in A_s$. We show that there exists $m \in A_s$ such that $t \notin C_m$ by contradiction. Suppose to the contrary that $t \in C_m$ for all $m \in A_s$, then $t \in \bigcap_{m \in A_s} C_m$. However, by Lemma 1, we have $\bigcap_{m \in A_s} C_m = \{s\}$. Hence $s = t$, which contradicts the assumption that $(s, t) \in B \times B - 1_B$. Therefore, $s \in C_m$ and $t \notin C_m$, that is, $(s, t) \in C_m \times C_m^c$. $\square$

As an illustration of the construction given in Theorem 2, we present a simple example.

Example 1. *Let* $n = 5$. *We have* $s(5) = 4$. *Consider* $A = \{1, 2, 3, 4\}$. *Take the antichain* $A_1 = \{1, 2\}, A_2 = \{1, 3\}, A_3 = \{1, 4\}, A_4 = \{2, 3\}, A_5 = \{3, 4\}$. *Then* $C_1 = \{1, 2, 3\}, C_2 = \{1, 4\}, C_3 = \{2, 4, 5\}$, *and* $C_4 = \{3, 5\}$. *We have*

$$\begin{aligned}(\{1,2,3,4,5\} \times \{1,2,3,4,5\}) - 1_B \quad = \quad & \{1,2,3\} \times \{4,5\} \cup \\ & \{1,4\} \times \{2,3,5\} \cup \\ & \{2,4,5\} \times \{1,3\} \cup \\ & \{3,5\} \times \{1,2,4\}.\end{aligned}$$

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