Global dimensions of rings with respect to a semidualizing module *†

Guoqiang Zhao[a]‡, Juxiang Sun[b]

[a] *School of Science, Hangzhou Dianzi University, Hangzhou, 310018, China*

[b] *School of Mathematics and Information Science, Shangqiu Normal University, Shangqiu, 476000, China*

E-mail address: gqzhao@hdu.edu.cn, sunjx8078@163.com

Abstract

In this paper, the notion of strongly G_C-projective and injective modules is introduced, where C is a semidualizing module. Using these modules we can obtain a new characterization of G_C-projective and injective modules, similar to the one of projective modules by the free modules. We then define and study the global dimensions of rings relative to a semidualizing module C, and prove that the global G_C-projective dimension of a ring R is equal to the global G_C-injective dimension of R.

1. Introduction

Over a Noetherian ring, Foxby [5] introduced the notion of semidualizing modules, which provided a common generalization of a dualizing module and a free module of rank one. Golod [6] and Vasconcelos [13] furthered the study of semidualizing modules. By using these modules, Golod defined the G_C-dimension, a refinement of projective dimension, for finitely generated modules. When $C = R$, this recovers the G-dimension introduced by Auslander and Bridger in [1]. Motivated by Enochs and Jenda$'$s extensions in [4] of G-dimension, Holm and Jϕgensen [8] have extended the G_C-dimension to arbitrary modules over a Noetherian ring (where they used the name of C-Gorenstein projective dimension). This also enables them to give the dual notion. Then White [14] extended these concepts to the non-Noetherian setting, named G_C-projective and G_C-injective dimension, and showed that they share many common properties with the Gorenstein homological dimensions extensively studied in recent decades.

It is well-known that, the classical global dimensions of rings play an important role in the theory of rings. Recently, Bennis and Mahadou [3] defined the global Gorenstein projec-

*2000 Mathematics Subject Classification: 13D02, 13D05, 13D07, 18G25.

†Keywords: semidualizing modules; Strongly G_C-projective and injective modules; G_C-projective and injective dimensions; G_C-(weak) global dimension of rings.

‡Corresponding author.

tive dimension of a ring R and the global Gorenstein injective dimension of R, and proved that they are equal for any associative ring by using the properties of strongly Gorenstein projective and injective modules. For more details of these two modules, see [2]. Based on the above results, in this paper we mainly study the global dimensions of a ring R with respect to a semidualizing module.

This paper is organized as follows.

In Section 2, we give some definitions and some preliminary results.

In Section 3, we introduce and study the strongly G_C-projective and injective modules. The main result of this section is that an R-module is G_C-projective (resp. injective) if and only if it is a direct summand of a strongly G_C-projective (resp. injective) R-module. We then give some equivalent characterizations of the strongly G_C-projective and injective modules, and show that the class of strongly G_C-projective (resp. injective) modules is between the class of projective, C-projective (resp. injective) modules and the class of G_C-projective (resp. injective) modules.

In Section 4, relative to a semidualizing module C, we firstly define the $\mathcal{P}_C$-projective and $\mathcal{I}_C$-injective dimension of a ring R, and prove that they are both equal to the classical global dimension of R. Next, we define and investigate the G_C-projective and G_C-injective dimension of a ring R. The main result of this paper is that the G_C-projective and G_C-injective dimension of a ring R coincide, and we call the common value the C-Gorenstein global dimension of R. Then we discuss the relations between the C-Gorenstein global dimension of a ring R with other global dimensions of R. At the end of this section, we study the behavior of modules over rings of finite C-Gorenstein global dimension, and give a partial answer to the question posed by Takahashi and White in [12].

2. Preliminaries

Throughout this work R is a commutative ring with unity. For an R-module M, we use $\mathrm{id}_R(M)$, $\mathrm{pd}_R(M)$ and $\mathrm{fd}_R(M)$ to denote the injective dimension, projective dimension and flat dimension of M, respectively. We use $\mathrm{gl.dim}(R)$ to denote the classical global dimension of R.

Semidualizing modules, defined next, form the basis for our categories of interest.

Definition 2.1 ([14]) An R-module C is *semidualizing* if

(a) C admits a degreewise finite projective resolution,

(b) The natural homothety map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism, and

(c) $\mathrm{Ext}^i_R(C, C) = 0$ for any $i \geq 1$.

From now on, C is a semidualizing module.

Definition 2.2 ([9]) An R-module is called *C-projective* if it has the form $C\otimes_R P$ for some projective R-module P. An R-module is called *C-injective* if it has the form $\mathrm{Hom}_R(C,I)$ for some injective R-module I. Set

$$\mathcal{P}_C = \mathcal{P}_C(R) = \{C \otimes_R P|_R P \textit{ is projective}\}, and$$

$$\mathcal{I}_C = \mathcal{I}_C(R) = \{\mathrm{Hom}_R(C,I)|_R I \textit{ is injective}\}.$$

Let $\mathcal{C}$ be a subclass of R-modules. Recall that a sequence of R-modules $\mathbf{L}$ is called $\mathrm{Hom}_R(-,\mathcal{C})$ (resp. $\mathrm{Hom}_R(\mathcal{C},-)$) exact if the sequence $\mathrm{Hom}_R(\mathbf{L},C')$ (resp. $\mathrm{Hom}_R(C',\mathbf{L})$) is exact for any $C' \in \mathcal{C}$.

Definition 2.3 ([14]) A complete $\mathcal{P}\mathcal{P}_C$-resolution is a $\mathrm{Hom}_R(-,\mathcal{P}_C)$ exact exact sequence of R-modules

$$\mathbf{X} = \cdots \to P_1 \to P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots$$

with P_i and P^i are projective R-modules. An R-module M is called *G_C-projective* if there exists a complete $\mathcal{P}\mathcal{P}_C$-resolution as above with $M \cong \mathrm{Coker}(P_1 \to P_0)$. Set

$$\mathcal{G}\mathcal{P}_C(R) = \text{the class of } \mathrm{G_C}\text{-projective R-modules.}$$

A complete $\mathcal{I}_C\mathcal{I}$-resolution and G_C-injective module are defined dually.

From [14, proposition 2.6], we know that every projective and C-projective R-module are G_C-projective. Now, we give a "non-trivial" example of G_C-projective R-module.

Example 2.4 Assume R is a Gorenstein artin algebra with $\mathrm{gl.dim}(R) = \infty$. Let $C = \oplus I_j$, where I_j are all the indecomposable and non-isomorphic direct summands of modules appeared in the minimal injective resolution of R. Then C is a semidualizing module. In this case, every finitely generated R-module is G_C-projective. While the class of finitely generated C-projective R-modules is just the class of all finitely generated injective R-modules. However, it is clear that there exists an R-module which is not projective and injective.

Let $\mathcal{G}^2\mathcal{P}_C(R) = \{$ A is an R-module | there exists a $\mathrm{Hom}_R(-,\mathcal{P}_C)$ exact exact sequence of R-modules $\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$ with all G_i and G^i in $\mathcal{G}\mathcal{P}_C(R)$ and $A \cong \mathrm{Im}(G_0 \to G^0)\}$.

The following result means that an iteration of the procedure used to define the G_C-projective modules yields exactly the G_C-projective modules.

Lemma 2.5 ([10, Theorem 2.9]) $\mathcal{G}^2\mathcal{P}_C(R) = \mathcal{G}\mathcal{P}_C(R)$.

Definition 2.6 ([14]) Let $\mathcal{X}$ be a class of R-modules and M an R-module. An $\mathcal{X}$-*resolution* of M is an exact sequence of R-modules as follows:

$$\cdots \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

with each $X_i \in \mathcal{X}$ for any $i \geq 0$. The $\mathcal{X}$-*projective dimension* of M is the quantity

$$\mathcal{X}\text{-}\operatorname{pd}_R(M) = \inf\{\sup\{n \geq 0 | X_n \neq 0\} | X \textit{is an } \mathcal{X}\textit{-resolution of } M\}.$$

The $\mathcal{X}$-*coresolution* and $\mathcal{X}$-*injective dimension* of M are defined dually. We write $G_C\text{-}\operatorname{pd}_R(M)$ $= \mathcal{GP}_C(R)\text{-}\operatorname{pd}_R(M)$.

Lemma 2.7 *If* $\sup\{G_C\text{-}\operatorname{pd}_R(M) | M$ *is an* R*-module*$\} < \infty$*, then, for an integer* n*, the following are equivalent:*

(1) $\sup\{G_C\text{-}\operatorname{pd}_R(M) | M$ *is an* R*-module*$\} \leq n$*,*

(2) $\operatorname{id}_R(N) \leq n$ *for every* R*-module* N *with finite* $\mathcal{P}_C$*-projective dimension.*

Proof. Use [14, Proposition 2.12] and [11, Theorem 9.8]. □

Definition 2.8 ([8]) M is called G_C-*flat* if there is an exact sequence of R-modules

$$\mathbf{X} = \cdots \to F_1 \to F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots$$

with F_i and F^i are flat R-modules, such that $M \cong \operatorname{Coker}(F_1 \to F_0)$ and $\operatorname{Hom}_R(C, I) \otimes_R \mathbf{X}$ is still exact for any injective R-module I. We define $G_C\text{-}\operatorname{fd}_R(M)$ analogously to $G_C\text{-}\operatorname{pd}_R(M)$.

3. Strongly G_C-projective and injective modules

We denote $\operatorname{Add}_R M$ the subclass of R-modules consisting of all modules isomorphic to direct summands of direct sums of copies of M. By [10, Proposition 2.4], $\mathcal{P}_C = \operatorname{Add}_R C$.

Definition 3.1 An R-module M is called *strongly* G_C*-projective*, if there exists an exact sequence of R-modules

$$\mathbf{D} = \cdots \xrightarrow{f} D \xrightarrow{f} D \xrightarrow{f} D \xrightarrow{f} \cdots$$

with $D \in \operatorname{Add}_R(C \oplus R)$, such that $M \cong \operatorname{Ker} f$ and $\operatorname{Hom}_R(\mathbf{D}, \mathcal{P}_C)$ is still exact.

When $C = R$, it is just the strongly Gorenstein projective module introduced in [2].

Strongly G_C-injective modules are defined dually. In the following, we only deal with the strongly G_C-projectivity of modules. The results about strongly G_C-injective modules have a dual version, and we omit them.

From definition, we immediately have:

Proposition 3.2 *The class of strongly* G_C*-projective modules is closed under direct sums.*

The principal role of these modules is to give a simple characterization of the G_C-projective modules, as follows:

Theorem 3.3 *An R-module is G_C-projective if and only if it is a direct summand of a strongly G_C-projective R-module.*

Proof. ($\Leftarrow$) From Lemma 2.5 and [14, Proposition 2.6], it is easy to see that every strongly G_C-projective module is G_C-projective. Since the class of G_C-projective modules is closed under direct summands by [14, Theorem 2.8], the assertion follows immediately.

($\Rightarrow$) Let M be a G_C-projective R-module. Then the definition gives rise to a $\mathrm{Hom}_R(-,\mathcal{P}_C)$ exact exact sequence of R-modules:

$$X = \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \longrightarrow \cdots$$

with all $C_i \in \mathrm{Add}R \cup \mathrm{Add}C$ and $M \cong \mathrm{Im}(C_0 \to C_{-1})$.

For each $n \in \mathbb{Z}$, let $\Sigma^n X$ be the exact complex obtained from X by increasing all index by n: $(\Sigma^n X)_i = X_{i-n}$ and $d_i^{\Sigma^n X} = d_{i-n}$ for all $i \in \mathbb{Z}$.

Then we obtain an exact complex

$$\bigoplus \Sigma^n X = \cdots \longrightarrow \oplus C_i \xrightarrow{\oplus d_i} \oplus C_i \xrightarrow{\oplus d_i} \oplus C_i \xrightarrow{\oplus d_i} \oplus C_i \longrightarrow \cdots$$

Clearly, $\oplus C_i \in \mathrm{Add}_R(C \oplus R)$. Since $\mathrm{Hom}_R(\bigoplus \Sigma^n X, \mathcal{P}_C) \cong \prod \mathrm{Hom}_R(\Sigma^n X, \mathcal{P}_C)$, the complex $\bigoplus \Sigma^n X$ is also $\mathrm{Hom}_R(-,\mathcal{P}_C)$ exact. Thus M is a direct summand of the strongly G_C-projective module $\mathrm{Im}(\oplus d_i)$, as desired. □

The next result gives a simple characterization of the strongly G_C-projective modules.

Proposition 3.4 *For any R-module M, the following are equivalent:*

(1)M is strongly G_C-projective;

(2)There exists a short exact sequence $0 \to M \to D \to M \to 0$, with $D \in \mathrm{Add}_R(C \oplus R)$, and $\mathrm{Ext}_R^1(M,J) = 0$ for any R-module J with finite $\mathcal{P}_C$-projective dimension (or for any C-projective R-module J);

(3)There exists a short exact sequence $0 \to M \to D \to M \to 0$, with $D \in \mathrm{Add}_R(C \oplus R)$, and $\mathrm{Ext}_R^i(M,J) = 0$ for some integer $i > 0$ and for any R-module J with finite $\mathcal{P}_C$-projective dimension.

Proof. (1) $\Rightarrow$ (2) Follows from definition and [14, Proposition 2.12], and (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let $0 \to M \to D \to M \to 0$ be the short exact sequence with $D \in \mathrm{Add}_R(C\oplus R)$. Then, for for any R-module J with finite $\mathcal{P}_C$-projective dimension and all $j > 0$, we have the long exact sequence:

$$0 = \mathrm{Ext}_R^j(D,J) \to \mathrm{Ext}_R^j(M,J) \to \mathrm{Ext}_R^{j+1}(M,J) \to \mathrm{Ext}_R^{j+1}(D,J) = 0$$

Then $\mathrm{Ext}^i_R(M,J)=0$ for some integer $i>0$ implies $\mathrm{Ext}^j_R(M,J)=0$ for all $j>0$. Gluing the short exact sequence $0\to M\to D\to M\to 0$, we get that M is strongly G_C-projective.□

Remark 3.5 From Theorem 3.3 and Proposition 3.4, we know that every projective and C-projective module are strongly G_C-projective. Indeed, suppose that M is projective or C-projective, then it is clear that $M\in \mathrm{Add}_R(C\oplus R)$. Moreover, we have a split exact sequence $0\to M\to M\oplus M\to M\to 0$.

4. Global dimensions of a ring relative to a semidualizing module

The $\mathcal{P}_C$-projective and $\mathcal{I}_C$-injective dimension of a ring R are defined as

$$\mathrm{P}_C\text{-PD}(R)=\sup\{\mathcal{P}_C\text{-}\,\mathrm{pd}_R(M)|M \textit{ is an } R\textit{-module}\}$$

$$\mathrm{I}_C\text{-ID}(R)=\sup\{\mathcal{I}_C\text{-}\,\mathrm{id}_R(M)|M \textit{ is an } R\textit{-module}\}.$$

When $C=R$, they are the classical homological dimensions of the ring R. It is natural to ask whether the $\mathcal{P}_C$-projective and $\mathcal{I}_C$-injective dimension of a ring R are equal.

Proposition 4.1 *For a ring R,* $\mathrm{P}_C\text{-PD}(R)=\mathrm{I}_C\text{-ID}(R)=\mathrm{gl.dim}(R)$.

Proof. Let M be an R-module, then $\mathcal{P}_C\text{-}\,\mathrm{pd}_R(M)=\mathrm{pd}_R(\mathrm{Hom}_R(C,M))$ from [12, Theorem 2.11]. So $\mathrm{P}_C\text{-PD}(R)\le \mathrm{gl.dim}(R)$, and the converse holds true since $\mathrm{pd}_R(M)=\mathcal{P}_C\text{-}\,\mathrm{pd}_R(C\otimes_R M)$. Therefore $\mathrm{P}_C\text{-PD}(R)=\mathrm{gl.dim}(R)$.

Similarly, we get $\mathrm{I}_C\text{-ID}(R)=\mathrm{gl.dim}(R)$. Thus $\mathrm{P}_C\text{-PD}(R)=\mathrm{I}_C\text{-ID}(R)$. □

Then one can get a new characterization of semisimple rings in terms of C-projective and C-injective modules.

Corollary 4.2 *For a ring R, the following are equivalent:*

(1) R is semisimple,

(2) Every R-module is C-projective,

(3) Every R-module is C-injective.

The G_C-projective and G_C-injective dimension of a ring R are defined as

$$\mathrm{G}_C\text{-PD}(R)=\sup\{G_C\text{-}\,\mathrm{pd}_R(M)|M \textit{ is an } R\textit{-module}\}$$

$$\mathrm{G}_C\text{-ID}(R)=\sup\{G_C\text{-}\,\mathrm{id}_R(M)|M \textit{ is an } R\textit{-module}\}.$$

The following lemma is useful in the proof of the main result, and its proof uses so-called Bass class techniques. Recall from [14] that, the Bass class with respect to C, denoted $\mathcal{B}_C(R)$, consists of all R-modules N satisfying

(a) $\mathrm{Ext}^{i\ge1}_R(C,N)=0$,

(b) $\mathrm{Tor}^R_{i\ge1}(C,\mathrm{Hom}_R(C,N))=0$, and

(c) The evaluation map $C \otimes_R \mathrm{Hom}_R(C, N) \to N$ is an isomorphism.

Lemma 4.3 *Let M be an R-module with $\mathrm{id}_R(M) < \infty$ and $G_C\text{-}\mathrm{pd}_R(M) < \infty$. Then $\mathcal{P}_C\text{-}\mathrm{pd}_R(M) = G_C\text{-}\mathrm{pd}_R(M) < \infty$.*

Proof. Since $G_C\text{-}\mathrm{pd}_R(M)$ is finite, by [10, Lemma 2.8], there is a short exact sequence of R-modules

$$0 \to M \to N \to G \to 0 \qquad (*)$$

such that $\mathcal{P}_C\text{-}\mathrm{pd}_R(N) < \infty$ and G is G_C-projective.

We claim that $\mathrm{Ext}^1_R(G, M) = 0$. In fact, as G is G_C-projective, there exists an exact sequence

$$\mathbf{X} = 0 \to G \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots$$

with P^i projective. Set $G_i = \mathrm{Ker}(C \otimes_R P^i \to C \otimes_R P^{i+1})$ for each $i \geq 0$. The finiteness of $\mathrm{id}_R(M)$ implies that $M \in \mathcal{B}_C(R)$ by [9, Corollary 6.6], and so $\mathrm{Ext}^{i\geq 1}_R(C, M) = 0$. Thus $\mathrm{Ext}^{i\geq 1}_R(C \otimes_R P, M) \cong \mathrm{Hom}_R(P, \mathrm{Ext}^{i\geq 1}_R(C, M)) = 0$ for each projective R-module P from [11, P.258, 9.20]. Applying $\mathrm{Hom}_R(-, M)$ to the sequence $\mathbf{X}$, we have $\mathrm{Ext}^1_R(G_0, M) \cong \mathrm{Ext}^{d+1}_R(G_d, M) = 0$ by dimension-shifting argument, where $d = \mathrm{id}_R(M)$, as claimed.

This implies the sequence $(*)$ splits, then $\sup\{\mathcal{P}_C\text{-}\mathrm{pd}_R(M), \mathcal{P}_C\text{-}\mathrm{pd}_R(G)\} = \mathcal{P}_C\text{-}\mathrm{pd}_R(N) < \infty$, and hence $\mathcal{P}_C\text{-}\mathrm{pd}_R(M) < \infty$. The equality $\mathcal{P}_C\text{-}\mathrm{pd}_R(M) = G_C\text{-}\mathrm{pd}_R(M)$ now follows from the result [14, Proposition 2.16]. □

Theorem 4.4 *For a ring R, $\mathrm{G}_C\text{-PD}(R) = \mathrm{G}_C\text{-ID}(R)$.*

Proof. Assume that $\mathrm{G}_C\text{-PD}(R)$ is finite and not more than n for some integer n.

Firstly, suppose that M is a strongly G_C-projective R-module. We claim that $G_C\text{-}\mathrm{id}_R(M) \leq n$. By Proposition 3.4, there exists an exact sequence $0 \to M \to D \to M \to 0$, with $D \in \mathrm{Add}_R(C \oplus R)$. Let $0 \to M \to I_0 \to I_1 \to \cdots$ be an injective resolution of M. By the dual version of [11, Lemma 6.20], we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & D & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_0 & \to & I_0 \oplus I_0 & \to & I_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_{n-1} & \to & I_{n-1} \oplus I_{n-1} & \to & I_{n-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_n & \to & G & \to & K_n & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Because G_C-$\mathrm{PD}(R) \leq n$, by Lemma 2.7, $\mathrm{id}_R(C \otimes_R P) \leq n$ for any projective module P. It follows from [12, Theorem 2.11] that $\mathcal{I}_C$-$\mathrm{id}_R(Q) = \mathrm{id}_R(C \otimes_R Q) \leq n$ for any projective module Q. Thus G_C-$\mathrm{id}_R(C \otimes_R P \oplus Q) = \sup\{G_C\text{-}\mathrm{id}_R(C \otimes_R P), G_C\text{-}\mathrm{id}_R(Q)\} \leq n$ since every injective and C-injective module are both G_C-injective. Therefore G_C-$\mathrm{id}_R(D) \leq n$, and so G is G_C-injective by the dual version of [14, Proposition 2.12].

So we obtain an exact sequence

$$\mathbf{Y} = \cdots \xrightarrow{f} G \xrightarrow{f} G \xrightarrow{f} G \xrightarrow{f} \cdots$$

with G is G_C-injective and $K_n \cong \mathrm{Ker}\ f$. Applying $\mathrm{Hom}_R(\mathrm{Hom}_R(C, I), -)$ to the sequence $\mathbf{Y}$ for any injective module I, we get $\mathrm{Ext}_R^i(\mathrm{Hom}_R(C, I), K_n) \cong \mathrm{Ext}_R^{i+1}(\mathrm{Hom}_R(C, I), K_n)$ for each $i > 0$. Because I is injective and G_C-$\mathrm{pd}_R(I) \leq n$, Lemma 4.3 implies $\mathcal{P}_C$-$\mathrm{pd}_R(I) \leq n$. Then $\mathrm{pd}_R(\mathrm{Hom}_R(C, I)) = \mathcal{P}_C$-$\mathrm{pd}_R(I) \leq n$ by [12, Theorem 2.11], and So $\mathrm{Ext}_R^1(\mathrm{Hom}_R(C, I), K_n)$ $\cong \mathrm{Ext}_R^{n+1}(\mathrm{Hom}_R(C, I), K_n) = 0$. Thus $\mathrm{Hom}_R(\mathcal{I}_C, -)$ leaves the sequence $\mathbf{Y}$ exact, and hence K_n is G_C-injective by the injective version of Lemma 2.5. So G_C-$\mathrm{id}_R(M) \leq n$ as claimed. This yields, from [14, Proposition 2.11] and Theorem 3.3, that G_C-$\mathrm{id}_R(N) \leq n$ for any G_C-projective R-module N.

Finally, let M be an arbitrary R-module. By hypothesis G_C-$\mathrm{pd}_R(M) \leq n$. We may assume that G_C-$\mathrm{pd}_R(M) \neq 0$. Then, there exists an exact sequence $0 \to K \to N \to M \to 0$ such that N is G_C-projective and $\mathcal{P}_C$-$\mathrm{pd}_R(K) \leq n-1$ from the proof of [14, Theorem 3.6]. By induction, G_C-$\mathrm{id}_R(K) \leq n$. It follows from the dual version of [10, Lemma 3.2] that G_C-$\mathrm{id}_R(M) \leq n$ since G_C-$\mathrm{id}_R(N) \leq n$.

Therefore, the right of the equality is not more than the left one, and the converse has a dual proof. □

In the special case $C = R$, this recovers the main result of [3, Theorem 1.1]:

Corollary 4.5 $\sup\{Gpd_R(M) | M \textit{ is an } R\textit{-module}\} = \sup\{Gid_R(M) | M \textit{ is an } R\textit{-module}\}$.

We call the common value of the quantities in the theorem the C-Gorenstein global dimension of R, and denote it by G_C-$\mathrm{gl.dim}(R)$. Similarly, we set

$$\mathrm{G}_C\text{-w gl.dim}(R) = \sup\{G_C\text{-}\mathrm{fd}_R(M) | M \textit{ is an } R\textit{-module}\}.$$

Corollary 4.6 *The following inequalities hold:*

(1) G_C-$\mathrm{w\,gl.dim}(R) \leq \mathrm{G}_C$-$\mathrm{gl.dim}(R)$,

(2) G_C-$\mathrm{gl.dim}(R) \leq \mathrm{gl.dim}(R)$, *and the equality holds if* $\mathrm{gl.dim}(R) < \infty$.

Proof. (1) We may assume that G_C-$\mathrm{gl.dim}(R) \leq n$. We claim that every G_C-projective R-module is G_C-flat. Similarly to the proof of [7, Proposition 3.4], by [14, Proposition 2.12], it suffices to show that the character module, $N^+ = \mathrm{Hom}_Z(N, Q/Z)$, of every C-injective R-module N has finite C-projective dimension. Indeed, by the dual version of Lemma 2.7, $\mathrm{pd}_R(N) \leq n$, and so $\mathrm{fd}_R(N) \leq n$. Then $\mathrm{id}_R(N^+) \leq n$ from [11, Theorem 3.52]. It follows from Lemma 4.3 that $\mathcal{P}_C\text{-}\,\mathrm{pd}_R(N^+) \leq n$ as desired. Therefore, $\mathrm{G}_C\text{-w gl.dim}(R) \leq \mathrm{G}_C\text{-}\,\mathrm{gl.dim}(R)$.

(2) The inequality holds true since every projective module is G_C-projective. If $\mathrm{gl.dim}(R) < \infty$, then by Proposition 4.1, $\mathrm{P}_C\text{-PD}(R) = \mathrm{gl.dim}(R) < \infty$, the assertion follows from [14, Proposition 2.16]. □

Now, we study the behavior of modules over rings of finite C-Gorenstein global dimension.

Proposition 4.7 *Let R be a ring, and M be an R-module. If* $\mathrm{G}_C\text{-}\,\mathrm{gl.dim}(R) < \infty$*, then*

(1) $\mathcal{P}_C\text{-}\,\mathrm{pd}_R(M) < \infty \Leftrightarrow \mathrm{id}_R(M) < \infty$.

(2) $\mathcal{I}_C\text{-}\,\mathrm{id}_R(M) < \infty \Leftrightarrow \mathrm{pd}_R(M) < \infty$.

Proof. (1) Simply combine Lemma 2.7 with Lemma 4.3.

(2) has a dual proof. □

Remark 4.8 Takahashi and White [12] posed the following question: When R is a local Cohen- Macaulay ring admitting a dualizing module and C is a semidualizing R-module, if M is an R-module of finite depth such that $\mathcal{P}_C\text{-}\,\mathrm{pd}_R(M)$ and $\mathcal{I}_C\text{-}\,\mathrm{id}_R(M)$ are finite, must R be Gorenstein? From Proposition 4.7 and the classical result, we know that for the rings of finite C-Gorenstein global dimension, the question is positive.

Acknowledgements The research was supported by National Natural Science Foundation of China (Grant No. 11126092) and Science Research Foundation of Hangzhou Dianzi University (Grant No. KYS075610050).

References

[1] Auslander, M., Bridger, M. (1969). Stable module theory. *Mem. Amer. Math. Soc.* 94, American Mathematical Society, Providence, Rhode Island.

[2] Bennis, D., Mahdou, N. (2007). Strongly Gorenstein projective, injective, and flat modules. *J. Pure Appl. Algebra* 210: 437-445.

[3] Bennis, D., Mahdou, N. (2010). Global Gorenstein dimensions. *Proc. Amer. Math. Soc.* 138: 461-465.

[4] Enochs, E. E., Jenda, O. M. G. (1995). Gorenstein injective and projective modules. *Math. Z.* 220: 611-633.

[5] Foxby, H.-B. (1972). Gorenstein modules and related modules. *Math. Scand.* 31: 267-284.

[6] Golod, E. S. (1984). G-dimension and generalized perfect ideals. *Trudy Mat. Inst. Steklov.* 165: 62-66, Algebraic geometry and its applications.

[7] Holm, H. (2004). Gorenstein homological dimensions. *J. Pure Appl. Algebra* 189: 167-193.

[8] Holm, H., Jϕrgensen, P. (2006). Semi-dualizing modules and related Gorenstein homological dimensions. *J. Pure Appl. Algebra* 205: 423-445.

[9] Holm, H., White, D. (2007). Foxby equivalence over associative rings. *J. Math. Kyoto Univ.* 47: 781-808.

[10] Liu, Z. F., Huang, Z. Y. and Xu, A. M. (2012). Gorensten projective dimension relative to a semidualizing bimodule. *Comm. Algebra* (to appear).

[11] Rotman, J. J. (1979). An Introduction to Homological Algebra. *Academic Press, New York*.

[12] Takahashi, R., White, D. Homological aspects of semidualizing modules. *Math. Scand.* (to appear), available from arXiv:math.AC/0703643.

[13] Vasconcelos, W. V. (1974). Divisor theory in module categories. *North-Holland Publishing Co., Am-sterdam, North-Holland Mathematics Studies*. No. 14, Notas de Matematica No. 53. [Notes on Mathematics, No. 53].

[14] White, D. (2010). Gorensten projective dimension with respect to a semidualizing module. *J. Comm. Algebra* 2: 111-137.