ON THE DIMENSION FILTRATION AND COHEN-MACAULAY FILTERED MODULES

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ABSTRACT. For a finitely generated A-module M we define the dimension filtration $\mathcal{M} = \{M_i\}_{0 \le i \le d}, d = \dim_A M$, where M_i denotes the largest submodule of M of dimension $\le i$. Several properties of this filtration are investigated. In particular, in case the local ring $(A, \mathfrak{m})$ possesses a dualizing complex, then this filtration occurs as the filtration of a spectral sequence related to duality. Furthermore, we call an A-module M a Cohen-Macaulay filtered module provided all of the quotient modules M_i/M_{i-1} are either zero or i-dimensional Cohen-Macaulay modules. We describe a few basic properties of these kind of generalized Cohen-Macaulay modules. In the case A posesses a dualizing complex it turns out – as one of the main results – that M is a Cohen-Macaulay filtered A-module if and only if for all $0 \le i < d$ the module of deficiency $K^i(M)$ is either zero or an i-dimensional Cohen-Macaulay module. Furthermore basic properties of Cohen-Macaulay filtered modules with respect to localizations, completion, passing to a non-zero divisor, flat extensions are investigated.

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1. Introduction

Let $(A, \mathfrak{m})$ denote a local Noetherian ring. For a finitely generated A-module M with $d = \dim_A M$ and an integer $0 \leq i \leq d$ define M_i the largest submodule of M such that $\dim_A M_i \leq i$. Because M is a Noetherian A-module the submodules M_i are well-defined. They form an increasing family of submodules. We call $\mathcal{M} = \{M_i\}_{0 \leq i \leq d}$ the dimension filtration of M. In the first Section of the paper we describe in more details the structure of the submodules M_i. It turns out, see 2.2, that they are described in terms of the reduced primary decomposition of 0 in M. For further investigations we introduce the notion of a distinguished system of parameters $\underline{x} = x_1, \ldots, x_d, d = \dim_A M$, see the Definition 2.5. It turns out that $M_i = 0 :_M (x_{i+1}, \ldots, x_d)$ for a distinguished system of parameters $\underline{x} = x_1, \ldots, x_d$, see 2.7. Moreover under the additional assumption that $\operatorname{Supp}_A M$ is a catenary subset of $\operatorname{Spec} A$ it follows that the dimension filtration localizes for all $\mathfrak{p} \in \operatorname{Supp}_A M$, see 2.5.

Suppose that the local ring $(A, \mathfrak{m})$ possesses a dualizing complex $D_A^{\cdot}$, see [H, Chapter V] for the definition and basic results. We normalize it in such a way that $D_A^{\cdot}$ is a bounded complex with finitely generated cohomology modules and

$$D_A^{-i} \simeq \oplus_{\mathfrak{p} \in \operatorname{Spec} A} E_A(A/\mathfrak{p})$$

for all $i \in \mathbb{N}$. Here $E_A(A/\mathfrak{p})$ denotes the injective hull of $A/\mathfrak{p}$. For a finitely generated d-dimensional A-module M the homology module

$$K^i(M) := H^{-i}(\operatorname{Hom}_A(M, D_A^{\cdot}))\ 0 \leq i < d$$

is called the i-th module of deficiency. Moreover $K(M) = H^{-d}(\operatorname{Hom}_A(M, D_A^{\cdot}))$ is called the canonical module of M. This is the generalization of the canonical module of a ring $(A, \mathfrak{m})$ introduced by J. Herzog and E. Kunz in [HK], see also [S1, 3.1] for the generalization. The basic property of the dualizing complex says that the natural homomorphism of complexes

$$M \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, D_A^{\cdot}), D_A^{\cdot})$$

induces an isomorphism in cohomology for a finitely generated A-module M. That is, the 0-th cohomology of the complex at the right hand side is isomorphic to M. Now there is a spectral sequence in order to compute the cohomology of this complex, see Section 3 for more details. In particular, it induces a filtration $\mathcal{F} = \{F^{-i}\}_{0 \leq i \leq d}$ on the A-module M.

Theorem 1.1. *Both of the filtrations $\mathcal{M}$ and $\mathcal{F}$ conincide, i.e. $M_i = F^{-i}$ for all $0 \leq i \leq d$.*

For the proof of Theorem 1.1 see Theorem 3.4. So the dimension filtration $\mathcal{M}$ occurs in a natural way as a by-product of the duality of the dualizing complex. Note that the initial terms of the spectral sequence are deficiency modules of the deficiency modules $K^i(K^j(M))$ of M. Note that the deficiency modules $K^i(M), i = 0, \dots, d-1$, measure the non-Cohen-Macaulayness of M in the sense that M is a Cohen-Macaulay module if and only if $K^i(M) = 0$ for all $i = 0, \dots, d-1$, as follows by the local duality theorem.

There are several approaches to study generalized Cohen-Macaulay modules from different point of views. We add here another one saying that a finitely generated A-module M is called a Cohen-Macaulay filtered module (CMF for short) whenever all the qoutients $\mathcal{M}_i = M_i/M_{i-1}$ of the dimension filtration are either zero or an i-dimensional Cohen-Macaulay module. Since a Cohen-Macaulay module M is unmixed it is a CMF module with $M = M_d, d = \dim_A M$, and $M_i = 0$ for all $0 \leq i < d$.

Examples of non-Cohen-Macaulay CMF modules are approximately Cohen-Macaulay modules, see Section 3 for the definition. It extends the notion of an approximately Cohen-Macaulay ring introduced by S. Gôto in [G]. In Section 3 we describe some basic properties of CMF modules. Among them their permanence properties with respect to localizations, completion and passing to a non-zero divisor. In particular it follows that a finite direct sum of Cohen-Macaulay modules is a CMF module. For more examples we refer to Section 6 of the paper. There we prove also by an example, see 6.1, that in general the CMF property does not descend from the completion $\hat{M}$ to M, where M denotes a finitely generated A-module.

There is a cohomological characterization of CMF modules in terms of the modules of deficiency. This is another main observation of the present investigations.

Theorem 1.2. *Suppose that the local ring $(A, \mathfrak{m})$ possesses a dualizing complex $D^{\cdot}_A$. Let M be a finitely generated A-module. Then M is a CMF module if and only if for all $0 \leq i < \dim_A M$ the module of defiency $K^i(M)$ is either zero or an i-dimensional Cohen-Macaulay module. Under this conditions the canonical module $K(M)$ is a Cohen-Macaulay module.*

This result will be shown in 5.5. Note the following: While the Cohen-Macaulayness is described by the vanishing of the modules of deficiency the property of being a CMF module is described in terms of the Cohen-Macaulay property of the modules of deficiency. Moreover it turns out that the canonical module $K(M)$ of a CMF module M is a Cohen-Macaulay module. Note that if $K(M)$ is a Cohen-Macaulay module, then in general M is not a Cohen-Macaulay module. So Theorem 1.2 provides another sufficient condition for $K(M)$ being a Cohen-Macaulay module.

In Section 6 we conclude with the behaviour of the CMF property under flat base extensions of the ground ring. As a basic reference of all of the unexplained terminology we use H. Matsumura's textbook [M]. For the results about Cohen-Macaulay rings and modules see also [BH]. Furthermore, a short introduction into the theory about dualizing complexes the interested reader might found also in [S2].

2. The Dimension Filtration

Let $(A, \mathfrak{m})$ denote a local Noetherian ring. Let M be a finitely generated A-module and $d = \dim_A M$. For an integer $0 \leq i < d$ let M_i denote the largest submodule of M such that $\dim_A M_i \leq i$. Because of the maximal condition of a Noetherian A-module the submodules M_i of M are well-defined. Moreover it follows that $M_{i-1} \subseteq M_i$ for all $1 \leq i \leq d$.

Definition 2.1. The inreasing filtration $\mathcal{M} = \{M_i\}_{0 \leq i \leq d}$ of submodules of M is called the dimension filtration of M. Put $\mathcal{M}_i = M_i / M_{i-1}$ for all $1 \leq i \leq d$.

As a first part of our investigations we give a more detailed description of the modules M_i. Note that $M_0 = H^0_{\mathfrak{m}}(M)$, where $H^0_{\mathfrak{m}}(\cdot)$ denotes the section functor with support in $\{\mathfrak{m}\}$. In order to generalize this observation let $0 = \cap_{j=1}^n N_j$ denote a reduced primary decomposition of 0 in M. That is, $0 \neq \cap_{j=1, j\neq k}^n N_j$ for all $k = 1, \ldots, n$, and N_j is a $\mathfrak{p}_j$-coprimary submodule of M such that the prime ideals $\mathfrak{p}_j$ are pairwise different and $\operatorname{Ass}_A M = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Hence $M_0 = \cap_{\dim A/\mathfrak{p}_j > 0} N_j$.

Both of these representations of M_0 will be generalized to $M_i, 0 \leq i \leq d$, in the following result. To this end let

$$\mathfrak{a}_i = \prod_{\mathfrak{p} \in \operatorname{Ass} M, \dim A/\mathfrak{p} \leq i} \mathfrak{p}.$$

In the case that $\{\mathfrak{p} \in \operatorname{Ass} M \mid \dim A/\mathfrak{p} \leq i\} = \emptyset$ put $\mathfrak{a}_i = A$.

Proposition 2.2. *Let M be a finitely generated A-module. Then*

$$M_i = H^0_{\mathfrak{a}_i}(M) = \cap_{\dim A/\mathfrak{p}_j > i} N_j$$

for all $0 \leq i \leq d$. Here $0 = \cap_{j=1}^n N_j$ denotes a reduced primary decomposition of 0 in M.

Proof. The equality of the last two modules in the statement follows by easy arguments about the primary decomposition of the zero submodule 0 of M. Now let us prove that $M_i = H^0_{\mathfrak{a}_i}(M)$ for all $0 \leq i \leq d$. Clearly we have $\operatorname{Supp} H^0_{\mathfrak{a}_i}(M) = \operatorname{Supp} M \cap V(\mathfrak{a}_i)$. Therefore it follows that $M_i \subseteq H^0_{\mathfrak{a}_i}(M)$ because any element of M_i is annihilated by an ideal of dimension $\leq i$. By the maximality of M_i this proves the equality. □

The previous result provides information about the associated prime ideals of M_i and $\mathcal{M}_i$ respectively.

Corollary 2.3. *Let* $\mathcal{M} = \{M_i\}_{0 \le i \le d}$ *denote the dimension filtration of* M. *Then*

a) $\operatorname{Ass}_A M_i = \{\mathfrak{p} \in \operatorname{Ass} M \mid \dim A/\mathfrak{p} \le i\}$,

b) $\operatorname{Ass}_A M/M_i = \{\mathfrak{p} \in \operatorname{Ass} M \mid \dim A/\mathfrak{p} > i\}$, *and*

c) $\operatorname{Ass}_A \mathcal{M}_i = \{\mathfrak{p} \in \operatorname{Ass} M \mid \dim A/\mathfrak{p} = i\}$

for all $0 \le i \le d$.

Proof. The two first equalities are obviously true by view of 2.2. Note that

$$\operatorname{Ass}_A H^0_{\mathfrak{a}_i}(M) = \{\mathfrak{p} \in \operatorname{Ass}_A M \mid \mathfrak{p} \in V(\mathfrak{a}_i)\}.$$

The third equality is a consequence of the embedding $\mathcal{M}_i \subseteq M/M_{i-1}$ and the short exact sequence

$$0 \to M_{i-1} \to M_i \to \mathcal{M}_i \to 0.$$

Here we use the containement relation

$$\operatorname{Ass}_A M_i \subseteq \operatorname{Ass}_A M_{i-1} \cup \operatorname{Ass}_A \mathcal{M}_i$$

for the associated prime ideals of the corresponding modules. □

In a certain sense the quotients $\mathcal{M}_i, 0 \le i \le d$, of the dimension filtration $\mathcal{M} = \{M_i\}_{0 \le i \le d}$ of M are a measure for the unmixedness of M. Note that the A-module M is unmixed if

$$\dim A/\mathfrak{p} = \dim_A M \text{ for all } \mathfrak{p} \in \operatorname{Ass}_A M.$$

In this case $\mathcal{M}_i = 0$ for all $i < \dim_A M = d$ and $M_d = M$. So the filtration is discret in the case M is unmixed.

More general let $\mathcal{M} = \{M_i\}_{0 \le i \le d}$ be the dimension filtration of M. Then $M_i = 0$ for all $i < \operatorname{depth}_A M$. This follows by 2.3 and the fact

$$\operatorname{depth}_A M \le \dim A/\mathfrak{p} \text{ for all } \mathfrak{p} \in \operatorname{Ass}_A M,$$

see [M, Theorem 17.2] for this inequality.

In the following we consider the question whether the dimension filtration behaves well under localizations.

Proposition 2.4. *Let* $\mathcal{M} = \{M_i\}_{0 \le i \le d}$ *be the dimension filtration of a finitely generated* A*-module* M. *Suppose that* $\operatorname{Supp}_A M$ *is a catenary subset of* $\operatorname{Spec} A$. *Let* $\mathfrak{p} \in \operatorname{Supp} M$ *denote a prime ideal. Define*

$$M_i' = M_{i+\dim A/\mathfrak{p}} \otimes_A A_{\mathfrak{p}} \text{ for all } 0 \le i \le \dim_{A_{\mathfrak{p}}} M_{\mathfrak{p}} = t.$$

Then $\mathcal{M}' = \{M_i'\}_{0 \le i \le t}$ *is the dimension filtration of the* $A_{\mathfrak{p}}$*-module* $M_{\mathfrak{p}}$.

Proof. First we mention that there is the bound

$$\dim_A M'_i \le (i + \dim A/\mathfrak{p}) - \dim A/\mathfrak{p} = i$$

for all $i \in \mathbb{Z}$. Next we recall the following statement about associated prime ideals

$$\operatorname{Ass}_{A_\mathfrak{p}} M_\mathfrak{p} = \{\mathfrak{q}A_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Ass}_A M, \mathfrak{q} \subseteq \mathfrak{p}\},$$

see [M, Theorem 6.2]. Now let $0 = \cap_{j=1}^n N_j$ be a reduced primary decomposition of 0 in M, where N_j is $\mathfrak{q}_j$-coprimary. Suppose that $\mathfrak{q}_j \subseteq \mathfrak{p}$ for all $j = 1, \ldots, m$ and $\mathfrak{q}_j \not\subseteq \mathfrak{p}$ for all $j = m+1, \ldots, n$. Then $0 = \cap_{j=1}^m (N_j \otimes_A A_\mathfrak{p})$ is a reduced primary decomposition of 0 in $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module. Therefore, by view of 2.2, it yields that

$$(M_\mathfrak{p})_i = \cap_{\dim A_\mathfrak{p}/\mathfrak{q}_j A_\mathfrak{p} > i} (N_j \otimes_A A_\mathfrak{p}).$$

Moreover by the localization of $M_{i+\dim A/\mathfrak{p}}$ we get the following equality

$$M'_i = \cap_{\dim A/\mathfrak{q}_j > i + \dim A/\mathfrak{p}} (N_j \otimes_A A_\mathfrak{p}).$$

Because $\operatorname{Supp}_A M$ is supposed to be a catenary subset of $\operatorname{Spec} A$ we get that

$$\dim A/\mathfrak{q}_j = \dim A/\mathfrak{p} + \dim A_\mathfrak{p}/\mathfrak{q}_j A_\mathfrak{p}.$$

First this proves that $d = t + \dim A/\mathfrak{p}$. Because of the above statement about the associated prime ideals it shows finally that $M'_i = (M_\mathfrak{p})_i$ for all $0 \le i \le t$, as required. □

In the following we consider a variation of the notion of a system of parameters of an A-module M.

Definition 2.5. Let $\underline{x} = x_1, \ldots, x_d, d = \dim_A M$, denote a system of parameters of M. Then $\underline{x} = x_1, \ldots, x_d$ is called a distinguished system of parameters of M provided $(x_{i+1}, \ldots, x_d)M_i = 0$ for all $i = 0, \ldots, d-1$.

In the next result let us prove the existence of distinguished systems of parameters of an A-module M.

Lemma 2.6. *Any finitely generated A-module M admits a distinguished system of parameters.*

Proof. First we show the existence of a parameter x_d of M such that $x_d M_i = 0$ for all $i = 0, \ldots, d-1$. To this end note that $\dim_A M_i \le i < d$ for all $i = 0, \ldots, d-1$. Put $\mathfrak{b} = \prod_{i=0}^{d-1} \operatorname{Ann}_A M_i$. Then $\mathfrak{b} \not\subseteq \mathfrak{p}$ for any associated prime ideal $\mathfrak{p} \in \operatorname{Ass}_A M$ with $\dim A/\mathfrak{p} = d$. Therefore there is an element $x_d \in \mathfrak{b}$ and $x_d \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_A M$ with $\dim A/\mathfrak{p} = d$. Whence x_d is a parameter with the desired property. Now pass to the factor module $M/x_d M$ and choose a parameter x_{d-1} of $M/x_d M$ such that $x_{d-1} M_i = 0$ for all $i = 0, \ldots, d-2$. Then an induction finishes the proof of the claim. □

It turns out that whenever $\underline{x} = x_1, \dots, x_d$ is a distinguished system of parameters of M, the elements $x_1, \dots, x_i$ generate an ideal of definition of $\mathcal{M}_i$. This follows since $M_i/\underline{x}M_i$ is an A-module of finite length. Therefore, whenever $\mathcal{M}_i \neq 0$, then $x_1, \dots, x_i$ is a system of parameters of $\mathcal{M}_i$.

Lemma 2.7. *A system of parameters $\underline{x} = x_1, \dots, x_d$ of M is a distinguished system of parameters if and only if $M_i = 0 :_M (x_{i+1}, \dots, x_d)$ for $i = 0, \dots, d-1$.*

Proof. Let $\underline{x} = x_1, \dots, x_d$ denote a system of parameters of M such that

$$M_i = 0 :_M (x_{i+1}, \dots, x_d) \text{ for all } i = 0, \dots, d-1.$$

Then $(x_{i+1}, \dots, x_d)M_i = 0$, i.e. $\underline{x}$ is a distinguished system of parameters.

Conversely let $\underline{x}$ be a distinguished system of parameters. Then

$$M_i \subseteq 0 :_M (x_{i+1}, \dots, x_d) \text{ for all } i = 0, \dots, d-1$$

as follows by the definition. Moreover there is the following expression for the associated prime ideals

$$\operatorname{Ass}_A(0 :_M (x_{i+1}, \dots, x_d)) = \{\mathfrak{p} \in \operatorname{Ass}_A M \mid \mathfrak{p} \in V(x_{i+1}, \dots, x_d)\}.$$

Let $\mathfrak{p}$ denote an associated prime ideal of $0 :_M (x_{i+1}, \dots, x_d)$. Then we obtain $\mathfrak{p} \in \operatorname{Supp}_A M/(x_{i+1}, \dots, x_d)M$ and therefore $\dim A/\mathfrak{p} \leq d - (d-i) = i$. That is, $\dim_A(0 :_M (x_{i+1}, \dots, x_d)) \leq i$. Because of the maximality of M_i the equality $M_i = 0 :_M (x_{i+1}, \dots, x_d)$ follows now. □

3. A Supplement to Duality

In this section let $(A, \mathfrak{m})$ denote a local ring possessing a dualizing complex $D_A^{\cdot}$. That is a bounded complex of injective A-modules D_A^i whose cohomology modules $H^i(D_A^{\cdot}), i \in \mathbb{Z}$, are finitely generated A-modules. We refer to [H, Chapter V, §2] or to [S2, 1.2] for basic results about dualizing complexes. Note that the natural homomorphism of complexes

$$M \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, D_A^{\cdot}), D_A^{\cdot})$$

induces an isomorphism in cohomlogy for any finitely generated A-module M. Moreover there is an integer $l \in \mathbb{Z}$ such that

$$\operatorname{Hom}_A(k, D_A^{\cdot}) \simeq k[l],$$

where $k = A/\mathfrak{m}$ denotes the residue field of A. Without loss of generality assume that $l = 0$. Then the dualizing complex $D_A^{\cdot}$ has the property

$$D_A^{-i} \simeq \oplus_{\mathfrak{p} \in \operatorname{Spec} A, \dim A/\mathfrak{p} = i} E_A(A/\mathfrak{p}),$$

where $E_A(A/\mathfrak{p})$ denotes the injectice hull of $A/\mathfrak{p}$ as A-module. Therefore $D_A^i = 0$ for $i < -\dim A$ and $i > 0$. The following modules were introduced in [S1, 3.1], see also [S2, 1.2].

Definition 3.1. Let M denote a finitely generated A-module and $d = \dim_A M$. For an integer $i \in \mathbb{Z}$ define

$$K^i(M) := H^{-i}(\operatorname{Hom}_A(M, D_A^{\cdot})).$$

The module $K(M) := K^d(M)$ is called the canonical module of M. For $i \neq d$ the modules $K^i(M)$ are called the modules of deficiency of M. Note that $K^i(M) = 0$ for all $i < 0$ or $i > d$.

By the local duality theorem, see [H, Chapter V, §6] or [S2, Theorem 1.11], there are the following canonical isomorphisms

$$H_{\mathfrak{m}}^i(M) \simeq \operatorname{Hom}_A(K^i(M), E), i \in \mathbb{Z},$$

where $E = E_A(A/\mathfrak{m})$. Recall that all of the $K^i(M), i \in \mathbb{Z}$, are finitely generated A-modules. Moreover M is a Cohen-Macaulay module if and only if $K^i(M) = 0$ for all $i \neq d$. Whence the modules of deficiencies of M measure the deviation of M from being a Cohen-Macaulay module. The canonical module $K(M)$ of M is a Cohen-Macaulay module provided M is a Cohen-Macaulay module. The converse does not hold in general, see [S2, Lemma 1.9] for the precise statements.

For an arbitrary A-module X and an integer $i \in \mathbb{N}$ let

$$(\operatorname{Ass}_A X)_i = \{\mathfrak{p} \in \operatorname{Ass}_A M \mid \dim A/\mathfrak{p} = i\}.$$

For the proof of the next result see [S1, 3.1] and [S2, Lemma 1.9].

Proposition 3.2. *Let M denote a d-dimensional A-module. Then the following results are true:*

a) $\dim_A K^i(M) \leq i$ *for all* $0 \leq i < d$ *and* $\dim_A K(M) = d$.
b) $\operatorname{Ass}_A K(M) = (\operatorname{Ass}_A M)_d$.
c) $(\operatorname{Ass}_A K^i(M))_i = (\operatorname{Ass}_A M)_i$ *for all* $0 \leq i < d$.
d) *Let M be a Cohen-Macaulay module. Then $K(M)$ is also a Cohen-Macaulay module.*

As mentioned above for a finitely generated A-module M the induced homomorphisms of the cohomology of the natural map

$$M \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, D_A^{\cdot}), D_A^{\cdot})$$

are isomorphisms. In order to compute the cohomology of the complex at the right hand side there is the following spectral sequence

$$E_1^{pq} = H^q(\operatorname{Hom}_A(\operatorname{Hom}_A(M, D_A^{\cdot}), D_A^p)),$$

see [E, Appendix 3, Part II] or [W, Section 5] for the details about spectral sequences used here in this section. Because D_A^p is an injective A-module the corresponding E_2-term has the following form

$$E_2^{pq} = H^p(\mathrm{Hom}_A(H^{-q}(\mathrm{Hom}_A(M, D_A^\cdot)), D_A^\cdot)).$$

With regard to our previous notation it follows that $E_2^{pq} = K^{-p}(K^q(M))$. Now we have to prove the following basic observation.

Lemma 3.3. *Let M denote a finitely generated A-module. Let $\mathfrak{p} \in \mathrm{Supp}_A M$ be a prime ideal with $t = \dim A/\mathfrak{p}$. Then there are the following isomorphisms*

$$K^i(K^j(M)) \otimes_A A_\mathfrak{p} \simeq K^{i-t}(K^{j-t}(M \otimes_A A_\mathfrak{p}))$$

for any pair $(i, j) \in \mathbb{Z}^2$.

Proof. First note that there is an isomorphism of dualizing complexes

$$D_A^\cdot \otimes_A A_\mathfrak{p} \simeq D_{A_\mathfrak{p}}^\cdot[t],$$

see e.g. [H, Chapter V, Proposition 7.1]. Now by the definition of the K^i's write

$$K^i(K^j(M)) \simeq H^{-i}(\mathrm{Hom}_A(H^{-j}(\mathrm{Hom}_A(M, D_A^\cdot)), D_A^\cdot)).$$

The localization functor $\cdot \otimes_A A_\mathfrak{p}$ is exact, i.e. it commutes with cohomology. Moreover let X denote a bounded complex of A-modules whose cohomology modules are finitely generated A-modules. Then there is the following isomorphism of complexes

$$\mathrm{Hom}_A(X, D_A^\cdot) \otimes_A A_\mathfrak{p} \simeq \mathrm{Hom}_{A_\mathfrak{p}}(X \otimes_A A_\mathfrak{p}, D_{A_\mathfrak{p}}^\cdot)[t],$$

see [H, Chapter II]. Putting together all of these ingredients the statement of the proposition follows now. □

Let M denote a finitely generated A-module with $d = \dim_A M$. Let us return to the above spectral sequence. Consider the stage $p + q = 0$, the onliest place in which non-zero cohomology occurs. Then the limit terms $E_\infty^{p,-p}$, $-d \le p \le 0$, are the quotients of a filtration

$$F^0 \subseteq F^{-1} \subseteq \ldots \subseteq F^{-d+1} \subseteq F^{-d} = M$$

of M. That is we have $F^p/F^{p+1} \simeq E_\infty^{p,-p}$ for all $-d \le p \le 0$. The natural question about the filtration $\mathcal{F} = \{F^{-i}\}_{0 \le i \le d}$ is its relationship to the dimension filtration of M. This is answered in the following result.

Theorem 3.4. *Let $\mathcal{M} = \{M_i\}_{0 \le i \le d}$ be the dimension filtration of M. Then it follows $M_i = F^{-i}$ for all $0 \le i \le d$.*

Proof. By the construction of the spectral sequence the term $E_{r+1}^{p,-p}$ is the cohomology at the middle of the following sequence of A-modules

$$E_r^{p-r,-p+r-1} \to E_r^{p,-p} \to E_r^{p+r,-p-r+1}.$$

The term on the left hand side is zero because it is a subquotient of

$$K^{-p+r}(K^{-p+r-1}(M)) = 0.$$

To this end recall that $\dim_A K^{-p+r-1}(M) \leq -p+r-1$, see 3.2.

First of all let us consider the case of $p = 0$. Then also the term on the right hand side is zero since it is a subqoutient of $K^{-r}(K^{r-1}(M)) = 0, r \geq 2$. That is we get a partial degeneration of the spectral sequence to the isomorphisms

$$F^0 \simeq E_\infty^{0,0} \simeq E_2^{0,0} \simeq K^0(K^0(M)).$$

Moreover the local duality theorem implies that $K^0(K^0(M)) \simeq H^0_{\mathfrak{m}}(M) = M_0$. To this end recall that

$$\operatorname{Hom}_A(H^0(\operatorname{Hom}_A(M, D_A^{\cdot})), D_A^{\cdot}) \simeq \operatorname{Hom}_A(H^0(\operatorname{Hom}_A(M, D_A^{\cdot})), E).$$

This proves the claim in the case $p = 0$.

For an arbitrary p the above considerations provide the following chain of inclusions

$$E_\infty^{p,-p} \subseteq E_{r+1}^{p,-p} \subseteq E_r^{p,-p} \subseteq E_2^{p,-p} = K^{-p}(K^{-p}(M))$$

for all $r \geq 2$. By view of 3.2 this implies that either $\dim_A E_\infty^{p,-p} = -p$ or $E_\infty^{p,-p} = 0$. Note that $\operatorname{Ass}_A K^{-p}(K^{-p}(M)) = (\operatorname{Ass}_A K^{-p}(M))_{-p}$ for all $-d \leq p \leq 0$, see 3.2. But now we have

$$E_\infty^{p,-p} \simeq F^p/F^{p+1} \text{ and } E_\infty^{0,0} \simeq F^0 = M_0.$$

Therefore $\dim_A F^p \leq -p$ and $F^p \subseteq M_{-p}$ for all $-d \leq p \leq 0$. In the final part of the proof we have to show equality.

We proceed by an induction on $d = \dim_A M$. As mentioned above the case $d = 0$, i.e. $M_0 = F^0$ is shown to be true. So let $d > 0$. It is known that $(A, \mathfrak{m})$ is a catenary local ring since it possesses a dualizing complex, see [H, Chapter V, Corollary 7.2]. By 2.4 it follows that $(M_{\mathfrak{p}})_{i+\dim A/\mathfrak{p}} = (M_i) \otimes_A A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \in \operatorname{Supp} M$. Next we want to prove a corresponding result for the filtration $\mathcal{F}$, i.e. $F^i(M) \otimes_A A_{\mathfrak{p}} = F^{i+\dim A/\mathfrak{p}}(M_{\mathfrak{p}})$. Here we refer to $F^{\cdot}(M)$ resp. $F^{\cdot}(M_{\mathfrak{p}})$ as the filtration induced by the A-module M resp. by the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$. By 3.3 it turns out that

$$E_2^{pq}(M) \otimes_A A_{\mathfrak{p}} \simeq E_2^{p+t,q-t}(M \otimes_A A_{\mathfrak{p}}), \quad t = \dim A/\mathfrak{p},$$

for all pairs $(p,q) \in \mathbb{Z}^2$. Because of the exactness of the localization functor $\cdot \otimes A_{\mathfrak{p}}$ and because of the functoriality of the spectral sequence this finally shows that $F^p(M) \otimes_A A_{\mathfrak{p}} \simeq F^{p+t}(M_{\mathfrak{p}})$ for all $p \in \mathbb{Z}$.

Now let us finish the proof. Because of the spectral sequence we know that $F^p = M_{-p}$ for $p = -d$. So let us assume statement for p in order to prove it for $p+1$. To this end consider the injection

$$0 \to M_{-p-1}/F^{p+1} \to F^p/F^{p+1} \simeq E_\infty^{p,-p}.$$

Note that $M_{-p-1} \subseteq M_p = F^p$. Because of the induction hypothesis and the previous considerations we have that $M_{-p-1} \otimes A_\mathfrak{p} = F^{p+1} \otimes_A A_\mathfrak{p}$ for all non-maximal prime ideals $\mathfrak{p} \in \operatorname{Supp}_A M$. Therefore the module at the left hand side of the above exact sequence has its support contained in $V(\mathfrak{m})$. Moreover by the spectral sequence we get that

$$E_\infty^{p,-p} \subseteq E_2^{p,-p} \simeq K^{-p}(K^{-p}(M)).$$

By virtue of 3.2 we now have the following inclusion

$$\operatorname{Ass}_A E_\infty^{p,-p} \subseteq \{\mathfrak{p} \in \operatorname{Ass}_A M \mid \dim A/\mathfrak{p} = -p\}.$$

Since M_{-p-1}/F^{p+1} is by induction hypothesis an A-module of finite length it is in fact zero, which completes the inductive step. □

It is worth to remark that in general the limit terms $E_\infty^{p,-p}$ of the spectral sequence considered above do not agree with $E_2^{p,-p} \simeq K^{-p}(K^{-p}(M))$. It would be interesting to find an explicit description of these modules.

4. Cohen-Macaulay Filtered Modules

Let M denote a finitely generated A-module, where $(A, \mathfrak{m})$ is a local Noetherian ring. Let $\mathcal{M} = \{M_i\}_{0 \le i \le d}$ denote the dimension filtration.

Definition 4.1. A finitely generated A-module M is called a Cohen-Macaulay filtered module (CMF module), whenever $\mathcal{M}_i = M_i/M_{i-1}$ is either zero or an i-dimensional Cohen-Macaulay module for all $0 \le i \le \dim_A M$.

Note that any Cohen-Macaulay module is also a CMF module. This follows because under this assumption $M_i = 0$ for all $i < \dim_A M$. Conversely an unmixed CMF module is also a Cohen-Macaulay module. Let M be an A-module such that $\operatorname{depth}_A M = 0$ and $M/H_\mathfrak{m}^0(M)$ is a Cohen-Maculay module. Then M is a CMF module as easily seen. For further examples see Section 6.

Related to the definition of a CMF module it will be useful to have the notion of a Cohen-Macaulay filtration.

Definition 4.2. Let M denote a finitely generated A-module with $d = \dim_A M$. An increasing filtration $C = \{C_i\}_{0 \le i \le d}$ of M is called a Cohen-Macaulay filtration whenever $M = C_d, d = \dim_A M$, and $\mathcal{C}_i = C_i/C_{i-1}$ is either zero or an i-dimensional Cohen-Macaulay module for all $1 \le i \le d$.

The following proposition is useful in order to characterize CMF modules. In fact it shows that a Cohen-Macaulay filtration coincides automatically with the dimension filtration.

Proposition 4.3. *Let* $C = \{C_i\}_{0 \le i \le d}$ *be Cohen-Macaulay filtration of* M*. Then* C *coincides with the dimension filtration.*

Proof. First of all it is easily seen that $\dim C_i \le i$ for all $0 \le i \le d$. Moreover it follows that

$$\operatorname{Ass}_A C_i \subseteq \{\mathfrak{p} \in \operatorname{Ass}_A M \mid \dim A/\mathfrak{p} \le i\}.$$

With the definition of $\mathfrak{a}_i$ – as done in Section 2 – this implies that $H^0_{\mathfrak{a}_i}(C_i) = C_i$ for all $0 \le i \le d$. Now fix i and let $j \ge i$ be an integer. Next consider the following short exact sequence

$$0 \to C_j \to C_{j+1} \to C_{j+1} \to 0.$$

Since C_{j+1} is either zero or a $(j+1)$-dimensional Cohen-Macaulay module it induces isomorphisms

$$H^0_{\mathfrak{a}_i}(C_j) \simeq H^0_{\mathfrak{a}_i}(C_{j+1}) \text{ for all } j \ge i.$$

Therefore $C_i = H^0_{\mathfrak{a}_i}(C_j)$ for all $j \ge i$. Because of $M = C_d$ this finally proves the claim, see 2.2. □

Let $u_M(0) = \cap_{\dim A/\mathfrak{p}_j = d} N_j$, where $0 = \cap_{j=1}^n N_j$ denotes a reduced primary decomposition of 0 in M as considered in Section 2.

Definition 4.4. A finitely generated A-module $M, d = \dim_A M$, is called an approximately Cohen-Macaulay module whenever $M/u_M(0)$ is a Cohen-Macaulay module and $\operatorname{depth}_A M \ge d - 1$.

This is the extension of the notion of an approximately Cohen-Macaulay ring introduced by S. Gôto, see [G]. Note that a Cohen-Macaulay module is always an approximately Cohen-Macaulay module. Next let us describe the relation of this notion to that of CMF modules.

Proposition 4.5. *Let* M *be a finitely generated* A*-module. Then* M *is approximately Cohen-Macaulay if and only if* M *is a CMF module and* $\operatorname{depth}_A M \ge \dim_A M - 1$.

Proof. First let M be an approximately Cohen-Macaulay module. Put $d = \dim_A M$. By [M, Theorem 17.2] it follows that

$$d - 1 \le \operatorname{depth}_A M \le \dim A/\mathfrak{p} \text{ for all } \mathfrak{p} \in \operatorname{Ass}_A M.$$

Therefore $M_i = 0$ for $i = 0, \ldots, d-2$ and $M_{d-1} = u_M(0)$, see 2.2. Now consider the short exact sequence

$$0 \to M_{d-1} \to M \to M/M_{d-1} \to 0.$$

Because M is approximately Cohen-Macaulay it follows that M/M_{d-1} is a d-dimensional Cohen-Macaulay module and $\operatorname{depth}_A M \geq d-1$. So the short exact sequence implies $\operatorname{depth}_A M_{d-1} \geq d-1$. Because of $\dim_A M_{d-1} \leq d-1$ it turns out that M_{d-1} is either zero or a $(d-1)$-dimensional Cohen-Macaulay module.

The reverse statement follows the same line of reasoning. Hence we omit the details. □

Before we shall present in Section 6 a general construction method for CMF modules there are a few results on permanence properties of CMF module. To this end $\hat{A}$ denotes the $\mathfrak{m}$-adic completion of A. Note that there is a natural isomorphism $M \otimes_A \hat{A} \simeq \hat{M}$ for a finitely generated A-module M.

Proposition 4.6. *Let M denote a CMF A-module. Then the following conditions are satisfied:*

a) $\operatorname{Supp}_A M$ *is a catenary subset of* $\operatorname{Spec} A$.
b) *Let* $\mathfrak{p} \in \operatorname{Supp}_A M$. *Then*

$$\dim A/\mathfrak{p} = \dim \hat{A}/\mathfrak{q} \text{ for all } \mathfrak{q} \in \operatorname{Ass}_{\hat{A}} \hat{A}/\mathfrak{p}\hat{A},$$

i.e. $A/\mathfrak{p}$ is formally unmixed for all $\mathfrak{p} \in \operatorname{Supp}_A M$.

Proof. Because M/M_{d-1} is a Cohen-Macaulay module and

$$\operatorname{Supp}_A M = \operatorname{Supp}_A M/M_{d-1}$$

both of the statements follow. For the first statement see [M, §17]. The second is a consequence of [N, (34.9)]. □

Note that a CMF ring A possesses a small Cohen-Macaulay module. That is a Cohen-Macaulay module X such that $\operatorname{depth} X = \dim A$. This follows since $A/A_{d-1}, d = \dim A$, is a d-dimensional Cohen-Macaulay module. Consequently for a CMF ring A all the homological conjectures are true.

Now we start with the permanence properties of the CMF property.

Theorem 4.7. *Let M denote a finitely generated A-module. Let $x \in \mathfrak{m}$ be an M-regular element. Then M is a CMF module if and only if M/xM is a CMF module.*

Proof. First note that whenever $x \in \mathfrak{m}$ is an M-regular element, then $M_0 = 0$ and x is also M/M_i-regular as well as $\mathcal{M}_i$-regular for all $i \geq 1$. Here $\mathcal{M} = \{M_i\}_{0 \leq i \leq d}$ denotes the dimension filtration and $\mathcal{M}_i = M_i/M_{i-1}$. In particular it follows that $M_i \cap xM = xM_i$ for all $1 \leq i \leq d$.

Now suppose that M is a CMF module. Let $i \geq 1$. Then

$$\mathcal{M}_i/x\mathcal{M}_i \simeq ((M_i, xM)/xM)/(M_{i-1}, xM)/xM)$$

is a $(i-1)$-dimensional Cohen-Macaulay module or zero. Therefore by 4.3 it follows that M/xM is a $(d-1)$-dimensional CMF module since

$$\{(M_{i+1}, xM/xM)\}_{0\le i<d}$$

is a Cohen-Macaulay filtration.

Conversely let M/xM be a CMF module. Then the dimension filtration $\{M'_i\}_{0\le i\le d}$ of M/xM has the property that $\mathcal{M}'_i = M'_i/M'_{i-1}$ is either zero or an i-dimensional Cohen-Macaulay module. Let $M_{i+1}, 0 \le i < d$, denote the preimage of M'_i in M and $M_0 = 0$. By the same isomorphism as above it follows now that $\mathcal{M}_i/x\mathcal{M}_i$ is either zero or a $(i-1)$-dimensional Cohen-Macaulay module. Since x is an $\mathcal{M}_i$-regular element, $\mathcal{M}_i$ is either zero or an i-dimensional Cohen-Macaulay module. By view of 4.1 this proves the claim. □

Another permanence poperty of CMF modules is the following result about the localization behaviour.

Proposition 4.8. *Let M denote a CMF module. Then $M_{\mathfrak{p}}$ is a CMF $A_{\mathfrak{p}}$-module for any prime ideal $\mathfrak{p} \in \operatorname{Supp}_A M$.*

Proof. Let $\mathcal{M} = \{M_i\}_{0\le i\le d}$ denote the dimension filtration of M. Let $\mathfrak{p} \in \operatorname{Supp}_A M$ and $t = \dim A/\mathfrak{p}$. Then we define $M'_i = (M_{i+t}) \otimes_A A_{\mathfrak{p}}$, for all $i \ge t$. Then

$$M'_i/M'_{i-1} \simeq (M_{i+t}/M_{i-1+t}) \otimes_A A_{\mathfrak{p}}$$

is either zero or a Cohen-Macaulay $A_{\mathfrak{p}}$-module of dimension i, see [M, §17]. By virtue of 4.3 this proves the claim. □

It is worth to note that we do not need 2.4 in order to prove 4.8. The result turns out because of

$$\dim_A X = \dim A/\mathfrak{p} + \dim_{A_{\mathfrak{p}}} X_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in \operatorname{Supp}_A X,$$

where X denotes a Cohen-Macaulay A-module. Moreover by 4.6 we know that $\operatorname{Supp}_A M$ is a catenary subset of $\operatorname{Spec} A$.

In the final part of this section consider the behaviour of the CMF property by passing to the completion.

Theorem 4.9. *Let M be a finitely generated A-module. Let M be a CMF A-module. Then $M \otimes_A \hat{A}$ is a CMF $\hat{A}$-module.*

Proof. Let $\mathcal{M} = \{M_i\}_{0\le i\le d}$ denote the Cohen-Macaulay filtration of the CMF A-module M. Then $\{M_i \otimes_A \hat{A}\}_{0\le i\le d}$ is clearly a Cohen-Macaulay filtration of the $\hat{A}$-module $M \otimes_A \hat{A}$. So by 4.3 $M \otimes_A \hat{A}$ is a CMF module over $\hat{A}$. □

The converse of the above statement is not true in general as we will show by an example, see Example 6.1. A more general statement is true for an arbitrary faithful flat extension $(A, \mathfrak{m}) \to (B, \mathfrak{n})$. For the precise statement see the considerations in Section 6.

5. Characterizations of Cohen-Macaulay Filtered Modules

In the first part of this section we consider a few algebraic properties of a CMF module. To this end let M denote a finitely generated A-module M, where $(A, \mathfrak{m})$ is an arbitrary local Noetherian ring. First we need a preliminary result. Here L_A denotes the length function on A-modules.

Lemma 5.1. *Let $\mathcal{M} = \{M_i\}_{0 \le i \le d}$ denote the dimension filtration of M. Then*

$$L_A(M/\underline{x}M) \le \sum_{i=0}^{d} L_A(\mathcal{M}_i/(x_1, \ldots, x_i)\mathcal{M}_i)$$

for any distinguished system of parameters $\underline{x} = x_1, \ldots, x_d, d = \dim_A M$, of M.

Proof. For $1 \le i \le d$ let us consider the following short exact sequences

$$0 \to M_{i-1} \to M_i \to \mathcal{M}_i \to 0.$$

Tensor it by $A/(x_1, \ldots, x_d)A$. Because of $x_i M_{i-1} = 0, 1 \le i \le d$, it induces an exact sequence

$$M_{i-1}/(x_1, \ldots, x_{i-1})M_{i-1} \to M_i/(x_1, \ldots, x_i)M_i \to \mathcal{M}_i/(x_1, \ldots, x_i)\mathcal{M}_i \to 0.$$

Because $\underline{x}$ is a distinguished system of parameters of M the elements $x_1, \ldots, x_i$ generate an ideal of definition of M_i. That is, the A-modules

$$M_i/(x_1, \ldots, x_i)M_i \text{ and } \mathcal{M}_i/(x_1, \ldots, x_i)\mathcal{M}_i, i = 0, \ldots, d,$$

are A-modules of finite length. Therefore

$$L_A(M_i/(x_1, \ldots, x_i)M_i) \le L_A(M_{i-1}/(x_1, \ldots, x_{i-1})M_{i-1}) + L_A(\mathcal{M}_i/(x_1, \ldots, x_i)\mathcal{M}_i)$$

for all $i = 1, \ldots, d$. Because of $M = M_d$ a recurrence proves the desired inequality. □

Note that the inequality of 5.1 is also true for any system of parameters $\underline{x} = x_1, \ldots, x_d$ of M. But in this case it might happen that the modules on the right hand side are not of finite length. In this case the estimate is trivially true.

Theorem 5.2. *Let $\mathcal{M} = \{M_i\}_{0 \le i \le d}$ denote the dimension filtration of a finitely generated A-module M with $d = \dim_A M$ and $t = \operatorname{depth}_A M$. Let $\underline{x} = x_1, \ldots, x_d$ be a distinguished system of parameters.*

Suppose that M is a CMF module. Then the following conditions are satisfied:

a) $L_A(M/(x_1,\dots,x_d)M) = \sum_{i=0}^d L_A(\mathcal{M}_i/(x_1,\dots,x_i)\mathcal{M}_i)$.
b) $M/(x_1,\dots,x_{d-t})M$ *is a* t*-dimensional Cohen-Macaulay module.*

The converse is true, i.e. the conditions a) and b) imply that M *is a CMF module, provided* $\operatorname{depth}_A M \geq d-1$.

Proof. Suppose that M is a CMF module. Then the factor modules $\mathcal{M}_i, 0 \leq i \leq d$, are either zero or i-dimensional Cohen-Macaulay modules. Since x_i is an $\mathcal{M}_i$-regular element there are the following short exact sequences

$$0 \to M_{i-1} \to M_i/x_iM_i \to \mathcal{M}_i/x_i\mathcal{M}_i \to 0.$$

Now apply the Koszul homology $H.(x_1,\dots,x_{i-1};\cdot)$ to this sequence. Because of

$$H_1(x_1,\dots,x_{i-1};\mathcal{M}_i/x_i\mathcal{M}_i) = 0,$$

note that $x_1,\dots,x_{i-1}$ is a $\mathcal{M}_i/x_i\mathcal{M}_i$-regular sequence, it induces a short exact sequence

$$0 \to M_{i-1}/(x_1,\dots,x_{i-1})M_{i-1} \to M_i/(x_1,\dots,x_i)M_i \to \mathcal{M}_i/(x_1,\dots,x_i)\mathcal{M}_i \to 0$$

for all $i = 1,\dots,d$. Because of $M = M_d$ a recurrence on i proves the equality of the statement in a).

For the proof of b) first note that $M_i = 0$ for all $i < t = \operatorname{depth}_A M$. Moreover M_t a t-dimensional Cohen-Macaulay module. Now we investigate the above short exact sequence

$$0 \to M_{i-1} \to M_i/x_iM_i \to \mathcal{M}_i/x_i\mathcal{M}_i \to 0.$$

for $i \geq t+1$. Applying the Koszul homology $H.(x_{t+1},\dots,x_{i-1};\cdot)$ to this sequence it induces – by similar arguments as above – a short exact sequence

$$0 \to M_{i-1}/(x_{t+1},\dots,x_{i-1})M_{i-1} \to M_i/(x_{t+1},\dots,x_i)M_i \to \mathcal{M}_i/(x_{t+1},\dots,x_i)\mathcal{M}_i \to 0$$

for all $i = t+1,\dots,d$. Note that $x_{t+1},\dots,x_{i-1}$ forms a regular sequence on the $(i-1)$-dimensional Cohen-Macaulay module $\mathcal{M}_i/x_i\mathcal{M}_i$. Moreover all of the modules considered in these sequences are either zero or of dimension t. By induction we prove now that the module in the middle is a t-dimensional Cohen-Macaulay module. First note that $M_t = \mathcal{M}_t$ is a t-dimensional Cohen-Macaulay module. Because $M_{i-1}/(x_{t+1},\dots,x_{i-1})M_{i-1}$ and $\mathcal{M}_i/(x_{t+1},\dots,x_i)\mathcal{M}_i$ are either zero or t-dimensional Cohen-Macaulay modules the short exact sequence implies that $M_i/(x_{t+1},\dots,x_i)M_i$ is also either zero or a t-dimensional Cohen-Macaulay module. Because of $M = M_d$ this proves the statement of the condition b).

In order to prove the reverse implication let $\operatorname{depth}_A M = d-1$. (In the Cohen-Macaulay case there is nothing to prove.) Therefore we get $M_i = 0$ for all $i <$

$d-1$. Then – as above – there is the short exact sequence

$$0 \to M_{d-1} \to M_d/x_1M_d \to \mathcal{M}_d/x_1\mathcal{M}_d \to 0.$$

Note that x_1 is an $\mathcal{M}_d$-regular element. Next apply the Koszul homology $H_\cdot(x_2,\dots,x_d;\cdot)$ to this sequence. Because $x_2,\dots,x_d$ is an M_d/x_1M_d-regular sequence we get the following exact sequence

$$\begin{aligned}0 \to\ & H_1(x_2,\dots,x_d;\mathcal{M}_d/x_1\mathcal{M}_d) \to \\ & \to M_{d-1}/(x_2,\dots,x_d)M_{d-1} \to M_d/\underline{x}M_d \to \mathcal{M}_d/\underline{x}\mathcal{M}_d \to 0.\end{aligned}$$

By the assumption on the length it turns out that

$$H_1(x_2,\dots,x_d;\mathcal{M}_d/x_1\mathcal{M}_d) = 0.$$

That is, $x_2,\dots,x_d$ forms an $\mathcal{M}_d/x_1\mathcal{M}_d$-regular sequence. So $\mathcal{M}_d$ is a d-dimensional Cohen-Macaulay module. Moreover the above short exact sequence implies that M_{d-1} is a $(d-1)$-dimensional Cohen-Macaulay module. That means, M is a CMF module, as required. □

It would be interesting to generalize the converse of 5.2 to a more general situation. In the case of $\operatorname{depth}_A M = d-1$ neither a) nor b) will be sufficient for M being a CMF module. For b) see S. Gôto's example [G, Remark 2.9]. For a) consider the ring $A = k[[w,x,y,z]]/(w,x)\cap(y,z)\cap(w^2,x,y^2,z)$, where k denotes a field. Then $\underline{x} = w-y, x-z$ is a distinguished system of parameters satisfying the equality in a) but A is not a CMF ring.

Another partial result in order to prove the converse is the following slight generalization of [G, Lemma 2.1].

Proposition 5.3. *Let M denote a finitely generated A-module with $d = \dim_A M$. Let $r \in \mathbb{N}$ denote an integer. Suppose that there is an element $x \in \mathfrak{m}$ satisfying the following two conditions:*

a) *$M/x^{r+1}M$ is a $(d-1)$-dimensional Cohen-Macaulay module.*
b) $0 :_M x^r = 0 :_M x^{r+1}$.

Then $\operatorname{depth}_A M \geq d-1$ *and M is a CMF module with $M_{d-1} = 0 :_M x^r$.*

Proof. Put $N := 0 :_M x^r = 0 :_M x^{r+1}$. We first claim that $\operatorname{depth}_A M/x^rM \geq d-1$. Suppose the contrary, i.e. $\operatorname{depth}_A M/x^rM =: t < d-1$. Then the short exact sequence

$$0 \to M/(xM,N) \to M/x^{r+1}M \to M/x^rM \to 0$$

implies that $\operatorname{depth}_A M/(xM,N) = t+1$. Because x is an M/N-regular element it follows that

$$\operatorname{depth}_A M/N = t+2 \text{ and } \operatorname{depth}_A M/(x^sM,N) = t+1 \text{ for all } s \geq 1.$$

Therefore the short exact sequence

$$0 \to N \to M/x^s M \to M/(x^s M, N) \to 0,$$

considered for $s = r+1$, provides that $\operatorname{depth}_A N = t+2$. Then the same sequence considered for $s = r$ yields that $\operatorname{depth}_A M/x^r M \geq t+1$, a contradiction.

Therefore $M/x^r M$ is a $(d-1)$-dimensional Cohen-Macaulay module. Now the first of the above short exact sequences proves that $M/(xM, N)$ and therefore also M/N is a Cohen-Macaulay module. Moreover the previous exact sequence considered for $s = r$ provides that N is a Cohen-Macaulay module of dimension $d-1$. By 2.2 this finishes the proof. □

Now we start the cohomological investigation of CMF modules. To this end at first we need a description of the local cohomology modules of a CMF module.

Lemma 5.4. *Let M denote a CMF module with $\mathcal{M} = \{M_i\}_{0 \leq i \leq d}$ its dimension filtration. Let i denote an integer with $0 \leq i \leq d$. Then*

$$H^i_{\mathfrak{m}}(M) \simeq H^i_{\mathfrak{m}}(M_i) \simeq H^i_{\mathfrak{m}}(\mathcal{M}_i).$$

In the case A possesses a dualizing complex it follows that $K^i(M) \simeq K^i(\mathcal{M}_i)$ for all $0 \leq i \leq d$.

Proof. First consider the short excat sequence $0 \to M_{i-1} \to M_i \to \mathcal{M}_i \to 0$. Because of $\dim M_{i-1} \leq i-1$ it induces an isomorphism $H^i_{\mathfrak{m}}(M_i) \simeq H^i_{\mathfrak{m}}(\mathcal{M}_i)$. Second for $j < i$ it yields isomorphims $H^j_{\mathfrak{m}}(M_i) \simeq H^j_{\mathfrak{m}}(M_{i-1})$. Note that $\mathcal{M}_i$ is either zero or an i-dimensional Cohen-Macaulay module. By induction it follows that

$$H^i_{\mathfrak{m}}(M) \simeq H^i_{\mathfrak{m}}(M_d) \simeq H^i_{\mathfrak{m}}(M_{d-1}) \simeq \ldots \simeq H^i_{\mathfrak{m}}(M_{i+1}) \simeq H^i_{\mathfrak{m}}(M_i),$$

which proves the statement about the local cohomology modules. The rest of the claim for $K^i(M)$ follows by similar arguments using the dualizing complex. □

Now we are prepared to prove the main result concerning a characterization of CMF modules in terms of the modules of deficiency $K^i(M), 0 \leq i < d$. Moreover there is an additional information about the canonical module.

Theorem 5.5. *Let $(A, \mathfrak{m})$ denote a local ring possessing a dualizing complex $D_A^{\cdot}$. Let M be a finitely generated A-module with $d = \dim_A M$. Then the following conditions are equivalent:*

(i) *M is a CMF A-module.*
(ii) *For all $0 \leq i < d$ the module of deficiency $K^i(M)$ is either zero or an i-dimensional Cohen-Macaulay module.*

(iii) *For all* $0 \leq i \leq d$ *the* A*-modules* $K^i(M)$ *are either zero or* i*-dimensional Cohen-Macaulay modules.*

Proof. First suppose that M is a CMF module. Then the dimension filtration $\mathcal{M} = \{M_i\}_{0 \leq i \leq d}$ has the property that for all $0 \leq i \leq d$ the quotient module $\mathcal{M}_i = M_i/M_{i-1}$ is eiher zero or an i-dimensional Cohen-Macaulay module. By view of 5.4 it follows that $K^i(M) \simeq K^i(\mathcal{M}_i)$ for all $0 \leq i \leq d$. Because $\mathcal{M}_i$ is either zero or an i-dimensional Cohen-Macaulay module we have that $K^i(\mathcal{M}_i)$ is either zero or the canonical module of the i-dimensional Cohen-Macaulay module $\mathcal{M}_i$. But then the canonical module of $\mathcal{M}_i$ is also an i-dimensional Cohen-Macaulay module. So $K^i(M)$ is either zero or an i-dimensional Cohen-Macaulay module. This proves the implication (i) ⇒ (ii) as well as (i) ⇒ (iii).

In order to prove (iii) ⇒ (i) consider the spectral sequence studied in the proof of 3.4. By view of Theorem 3.4 it will be enough to prove that all the quotients $F^p/F^{p+1} \simeq E_\infty^{p,-p}$ are either zero or $(-p)$-dimensional Cohen-Macaulay modules. We first claim that $E_\infty^{p,-p} \simeq E_2^{p,-p}$ for all $-d \leq p \leq 0$. To this end consider the subsequent stages of the spectral sequence

$$E_r^{p-r,-p+r-1} \to E_r^{p,-p} \to E_r^{p+r,-p-r+1}.$$

The left term is zero because it is a subquotient of $K^{-p+r}(K^{-p+r-1}(M)) = 0$. To this end recall that $\dim_A K^{-p+r-1}(M) \leq -p+r-1$, see 3.2. The term on the right hand side is a subquotient of $K^{-p-r}(K^{-p-r+1}(M))$. By our assumption we have that $K^{-p-r+1}(M)$ is either zero or an $(-p-r+1)$-dimensional Cohen-Macaulay module. But then the $(-p-r)$-th module of deficieny $K^{-p-r}(K^{-p-r+1}(M))$ is zero. That is, the modules at the right are always zero. But this implies that

$$F^p/F^{p+1} \simeq E_2^{p,-p} \simeq K^{-p}(K^{-p}(M))$$

for all $-d \leq p \leq 0$. We have to finish the proof by showing that $K^{-p}(K^{-p}(M))$ is either zero or a $(-p)$-dimensional Cohen-Macaulay module. By our assumption $K^{-p}(M)$ is either zero or an $(-p)$-dimensional Cohen-Macaulay module. Therefore $K^{-p}(K^{-p}(M))$ is either zero or – as the canonical module of $K^{-p}(M)$ – also a $(-p)$-dimensional Cohen-Macaulay module. By view of 4.3 this proves the claim of (i).

Finally we have to show that (ii) ⇒ (iii). That is, we have to show that the canonical module $K(M) = K^d(M)$ is a Cohen-Macaulay module provided for all $0 \leq i < d$ the module of deficiency $K^i(M)$ is either zero or an i-dimensional Cohen-Macaulay module. Now we have that $(\operatorname{Hom}_A(M, D_A^{\cdot}))^i = 0$ for all $i < -d$ and $H^{-d}(\operatorname{Hom}_A(M, D_A^{\cdot})) = K(M)$. So there is a short exact sequence of complexes

$$0 \to K(M)[d] \to \operatorname{Hom}_A(M, D_A^{\cdot}) \to C^{\cdot}(M) \to 0,$$

where $C^\cdot(M)$ denotes the cokernel of the natural embedding. So the complex $C^\cdot(M)$ carries as cohomology modules the modules of deficiencies, i.e.

$$H^i(C^\cdot(M)) \simeq \begin{cases} K^{-i}(M) & \text{for } -d < i \leq 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Now the natural homomorphism of complexes $M \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, D_A^\cdot), D_A^\cdot)$ induces an isomorphism in cohomology for a finitely generated A-module M. Therefore by applying $\operatorname{Hom}_A(\cdot, D_A^\cdot)$ to the above short exact sequence of complexes it induces a short exact sequence

$$0 \to H^0(\operatorname{Hom}_A(C^\cdot(M), D_A^\cdot)) \to M \to K(K(M)) \to H^1(\operatorname{Hom}_A(C^\cdot(M), D_A^\cdot)) \to 0$$

and isomorphisms $H^i(\operatorname{Hom}_A(C^\cdot(M), D_A^\cdot)) \simeq K^{d-i+1}(K(M))$ for all $i \geq 2$. In order to prove that $K(M)$ is a Cohen-Macaulay module – by local duality – it is enough to prove that $K^{d-i+1}(K(M)) = 0$ for $i \geq 2$. Whence it will be enough to show the vanishing of $H^i := H^i(\operatorname{Hom}_A(C^\cdot(M), D_A^\cdot))$ for all $i \geq 1$. To this end take the corresponding spectral sequence

$$E_2^{pq} = H^p(\operatorname{Hom}_A(H^{-q}(C^\cdot(M)), D_A^\cdot)) \Rightarrow E_\infty^{p+q} = H^{p+q},$$

derived in the same way as the spectral sequence studied in Section 3. Because of $E_2^{pq} = K^{-p}(K^q(M)) = 0$ for all q and all $p \neq -q$, note that $K^q(M), 0 \leq q < d$, is either zero or a q-dimensional Cohen-Macaulay module, there is a partial degeneration to $H^i = 0$ for all $i > 0$. This completes the proof. □

Looking at the second part of Theorem 5.5 there is another sufficient criterion for the canonical module $K(M)$ of M being a Cohen-Macaulay module. Moreover the filtration induced by the spectral sequence for the computation of $H^0(\operatorname{Hom}_A(C^\cdot(M), D_A^\cdot))$ is just the truncated dimension filtration, i.e. it follows $H^0(\operatorname{Hom}_A(C^\cdot(M), D_A^\cdot)) \simeq M_{d-1}$ and $K(K(M)) \simeq M/M_{d-1}$.

6. Faithful Flat Extensions and Examples

Let M denote a finitely generated A-module, $(A, \mathfrak{m})$ a local Noetherian ring. As mentioned in Section 2 in general M is not a CMF module in case $M \otimes \hat{A}$ is a CMF $\hat{A}$-module. In particular this is not even true for the ring itself as follows by the next example.

Example 6.1. Let $(A, \mathfrak{m})$ denote the 2-dimensional local domain considered by M. Nagata in [N, Example 2]. Clearly it is not a Cohen-Macaulay ring. For the multiplicity $e(\mathfrak{m}, A)$ it is shown that $e(\mathfrak{m}, A) = 1$. Therefore it implies that

$$1 = e(\mathfrak{m}, A) = e(\hat{\mathfrak{m}}, \hat{A}) = e(\hat{\mathfrak{m}}, \hat{A}/u_{\hat{A}}(0)).$$

By the view of [N, (40.6)] it yields that $\hat{A}/u_{\hat{A}}(0)$ is a regular local ring, in particular a 2-dimensional Cohen-Macaulay ring. Moreover since $\operatorname{depth} A = \operatorname{depth} \hat{A} = 1$ the ideal $u_{\hat{A}}(0)$ is – considered as an $\hat{A}$-module – a 1-dimensional Cohen-Macaulay module. But this means that $\hat{A}$ is a CMF ring or equivalently an approximately Cohen-Macaulay ring. But this is not true for A. Otherwise A would be a Cohen-Macaulay ring since it is a domain.

Before we shall formulate our next result let us recall the definition of a Cohen-Macaulay filtration, 4.2. An increasing filtration $C = \{C_i\}_{0 \le i \le d}$ of M is called a Cohen-Macaulay filtration whenever $M = C_d, d = \dim_A M$, and $C_i = C_i/C_{i-1}$ is either zero or an i-dimensional Cohen-Macaulay module for all $1 \le i \le d$. As it was mentioned in 4.3 a Cohen-Macaulay filtration coincides automatically with the dimension filtration.

Now let $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a faithful flat homomorphism of local rings. Let M be a finitely generated A-module with $d = \dim_A M$. Let $C = \{C_i\}_{0 \le i \le d}$ denote an increasing filtration of M such that $M = C_d$. Let $C_B = \{(C_B)_i\}_{0 \le i \le n}$ denote the induced filtration defined by $(C_B)_i = C_{i+t} \otimes_A B$, where $t = \dim B/\mathfrak{m}B$ denotes the dimension of the fibre ring.

Theorem 6.2. *Let $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a faithful flat homomorphism of local rings. Let M be a finitely generated A-module with $d = \dim_A M$. Then the following conditions are equivalent:*

(i) *The filtration C is a Cohen-Macaulay filtration of M and the fibre ring $B/\mathfrak{m}B$ is a Cohen-Macaulay ring.*
(ii) *The induced filtration C_B is a Cohen-Macaulay filtration of the B-module $M \otimes_A B$.*

Proof. Let X denote an arbitrary finitely generated A-module. By virtue of [M, Theorem 15.1] and [M, Theorem 23.3] the following two equalities are true

$$\dim_B X \otimes_A B = \dim_A X + \dim B/\mathfrak{m}B \text{ and } \operatorname{depth}_B X \otimes_A B = \operatorname{depth}_A X + \operatorname{depth} B/\mathfrak{m}B.$$

First of all this proves that $\dim_B X \otimes_A B = d + t$, i.e. $(C_B)_{d+t} = M \otimes_A B$.

Now suppose that condition (i) is satisfied. Then the above equalities show that each of the B-modules

$$(C_B)_i/(C_B)_{i-1} \simeq (C_{i-t}/C_{i-1-t}) \otimes_A B$$

are either zero or i-dimensional Cohen-Macaulay modules. The converse follows the same line of resoning. Hence we omit it. □

Note that the previuos result 6.2 does not apply to the example considered in 6.1. In the example there does not exist a Cohen-Macaulay filtration in A, while there is one in $\hat{A}$. The Cohen-Macaulay filtration in $\hat{A}$ does not occur as the extension of a Cohen-Macaulay filtration of A.

In the following we want to sum up the examples of CMF modules and rings showing that the occurance of them is quite natural.

Example 6.3. a) Let M be a Cohen-Macaulay module. Then M is also a CMF module.
b) Let $(A, \mathfrak{m})$ be a local ring with $d = \dim A$. Let $N_i, i = 0, \ldots, d$, be a family of A-modules such that either $N_i = 0$ or N_i is an i-dimensional Cohen-Macaulay module. Then $M = \oplus_{i=0}^{d} N_i$ is a CMF module over A. This follows easily by 4.3 since M admitts a filtration $M_i = \oplus_{j=0}^{i} N_j$ such that $M_i/M_{i-1} \simeq N_i, i = 0, \ldots, d$, is either zero or an i-dimenional Cohen-Macaulay module.
c) Let $(A, \mathfrak{m})$ denote a local ring. Let M be a finitely generated A-module. Then consider $A \ltimes M$, the idealization of M over A. That is, the additive group of $A \ltimes M$ coincides with the direct sum of the abelian groups A and M. The muliplication is given by

$$(a, m) \cdot (b, n) := (ab, an + bm).$$

Then $A \ltimes M$ is a d-dimensional local ring, see [N, (1.1)] or [BH, 3.3.22] for these and related facts.

Now suppose that $(A, \mathfrak{m})$ is a d-dimensional Cohen-Macaulay ring. Let M be a CMF module with $\dim M = t < d$. Then $A \ltimes M$ is a d-dimensional CMF ring. To this end let $\mathcal{M} = \{M_i\}_{0 \le i \le t}$ denote the dimension filtration of M. Now put

$$R_i = \begin{cases} A \ltimes M & \text{for} \quad i = d, \\ 0 \ltimes M & \text{for} \quad i = t+1, \ldots, d-1, \text{ and} \\ 0 \ltimes M_i & \text{for} \quad i = 0, \ldots, t. \end{cases}$$

Then $\{R_i\}_{0 \le i \le d}$ is a filtration of $R = A \ltimes M$ such that $R_d = A \ltimes M$ and R_i/R_{i-1} is either zero or an i-dimensional Cohen-Macaulay module. Note that

$$R_i/R_{i-1} \simeq \begin{cases} A & \text{for} \quad i = d, \\ 0 & \text{for} \quad i = t+1, \ldots, d-1, \text{ and} \\ M_i/M_{i-1} & \text{for} \quad i = 1, \ldots, t. \end{cases}$$

By view of 4.3 this proves the claim.
d) Let $A[[x]]$ denote the formal power series ring in one variable x over the local ring $(A, \mathfrak{m})$. Then a finitely A-module M is a CMF module if and only if $M[[x]]$ is a CMF module over the ring $A[[x]]$.
e) Let M be a finitely generated A-module such that $H^i_{\mathfrak{m}}(M), i \neq \dim_A M$, is a finitely generated A-module. Then M is a CMF module if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all $0 < i < \dim_A M$. In particular, under these circumstances M is a Cohen-Macaulay module if and only if M is a CMF module with $\operatorname{depth}_A M > 0$.

f) Every 1-dimensional A-module M is a CMF module. Therefore for any d-dimensional Cohen-Macaulay ring with $d \geq 2$ and a 1-dimensional A-module M the idealization $A \ltimes M$ is a d-dimensional CMF ring.

It would be of some interest to understand the descend of the CMF property from $M \otimes_A \hat{A}$ to M. What are suffficient condition on A? The Example 6.1 does not has Cohen-Macaulay formel fibres. Is it enough to suppose that the homomorphism $A \to \hat{A}$ has Cohen-Macaulay formel fibres ?

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