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Solving Polynomial Systems for the Kinematic Analysis and Synthesis of Mechanisms and Robot Manipulators

Problems in mechanisms analysis and synthesis and robotics lead naturally to systems of polynomial equations. This paper reviews the state of the art in the solution of such systems of equations. Three well-known methods for solving systems of polynomial equations, viz., Dialytic Elimination, Polynomial Continuation, and Grobner bases are reviewed. The methods are illustrated by means of simple examples. We also review important kinematic analysis and synthesis problems and their solutions using these mathematical procedures.

1 Introduction

Many problems in mechanisms analysis and synthesis and robotics lead naturally to systems of polynomial equations. Typical constraints state that two points on a rigid body must remain a fixed distance apart or that the angle between two lines in a rigid body must remain constant. Such constraints are generally expressed by vector dot and cross-products and they result in polynomial equations usually of the second degree. This paper reviews the state of the art in the solution of such systems of equations. We also review important kinematic analysis and synthesis problems and their solutions using these mathematical procedures. Three well-known methods for solving systems of polynomial equations, viz., Dialytic Elimination, Polynomial Continuation, and Grobner bases are reviewed. Let us begin with some mathematical preliminaries.

Polynomial Equations. A polynomial equation is one in which the variables appear in positive integral powers. For example the equation

$$5x^2 + 2xy^2 + 3y + 8 = 0 \tag{1}$$

is a polynomial equation in the variables x and y. The degree of each term of the polynomial is the sum of its exponents. The terms x^2, xy^2 and y appearing in Eq. (1) are respectively of degrees 2, 3, and 1. The degree of the polynomial is equal to the degree of its highest degree term.

2 Number of Solutions

Problems in kinematic analysis and synthesis may be posed as systems of polynomial equations to be solved simultaneously for the unknowns. The unknowns generally represent kinematic information such as the joint angles and displacements of a robot manipulator for a given gripper location. The polynomial systems are typically comprised of nonlinear equations. Such systems have multiple solutions in contrast to systems of linear polynomial equations. In a kinematic analysis problem, multiple solutions represent the different possible poses of the mechanism under the stated constraints. The total degree of a system of polynomial equations is the product of the degrees of its individual polynomial equations.

Bezout's Theorem. Bezout's Theorem states that the total number of solutions of a polynomial system is equal to the total degree of the system. This number, called the Bezout number or bound, includes both the finite solutions as well as the "so-called" solutions at infinity. The Bezout bound is usually a loose upper bound on the number of finite solutions. Consider the system of polynomial equations:

$$f_1: \quad a_{11}xy + a_{12}x + a_{13} = 0,$$
$$f_2: \quad a_{21}xy + a_{22}x + a_{23} = 0. \tag{2}$$

f_1 is of degree 2 because it contains xy, a term of degree 2. Similarly f_2 is also of degree 2. Therefore Bezout's Theorem states that this system has $2 \times 2 = 4$ solutions.

The M-Homogeneous Bezout Number. An alternative to the total degree theorem stated above is the *multi-homogeneous* (m-homogeneous) form of Bezout's Theorem which may be stated as follows: (see [1]). Given n polynomial equations $f_1, f_2, \ldots, f_n$ in n unknowns $x_1, x_2, \ldots, x_n$, the unknowns are first divided into m groups $\{x_{11}, \ldots, x_{1k_1}\}$, $\{x_{21}, \ldots, x_{2k_2}\}$, $\ldots$, $\{x_{m1}, \ldots, x_{mk_m}\}$ where k_i is the number of elements in the ith group and $k_1 + k_2 + \ldots + k_m = n$. Let the degree of equation l with respect to the variables of group j be d_{jl}. The m-homogeneous Bezout

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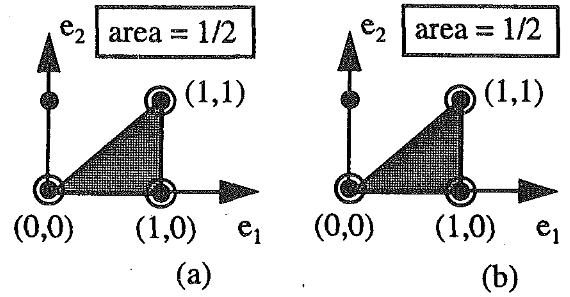

Fig. 1

number is equal to the coefficient of $\Pi_{j=1}^{m}\alpha_j^{k_j}$ in the product $\Pi_{l=1}^{n}(\Sigma_{j=1}^{m} d_{jl}\alpha_j)$. The value of the m-homogeneous Bezout number depends on the grouping of the variables. Many such groupings are possible, each one leading to its associated m-homogeneous number. The computation of the m-homogeneous Bezout number is illustrated with Eq. (2). Here there are only two possibilities: either we have one group $\{x_1, x_2\}$ or we have two groups $\{x_1\}$, $\{x_2\}$. Let us assign the variables x_1 and x_2 to separate groups: $\{x_1\}$, $\{x_2\}$. For this case $k_1 = 1$, $k_2 = 1$, and $m = 2$. d_{11}, the degree of f_1 in the variable in group 1, viz., $\{x_1\}$, is clearly 1. Similarly d_{21}, the degree of f_1 in the variable in group 2, viz., $\{x_2\}$, is 1. Likewise d_{12} and d_{22} are both 1. Therefore the product $\Pi_{l=1}^{n}(\Sigma_{j=1}^{m} d_{jl}\alpha_j)$ for this example, is $(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2)$. The 2-homogeneous Bezout number is the coefficient of $\alpha_1^1\alpha_2^1$ in this product, and is equal to 2. In numerous situations the use of multihomogeneous variables can reduce the number of solutions at infinity and provide a good bound on the number of finite solutions.

The BKK Bound. An alternative to the above Bezout numbers is a computation due to Bernstein [2], Khovanskii [3], and Kushnirenko [4] known as the BKK bound. The form of the computations presented in the following is adapted from the papers by Canny and Emiris [5], [6]. To understand this procedure we must first familiarize ourselves with some simple mathematical concepts. The exponent vector of the term $x_1^{e1}x_2^{e2} \ldots x_n^{en}$ is the vector $(e_1\ e_2\ \ldots\ e_n)^T$. Given a polynomial f_1 in the variables $x_1, x_2, \ldots, x_n$, the exponent vector set of f_1 is the set of all exponent vectors of the terms of f_1. For example the exponent vector set of the polynomial equation $5x_1^2x_2^3x_3 + 6x_3^2 + 2x_1 + 8 = 0$ in the variables x_1, x_2, x_3 is {(2 3 1), (0 0 2), (1 0 0), (0 0 0)}. The first element (2 3 1) corresponds to the term $x_1^2x_2^3x_3$, the term (0 0 2) corresponds to the term x_3^2, etc.

Another useful concept is that of the Minkowski Sum. Let A_1 and A_2 be two exponent vector sets. Let Q_1 and Q_2 be their convex hulls. By convex hull, we mean the smallest convex polyhedron containing a given set of points. The Minkowski Sum of Q_1 and Q_2 is the set $\{a + b | a \in Q_1, b \in Q_2\}$. The BKK bound on the number of solutions of a given system of polynomials $f_1, f_2, \ldots, f_n$ in the unknowns $x_1, x_2, \ldots, x_n$ may be constructed as follows. First construct the exponent vector sets $A_1, \ldots, A_n$ of the given system $f_1, f_2, \ldots, f_n$. Then construct the convex hulls $Q_1, \ldots, Q_n$ of the sets $A_1, \ldots, A_n$. The BKK bound is given by the formula $\Sigma_{I \subset \{1,\ldots,n\}}(-1)^{n-|I|}Vol(\Sigma^{\approx}_{i \in I} Q_i)$, where I ranges over all subsets of $\{1, \ldots, n\}$, $|I|$ is the cardinality of I (i.e., the number of elements in I), $\Sigma^{\approx}$ represents the Minkowski Sum, $Vol(Q_i)$ is the volume of the convex hull Q_i. Let us compute the BKK bound for the system of Eq. (2). The exponent vector sets of f_1 and f_2 are respectively {(1 1), (1 0), (0 0)} and {(1 1), (1 0), (0 0)}. Their convex hulls are shown in Figs. 1(*a*) and 1(*b*). The set of all subsets of the set {1, 2} is {{1}, {2}, {1, 2}}. The BKK formula applied to this example is

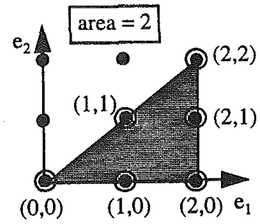

Fig. 2

$$(-1)^{(2-|\{1\}|)}Vol(Q_1) + (-1)^{(2-|\{2\}|)}Vol(Q_2) + (-1)^{(2-|\{1,2\}|)}(Vol(Q_1 + Q_2)) \quad (3)$$

Figure 2 shows the convex hull of the Minkowski Sum of Q_1 and Q_2. Substituting from Figs. 1 and 2 into Eq. (3) we get a BKK bound of $-1/2 - 1/2 + 2 = 1$.

3 Dialytic Elimination

Sylvester's Dialytic Elimination procedure [7] has been used in kinematics for eliminating one or two unknowns from small sets of equations. Recently we have been able to modify the dialytic method in order to make it part of a more practical approach. In the following, first the basic dialytic method is described, then its limitations are pointed out with suggestions on how these may be overcome.

There are six basic steps in using the Dialytic Elimination method to solve a set of nonlinear equations. Although the steps are easy to explain, the ideas behind Step 2 and Step 3 at first-sight seem strange, and even incorrect. The basic steps are:

(*1*) Rewrite equations with one variable suppressed.

(*2*) Define the remaining power products as new linear homogeneous unknowns.

(By power products, we mean the variable groups within each nominal of a polynomial; for example the power products of $5x_1^2x_2^3 + 3x_2^2x_3^4 - 8x_4$ are $x_1^2x_2^3$, $x_2^2x_3^4$ and x_4.)

(*3*) Obtain new linear equations so as to have as many linearly independent homogeneous equations as linear unknowns.

(*4*) Set the determinant of the coefficient matrix, of the set of equations formed from steps 2 and 3, to zero and obtain a polynomial in the suppressed variable. (If one is interested in only numerical solutions, this step can be omitted if we calculate eigenvalues in Step 5.)

(*5*) Determine the roots of the characteristic polynomial or the eigenvalues of the matrix. (This yields all possible values for the suppressed variable.)

(*6*) Substitute (one of the roots or eigenvalues) for the suppressed variable and solve the linear system for the remaining unknowns. Repeat this for each value of the suppressed variable.

These steps can best be explained with a simple example. Consider the following two nonlinear equations:

$$axy^3 + bx^3 + cy^3 + dx^2y + ex^2 + f = 0,$$

$$gx^4 + hxy^3 + ix^3 + jx^2 + kxy^2 + lx + m = 0. \quad (4)$$

Here, a, b, c, d, e, f, g, h, i, j, k, l and m are known coefficients, and x and y are the unknowns. For elimination theory it is useful to first suppress one of the variables, say, x and rewrite the system as:

$$(ax + c)y^3 + (dx^2)y + (bx^3 + ex^2 + f) = 0,$$

$$(hx)y^3 + (kx)y^2 + (gx^4 + ix^3 + jx^2 + lx + m) = 0. \quad (5)$$

where the entities in brackets are treated as coefficients. The two equations are now of the form:

$$Ay^3 + By + C = 0,$$
$$Dy^3 + Ey^2 + F = 0, \qquad (6)$$

where the new coefficients A, B, C, D, E and F, contain the suppressed variable x.

In Step 2, each power product is considered as a separate independent linear unknown. So our system is rewritten as

$$Ay_1 + By_2 + Cy_3 = 0,$$
$$Dy_1 + Ey_4 + Fy_3 = 0. \qquad (7)$$

In Step 3, we obtain additional equations by multiplying Eqs. (7) by y and then y^2. The results are four new equations with only two new power products. Using the concept of Step 2, i.e., labeling every power product as an independent linear unknown ($y^4 = y_5$, $y^5 = y_6$) we can write these new equations as:

$$Ay_5 + By_4 + Cy_2 = 0,$$
$$Dy_5 + Ey_1 + Fy_2 = 0,$$
$$Ay_6 + By_1 + Cy_4 = 0,$$
$$Dy_6 + Ey_5 + Fy_4 = 0. \qquad (8)$$

Combining Eqs. (7) and (8) we have a system of six homogeneous linear equations in six linear unknowns. The newly manufactured equations in (8) are dependent on the original equations but their dependence is not linear and hence they are linearly independent. These six equations in matrix form are as follows.

$$\begin{bmatrix} A & B & C & 0 & 0 & 0 \\ D & 0 & F & E & 0 & 0 \\ 0 & C & 0 & B & A & 0 \\ E & F & 0 & 0 & D & 0 \\ B & 0 & 0 & C & 0 & A \\ 0 & 0 & 0 & F & E & D \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = 0 \qquad (9)$$

Since we know that $y_3 = 1$, it is clear that the trivial solution of $y_i = 0$, ($i = 1, 2, \ldots, 6$) is not admissible, and therefore the determinant of the coefficient matrix must be equal to zero. In Step 4, we expand the determinant of the coefficient matrix and set the result to zero, to get a polynomial equation in A, B, C, D, E, F. If we then substitute into this polynomial the expressions for the suppressed variable (viz.: $A = ax + c$, $B = dx^2$, $C = bx^3 + ex^2 + f$, $D = hx$, $E = kx$, $F = gx^4 + ix^3 + jx^2 + lx + m$) we obtain a polynomial in the suppressed variable, x, and the original coefficients, a, b, c, d, e, f, g, h, i, j, k, l, m.

In Step 5, we obtain all values of the suppressed variable by using a root-finder routine on this polynomial. Finally in Step 6, we substitute the computed values of the suppressed variable into the linear set of equations and solve for the other original variables. In our example, we can use Eq. (9) to obtain y as follows; substitute one root of x into (9) and set $y_3 = 1$, and then use any five of the resulting equations to solve for y_2, etc. Then we have y since it is y_2. Similarly, all the other pairs (x, y) are obtained by repeating Step 6 for each of the roots from Step 5.

In principle, the foregoing procedure will always work if enough new equations can be determined from the original equations to obtain an $N \times N$ system of linear homogeneous equations whose determinant is not identically zero. However, it is important that the value of N be as small as possible. If the procedure introduces extraneous roots, N is larger than its minimum value and the suppressed variable polynomial is of higher degree than is necessary. One approach to obtaining a minimal value of N for problems in kinematic analysis is to use the trigonometric relations that exist among the coefficients of the governing equations to generate new linearly independent equations with same power products. This was demonstrated in the works of Lee and Liang [17] and Raghavan and Roth [18]. Recent papers by B. Roth [8], [9], and Innocenti [36], give alternative procedures for manufacturing additional equations to permit the construction of a minimal $N \times N$ system. The computation of the univariate polynomial in the suppressed variable requires the expansion of an N by N determinant with polynomial entries. Efficient procedures for effecting determinant expansion are presented in [10] and [11], Chapter 20.

Dialytic Elimination and the Eigenvalue Problem. Once an N by N matrix with polynomial entries has been created by the Dialytic Elimination procedure the determinant expansion step may be eliminated by setting up the problem as an eigenvalue problem. The procedure for effecting this was developed by Prof. Gene Golub of the Computer Science Department, Stanford University. We illustrate this for the case where the suppressed variable is of degree at most 2 in the entries of the matrix. Let the matrix be A and the suppressed variable x_1. Then

$$A = Bx_1^2 + Cx_1 + D, \qquad (10)$$

where A, B, C, and D are matrices with constant entries.

$$\det(A) = 0 \Rightarrow \det(Bx_1^2 + Cx_1 + D) = 0. \qquad (11)$$

This means that we seek those values of x_1 for which the system $(Bx_1^2 + Cx_1 + D)y = 0$ has a nontrivial solution vector y. Eq. (11) may be written as

$$Bx_1 z + Cz + Dy = 0, \qquad (12)$$

where $z = x_1 y$. The values of x_1 and z may be computed by setting up Eq. (12) as a generalized eigenvalue problem of the form $G\bar{x} = \lambda H\bar{x}$, as follows.

$$\begin{pmatrix} 0 & I \\ D & C \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = x_1 \begin{pmatrix} I & 0 \\ 0 & -B \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \qquad (13)$$

The eigenvalues x_1 may be computed using the QZ-algorithm described in [12], pages 251–264. The eigenvectors $\begin{pmatrix} y \\ z \end{pmatrix}$ may be computed using inverse iteration as described in [12], pages 238–239. Since the eigenvectors are comprised of the power products of the unsuppressed variables, x_2, x_3, ..., etc., the values of these variables may be computed once the eigenvectors are computed. When the matrix B is invertible we may write Eq. (13) as

$$\begin{pmatrix} 0 & I \\ -B^{-1}D & -B^{-1}C \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = x_1 \begin{pmatrix} y \\ z \end{pmatrix}. \qquad (14)$$

This is in the form of the standard eigenvalue problem which may be solved easily using the QR-algorithm also described in [12], pages 228–238. Recent examples of applications of this approach to problems in robotics and mechanisms are presented in [13] and [14].

Application to Linkage Analysis Problems. Dialytic Elimination has been used extensively by Duffy and his co-workers for the analysis of various single-loop mechanisms. In [15] an input-output equation of degree 32 in the tan-half-angle of the output angular displacement is obtained for the general spatial single-loop 7R mechanism. This problem was first posed to the mechanisms community by Prof. F. Freudenstein at the 1972 ASME Mechanisms Conference and in a subsequent paper. The application of this procedure to various classes of single-loop single degree-of-freedom linkages with R, P, and C joints is presented in [16]. Lee and Liang [17] derive a 16th degree polynomial input-output equation in the tan-half-angle of the output angular displacement of the general spatial 7-link 7R mechanism. The analysis is performed using an extension of Duffy's method.

Raghavan and Roth [18] developed a solution to the same problem using the Denavit-Hartenberg notation and the 4 by 4 transformation matrices commonly used in the robotics literature. They used a new systematic procedure based on elimination theory and its basic ideas. They went on to elaborate their method to include all six-degree-of-freedom open-loop series chains and by implication all closed-loop series one-degree-of-freedom mechanisms [19]. What we now know is that a series six-degrees-of-freedom chain has at most 16 ways to reach a pose if the chain is composed of all revolute joints (6R) or has one prismatic joint and five revolute joints (5R1P). If the chain has two prismatic joints (4R2P) the maximum drops to 8. While if it has three prismatic joints (3R3P) the number drops to two. Furthermore any time three neighboring axes intersect or are parallel, the number is at most 8. By using special structural restrictions, such as parallel and intersecting joint axes, it is also possible to get 2, 4, 8, and 12 maximum assembly variants at each pose [20]. Most recently the methods of Raghavan and Roth have also led to studies of special manipulator geometry which lead to overconstrained mechanisms. Mavroidis and Roth [21] have developed a systematic method for discovering new overconstrained mechanisms, verifying previously known overconstrained mechanisms, and computing their input-output equations.

Studies on the analysis of serial linkage manipulators are also presented in [14] and [22]. After eliminating all but one variable they obtain a single polynomial equation in one variable in the form of a determinant. They compute the roots of this polynomial using the eigenvalue formulation described earlier to minimize computation time. Similar eigenvalue formulations have also been studied by M. Ghazvini [23].

There is currently a lot of interest in different in-parallel and hybrid configurations [24], [25]. Husian and Waldron [26] present the equations for the inverse and direct kinematics problems of a mixed mechanism with two actuated joints and four passive joints in each of three parallel branches. The so-called generalized Stewart platform direct kinematics problem has attracted a lot of attention. It is a six degree-of-freedom in-parallel mechanism with a base link, a coupler link, and six entensible legs connecting base and coupler. The direct kinematics problem involves computing the position of the coupler given the leg lengths. This problem was first posed to the mechanisms community by Prof. K. Waldron at a panel session at the 1990 ASME Mechanisms Conference. The problem in its most general form (attachment points of the legs to the base and coupler may be arbitrarily located) is algebraically quite difficult to solve. Special cases of the problem where the locations of the leg attachment points satisfy geometric constraints are algebraically less difficult and have been tackled by various researchers. For example Lin, Griffis, and Duffy [27] use variable elimination to effect an algebraic solution to the direct kinematics problem for the 4-4 Stewart platforms (a special class). Zhang and Song [28] present a closed-form solution of the forward position analysis for the case where the ball joints are in planes in the base and coupler. Innocenti and Parenti-Castelli [29] present a closed-form solution for the 5-5 Stewart platform. Their analysis results in a 40th degree polynomial equation in one unknown. Lazard and Merlet [56] use a combination of geometric and algebraic techniques to show that a particular three-legged version of the Stewart platform has twelve assembly configurations.

The problem of solving the direct kinematics of the most general form of the generalized Stewart platform has been approached by various methods. The original solution and proof that there are 40 solutions is due to Raghavan [30], who used the polynomial continuation method (to be discussed in Section 4). Wampler [54], Mourrain [31] and Lazard [32] proved the same result using algebraic techniques. Their proofs were nonconstructive, i.e., they did not reduce the problem to a single polynomial equations in one variable. Recently M. Husty [33] has succeeded in developing an elimination procedure which reduces the problem to a 40th polynomial equation in one variable. A crucial factor in the success of his approach is the use of Soma coordinates or equivalently quaternions to represent the displacement of a rigid body. The use of these coordinates in in-parallel linkage problems significantly reduces the overall Bezout count of the problem by eliminating numerous solutions at infinity which would be present if alternative representations such as Euler angles were used. Husty effects his solution by a sequential elimination of variables and by the repeated use of (*a*) Sylvester's Resultant for eliminating one variable and (*b*) polynomial GCD computations. The problem of identifying a general Stewart platform with 40 real configurations is as yet unsolved.

While on the subject of Dialytic Elimination and in-parallel mechanisms, it is worth mentioning some important problems in mechanisms synthesis. The planar Burmester problem, viz., finding the points on a plane lying on a circle for several positions of the plane, is important for the design of four-bar linkages for rigid-body guidance, function generation, and for point-angle syntheses. It is solved algebraically by variable elimination in [34], pages 251–253. The spatial version of this problem, viz., finding lines in a body which have a constant angle and distance relative to a fixed line for up to five positions of a body, was solved by Roth [34]. Similar problems involving points on circles, cylinders, planes, etc. and including finding points on a rigid body lying on a sphere for several positions of the rigid body were solved numerically by Roth [35]. Recently the first algebraic solution of one of those problems was effected by *C*. Innocenti using Dialytic Elimination [36]. These problems are also examined in [1] where the m-homogenous Bezout number is used to get a tight bound on the number of finite solutions.

Variable Elimination by Resultants. The determinantal polynomial equation in one variable obtained by Dialytic Elimination may also be obtained by a more structured and rigorous construction of an entity known as the resultant. The essential facts on variable elimination by means of resultants are presented in [37] and [38]. Alternative procedures for resultant construction, due to *A*. Cayley [7], page 87, and F. Macaulay [39] are also very interesting. All of these procedures generate enormous matrices for problems of moderate size and are generally considered impractical with current computing capabilities. In order to exploit sparsity in a given system of polynomial equations, Gelfand, Kapranov and Zelevinsky have modified the resultant construction procedure and laid the foundations for defining an entity now generally known as the sparse mixed resultant [40], [41]. Algorithms for its construction have been proposed by Pedersen and Sturmfels [42] and the computational aspects of this problem have been investigated by Emiris and Canny [5], [6]. The algorithms proceed to construct the determinantal equation similar to Dialytic Elimination except that the selection of power products used as multipliers to generate new equations is done in a systematic manner. This ensures that the coefficients of each equation appear in the determinant a prescribed number of times. For the ith equation this number N_i is given by

$$N_i = \sum_{I \subset \{1, \ldots, i-1, i+1, \ldots, n\}} (-1)^{n-|I|} Vol\left(\sum_{j \in I} Q_j \right), \quad (15)$$

where Q_j is the convex polytope associated with the jth polynomial. This approach ensures that the final determinantal equation in one variable contains no extraneous roots.

4 Polynomial Continuation

Polynomial Continuation is a numerical procedure for computing all solutions to a system of polynomial equations. The basis for this method is the fact that, generally, small changes in the parameters of the system of equations result in small changes in the numerical values of the solutions. Therefore if we know the solution to a system of polynomial equations **A**, we may compute the solutions of a similar system **B** by tracking the solutions of **A** as we gradually modify its coefficients to those of **B** in small increments. A detailed tutorial on the subject by Wampler et al. is available in [1]. The essential ideas are as follows. Consider the following system of two polynomial equations in the two variables x_1, x_2:

$$f_1: \ (x_1 - 5)^2 + x_2^2 = 5^2,$$
$$f_2: \ x_1^2 + x_2^2 = 4^2.$$

Simplification and Initial Solution. We omit some terms of the two polynomial and add new ones, getting the following decoupled system:

$$g_1: \ x_1^2 + (-49 + 23i) = 0,$$
$$g_2: \ x_2^2 + (-144 - 16i) = 0.$$

The roots of g_1 are $\pm(-7.18 + 1.6i)$. The roots of g_2 are $\pm(12.018 + 0.665i)$. We thus have the following four pairs as starting (x_1, x_2) values for the "deformation phase":

$$(-7.18 + 1.6i, 12.018 + 0.665i),$$
$$(-7.18 + 1.6i, -12.018 - 0.665i),$$
$$(7.18 - 1.6i, 12.018 + 0.665i),$$
$$(7.18 - 1.6i, -12.018 - 0.665i).$$

Next we combine f_1, f_2 and g_1, g_2 with an additional parameter t (the parameter which controls the deformation) as follows:

$$h_1 = t(f_1) + (1 - t)(g_1),$$
$$h_2 = t(f_2) + (1 - t)(g_2). \tag{16}$$

When $t = 0$, we get the simplified system g_1, g_2, and when $t = 1$, we get the original system f_1, f_2. We vary t from 0 to 1 in small steps and use the Newton-Raphson iteration procedure to find the solutions of the deformed polynomials at each step of the deformation. The solutions from the previous deformation step serve as the initial guess for the current step. The solutions at $t = 1$ are the solutions of the original system of polynomials.

A systematic procedure for constructing the start system (system **A** above) to solve a system of polynomials $f_1, \ldots, f_n$ for the variables $x_1, \ldots, x_n$ is as follows. Create a new system of polynomials $g_i = p_i^{d_i} x_i^{d_i} - q_i^{d_i}$, where d_i is the degree of f_i and p_i, q_i are random (non-zero) numbers. The initial guess for the ith variable x_i then consists of the set $\left\{\alpha_k \dfrac{q_i}{p_i}, k = 1, \ldots, d_i\right\}$ where α_k are the d_i^{th} roots of unity.

Start Systems for Multi-Homogeneous Systems. In Section 2 we stated that the m-homogeneous form often eliminates solutions at infinity. If the m-homogeneous Bezout number for a given system $f_1, \ldots, f_n$ is smaller than the general Bezout number for a particular grouping of variables we would like to construct a start system with the same m-homogeneous Bezout number and perform a coefficient deformation on this system to compute the solutions of the given system $f_1, \ldots, f_n$. The advantage is this approach is that we follow fewer paths than if we use the form with the regular Bezout number.

Continuing with the notation of Section 2, let d_{jl} be the degree of equation l with respect to group j. Then the corresponding start equation is a product of factors $\prod_{j=1}^{m} f_{jl}(x_{j1}, \ldots, x_{jk}) = 0$, where the degree of f_{jl} is d_{jl}. This gives a start system with an identical multi-homogeneous structure as the target system. Let us demonstrate this with the sample problem from the previous section.

$$f_1: \ (x_1 - 5)^2 + x_2^2 = 5^2,$$
$$f_2: \ x_1^2 + x_2^2 = 5^2.$$

For the variable grouping $\{\mathbf{x}_1\}$, $\{\mathbf{x}_2\}$, the table of degrees of the equations in the two groups is:

	Group 1 $\{\mathbf{x}_1\}$	Group 2 $\{\mathbf{x}_2\}$
f_1	2	2
f_2	2	2

So we may construct a start system g_1, g_2 as follows.

$$g_1 = (x_1^2 - 1)(x_2^2 - 1),$$
$$g_2 = (x_1^2 - 4)(x_2^2 - 4).$$

The degree table of g_1, g_2 is the same as that of f_1, f_2. The solutions to g_1, g_2 may be written by inspection by noting that at least one factor from each equation must vanish at a solution. The solutions are (1, 2), (1, −2), (−1, 2), (−1, −2), (2, 1), (2, −1), (−2, 1), (−2, −1). The Polynomial Continuation procedure offers no a priori information regarding the number of real solutions to a given system of equations. A system of polynomials with real-valued coefficients may have complex-valued solutions. It is therefore generally advisable to architect a start system such that all the start solutions are complex-valued. By doing this we ensure that all solutions of the target system (the system to be solved) are "hit" by the solution-tracking procedure. This may be achieved by constructing a start system with random complex-valued coefficients.

Another trick to safeguard the numerical stability of the process of deforming the start system **A** into the target system **B** is to use a scheme of the form

$$H(x, t) = (1 - t)e^{i\Theta}A(x) + tB(x),$$

where Θ is a random real number. Note that the factor $e^{i\Theta}$ could also be used in Eqs. (16) in the illustrative example. This factor improves the numerical stability of the continuation process by avoiding situations where the Jacobian of $H(x, t)$ is singular. Singularity of the Jacobian causes a failure of the Newton-Raphson based solution-following procedure and therefore prevents the computation of all solutions of system **B**.

The Projective Transformation. A crucial detail in the successful implementation of Polynomial Continuation is the use of the projective transformation to comfortably track solutions at infinity. To do this we associate one homogenizing variable y_{j0} to each of the variable groups of a m-homogeneous grouping and make the substitution $(x_{ij} \leftarrow x_{ij}/y_{j0})$. Then clearing denominators we get a system of n polynomials in $n + m$ variables $\{x_{11}, \ldots, x_{1k_1}, y_{10}\}$, $\{x_{21}, \ldots, x_{2k_2}, y_{20}\}$, $\ldots$, $\{x_{m1}, \ldots, x_{mk_m}, y_{m0}\}$. This is an indeterminate system or underconstrained system. This indeterminacy may be resolved by a procedure due to Morgan [43]. We introduce an inhomogeneous linear equation for each homogeneous group, i.e.,

$$c_{j0}y_{j0} + c_{j1}x_{j1} \dots + c_{jk_j}x_{jk_j} - 1 = 0, \quad (j = 1, \dots, m) \quad (17)$$

where y_{j0} is the homogeneous variable for group j and the coefficients c are random complex numbers. These equations are used to eliminate the homogeneous variables from the system of n homogeneous equations in $n + m$ variables to obtain a new system of n inhomogeneous equations in n unknowns. This new system may be solved by deforming the coefficients of an appropriate start system and tracking the solutions. These computed solutions must then be used in Eqs. (17) to compute the corresponding values of y_{j0}, $j = 1, \dots, m$. If $y_{j0} = 0$, then the corresponding solution of the original system of equations is at infinity. If y_{j0} is non-zero the corresponding finite solution is given by x_{ij}/y_{j0}, $j = 1, \dots, k_j$.

Coefficient Continuation and Parameter Continuation. Coefficient Continuation is the name given to the process of solving new instances of a problem, a general case of which we have already solved. Here the general case becomes the start system and the new instance becomes the target or end system. We track only the paths beginning at the finite solutions of the start system. These converge to the finite solutions of the end system. By avoiding all of the paths corresponding to the solutions at infinity, this procedure results in significant computational savings. If the coefficients of a given system of equations are functions of certain linkage structural parameters (as is often the case in linkage analysis problems) we may use a procedure called Parameter Polynomial Continuation. First we solve a generic instance of the problem using Polynomial Continuation. This system and its solutions become the start system for all other instances of the problem. We solve new instances by deforming the structural parameters of the start system into those of the end system. A detailed presentation of these topics is available in [1].

Applications. Continuation has been used to solve a variety of problems in kinematic analysis and synthesis. It's earliest known application in mechanisms is the so-called *Bootstrap Method* developed in the 1960s by Roth and Freudenstein [44]. This was applied to the synthesis of geared five-bar mechanisms for path generation. Roth and Freudenstein invented several techniques to avoid nonconvergence and to direct the continuation towards desirable solutions. Many advances have been made in the subject since then and the polynomial continuation procedures presented in the preceding sections represent the current state of the art. A thorough and complete treatment of the subject may be found in [1]. In recent times two of the most sought after pieces of information in robot kinematics have come to us from this method. The numerical proof that the inverse kinematics of a general 6R manipulator has 16 solutions [45] and the proof that the so-called generalized Stewart platform has 40 solutions for the direct kinematics problem [30] were first obtained by the continuation method. Most importantly, these were done while the true numbers were still in doubt, and after some "proofs" of incorrect misleading numbers had been obtained by other methods. This speaks very highly of this method. Another noteworthy application of Polynomial Continuation was the complete solution of the nine-position four-bar linkage synthesis problem [46]. This involves finding all four-bar linkages whose coupler curve passes through nine prescribed points in a plane. Wampler, Morgan, and Sommese showed that there are generally 1442 solutions to this problem. These solutions along with their Robert's cognates give a total of 4326 solutions. Other important applications are the solution of the planar and spatial Burmester problems. The planar Burmester problem requires the computation of points on a plane which lie on a circle for five given positions of the plane. In general, there are four points satisfying this requirement and they are known as the Burmester points. The spatial version of this problem requires the computation of lines in a body which remain at a fixed distance and angle from a fixed line, there are six such lines. There are many other spatial synthesis problems that are easily solved by polynomial continuation. One of the most important is the computation of points in a rigid body which lie on a sphere for seven given positions of the rigid body [1]. In general, there are twenty such points.

5 Grobner Bases

The Grobner bases method makes it possible to convert a set of polynomials $f_1, f_2, \dots, f_n$, into an equivalent set $g_1, g_2, \dots, g_m$, such that the new set has desirable properties which may be exploited in the solution of various problems concerning the *ideal* generated by $f_1, f_2, \dots, f_n$. The *ideal* generated by a set of polynomials $f_1, f_2, \dots, f_n$ in the variables $x_1, x_2, \dots, x_r$, may be loosely defined as the set of all elements of the form $f_1b_1 + f_2b_2 + \dots + f_nb_n$ where $b_1, b_2, \dots, b_n$ are arbitrary elements of the set of all polynomials in $x_1, x_2, \dots, x_r$.

The two sets $f_1, f_2, \dots, f_n$ and $g_1, g_2, \dots, g_m$ are equivalent "bases" in the sense that they generate the same ideal and therefore have the same set of zeros. For our purposes, it is sufficient to understand that when the terms of $f_1, f_2, \dots, f_n$, and every intermediate result of the algorithm, are arranged in the so-called lexicographic order (to be described) the set $g_1, g_2, \dots, g_m$ is triangularized, i.e., g_1 contains only x_1 and each subsequent polynomial contains at most one new variable. This reduces the task of solving a system of multivariate polynomial equations to that of solving a sequence of univariate polynomial equations. The following material is adapted from the work of B. Buchberger [47].

The **lexicographic ordering** states that $x_1^{i_1}x_2^{i_2}\dots x_{r-1}^{i_{r-1}}x_r^{i_r} < x_1^{j_1}x_2^{j_2}\dots x_{r-1}^{j_{r-1}}x_r^{j_r}$ if the leftmost nonzero entry in the difference of the exponent vectors (i.e., $(i_1, i_2, \dots, i_r) - (j_1, j_2, \dots, j_r)$) is negative. For example, $x_1^0x_2^3x_3^4 < x_1^1x_2^2x_3^0$, $x_1^3x_2^2x_3^1 < x_1^3x_2^2x_3^4$. There are other orderings such as reverse lexicographic ordering, graded lexicographic ordering, etc. A good description of these is presented in [48].

Given two polynomials f and g, we say f may be **reduced modulo** g, if and only if a term in f may be deleted by subtracting an appropriate multiple of g from f. For this the power product of the leading term of g must divide the power product of some term in f. For example if $f = 3x_1^2x_2 + 5x_1x_2^2 + 2x_1 + 8$ and $g = 7x_1^2 + 4x_2 + 14$, we may reduce f modulo g as follows:

$$f' = f - \left(\frac{3}{7}\right)x_2g + 5x_1x_2^2 + 2x_1 - \left(\frac{12}{7}\right)x_2^2 - 6x_2 + 8$$

Given a set of polynomials G and another polynomial f, we say f is in **normal form** modulo G, if no further reduction of f modulo the polynomials in G is possible.

Definition: Grobner Basis. A set of polynomials G is called a Grobner basis if and only if for all f, if h_1 and h_2 are normal forms of f modulo G, then $h_1 = h_2$.

The **S-polynomial** of a pair of polynomials f_1, f_2, is defined as a polynomial h such that

$$h = (b/s_1)f_1 - (c_1/c_2)(b/s_2)f_2,$$

where c_i is the leading coefficient of f_i, s_i is the leading power product of f_i, and b is the least common multiple of (s_1, s_2).

Buchberger's algorithm [47] for computing the Grobner basis G of a given set of polynomials F is as follows. We assume that the terms of F are arranged such that the previously stated requirements on hierarchy and ordering of terms are satisfied and further that the terms are arranged in decreasing order from left to right.

(*1*) Assign all of the elements of F to G, i.e., $G = F$.
(*2*) Construct a set B of all pairs of polynomials in G, i.e., $B = \{(f_i, f_j) | f_i, f_j \in G, f_i \neq f_j\}$.
(*3*) While B is not empty perform the following:
 (*a*) Pick an element (f_i, f_j) of B.
 (*b*) Remove this element from B, i.e., $B = B - \{(f_i, f_j)\}$.
 (*c*) Compute h, the S-polynomial of (f_i, f_j).
 (*d*) Compute h', the normal form of h modulo G.
 (*e*) If h' is not equal to 0 perform the following:
 (*f*) Include h' in G, i.e., $G = G \cup \{h'\}$;
 (*g*) Update B, i.e., $B = B \cup \{(g, h') | g \in G\}$.

The above steps may be illustrated by the following example.

$$f_1: \quad (x_1 - 5)^2 + x_2^2 = 5^2,$$
$$f_2: \quad x_1^2 + x_2^2 = 4^2.$$

The equations may be rewritten in lexicographic order as follows.

$$f_1: \quad x_2^2 + x_1^2 - 10x_1,$$
$$f_2: \quad x_2^2 + x_1^2 - 16.$$

$G = \{f_1, f_2\}$.
$B = \{(f_1, f_2)\}$.
Let us operate on the element (f_1, f_2) of B. $B = B - \{(f_1, f_2)\} = \{\}$. The S-polynomial of (f_1, f_2) is $f_1 - f_2 = -10x_1 + 16$. This is in normal form modulo G. Let us call it f_3. We then update G and B. $G = \{f_1, f_2, f_3\}$, $B = \{(f_1, f_3), (f_2, f_3)\}$. Next let us operate on the element (f_1, f_3) of B. B is set to $B - \{(f_1, f_3)\} = \{(f_2, f_3)\}$. The S-polynomial of (f_1, f_3) is $x_1 f_1 + (x_2^2/10) \; f_3 = (16/10)x_2^2 + x_1^3 - 10x_1^2$. This may be further reduced modulo the elements of G. Subtracting $(16/10)\ f_1$ from it we get $x_1^3 - (116/10)\ x_1^2 + 16x_1$. Adding $(x_1^2/10)\ f_3$ to this polynomial we get $-10x_1^2 + 16x_1$. Subtracting x_1 times f_3 from this polynomial we get zero. Therefore the normal form of the S-polynomial of (f_1, f_3) modulo G is 0. Next we may examine the element (f_2, f_3) of B. B is set to $B - \{(f_2, f_3)\} = \{\}$. The S-polynomial of (f_2, f_3) is $(16/10)x_2^2 + x_1^3 - 16x_1$. Subtracting $(16/10)f_1$ from this polynomial gives $x_1^3 - (16/10)x_1^2$. Adding $(x_1^2/10)f_3$ to this polynomial gives zero. Therefore the normal form of the S-polynomial of (f_2, f_3) modulo G is 0. Since B is empty, the algorithm has terminated and G is a Grobner basis. The element f_3 when set equal to zero is a univariate polynomial with the root $x_1 = 1.6$. Substituting this value of x_1 into f_1 and f_2 gives the following polynomials: f_1: $x_2^2 - 13.44 = 0$, f_2: $x_2^2 - 13.44 = 0$. They both vanish for the same values of x_2, i.e., ± 3.66. So the solutions to the given system of equations are $(1.6, \pm 3.66)$ and this is in agreement with the other methods. If upon substitution of the value of x_1 into f_1 and f_2 we got two different polynomials, we would have to compute their greatest common divisor (GCD). The roots of this GCD would give the acceptable values of x_2 corresponding to the solutions of the system f_1, f_2.

Applications. The existence of Grobner bases was shown in 1964 by H. Hironaka [49] who use the term "standard bases." Grobner bases have been applied to various problems including the solution of multivariate polynomial equations, geometrical theorem proving, and curve and surface implicitization [50], [51]. To date this method has proved to be inefficient for the types of problems commonly encountered in mechanism analysis because it generates a large number of intermediate polynomials and excessive computational time. However variants using the reverse lexicographical ordering have proved to be useful in determining the number of solutions theoretically possible in various kinematic analysis problems.

The most notable recent applications of Grobner bases to mechanisms problems are the works of Mourrain [31], Lazard [32], and Faugere and Lazard [52] in connection with the direct kinematics problem of the general Stewart platform. Lazard represents rigid-body motions as a subvariety of R^{15} where the coordinates are the coefficients of R, the rotation matrix, T the translation vector, and a vector U defined as R^tT. The Grobner basis of the ideal of the equations of this variety is computed and shown to be of dimension 6 and degree 20. The intersection of this variety with the 6 constraint equations representing the legs of the Stewart platform is then shown to be a set of dimension 0 and degree 40. Mourrain uses Euler parameters to represent rotations and a reverse lexicographical ordering to compute the Grobner basis of the ideal of the 6 constraint equations representing the 6 legs of the Stewart platform. This Grobner basis is shown to have codimension 6 and degree 40. Using a Grobner bases computer implementation package named *Gb*. Faugere and Lazard have determined the generic number of assembly configurations for in-parallel platform manipulators when special classes are created by requiring that two or more of the six spherical joints (in either the base or the platform) coincide. They also show that the condition that some or all of the centers of the spherical joints lie in a single plane does not change the generic configurations count for each special class.

6 Summary and Conclusions

We have reviewed three methods for solving systems of polynomial equations arising in kinematic analysis. These are the Dialytic Elimination method, Polynomial Continuation, and Grobner bases. In our experience, the Dialytic Elimination procedure is very useful for small problems with up to 6 variables. It gives the analyst algebraic and geometric insight into the problem by permitting studies of the solution space as a function of a linkage's structural parameters. When combined with the eigenvalue formulation, the Dialytic Elimination procedure often yields computationally-fast solution algorithms. This is most beneficial in time-critical applications such as computer-aided design (CAD) systems, off-line manufacturing simulation (CAM) systems, and "real-time" motion planning and navigation systems. For example, Manocha and Canny [13] solve the general $6R$ inverse kinematics problem in 11 milliseconds on an IBM RS/6000 workstation using 64 bit IEEE floating point arithmetic.

Polynomial Continuation is most appropriate for large problems with many solutions. It does not readily provide the insight available through algebraic methods but it may be adapted to provide equivalent information. Conceptually, this procedure is very easy to understand and use. The traditional complaint against this procedure has been that it is very time-consuming and computation intensive. However, recent developments such as the use of the projective transformation, the multi-homogeneous formulation, and Parameter Continuation put it on par with any of the other solution methods. For example, Parameter Continuation is used to compute all 16 solutions of the general $6R$ inverse kinematics problem in 10 seconds on an IBM 370-3090 using double precision arithmetic [53], and all 40 solutions of the general Stewart platform forward position problem in 14 seconds on an IBM RS/6000 using double precision [54]. Another advantage of continuation is that it usually can handle special cases without the need for special logic or derivations.

We have had very limited experience with the Grobner bases method and we have not used it to solve any significant problem in kinetic analysis. Our colleagues who have extensive experience with this method indicate that it suffers from exploding computations due to the large number of intermediate polynomials generated. Therefore its use in the industrial setting is questionable at the present time. Recent improvements seem to have considerably reduced the computation times [52], still the reported times are orders of magnitude greater than for the other methods.

There is a lot of interest in these three methods and various researchers are developing analytical, graphical [55], and numerical improvements. Based on our experience with these methods, we recommend that the analyst use a combination of Dialytic Elimination and Polynomial Continuation because they complement each other. Small problems may be solved easily using Dialytic Elimination. For larger problems, Polynomial Continuation may be used to determine the number of finite solutions. If this number is not too large (say, under 30) one may try to construct a solution using Dialytic Elimination. The eigenvalue formulation may be used for time-critical applications. If the number of solutions is large, it is advisable to stick with Parameter Continuation or Coefficient Continuation.

7 Open Problems

The following is a list of topics and directions for future research that may be of interest to the mechanisms community.

(1) Identification of a general 6-6 Stewart platform with 40 distinct real-valued configurations.

(2) Development of a Dialytic Elimination-based solution procedure for the general 6-6 Stewart platform forward position problem. M. Husty [33] has indicated in his work that he generates intermediate polynomials of high degree (320) but that his final polynomial in one variable is of degree 40. This suggests the presence of ideal properties similar to the **p.p, p.l, p × l, (p.p)l − 2(p.l)p** properties used in the general $6R$ inverse kinematics solution. The identification and use of such properties in constructing a univariate determinantal equation for this problem would be most interesting.

(3) Development of a Unified Solution Procedure for all Stewart Platforms. Our work on the general $6R$ inverse kinematics problem yielded a solution algorithm which we then adapted to arbitrary series chains with R and P joints. This resulted in a unified solution procedure for the kinematic analysis of arbitrary series chains. The development of a similar theory for arbitrary Stewart platforms would be most useful and enlightening.

(4) Evaluation of Sparse Elimination Theory for Kinematic Analysis. The sparse resultant was presented briefly in Section 3. The theory is fairly new and its appropriateness for kinematic analysis is as yet unexplored. An objective evaluation of this new tool would be most beneficial to the mechanisms community.

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