# Mean Codeword Lengths and Their Correspondence with Entropy Measures

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*Abstract*—The objective of the present communication is to develop new genuine exponentiated mean codeword lengths and to study deeply the problem of correspondence between well known measures of entropy and mean codeword lengths. With the help of some standard measures of entropy, we have illustrated such a correspondence. In literature, we usually come across many inequalities which are frequently used in information theory. Keeping this idea in mind, we have developed such inequalities via coding theory approach.

*Keywords*—Codeword, Code alphabet, Uniquely decipherable code, Mean codeword length, Uncertainty, Noiseless channel

## I. INTRODUCTION

IN coding theory, one of the many applications of notion of uncertainty will be to the problem of efficient coding of messages to be sent over a noiseless channel, that is, our only concern is to maximize the number of messages that can be sent over the channel in a given time. Let us assume that the messages to be transmitted are generated by a random variable X and each value $x_i$, i = 1, 2, ...., n of X must be represented by a finite sequence of symbols chosen from the set $\{ a_1, a_2, ....., a_D \}$. This set is called code alphabet or set of code characters and sequence assigned to each $x_i$, i = 1, 2, ....., n is called code word. Let $n_i$ be the length of code word associated with $x_i$ satisfying Kraft's [10] inequality given by the following mathematical expression:

$$D^{-l_1} + D^{-l_2} + ... + D^{-l_n} \leq 1 \qquad (1)$$

Where, D is the size of alphabet. In calculating the long run efficiency of communications, we choose codes to minimize average code word length, given by

$$L = \sum_{i=1}^{n} p_i \, n_i \qquad (2)$$

Where, $p_i$ is the probability of occurrence of $x_i$. For uniquely decipherable codes, Shannon's [13] noiseless coding theorem which states that

$$\frac{H(P)}{\log D} \leq L < \frac{H(P)}{\log D} + 1 \qquad (3)$$

Determines the lower and upper bounds on L in terms of Shannon's [13] entropy $H(P)$. Campbell [5] for the first

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time introduced the idea of exponentiated mean codeword length for uniquely decipherable codes and proved a noiseless coding theorem. He considered a special exponentiated mean of order α given by

$$L_\alpha = \frac{\alpha}{1-\alpha} \log_D \left[ \sum_{i=1}^{n} p_i \, D^{(1-\alpha)\,n_i/\alpha} \right] \qquad (4)$$

and showed that its lower bound lies between $R_\alpha(P)$ and $R_\alpha(P) + 1$ where $R_\alpha(P)$ is expressed as:

$$R_\alpha(P) = (1-\alpha)^{-1} \log_D \left[ \sum_{i=1}^{n} p_i^{\,\alpha} \right]; \alpha > 0 , \alpha \neq 1 \quad (5)$$

The above is Renyi's [12] measure of entropy of order $\alpha$. As $\alpha \to 1$, it is easily shown that $L_\alpha \to L$ and $R_\alpha(P)$ approaches $H(P)$.

Guiasu and Picard [6] defined the weighted average length for a uniquely decipherable code as

$$\overline{L} = \sum_{i=1}^{n} \left( \frac{u_i \, n_i \, p_i}{\sum_{i=1}^{n} u_i \, p_i} \right) \qquad (6)$$

Longo [11] interpreted (6) as the average cost of transmitting letters $x_i$ with probability $p_i$ and utility $u_i$ and provided some practical interpretation of this length and also derived the lower and upper bounds for the cost function (6).

In the literature of coding theory, for the given value of an inequality, say, Kraft's [10] inequality, we have a pair of problems. For a given mean codeword length, we can find its lower bounds for all uniquely decipherable codes. In the inverse problem, we can find the mean value for the given pair of lower bounds. The direct problem has a unique answer, though it may not always be easy to find an analytical expression for it. However, the inverse problem has no unique answer in the sense that the same lower bounds may arise for a number of means. The challenge is to find as many of the means as possible, which have the given pair of values as lower bounds and this is the theme of the present communication.

In section II, we have developed two new mean codeword lengths with the help of divergence measures. In section III, we have illustrated the correspondence between standard measures of entropy and the codeword lengths. The development of new inequalities has been made in section IV.

## II. DEVELOPMENT OF TWO NEW MEAN CODEWORDS

*A. For the development of first mean, we consider Sharma and Mittal's [15] measure of directed divergence given by*

$$D_r^s\left(P \Box Q\right) = \frac{1}{s-1}\left[\left(\sum_{i=1}^{n} p_i^{\,r} q_i^{\,1-r}\right)^{\frac{s-1}{r-1}} - 1\right] \geq 0 \qquad (7)$$

Putting $q_i = \dfrac{D^{-l_i}}{\sum_{i=1}^{n} D^{-l_i}}$ in equation (7), we get

$$\left(\sum_{i=1}^{n} p_i^{\,r} D^{-l_i(1-r)}\right)^{\frac{s-1}{r-1}} \geq \left(\sum_{i=1}^{n} D^{-l_i}\right)^{-(s-1)} \qquad (8)$$

Taking logarithms and then dividing both sides of equation (8) by $(s-1)$, we get the following expression:

$$\frac{1}{r-1}\log_D\left(\frac{\sum_{i=1}^{n} p_i^{\,r} D^{l_i(r-1)}}{\sum_{i=1}^{n} p_i^{\,r}}\right) + \log_D \sum_{i=1}^{n} D^{-l_i}$$

$$\geq \frac{1}{1-r}\log_D\left(\sum_{i=1}^{n} p_i^{\,r}\right)$$

that is

$$\bar{L}_r + \log_D \sum_{i=1}^{n} D^{-l_i} \geq R_r(P)$$

where

$$\bar{L}_r = \frac{1}{r-1}\log_D\left(\frac{\sum_{i=1}^{n} p_i^{\,r} D^{l_i(r-1)}}{\sum_{i=1}^{n} p_i^{\,r}}\right) \qquad (9)$$

Since $\sum_{i=1}^{n} D^{-l_i}$ lies between $D^{-1}$ and 1, so lower bound of

$\bar{L}_r$ lies between $R_r(P)$ and $R_r(P)+1$ where $R_r(P)$ is a Renyi's [12] measure of entropy of order $r$.

It can easily be proved that: (i) When $l_1 = l_2 = ... l_n = l$,

then $\bar{L}_r = l$

(ii) $\bar{L}_r$ lies between minimum and maximum values of $l_1, l_2, ..., l_n$

(iii) When $r \rightarrow 1$, then $\bar{L}_r \rightarrow L$ where $L = \sum_{i=1}^{n} p_i l_i$ (9)

Thus, the mean codeword length introduced in equation is a genuine mean codeword length as it satisfies the essential properties of being a mean codeword length.

*B. For the development of second mean, we consider Bhattacharya's [2] measure of directed divergence is given by*

$$D(P,Q) = \sum_{i=1}^{n}\left(\sqrt{p_i} - \sqrt{q_i}\right)^2$$

or

$$D(P,Q) = 1 - \sum_{i=1}^{n} \sqrt{p_i}\,\sqrt{q_i} \geq 0 \qquad (10)$$

Putting $q_i = \dfrac{D^{-l_i}}{\sum_{i=1}^{n} D^{-l_i}}$ in equation (10), we get

$$\left(\sum_{i=1}^{n} D^{-l_i}\right)^{\frac{1}{2}} \geq \sum_{i=1}^{n} \sqrt{p_i}\left(D^{-l_i}\right)^{\frac{1}{2}} \qquad (11)$$

Taking Logarithms on both sides of (11), we get

$$\log_D\left(\sum_{i=1}^{n} D^{-l_i}\right) \geq$$

$$2\log_D\left(\frac{\sum_{i=1}^{n} \sqrt{p_i}\left(D^{-l_i}\right)^{\frac{1}{2}}}{\sum_{i=1}^{n} \sqrt{p_i}}\right) + 2\log_D\left(\sum_{i=1}^{n} \sqrt{p_i}\right)$$

or

$$\frac{1}{\frac{1}{2}-1}\log_D\left(\frac{\sum_{i=1}^{n} p_i^{\frac{1}{2}} D^{l_i\left(\frac{1}{2}-1\right)}}{\sum_{i=1}^{n} p_i^{\frac{1}{2}}}\right) + \log_D\left(\sum_{i=1}^{n} D^{-l_i}\right)$$

$$\geq \frac{1}{1-\frac{1}{2}}\log_D\left(\sum_{i=1}^{n} p_i^{\frac{1}{2}}\right)$$

that is

$$\bar{L}_{\frac{1}{2}} \geq R_{\frac{1}{2}}(P) - \log_D\left(\sum_{i=1}^{n} D^{-l_i}\right)$$

where

$$\bar{L}_{\frac{1}{2}} = \frac{1}{\frac{1}{2}-1}\log_D\left(\frac{\sum_{i=1}^{n} p_i^{\frac{1}{2}} D^{l_i\left(\frac{1}{2}-1\right)}}{\sum_{i=1}^{n} p_i^{\frac{1}{2}}}\right) \qquad (12)$$

Since $\sum_{i=1}^{n} D^{-l_i}$ lies between $D^{-1}$ and 1, so lower bound of

$\bar{L}_{\frac{1}{2}}$ lies between $R_{\frac{1}{2}}(P)$ and $R_{\frac{1}{2}}(P)+1$ where $R_{\frac{1}{2}}(P)$ is

a Renyi's [12] measure of entropy of order $\frac{1}{2}$ .

Thus, the mean codeword length introduced in equation (12) is a genuine mean codeword length as it can easily be proved that it satisfies the essential properties of being a mean codeword length.

### III. CORRESPONDENCE BETWEEN MEAN CODEWORD LENGTHS AND THE ENTROPY MEASURES

The object of the present paper is to go deeper into the problem of correspondence between well known measures of entropy and mean codeword lengths. We state the results in a broader framework as follows:

(a) To every mean codeword length, there corresponds a measure of entropy or a monotonic increasing function of a measure of entropy.

(b) To every measure of entropy, there corresponds a mean codeword length or a monotonic increasing function of the mean codeword length.

For many purposes, especially for maximization of entropy purposes, every monotonic increasing function of a measure of entropy is as good as a measure of entropy and for such purposes; all such functions should be regarded as equivalent. A monotonic increasing function of mean codeword lengths is not the same as a mean codeword length, but minimizing a monotonic increasing function of a mean codeword length gives the same results as minimizing the mean codeword length itself. Thus, we do not lose anything significant from our results by using monotonic increasing functions of entropy and mean codeword lengths. Below, we illustrate the correspondence between standard measures of entropy and the codeword lengths:

*Theorem*: For all uniquely decipherable codes, the lower bound of exponentiated mean codeword length

$$L(\alpha,\beta) = \frac{\beta-\alpha+1}{\alpha-\beta} \log_D \left( \sum_{i=1}^{n} p_i D^{l_i \left( \frac{\beta-\alpha+1}{\alpha-\beta} \right)} \right)$$

lies between

$$\frac{1}{\alpha-\beta} \log_D \left( \sum_{i=1}^{n} p_i^{\beta-\alpha+1} \right) \text{ and}$$

$$\frac{1}{\alpha-\beta} \log_D \left( \sum_{i=1}^{n} p_i^{\beta-\alpha+1} \right) + 1,$$

$$\alpha-1 < \beta < \alpha, \ \alpha \geq 1$$

where

$$\frac{1}{\alpha-\beta} \log_D \left( \sum_{i=1}^{n} p_i^{\beta-\alpha+1} \right)$$

is the Varma's [16] measure of entropy.

Proof: Here we use Holder's inequality

$$\sum_{i=1}^{n} x_i y_i \geq \left( \sum_{i=1}^{n} x_i^{p} \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} y_i^{q} \right)^{\frac{1}{q}}, \frac{1}{p} + \frac{1}{q} = 1,$$

$$p \ or \ q < 1 \tag{13}$$

Substituting

$$x_{i=} p_i^{-\left( \frac{\beta-\alpha+1}{\alpha-\beta} \right)} D^{-l_i}, \ y_{i=} p_i^{\left( \frac{\beta-\alpha+1}{\alpha-\beta} \right)}$$

$$\frac{1}{p} = \frac{\alpha-\beta-1}{\alpha-\beta},$$

$$\frac{1}{q} = \frac{1}{\alpha-\beta} \text{ so that}$$

$$\frac{1}{p} + \frac{1}{q} = 1 \text{ and either } p \text{ or } q < 1 \text{ in equation (13), we get}$$

$$\sum_{i=1}^{n} D^{-l_i} \geq$$

$$\left[ \sum_{i=1}^{n} \left( p_i D^{-l_i \left( \frac{\alpha-\beta}{\alpha-\beta-1} \right)} \right) \right]^{\frac{\alpha-\beta-1}{\alpha-\beta}} \left[ \sum_{i=1}^{n} p_i^{(\beta-\alpha+1)} \right]^{\frac{1}{\alpha-\beta}} \tag{14}$$

Taking log on both sides of (14), we get

$$\left( \frac{\alpha-\beta-1}{\beta-\alpha} \right) \log_D \sum_{i=1}^{n} \left( p_i D^{l_i \left( \frac{\beta-\alpha}{\alpha-\beta-1} \right)} \right)$$

$$\geq \frac{1}{\alpha-\beta} \log_D \sum_{i=1}^{n} p_i^{(\beta-\alpha+1)} \tag{15}$$

So that, if $l_i's$ are to be integers, the lower bound for

$L(\alpha,\beta)$ lies between $\frac{1}{\alpha-\beta} \log_D \sum_{i=1}^{n} p_i^{(\beta-\alpha+1)}$ and

$$\frac{1}{\alpha-\beta} \log_D \sum_{i=1}^{n} p_i^{(\beta-\alpha+1)} + 1$$

The R.H.S of (15) is Varma's [16] measure of entropy and L.H.S is a genuine mean codeword length as it satisfies the essential properties of being a mean codeword length.

It has been proved that Shannon's [13] entropy, Renyi's [12] entropy of order , Kapur's [8] entropy of order and type all provide lower bounds for different mean codeword lengths, while Havrada and Charvat's [7], Arimoto's [1] and Behara and Chawla's [3] measures of entropy provide lower bounds for some monotonic increasing functions of mean codeword lengths but not for mean codeword length's themselves. Below, we discuss the correspondence between standard measures of entropy and the possible lower bounds:

#### A. Kapur's [8] measure of entropy as a possible lower bound

Kapur's [8] measure of entropy of order $\alpha$ and type $\beta$ is given by

$$K_\alpha^{\beta}(P) = \frac{1}{\alpha+\beta-2} \left( \sum_{i=1}^{n} p_i^{\alpha} + \sum_{i=1}^{n} p_i^{\beta} - 2 \right) \tag{16}$$

We know that

$$\frac{1}{1-\alpha}\log_D \sum_{i=1}^{n} p_i^{\alpha} \le \frac{1}{\alpha-1}\log_D\left(\frac{\sum_{i=1}^{n} p_i^{\alpha} D^{(\alpha-1)l_i}}{\sum_{i=1}^{n} p_i^{\alpha}}\right)$$

Where L.H.S is Renyi's [12] entropy of order $\alpha$ and R.H.S is an exponentiated mean of order $\alpha$ developed in equation (9).

that is,

$$\log_D \sum_{i=1}^{n} p_i^{\alpha} \le -\log_D\left(\frac{\sum_{i=1}^{n} p_i^{\alpha} D^{(\alpha-1)l_i}}{\sum_{i=1}^{n} p_i^{\alpha}}\right)$$

which gives

$$\sum_{i=1}^{n} p_i^{\alpha} \le \frac{\sum_{i=1}^{n} p_i^{\alpha}}{\sum_{i=1}^{n} p_i^{\alpha} D^{(\alpha-1)l_i}}$$

Similarly,

$$\sum_{i=1}^{n} p_i^{\beta} \le \frac{\sum_{i=1}^{n} p_i^{\beta}}{\sum_{i=1}^{n} p_i^{\beta} D^{(\beta-1)l_i}}$$

Equality sign holds in both cases when $D^{-l_i} = p_i$. Thus, we have

$$\left(\sum_{i=1}^{n} p_i^{\alpha} + \sum_{i=1}^{n} p_i^{\beta} - 2\right)$$

$$\le \left(\frac{\sum_{i=1}^{n} p_i^{\alpha}}{\sum_{i=1}^{n} p_i^{\alpha} D^{(\alpha-1)l_i}} + \frac{\sum_{i=1}^{n} p_i^{\beta}}{\sum_{i=1}^{n} p_i^{\beta} D^{(\beta-1)l_i}} - 2\right)$$

Dividing both sides by $\alpha + \beta - 2$, we get

$$\frac{1}{\alpha+\beta-2}\left(\sum_{i=1}^{n} p_i^{\alpha} + \sum_{i=1}^{n} p_i^{\beta} - 2\right)$$

$$\le \frac{1}{\alpha+\beta-2}\left(\frac{\sum_{i=1}^{n} p_i^{\alpha}}{\sum_{i=1}^{n} p_i^{\alpha} D^{(\alpha-1)l_i}} + \frac{\sum_{i=1}^{n} p_i^{\beta}}{\sum_{i=1}^{n} p_i^{\beta} D^{(\beta-1)l_i}} - 2\right) \quad (17)$$

The L.H.S. of equation (17) is Kapur's [8] measure of entropy of order $\alpha$ and type $\beta$ but R.H.S. is neither a mean codeword length nor a monotonic increasing unction of mean codeword length.

## B. Sharma and Mittal's [14] measure of entropy as a possible lower bound

Sharma and Mittal's [14] measure of entropy is given by

$$\phi(P) = \frac{1}{2^{1-\alpha}-1}\left[\left(\sum_{i=1}^{n} p_i^{\beta}\right)^{\frac{\alpha-1}{\beta-1}} - 1\right], \alpha \ne 1, \beta \ne 1$$

$$\alpha > 0, \beta > 0 \quad (18)$$

From equation (9), we have

$$\frac{1}{1-\beta}\log_D \sum_{i=1}^{n} p_i^{\beta} \le \frac{1}{\beta-1}\log_D\left(\frac{\sum_{i=1}^{n} p_i^{\beta} D^{(\beta-1)l_i}}{\sum_{i=1}^{n} p_i^{\beta}}\right)$$

If $\beta < 1$, we have

$$\log_D \sum_{i=1}^{n} p_i^{\beta} \le -\log_D\left(\frac{\sum_{i=1}^{n} p_i^{\beta} D^{(\beta-1)l_i}}{\sum_{i=1}^{n} p_i^{\beta}}\right)$$

which further gives

$$\Rightarrow \left[\left(\sum_{i=1}^{n} p_i^{\beta}\right)^{\frac{\alpha-1}{\beta-1}} - 1\right] \le \left[\left(\frac{\sum_{i=1}^{n} p_i^{\beta}}{\sum_{i=1}^{n} p_i^{\beta} D^{(\beta-1)l_i}}\right)^{\frac{\alpha-1}{\beta-1}} - 1\right]$$

Dividing both sides by $2^{1-\alpha}-1$, we get

$$\frac{1}{2^{1-\alpha}-1}\left[\left(\sum_{i=1}^{n} p_i^{\beta}\right)^{\frac{\alpha-1}{\beta-1}} - 1\right]$$

$$\le \frac{1}{2^{1-\alpha}-1}\left[\left(\frac{\sum_{i=1}^{n} p_i^{\beta}}{\sum_{i=1}^{n} p_i^{\beta} D^{(\beta-1)l_i}}\right)^{\frac{\alpha-1}{\beta-1}} - 1\right] \quad (19)$$

The L.H.S. of equation (19) is Sharma and Mittal's [14] measure of entropy but R.H.S is neither a mean codeword length nor a monotonic increasing unction of mean codeword length.

In the literature of information theory, we usually come across many inequalities which are frequently applicable and used for the manipulation of mathematical results. The following section deals with development of such inequalities and relations between measures of entropy via coding theory approach.

## IV. NEW INEQUALITIES VIA CODING THEORY APPROACH

In this section, we generate the following inequalities by using well known measures of directed divergence:

*A. Burg's [4] measure of entropy is given by*

$$\varphi_1(p) = \sum_{i=1}^{n} \log_D p_i \qquad (20)$$

Also, Burg's [4] measure of directed divergence is given by

$$D(P,Q) = \sum_{i=1}^{n}\left(\frac{p_i}{q_i} - \log_D \frac{p_i}{q_i} - 1\right) \geq 0 \qquad (21)$$

Putting $q_i = \dfrac{D^{-l_i}}{\sum\limits_{i=1}^{n} D^{-l_i}}$ in equation (21), we get

$$\sum_{i=1}^{n}\left( p_i\left(\frac{\sum\limits_{i=1}^{n} D^{-l_i}}{D^{-l_i}}\right) - \log_D p_i \left(\frac{\sum\limits_{i=1}^{n} D^{-l_i}}{D^{-l_i}}\right) - 1\right) \geq 0$$

or

$$\sum_{i=1}^{n}\left(\frac{p_i}{D^{-l_i}}\sum_{i=1}^{n} D^{-l_i}\right) -$$

$$\sum_{i=1}^{n}\log_D p_i - \sum_{i=1}^{n}\log_D\left(\frac{\sum\limits_{i=1}^{n} D^{-l_i}}{D^{-l_i}}\right) - n \geq 0 \qquad (22)$$

If we take $l_1 = l_2 = ... = l_n = l$ in (22), we get

$$\log_D\left(\frac{1}{n^n}\right) \geq \sum_{i=1}^{n}\log_D p_i$$

Thus, we have

$$\varphi_1(p) \leq \log_D\left(\frac{1}{n^n}\right)$$

which is a new inequality.

*B. Jensen's divergence is given by*

$$J(P \square Q) = \sum_{i=1}^{n}(p_i - q_i)\log_D \frac{p_i}{q_i}$$

We know that

$$J(P \square Q) = \sum_{i=1}^{n}(p_i - q_i)\log_D \frac{p_i}{q_i} \geq 0 \qquad (23)$$

Putting $q_i = \dfrac{D^{-l_i}}{\sum\limits_{i=1}^{n} D^{-l_i}}$ in equation (23), we get

$$\sum_{i=1}^{n}\left( p_i - \frac{D^{-l_i}}{\sum\limits_{i=1}^{n} D^{-l_i}}\right)\left(\log_D p_i + \log_D \frac{\sum\limits_{i=1}^{n} D^{-l_i}}{D^{-l_i}}\right) \geq 0$$

or

$$\sum_{i=1}^{n} p_i \log_D p_i + \sum_{i=1}^{n} p_i \log_D\left(\frac{\sum\limits_{i=1}^{n} D^{-l_i}}{D^{-l_i}}\right)$$

$$-\sum_{i=1}^{n}\left(\frac{D^{-l_i}}{\sum\limits_{i=1}^{n} D^{-l_i}}\right)\log_D p_i \qquad (24)$$

$$-\sum_{i=1}^{n}\left(\frac{D^{-l_i}}{\sum\limits_{i=1}^{n} D^{-l_i}}\right)\log_D\left(\frac{\sum\limits_{i=1}^{n} D^{-l_i}}{D^{-l_i}}\right) \geq 0$$

If we take $l_1 = l_2 = ... = l_n = l$ in (24), we get

$$\log_D n - \frac{1}{n}\sum_{i=1}^{n}\log_D p_i - \frac{1}{n}\sum_{i=1}^{n}\log_D n \geq -\sum_{i=1}^{n} p_i \log_D p_i$$

or

$$-\sum_{i=1}^{n} p_i \log_D p_i + \frac{1}{n}\sum_{i=1}^{n}\log_D p_i \leq 0$$

Thus, we have

$$nH(P) + \varphi_1(p) \leq 0$$

which gives the relation between Shannon's [13] measure of entropy and Burg's [4] entropy.

*C. Kapur's [9] measure of directed divergence is given by*

$$D_\alpha(P,Q) = \frac{1}{1-\alpha}\log_D\left(\frac{\left(\sum\limits_{i=1}^{n} p_i q_i^{\alpha-1}\right)^\alpha}{\left(\sum\limits_{i=1}^{n} q_i^\alpha\right)^{\alpha-1}\sum\limits_{i=1}^{n} p_i^\alpha}\right) \geq 0 \qquad (25)$$

Putting $q_i = \dfrac{D^{-l_i}}{\sum\limits_{i=1}^{n} D^{-l_i}}$ in equation (25), we get

or

$$\frac{\alpha}{1-\alpha}\log_D \sum_{i=1}^{n} p_i D^{l_i(1-\alpha)}$$

$$+ \log_D \sum_{i=1}^{n} D^{-l_i\alpha} \geq \frac{1}{1-\alpha}\log_D \sum_{i=1}^{n} p_i^\alpha \qquad (26)$$

If we take $l_1 = l_2 = ... = l_n = l$ in (26), we get

$$\frac{\alpha}{1-\alpha}\log_D D^{l(1-\alpha)} + \log_D D^{-l\alpha} n$$

$$\geq \frac{1}{1-\alpha}\log_D \sum_{i=1}^{n} p_i^{\alpha}$$

or

$$\frac{1}{1-\alpha}\log_D \sum_{i=1}^{n} p_i^{\alpha} \leq \log_D n$$

or $R_r(P) \leq \log_D n$,

This is a well known inequality, already existing in the literature of information theory and has been proved by some other technique.

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