Low Density Parity Check Codes for the Relay Channel

Arnab Chakrabarti, *Student Member, IEEE,* Alexandre de Baynast, *Member, IEEE,*
Ashutosh Sabharwal, *Senior Member, IEEE,* and Behnaam Aazhang, *Fellow, IEEE*

***Abstract*— We propose Low Density Parity Check (LDPC) code designs for the half-duplex relay channel. Our designs are based on the information theoretic random coding scheme for decode-and-forward relaying. The source transmission is decoded with the help of side information in the form of additional parity bits from the relay. We derive the exact relationships that the component LDPC code profiles in the relay coding scheme must satisfy. These relationships act as constraints for the density evolution algorithm which is used to search for good relay code profiles. To speed up optimization, we outline a Gaussian approximation of density evolution for the relay channel. The asymptotic noise thresholds of the discovered relay code profiles are a fraction of a decibel away from the achievable lower bound for decode-and-forward relaying. With random component LDPC codes, the overall relay coding scheme performs within 1.2 dB of the theoretical limit.**

***Index Terms*— Communication channels, multiuser channels, information rates, channel coding, relays.**

I. INTRODUCTION

THIS paper presents Low Density Parity Check (LDPC) code designs for the half-duplex relay channel. We focus on a key question - from a coding theoretic perspective, what are the essential steps in building a relay coding scheme that approaches fundamental limits? Since the capacity of the relay channel is not known, we do not know what the optimal coding strategy is. However, the decode-and-forward protocol outperforms other relay protocols when the source-relay link is strong [1]–[4]; therefore, we invent an LDPC coding scheme that can approach the rates theoretically achievable with the decode-and-forward protocol.

The following are the four main contributions of this paper. First, inspired by the information theoretic random coding scheme for decode-and-forward half-duplex relaying [5], we propose a near-optimal LDPC coding scheme in which side information is conveyed through additional parity bits. The key challenge of relay code design lies in the utilization of side information from the relay to decode the source transmission. In addition to the aforementioned coding scheme [5], there exist other information theoretic decode-and-forward strategies [6], and schemes with practical advantages [7], [8], all of which basically utilize side information from the relay. The relay LDPC optimization technique presented in Section V is generic and can be used for any decode-and-forward relay coding scheme.

Second, we propose three simplifications to reduce complexity of encoding and decoding without significantly compromising performance. First, we observe that the gain of relaying over direct communication is maximum at low SNR, which motivates the use of binary modulation in conjunction with binary codes on each channel dimension. Second, we argue that the relay coding scheme in multiple access mode is a superposition of two fundamental extremes. In one, the source and the relay send identical messages, and combine their signals coherently at the destination. In the other, the source and relay send completely independent information. Any source-relay correlation can be achieved by a combination of these two cases, but the interesting observation is that excellent performance can be achieved if we can simply choose the better of these two schemes. Third, we argue that successive decoding is optimal in MAC mode of the relay channel, an observation that simplifies decoder design.

The graphs of LDPC matrices can be structured to emulate a random coding scheme of any rate. Moreover, their profiles can be optimized using the density evolution algorithm. The new challenge is that relay code design requires joint optimization of multiple constituent LDPC code profiles. Our third contribution is the derivation of relationships between the profiles of constituent codes. These relationships act as additional constraints in the profile optimization problem that is solved using density evolution.

In the optimization of constituent LDPC codes, several useful simplifications that can be made in implementing density evolution for single-user links (such as assuming concentrated check node distributions) no longer remain valid. The resulting increase in the complexity of density evolution poses a significant challenge. Our final contribution is to reduce the complexity of density evolution by adapting the Gaussian approximation of density evolution [9]–[13] to the relay channel. The Gaussian approximation reduces the infinite dimensional problem of tracking densities to a one-dimensional problem of tracking means that is readily addressed with linear programming tools [14].

The information theoretic relay channel was first studied in [15]. Shortly afterwards, several fundamental capacity results on relaying were published in [1]. After the initial interest, however, the idea of relaying received little attention

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The authors are with the Department of Electrical and Computer Engineering, Rice University, Houston, TX 77005, USA (e-mail: {arnychak, debaynas, ashu, aaz}@rice.edu).

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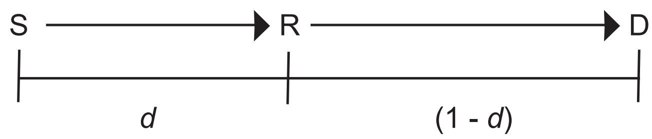

Fig. 1. Relay channel with source, relay and destination in a straight line.

for nearly two decades. Recent years have seen a renewed interest in relaying in the context of wireless networks [2], [5], [16]–[21]. It is worth noting that several of the above research contributions are based on the premise of half-duplex relaying [5], [19]–[22]. With significant advances in technology, the promise of relaying is very real. A large body of research is currently geared towards developing practical user-cooperation schemes to harvest the gains predicted by information theory. Solutions in this direction include [17], [22]–[32].

The remainder of this paper is organized is follows. In Section II, we describe the framework and assumptions on which our code design is based. Section III presents a coding theoretic interpretation of the information theoretic decode-and-forward random coding scheme. Section IV introduces the LDPC coding schemes. The problem of LDPC relay code profile optimization is addressed in Section V. We present numerically calculated noise thresholds for relay LDPC codes, and error rates of randomly generated LDPC codes in Section VI. Concluding remarks are presented in Section VII.

II. System Description

In the relay channel (see Fig. 1), the source (S) sends data to the destination (D), and in doing so it is aided by the relay (R), which does not have data of its own to transmit. In the particular case of a half-duplex relay channel, the relay cannot transmit and receive simultaneously in the same band.[1] We concentrate on time-division half-duplex relaying, where communication takes place over two time slots of (normalized) durations t and $t' = (1-t)$. In the first time slot, S transmits information, which is received by both R and D. We call this the broadcast (BC) mode of communication. In the second time slot, both S and R transmit information to D. We refer to this as the multiple-access (MAC) mode. These two modes are depicted in Fig. 2. In the rest of this paper, whenever we mention the relay channel, we will be implicitly referring to the time-division half-duplex relay channel.

Throughout this paper, we adopt the following conventions. We use X, V, W and Y to denote the source transmitted signal, the relay received signal, the relay transmitted signal, and the destination received signal respectively (see Fig. 2). Subscript 1 denotes BC mode, and 2 denotes MAC mode. Alphabets S, R, and D denote the source, relay and destination respectively, and SR channel, for instance, denotes the source-to-relay channel. With the above conventions, we introduce the

[1]Full-duplex operation is impractical because it requires shielding and accurate interference cancellation between transmitted and received signals that differ in power typically by more than 100 dB.

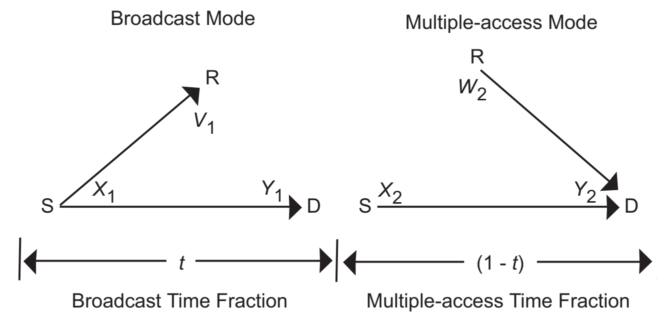

Fig. 2. Half-duplex relay modes.

following channel model

$$\begin{aligned} V_1 &= h_{SR}X_1 + N_{R_1}, \\ Y_1 &= h_{SD}X_1 + N_{D_1}, \\ Y_2 &= h_{SD}X_2 + h_{RD}W_2 + N_{D_2}. \end{aligned} \tag{1}$$

In the above model, h_{SR} is the source-to-relay channel realization, and N_{R_1} is the noise realization at the relay receiver in BC mode. The SR channel gain is denoted $\gamma_{SR} = |h_{SR}|^2$. The remaining expressions can be similarly interpreted. All noises are zero mean and unit variance Gaussians.

We consider static one-dimensional additive white Gaussian noise (AWGN) channels; however, extension to circularly symmetric AWGN channels is straightforward. Perfect global channel knowledge is assumed at all nodes.

An average global transmission power constraint is imposed on the nodes, denoted by the symbol Θ,

$$\Theta : tP_{S_1} + t'(P_{S_2} + P_{R_2}) \leq P, \tag{2}$$

where $P_{S_1} = E[X_1^2]$, for example, denotes the source transmission power in BC mode, and P represents the total system transmission power. Since noise power is normalized to unity, P is also the equivalent relay channel signal-to-noise ratio (SNR) in our plots. We choose a global power constraint as opposed to a per-node power constraint because it affords greater flexibility of power allocation, and leads to higher achievable rates.

We compare relaying with direct communication. For fair comparison, we ensure that the sum of the source and relay transmission powers in the relay channel equals the source transmission power for the direct link.

The relay position is described as follows. The distance between S and D is normalized to unity, and R is assumed to lie on the straight line joining S and D (see Fig. 1). The relay position, denoted d, represents its distance from the source. The collinearity of S, R and D does not affect the derivation of any of our results, but it enables a simple characterization of the relay position. In the above setting, the SD channel gain is $\gamma_{SD} = 1$, the SR gain is $\gamma_{SR} = \frac{1}{d^\alpha}$, and the RD gain is $\gamma_{RD} = \frac{1}{(1-d)^\alpha}$, where α is the channel attenuation exponent. We use $\alpha = 2$ in this paper.

III. Structure of Relay Channel Codes

In this section, we describe the information theoretic decode-and-forward coding scheme [5] that inspires the proposed codes, and motivate the simplifications leading to the LDPC code design discussed in the next section.

A. Achievable Rate and Code Structure

For the general half-duplex relay channel, the decode-and-forward protocol achieves the following rate [5]

$$R_{DF} = \sup_{0\le t\le 1} \min\{tI(X_1;V_1) + t'I(X_2;Y_2|W_2), tI(X_1;Y_1) + t'I(X_2,W_2;Y_2)\}. \quad (3)$$

The above mutual information expression can be evaluated for AWGN channels with Gaussian and binary inputs as shown in Appendix I. However, in the following discussion we will refer to the rates of component codes in mutual information terms because it conveys more intuition.

The following coding scheme achieves the aforementioned rate. A total of N symbols are transmitted of which tN are transmitted in BC mode, and the rest are sent in MAC mode.[2] The information at the source is first divided into two independent parts (ω, ν).

1) Encoding in BC mode: In BC mode, the source encodes ω to generate a tN symbol-long codeword $c_{SR_1} \in \mathcal{C}_{SR_1}$ with rate

$$R_{SR_1} = I(X_1;V_1). \quad (4)$$

2) Decoding the BC mode signals: The codeword c_{SR_1} is corrupted and received by both the relay and the destination. The relay decodes c_{SR_1} reliably since R_{SR_1} equals the capacity of the SR link. However, the destination cannot decode because the capacity of the SD link is less than that of the SR link.[3] The destination stores the received codeword for decoding at the end of MAC mode.

3) Encoding in MAC mode: The destination already has $tNI(X_1;Y_1)$ bits of information in the form of the undecodable noisy codeword c_{SR_1}. However, it still needs an additional $tN(I(X_1;V_1) - I(X_1;Y_1))$ bits to reliably decode c_{SR_1} [33]. These additional bits needed to decode c_{SR_1} are transmitted jointly by the source and the relay in a codeword $c_{RD_2} \in \mathcal{C}_{RD_2}$ of rate[4]

$$R_{RD_2} = \tfrac{t}{t'}(I(X_1;V_1) - I(X_1;Y_1)). \quad (5)$$

The second part of the information, ν, is also sent in MAC mode using a codeword $c_{SD_2} \in \mathcal{C}_{SD_2}$ to utilize the remaining capacity of the multiple access channel constituted by the source and relay as the two transmitters and the destination as the receiver. This information is sent by the source alone, since the relay does not have access to new information. The amount of new information is bounded by the capacity region of the multiple-access channel, and this information is therefore sent at a rate

$$R_{SD_2} = \min\{I(X_2,W_2;Y_2) - \tfrac{t}{t'}(I(X_1;V_1) - I(X_1;Y_1)), I(X_2;Y_2|W_2)\}. \quad (6)$$

[2]We assume tN to be an integer.

[3]It can be inferred from (3) that decode-and-forward relaying outperforms direct communication only if the SR link is better than the SD link.

[4]There is a slight abuse of notation here. The codeword c_{RD_2} is, in general, sent jointly by S and R to D, and not by R alone to D as the name suggests.

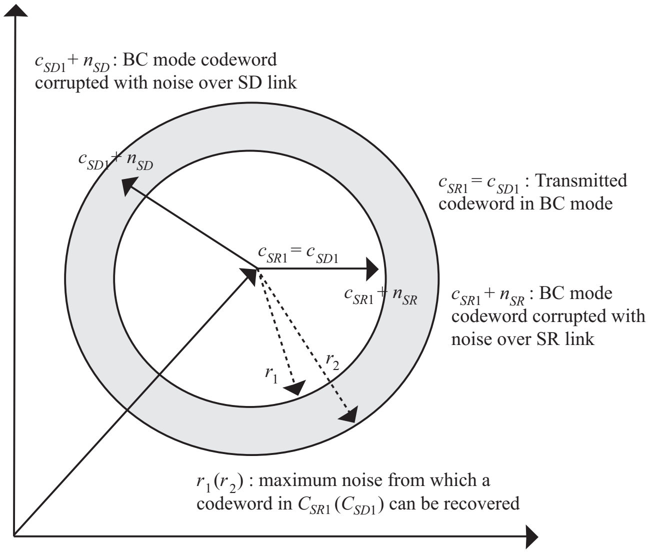

Fig. 3. Coding for decode-and-forward: S transmits codeword $c_{SR_1} = c_{SD_1}$ in BC mode. The inner circle denotes the maximum noise from which a codeword in $\mathcal{C}_{SR_1}$ can be recovered. The received signal at R is inside this circle, therefore R can decode correctly, but the signal at D is not decodable. In MAC mode, S and R send additional information to D about c_{SR_1}. Consequently, the same codeword c_{SD_1} now effectively belongs to a lower rate code $\mathcal{C}_{SD_1}$, and can be recovered from more noise (shown by the outer circle) at D.

4) Decoding at the end of MAC mode: The destination first decodes the codewords c_{RD_2} and c_{SD_2} transmitted in MAC mode. These codes are transmitted at rates belonging to the capacity region of the MAC. It is known that the corner points and the sides of the capacity region of the two-user multiple-access channel are achievable by successive decoding (also known as onion peeling, stripping, or superposition coding) with a pair of codes [34], [35]. However, to achieve a general point on the capacity region requires either joint decoding, time-sharing [36] or rate-splitting using at least three codes [37]. We will argue in Section III-B.3 that the source and relay rates in MAC mode correspond to a point on the multiple-access capacity region that can be achieved by a pair of single-user codes.

After decoding c_{SD_2} and c_{RD_2}, the destination can decode the corrupted codeword c_{SR_1} from BC mode using the information carried by c_{RD_2} as side information. For decoding c_{SR_1}, the destination treats it as a codeword $c_{SD_1} \in \mathcal{C}_{SD_1}$ of rate

$$R_{SD_1} = I(X_1;Y_1), \quad (7)$$

lower rate because of the side information carried by c_{RD_2}. For example, if all codes were binary codes, then the information bits in c_{RD_2} would act as additional parity information for c_{SR_1}. The use of side information is explained with the help of a diagram in Fig. 3.

B. Simplifications for Practical Code Design

This section is devoted to three observations that simplify relay code design in practice. First, we note that the gain due to relaying is most prominent at low SNRs and negligible at high SNRs, which motivates the use of binary modulation. Second, although there is an optimum value of correlation between

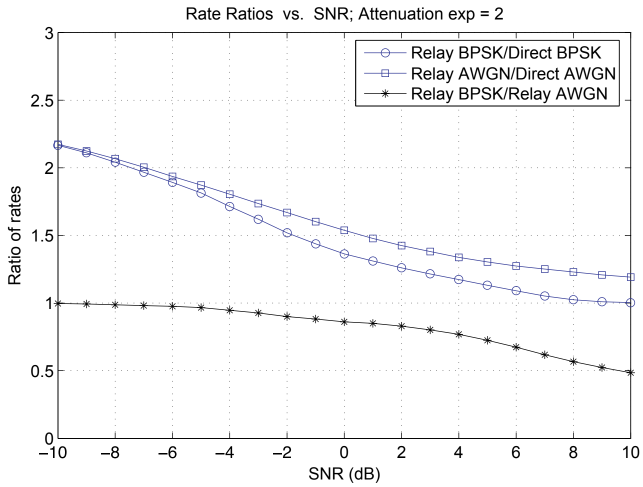

Fig. 4. Ratios of Rates vs. SNR(dB) for direct and relay channels.

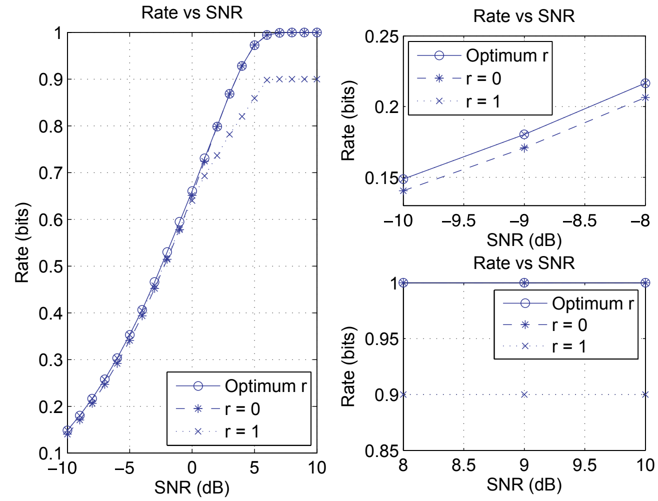

Fig. 5. Achievable rates for the relay channel with $r = 0, 1$ and the optimum value of r ($d = 0.5$). The subplots on the right show the behavior at low and high SNRs.

source and relay codewords in MAC mode that maximizes the achievable rate for Gaussian relay channels, we observe that the codebooks can be either completely correlated or completely independent without significant rate loss. Last, we argue that the source and relay rates in MAC mode correspond to a point on the capacity region of the multiple-access channel that can be achieved by a successive interference canceling decoder at the destination.

1) Motivation for binary coding: At high SNR, the achievable rate of decode-and-forward relaying on Gaussian channels exceeds that of direct communication only by a constant independent of the SNR (proved in Appendix II). Resultantly, the ratio of the decode-and-forward rate to the single-user capacity approaches unity at high SNR. The relaying gain is maximum at low SNR, where binary modulation for each channel dimension is near optimum.

To illustrate, Fig. 4 compares the relay rate with that of direct communication for Gaussian as well as BPSK signaling (rate expressions given in Appendix I) when the relay is at $d = 0.5$. The rate of BPSK relaying is also plotted as a fraction of the rate of AWGN relaying at different SNRs. The plot is in agreement with our claim that relaying is most beneficial in the low SNR regime, where BPSK achieves a significant fraction of the AWGN relay rate.

2) Insensitivity to MAC mode correlation: The achievable rate of decode-and-forward relaying over AWGN links is maximized when the source and relay codebooks are optimally correlated in MAC mode (see Appendix I). The correlation r reflects the fact that the source and the relay have common information since the relay has decoded the ω portion of the total information in BC mode.

Arbitrary correlation between source and relay transmissions in MAC mode can be achieved using a pair of independent binary codebooks. Let c_{RD_2}, c_{SD_2} be a pair of binary codewords from independent codebooks $\mathcal{C}_{RD_2}, \mathcal{C}_{SD_2}$ respectively. If the relay sends c_{RD_2}, and the source sends $rc_{RD_2} + (1-r)c_{SD_2}$, then the transmissions have correlation r. For the purpose of decoding, the destination can proceed as if there were two transmitters sending independent codewords c_{RD_2}, c_{SD_2} with transmission powers appropriately adjusted.

It is clear that $r = 0, 1$ are two fundamental extremes, and all intermediate correlations can be achieved by superposing the two in the right proportion. When $r = 1$, the source and the relay send identical information in MAC mode, consequently the source sends nothing new and there is no code $\mathcal{C}_{SD_2}$. For $r = 0$, the source and the relay send independent information in MAC mode, meaning that the source sends only new information through the codeword c_{SD_2}, whereas the relay alone sends c_{RD_2} to help the destination decode the BC mode codeword c_{SR_1}. The $r = 0$ scheme is sometimes called partial decode-and-forward in relay literature.

An interesting observation made in this context is that the achievable rate is fairly insensitive to r, and in particular, if we can choose the better of $r = 0, 1$, then the rate loss in comparison to a system that uses optimum correlation is negligible [38]. The effect of correlation on the achievable rate is shown in the plot in Fig. 5. Guided by the above observation, we limit ourselves to designing codes for the limiting cases of $r = 0, 1$.

3) Optimality of successive decoding in MAC mode: In our discussion of the information theoretic relay coding scheme in Section III-A, we mentioned that in MAC mode, the source transmits additional information ν at a rate permitted by the capacity of the multiple-access channel. This rate is the smaller of $I(X_2; Y_2|W_2)$ and $I(X_2, W_2; Y_2) - \frac{t}{t'}(I(X_1; V_1) - I(X_1; Y_1))$. Operation at the corner point of the capacity region corresponds to equality of the two aforementioned rates, which is the same as equating the two arguments of the $\min(.)$ function in (3). The above equality can be proved with the following power constraint

$$\begin{aligned} P_{S_2} + P_{R_2} &\leq P, \\ P_{S_1} &\leq P, \end{aligned} \tag{8}$$

which is a special case of (2). When (8) is true, the input distributions, and consequently the mutual information expressions, are independent of t. Since $I(X_2; Y_2|W_2) \leq I(X_2, W_2; Y_2)$, and $I(X_1; V_1) \geq I(X_1; Y_1)$ for decode-and-forward relaying, we conclude that the best choice of t equates the two arguments of the $\min(.)$ function.

Numerical results indicate that the aforementioned equality holds even for the more general power constraint (2). Therefore, from this point on, we will assume that

$$\begin{aligned} R_{SD_2} &= I(X_2, W_2; Y_2) - \frac{t}{t'}(I(X_1; V_1) - I(X_1; Y_1)) \\ &= I(X_2; Y_2 | W_2). \end{aligned} \quad (9)$$

IV. LDPC Code Design

We are now ready to present the binary LDPC coding schemes for $r = 0, 1$. The two schemes differ only in MAC mode. We first describe the full scheme for $r = 1$, and then explain what is different for $r = 0$. We use the same names for the component LDPC codes as we did in describing the general relay coding scheme of Section III-A, to ensure that the role of each component code is clear. The reader can also apply the interpretation of Fig. 3 to the LDPC coding schemes.

A. LDPC Code Design for $r = 1$

In BC mode, the source S uses an LDPC code $\mathcal{C}_{SR_1}$ with an $Nt \times Nt(1 - R_{SR_1})$ parity check matrix to transmit information. The relay decodes this codeword. The destination D stores it for future decoding. In MAC mode, S and R use the BC mode codeword as the basis for cooperation. Both nodes multiply the BC mode codeword with an $Nt \times Nt(R_{SR_1} - R_{SD_1})$ matrix to generate $Nt(R_{SR_1} - R_{SD_1})$ additional parity bits (side information to help the destination). These additional parity bits are then channel coded using an LDPC code $\mathcal{C}_{RD_2}$ with an $Nt' \times Nt'(1 - R_{RD_2})$ parity check matrix and transmitted from both S and R in a phase synchronized manner to the destination. The bits communicated by $\mathcal{C}_{RD_2}$, in addition to the parity bits of the original code $\mathcal{C}_{SR_1}$, form a code $\mathcal{C}_{SD_1}$ of lower rate R_{SD_1} that is decodable by D. Fig. 6 shows the LDPC code structure for $r = 1$ (with the switch in the figure open, implying the absence of code $\mathcal{C}_{SD_2}$ in MAC mode).

B. LDPC Code Design for $r = 0$

The BC mode remains unchanged. In MAC mode, the source and the relay transmit independent information. The relay first generates additional parity bits from the BC mode codeword by multiplying with a $Nt \times Nt(R_{SR_1} - R_{SD_1})$ matrix (same as for $r = 1$). It then uses an LDPC code $\mathcal{C}_{RD_2}$ with an $Nt' \times Nt'(1 - R_{RD_2})$ parity check matrix to encode and transmit the additional parity information in MAC mode. The information carried by this codeword enables decoding of the BC mode codeword at the end of MAC mode. The source, in MAC mode, uses an LDPC code $\mathcal{C}_{SD_2}$ with an $Nt' \times Nt'(1 - R_{SD_2})$ parity check matrix to send new information to the destination. At the end of MAC mode, D uses successive decoding to recover both the additional parity information and the new source information. Finally, the additional parity bits received in MAC mode are used to decode the received BC mode codeword at D. Fig. 6 shows a block diagram of the overall coding scheme for $r = 0$ (with the switch in the figure closed).

V. Code Profile Optimization

The main challenge in the proposed coding schemes is that of jointly designing the profiles of the two codes $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$. We start with a brief introduction to LDPC codes, following which, we will solve the joint code profile optimization problem.

A. Introduction to LDPC Codes

A binary LDPC code is a linear block code with a sparse binary parity-check matrix. This $n \times m$ parity check matrix can be represented by a bipartite graph with n variable nodes corresponding to rows (bits in the codeword) and m check nodes corresponding to columns (parity check equations). A one in a certain row and column of the parity check matrix denotes an edge between the respective variable and check node in the graph, whereas a zero indicates the absence of an edge.

An LDPC code *ensemble* is characterized by its variable and check degree distributions (or profiles) $\lambda = [\lambda_2 ... \lambda_{d_v}]$ and $\rho = [\rho_2 ... \rho_{d_c}]$ respectively, where λ_i (ρ_i) denotes the fraction of *edges* connected to a variable (check) node of degree i, and d_v (d_c) is the maximum number of edges connected to any variable (check) node. An equivalent representation of LDPC code profiles uses generating functions

$$\lambda(x) = \sum_{i=2}^{d_v} \lambda_i x^{i-1}, \quad (10)$$

$$\rho(x) = \sum_{i=2}^{d_c} \rho_i x^{i-1}. \quad (11)$$

The design rate of an ensemble is given in terms of $\lambda(x)$ and $\rho(x)$ by

$$R = 1 - \frac{m}{n} = 1 - \frac{\int_0^1 \rho(x)dx}{\int_0^1 \lambda(x)dx}. \quad (12)$$

Our formulation will require an equivalent representation of the degree profiles from a node perspective. To distinguish, we will use the superscript N for node, and call $\lambda^N(x)$ the variable *node* degree distribution as contrasted to $\lambda(x)$, which is the variable *edge* degree distribution. The following equation relates λ_i^N, the fraction of variable nodes of degree i, to λ_i

$$\lambda_i^N = \frac{\lambda_i / i}{\int_0^1 \lambda(x)dx}. \quad (13)$$

LDPC codes can be decoded by a variety of message passing algorithms, of which we will consider only belief propagation [39]. The density evolution algorithm tracks message densities over successive decoding iterations, and discovers a noise threshold below which decoding succeeds with high probability for any code within a given ensemble. Consequently, density evolution can be used to search for code profiles with excellent noise thresholds. Unfortunately, tracking entire densities over thousands of iterations is computationally intensive. To reduce the computational burden of threshold determination, it is common to approximate the messages as Gaussians and track their means. Gaussian approximation, therefore reduces the infinite dimensional problem of tracking

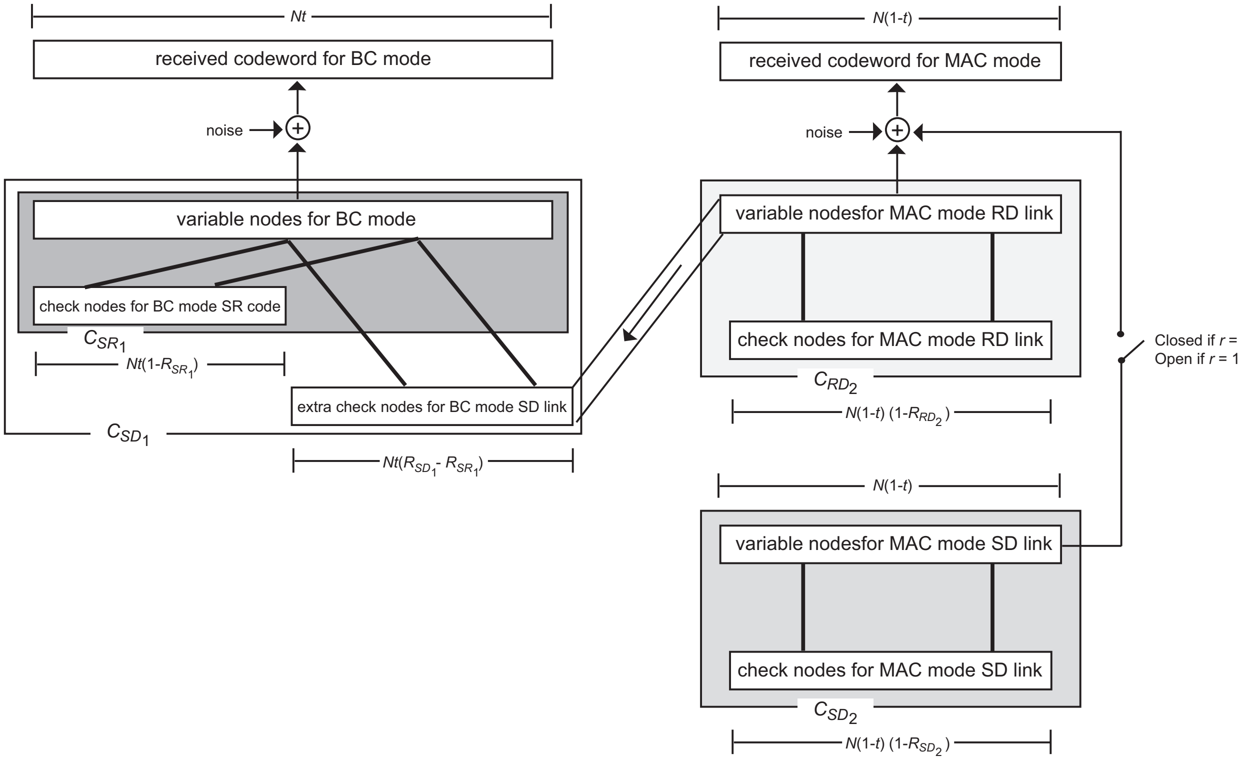

Fig. 6. LDPC code structure for $r = 0, 1$.

entire densities to a one-dimensional problem of tracking means. We use the Gaussian approximation to find good code profiles and to obtain approximate thresholds for relay channel codes. For brevity, we will present the procedure for determining the threshold without the steps in its derivation, which can be found in [9]. For a given $\rho(x)$ and $\lambda(x)$, and given channel noise variance σ_c^2, decoding succeeds in the limit of infinite blocklength and infinite iterations iff

$$r > h(s,r) \qquad \forall r \in (0, \phi(s)), \tag{14}$$

where

$$h(s,r) = \sum_{i=2}^{d_v} \lambda_i h_i(s,r), \tag{15}$$

$$h_i(s,r) = \phi\left(s + (i-1)\sum_{j=2}^{d_c} \rho_j \phi^{-1}\big(1-(1-r)^{j-1}\big)\right),$$

with $s = 2/\sigma_c^2$, where σ_c^2 is the channel noise variance, and

$$\phi(x) = \begin{cases} 1 - \frac{1}{\sqrt{4\pi x}} \int_{\mathbb{R}} \tanh \frac{u}{2} e^{-\frac{(u-x)^2}{4x}} du & \text{if } x > 0 \\ 1 & \text{if } x = 0. \end{cases} \tag{16}$$

For numerical purposes, we use the following approximation [9] for the function ϕ

$$\phi(x) \simeq \begin{cases} e^{(-0.4527x^{0.86}+0.0218)} & \text{if } x \in [0,10] \\ \sqrt{\frac{\pi}{x}} e^{-\frac{x}{4}} \left(1 - \frac{20}{7x}\right) & \text{if } x > 10. \end{cases} \tag{17}$$

We use (14) to find the largest σ_c^2 for which decoding is successful. In addition, we also satisfy the stability condition (see [9], [40] for details)

$$\lambda_2 < e^{1/2\sigma_c^2} / \sum_{j=2}^{d_c} \rho_j (j-1). \tag{18}$$

Finding the optimal degree distribution for a given rate is a search for the $(\lambda(x), \rho(x))$ pair that yields the largest noise threshold. For single-user codes, it has been demonstrated [9], [40], [41] that good check node distributions are concentrated, i.e. all parity check nodes should have equal or nearly equal degrees. The variable node distribution, on the other hand, is not so easily characterized. Therefore, the search for a good $(\lambda(x), \rho(x))$ pair for a single-user code is carried out in practice by choosing several concentrated $\rho(x)$ and finding the best $\lambda(x)$ for each, followed by selecting the $\rho(x)$ for which the best noise threshold was obtained. An advantage of this approach is that the best $\lambda(x)$ for a given $\rho(x)$ can be found by linear programming [9].

The following is an outline of the linear programming technique. We are interested in finding $\lambda(x)$ that maximizes the noise variance σ_c^2 for a given rate and $\rho(x)$. Instead, we solve the equivalent problem of finding $\lambda(x)$ that maximizes the rate for a given noise variance and $\rho(x)$, and slowly increasing the variance until the design rate is barely achievable. From (12), we see that for fixed $\rho(x)$, maximizing rate is equivalent to maximizing $\int_0^1 \lambda(x)dx = \sum_{i=2}^{d_v} \frac{\lambda_i}{i}$. The constraints are $\lambda(1) = 1$, and (14), which are both linear. This linear program with inequality constraints can be solved quickly and accurately using available optimization tools.

We conclude our brief summary of LDPC codes, density evolution and its Gaussian approximation here. The interested reader can find detailed discussions on these in [9]–[13], [39]–[43].

B. Relay Code Profile Optimization

The novel challenge in the context of half-duplex relaying is identical for both correlations $r = 0$ and $r = 1$, and it requires building two LDPC codes $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$ that are both excellent single-user codes of disparate rates R_{SR_1} and R_{SD_1} respectively, such that the bipartite graph of $\mathcal{C}_{SR_1}$ is a subgraph of $\mathcal{C}_{SD_1}$. The LDPC codes are represented by rectangles with rounded edges in Fig. 6. Both $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$ must produce codewords of length tN symbols. Therefore, $\mathcal{C}_{SR_1}$ has $tN(1-R_{SR_1})$ parity check nodes, and $\mathcal{C}_{SD_1}$ has $tN(1-R_{SD_1})$ parity nodes that are a superset of those belonging to $\mathcal{C}_{SR_1}$. As a result, the check degree distributions must satisfy a relationship given by the following simple theorem.

Theorem 5.1: If $\mathcal{C}_{SR_1}$ has a check node degree distribution $\rho^N_{SR_1}(x) = \sum_{i=2}^{d_{c1}} \rho^N_{SR_1,i} x^{i-1}$, and $\mathcal{C}_{SD_1}$ has a check node degree distribution $\rho^N_{SD_1}(x) = \sum_{i=2}^{d_{c2}} \rho^N_{SD_1,i} x^{i-1}$, then the following relationships must hold

$$\begin{aligned} &\frac{(1-R_{SR_1})}{(1-R_{SD_1})}\rho^N_{SR_1,i} \le \rho^N_{SD_1,i} \\ &\forall i = 2,3,...,\max(d_{c1},d_{c2}). \end{aligned} \quad (19)$$

Proof: *The number of check nodes of degree i in $\mathcal{C}_{SR_1}$ is $Nt(1-R_{SR_1})\rho^N_{SR_1,i}$, whereas the number of check nodes of degree i in $\mathcal{C}_{SD_1}$ is $Nt(1-R_{SD_1})\rho^N_{SD_1,i}$. Since $\mathcal{C}_{SD_1}$ is formed by adding check nodes to $\mathcal{C}_{SR_1}$, the latter must equal or exceed than the former, which gives us the result.* ◇

The relationship between variable nodes and their degree distributions is stated in the following theorem.

Theorem 5.2: If $\mathcal{C}_{SR_1}$ has a variable node degree distribution $\lambda^N_{SR_1}(x) = \sum_{i=2}^{d_{v1}} \lambda^N_{SR_1,i} x^{i-1}$, and $\mathcal{C}_{SD_1}$ has a variable node degree distribution $\lambda^N_{SD_1}(x) = \sum_{i=2}^{d_{v2}} \lambda^N_{SD_1,i} x^{i-1}$, then the following relationships must hold

$$\begin{aligned} &\sum_{i=j}^{\max(d_{v1},d_{v2})} \lambda^N_{SR_1,i} \le \sum_{i=j}^{\max(d_{v1},d_{v2})} \lambda^N_{SD_1,i} \\ &\forall j = 2,3,...,\max(d_{v1},d_{v2}). \end{aligned} \quad (20)$$

Proof: *The proof is a consequence of Hall's marriage theorem [44], which is stated as follows:*
Let $S = \{S_1, S_2, ..., S_m\}$ be a finite collection of finite sets. Then there exists a system of distinct representatives of S if and only if the following condition holds for all $T \subseteq S$

$$|\cup T| \ge |T|, \quad (21)$$

where $|.|$ represents cardinality.

Let us number the variable nodes $1,2,...,m$ in arbitrary order. Let d_i be the degree of variable node i in $\mathcal{C}_{SR_1}$, and let S_i denote the set of variable nodes in $\mathcal{C}_{SD_1}$ with degree $\ge d_i$. Note that in this way, if $d_i = d_j$, then $S_i = S_j$. When more check nodes and their connecting edges are added to $\mathcal{C}_{SR_1}$ to form $\mathcal{C}_{SD_1}$, then a variable node of degree k in $\mathcal{C}_{SR_1}$ maps to a node of degree $\ge k$ in $\mathcal{C}_{SD_1}$. In other words, node i in $\mathcal{C}_{SR_1}$ would be mapped to a node in S_i. Since both $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$ have the same variable nodes, there exists the trivial bijective identity mapping between variable nodes in the two graphs. Therefore, $S = \{S_1, S_2, ..., S_m\}$ has a system of distinct representatives, the representatives being the variable nodes in $\mathcal{C}_{SD_1}$ to which nodes $1 = 1,2,...,m$ in $\mathcal{C}_{SR_1}$ are mapped. Condition (20) now follows from (21). ◇

Equations (19) and (20) are necessary conditions relating the profiles of $\mathcal{C}_{SD_1}$ and $\mathcal{C}_{SR_1}$. In addition, starting with an arbitrary graph with the profile for $\mathcal{C}_{SR_1}$ and adding extra check nodes and edges, it is possible to obtain a graph with the profile of $\mathcal{C}_{SD_1}$. To that extent the above conditions may also be considered sufficient.

We are now ready to summarize the constraints of density evolution for the joint design of $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$. In addition to inequalities (19) and (20), we have the usual constraints

$$\begin{aligned} \lambda_{SR_1}(1) = \lambda_{SD_1}(1) = 1, \\ \rho_{SR_1}(1) = \rho_{SD_1}(1) = 1. \end{aligned} \quad (22)$$

An apparent difference from the single-user case is that now we have two equivalent noise variances that are related by the relative channel strengths

$$\sigma^2_{SD_1} = \frac{\gamma_{SR}}{\gamma_{SD}}\sigma^2_{SR_1}, \quad (23)$$

where $\sigma^2_{SD_1}$ is the noise variance for $\mathcal{C}_{SD_1}$ and $\sigma^2_{SR_1}$ is the noise variance for $\mathcal{C}_{SR_1}$. We treat only $\sigma^2_{SD_1}$ as a variable, since it completely determines $\sigma^2_{SR_1}$.

The set of inequality constraints (14) must be satisfied for both $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$, as in the single-user case. Finally, the stability condition (18) should be fulfilled.

With the above constraints, it is possible to perform density evolution and search for good degree distributions for $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$ for a given pair of rates. However, this search is computation intensive, more so because now we must find a pair of codes instead of one, making the search space much larger. Our next step is to modify the Gaussian approximation procedure to make it applicable in the relay setting.

C. Gaussian Approximation for the Relay Channel

Our aim is to use a Gaussian approximation to pose the search for good code profiles as a linear program using the means of the messages instead of entire densities [9]. However, extending the above procedure to relay channels is not trivial for the following reasons.

First, usually the rates of $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$ are significantly different. Consequently, the average check degrees for the two codes to be individually good are also not close. As a result, we do not have the luxury of assuming that ρ_{SR_1} and ρ_{SD_1} are both concentrated, which was the case for single-user LDPC codes. To reduce the exploration space for our search, we restrict the search for good codes to ρ_{SR_1} profiles concentrated at a single degree j and ρ_{SD_1} supported on two degrees i and j only with $i < j$

$$\begin{aligned} &\rho_{SR_1}(x) = x^{j-1}, \\ &\rho_{SD_1}(x) = \frac{ax^{i-1} + bx^{j-1}}{(a+b)}, \\ &a = i(R_{SR_1} - R_{SD_1}), \qquad b = j(1 - R_{SR_1}). \end{aligned} \quad (24)$$

There is no analytical justification for reducing our search space in this way, but the intuition is that we need at least two different check degrees for both $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$ to perform well. The thresholds of the codes that we discover in this way indicate that good code pairs can be found with the above constraint.

Second, (20) is not linear in the variable edge degrees but is linear in the node degrees. Fortunately, the other constraints remain linear when stated in terms of node degrees instead of edge degrees. Therefore, it is natural to pose the code profile optimization in terms of the distribution of variable node degrees.

Third, for single-user LDPC codes, the optimization was easily posed as a rate maximization problem. But now, there are two code rates. To tackle this, we fix one of the LDPC code rates to its design rate (we fix R_{SR_1}), and maximize the other (R_{SD_1}). By fixing R_{SR_1}, we obtain the following equality constraint

$$R_{SR_1} = 1 - \frac{\int_0^1 \rho_{SR_1}(x)dx}{\int_0^1 \lambda_{SR_1}(x)dx}. \quad (25)$$

For the sake of clarity, we summarize all the constraints and the objective of our linear program below. We assume that the check degree distributions are given by (24). Our objective is to find the maximum noise variance $\sigma^2_{SD_1}$, for which the achievable R_{SD_1} equals or exceeds the target value. As in the single-user case, we equivalently maximize the rate R_{SD_1} for a given $\sigma^2_{SD_1}$. Since

$$\begin{aligned} R_{SD_1} &= 1 - \frac{\int_0^1 \rho_{SD_1}(x)dx}{\int_0^1 \lambda_{SD_1}(x)dx} \\ &= 1 - \left(\int_0^1 \rho_{SD_1}(x)dx\right)\left(\sum_{i=2}^{d_v} i\lambda_i^N\right), \quad (26) \end{aligned}$$

our objective of maximizing R_{SD_1} is equivalent to that of minimizing $\sum_{i=2}^{d_v} i\lambda_i^N$, where d_v is the maximum allowed variable node degree in the search. The following are the remaining constraints, stated in terms of variable *node* degrees. First, we have

$$\lambda^N_{SR_1}(1) = \lambda^N_{SD_1}(1) = 1. \quad (27)$$

The corresponding equality for check degrees is automatically satisfied by (24). The stability condition (18) can be restated in terms of variable node degrees as follows

$$\frac{2\lambda^N_{\zeta,2}}{\sum_{i=2}^{d_v} i\lambda^N_{\zeta,i}} < \frac{e^{1/2\sigma^2_\zeta}}{\sum_{j=2}^{d_c} \rho_{\zeta,j}(j-1)}, \quad {\scriptstyle \zeta=SR_1,SD_1}, \quad (28)$$

which is a linear inequality constraint in λ^N for fixed ρ. Additionally, relation (14) imposes the following constraints

$$\sum_{i=2}^{d_v} i\lambda^N_{\zeta,i}(h_i(s,r) - r) < 0, \ \forall r \in (0, \phi(s)), \ {\scriptstyle \zeta=SR_1,SD_1}. \quad (29)$$

Inequalities (19) and (20) must be satisfied. Finally, (25) yields the following equality constraint

$$R_{SR_1} = 1 - \left(\int_0^1 \rho_{SR_1}(x)dx\right)\left(\sum_{i=2}^{d_v} i\lambda^N_{SR_1,i}\right). \quad (30)$$

VI. Numerical Results

Using the density evolution technique discussed in Section V, we calculate thresholds for the relay code profiles for $r = 0, 1$. Since we are also interested in the performance of the optimized LDPC codes for finite block size, we present bit error rate (BER) simulation results with randomly generated component LDPC codes for a single point of SNR.

A. Asymptotic Noise Thresholds

For a given total power, we numerically calculate the rates of constituent codes, optimal values of the BC mode time fraction t, MAC mode correlation r, as well as the power allocation for the source and the relay in BC and MAC modes. We then design LDPC codes with appropriate rates. The code profiles for $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$ are jointly optimized using the procedure in Section V-C. The code $\mathcal{C}_{RD_2}$ is a single-user code when $r = 1$, for which we use a good degree profile from [45]. When $r = 0$, we use a pair of codes $\mathcal{C}_{RD_2}$ and $\mathcal{C}_{SD_2}$ with good single-user profiles of the respective rates [45].

To compare the performance of the overall relay coding scheme with the theoretical bound, we perform the following steps. Replacing the mutual information expressions in (3) with rates of the corresponding LDPC codes, we obtain the achievable rate of the LDPC coding scheme. The total power necessary to achieve the above rate asymptotically is obtained from the noise thresholds of the component LDPC codes. Here, by noise threshold, we mean the maximum noise standard deviation that an LDPC code can recover from when the noiseless coded binary symbols are ± 1. If T_ζ denotes the noise threshold of $\mathcal{C}_\zeta$, where $\zeta \in \{SR_1, SD_1, RD_2, SD_2\}$, then the total transmission power (assuming unit noise power at the receivers) is

$$\begin{aligned} P_{r=0} &= \frac{t}{T^2_{SD_1}\gamma_{SD}} + \frac{t'}{T^2_{RD_2}\gamma_{RD}}\left(1 + \frac{T^2_{RD_2}\gamma_{RD}}{T^2_{SD_2}\gamma_{SD}} + \frac{1}{T^2_{SD_2}}\right), \\ P_{r=1} &= \frac{t}{T^2_{SD_1}\gamma_{SD}} + \frac{t'}{T^2_{RD_2}}\left(\frac{1}{\gamma_{RD}+\gamma_{SD}}\right), \quad (31) \end{aligned}$$

where we use the fact that optimal power allocation in MAC mode is governed by the principle of maximal ratio combining for $r = 1$, and we assume successive decoding for $r = 0$. The absence of T_{SR_1} in the above expressions is due to the fact that the thresholds T_{SD_1} and T_{RD_1} are related by (23).

In Fig. 7, we plot the limiting performance (rate vs. E_b/N_0) of the LDPC coding schemes calculated with the above procedure, and compare them with the theoretical performance for binary signaling. For a maximum variable node degree $d_v = 25$, the thresholds of the overall relay coding scheme are approximately 0.4 dB from the decode-and-forward bound if we choose the better of $r = 0, 1$. The reader is reminded that the thresholds of some of the component codes have been calculated using the Gaussian approximation, therefore they are not precise. Since it has been shown that the error in using the Gaussian approximation is small [9], the profiles are asymptotically good. However, it is challenging to accurately estimate the gap to the theoretical limit, since the gap may be of the same order as the approximation error.

B. Bit and Frame Error Rates of Randomly Generated Codes

BER simulations are performed for $P = -1$ dB where $t = 0.65$ yields the best rate; therefore, maintaining the BC mode time fraction, we choose the BC mode codeword to be $1.3e5$ bits and the MAC mode codewords to be $0.7e5$ bits

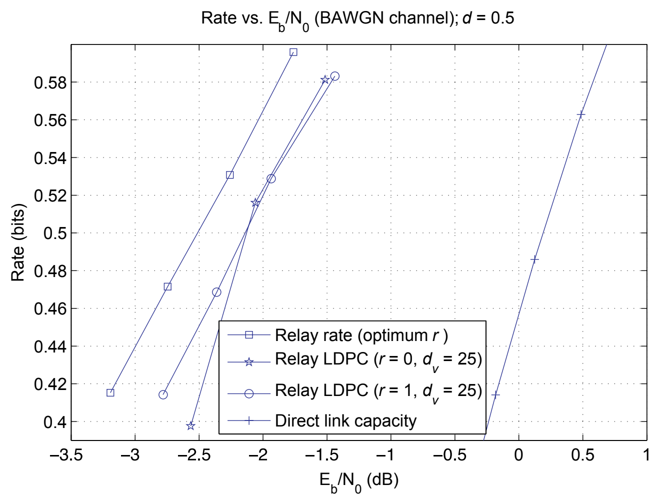

Fig. 7. Rate vs. E_b/N_0 (d=0.5,d_v=25). Theoretical limits and LDPC performance based on thresholds.

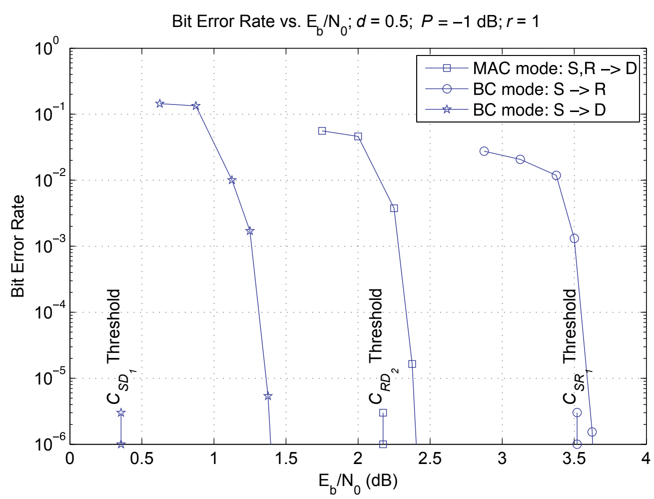

Fig. 8. BER vs. E_b/N_0 for component LDPC codes($P = -1dB, r = 1$). Theoretical limits and noise thresholds are indicated on the x-axis.

long. We show results for 300 decoding iterations. The codes are randomly generated from their profiles, the same profiles for which the overall thresholds are given in Fig. 7. No cycle removal is performed with the exception of removing double edges between node pairs.

Fig. 8 plots the BER vs. E_b/N_0 for each of the three constituent codes. The gap to the asymptotic threshold is nearly 1 dB for the code $\mathcal{C}_{SD_1}$, whereas it is significantly less for single-user codes of comparable profiles. The reason is that $\mathcal{C}_{SD_1}$ does not have a concentrated check degree. It is known that concentrated check degrees are decoded in the fewest iterations, whereas codes with large variance of check degrees take much longer to achieve the same decoding performance [46].

For the case of $r = 0$, the BER performance of the component codes is similar to that of the $r = 1$ codes. The only difference is that there are two codes $\mathcal{C}_{SD_2}$ and $\mathcal{C}_{RD_2}$ in MAC mode, and the latter is decoded treating the former as interference. The two MAC codes should be jointly designed for optimal performance, but discussing their joint design is beyond the scope of this paper. Using a pair of

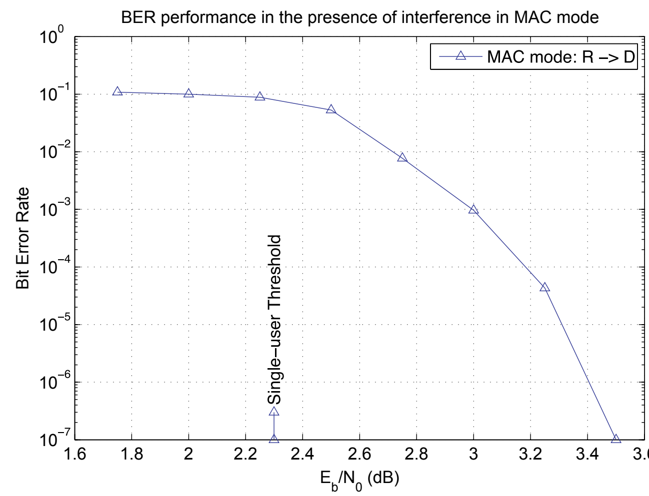

Fig. 9. Performance of the MAC code $\mathcal{C}_{RD_2}$ after 100 decoding iterations with interference from $\mathcal{C}_{SD_2}(P = -1dB, r = 1)$. The single-user noise threshold is indicated on the x-axis.

codes with good single-user profiles often incurs little loss of performance. Fig. 9 shows the BER performance of decoding $\mathcal{C}_{RD_2}$ in the presence of interference. The interference is modeled by a random bit sequence. Once $\mathcal{C}_{RD_2}$ has been decoded and subtracted out, the performance of $\mathcal{C}_{SD_2}$ will be the same as that of a single-user code in Gaussian noise.

In the proposed relay coding scheme, the relay can forward information only after successful decoding, which requires the entire codeword to be correct. Therefore, the component codes must also have excellent frame (codeword) error rates (FERs). Random LDPC codes tend to have excellent BER, but their FER is not always good due to the presence of small cycles in the graph. The problem can be addressed either by optimizing the parity check matrix to eliminate cycles [47]–[49], or by using a very high rate outer code, such as a BCH or Reed-Solomon code to bring the FER down to zero with a small rate penalty. The end-to-end FER performance of the relay coding scheme, in conjunction with a $(1023, 993)$ BCH outer code of rate 0.97 is shown in Figure 10. Here, by frame we mean an entire block of information that travels from the source to the destination in a single transmission comprising both BC and MAC modes. The relay code performs significantly better than the capacity of the single-user link.

VII. Conclusions and Future Directions

We have presented LDPC code designs for decode-and-forward relaying. Several interesting questions arise in the context of this work. First, why is there a large gap between the asymptotic and the simulated performance of the LDPC codes? It is likely that convergence to the threshold is extremely slow. We believe that the check node profile holds clues to the answer; an answer that may yield new insights into the structure of optimal profiles and decoding algorithms for finite iterations. Note that this code optimization problem is unique here in that it disallows concentrated check node degrees unlike a single-user code. Second, how can these codes be applied to fading channels? Achieving good performance in a fading environment requires a strategy that can exploit

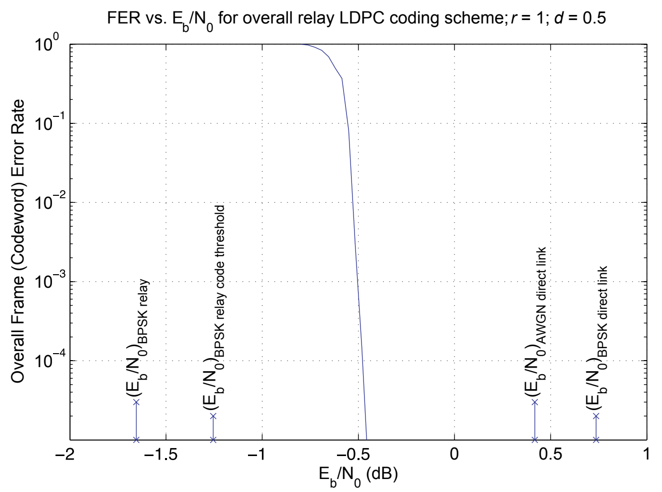

Fig. 10. End-to-end relay FER performance with a $(1023, 993)$ BCH outer code at an SNR of -1 dB.

the channel variations to its advantage. The proposed code designs can be used in conjunction with power or rate control in a fading environment. Finally, decode-and-forward is just one relay protocol; code designs for other protocols, such as estimate-and-forward, require further investigation.

Appendix I
Achievable Rates of Gaussian and BPSK Gaussian Relay Channels

For a Gaussian relay channel, the achievable rate is [5]

$$R_G = \sup_{\Theta, 0\le t, r \le 1} \min\{tC(P_{SR}) + t'C((1-r^2)P_{SD_2}), \\ tC(P_{SD_1}) + t'C(P_{SD_2} + P_{RD} + 2r\sqrt{P_{SD_2}P_{RD}})\}. \quad (32)$$

where r is the correlation between the source and relay signals in MAC mode, $C(x) = \frac{1}{2}\log(1+x)$ is the capacity of a Gaussian link, and the following are notations for received power

$$P_{SR} = P_{S_1}\gamma_{SR} \quad , \quad P_{SD_1} = P_{S_1}\gamma_{SD} \ , \\ P_{RD} = P_{R_2}\gamma_{RD} \quad , \quad P_{SD_2} = P_{S_2}\gamma_{SD} \ . \quad (33)$$

We also present the achievable rate of decode-and-forward relaying using BPSK modulation over an AWGN relay channel. The rate is given by (3) and the following mutual information terms [4]

$$I(X_1; V_1) = H(V_1) - H(Z), \\ I(X_1, Y_1) = H(Y_1) - H(Z), \\ I(X_2; Y_2|W_2) = \frac{1}{2}\left\{\sum_{b_2=\pm 1} H(Y_2|W_2 = b_2)\right\} - H(Z), \\ I(X_2, W_2; Y_2) = H(Y_2) - H(Z). \quad (34)$$

where

$$f_{V_1}(v_1) = \sum_{b_1=\pm 1} f_{X_1}(b_1) f_Z(v_1 - b_1\sqrt{P_{SR}}), \quad (35)$$

$$f_{Y_1}(y_1) = \sum_{b_1=\pm 1} f_{X_1}(b_1) f_Z(y_1 - b_1\sqrt{P_{SD1}}), \quad (36)$$

$$f_{Y_2}(y_2) = \sum_{b_1=\pm 1}\sum_{b_2=\pm 1} f_{X_2,W_2}(b_1, b_2) \\ f_Z(y_2 - b_1\sqrt{P_{SD2}} - b_2\sqrt{P_{RD}}), \quad (37)$$

$$f_{Y_2|W_2}(y_2|b_2) = \sum_{b_1=\pm 1} f_{X_2|W_2}(b_1|b_2) \\ f_Z(y_2 - b_1\sqrt{P_{SD2}} - b_2\sqrt{P_{RD}}), \quad (38)$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{z^2}{2\sigma^2}\right). \quad (39)$$

In the above expressions, $H(.)$ is the entropy function, $f_{X_1}(-1) = f_{X_1}(1) = \frac{1}{2}$, and the optimal input distribution $f_{X_2,W_2}(b_1, b_2)$ is

$$f_{X_2,W_2}(1,1) = f_{X_2,W_2}(-1,-1) = \frac{1+r}{4}, \\ f_{X_2,W_2}(1,-1) = f_{X_2,W_2}(-1,1) = \frac{1-r}{4}. \quad (40)$$

Appendix II
Relationship Between Relay and Single-User Rates at High SNR

We prove that the achievable rate of decode-and-forward relaying at high SNR exceeds the capacity of the direct link by at most a constant independent of the SNR. In the high SNR regime, the capacity function $C(x)$ approaches $\frac{1}{2}\log(x)$. Therefore, in the limit

$$\lim_{P\to\infty} R_G \\ = \sup_{\Theta, 0\le t, r\le 1} \min\{\frac{t}{2}\log(P_{SR}) + \frac{t'}{2}\log((1-r^2)P_{SD_2}), \\ \frac{t}{2}\log(P_{SD_1}) + \frac{t'}{2}\log(P_{SD_2} + P_{RD} + 2r\sqrt{P_{SD_2}P_{RD}})\} \\ \le \sup_{\Theta, 0\le t\le 1} \frac{t}{2}\log(P_{SR}) + \frac{t'}{2}\log(2(P_{SD_2} + P_{RD})), \quad (41)$$

where the notation is the same as in Appendix I. The last expression is easily maximized with respect to t subject to the power constraint Θ defined in (2), and the maximum equals $\frac{1}{2}\log(P) + \textbf{constant terms}$.

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Arnab Chakrabarti received his MS in Electrical and Computer Engineering from Rice University in 2003, where he is currently working toward his Ph.D. He was a recipient of the TI fellowship at Rice University. Earlier he received his BTech (Hons.) from IIT Kharagpur in 2001. His research interests include communications, information and coding theory, and wireless sensor networks. He is a student member of IEEE.

Alexandre de Baynast received the diploma degree in Electrical and Computer Engineering from ESME Sudria, Paris, France, in 1998, and the M.S. and Ph.D. degrees in Electrical Engineering from the University of cergy-Pontoise, France, in 1999 and 2002 respectively. Since 2002, he has been with the Department of Electrical and Computer Engineering, Rice University. His research interests span the broad area of wireless communication and networking with special emphasis on relay techniques, signal processing for communication, and architecture design for coding theory applications.

Ashutosh Sabharwal (S'90, M'99, SM'06) is a research faculty of Electrical and Computer Engineering and the Director of Center for Multimedia Communications at Rice University. He received the B.Tech. degree from Indian Institute of Technology, New Delhi in 1993 and the M.S. and Ph.D. degrees from the Ohio State University in 1995 and 1999 respectively. He was the recipient of Presidential Dissertation Fellowship award in 1998.

He is currently an Associate Editor for *IEEE Transactions on Wireless Communications*. He served as technical co-chair of the wireless personal communication systems symposium at IEEE VTC 2005, and has served on the program committee of several conferences including IEEE INFOCOM, SIGCOMM Workshops, WinMee and GLOBECOM. His research interests are in the areas of information theory, protocols and algorithms for high-performance wireless systems.

Behnaam Aazhang received his B.S. (with highest honors), M.S., and Ph.D. degrees in Electrical and Computer Engineering from University of Illinois at Urbana-Champaign in 1981, 1983, and 1986, respectively.

From 1981 to 1985, he was a Research Assistant in the Coordinated Science Laboratory, University of Illinois. In August 1985, he joined the faculty of Rice University, Houston, Texas, where he is now the J.S. Abercrombie Professor, and Chair of the Department of Electrical and Computer Engineering. He has served as the founding director of Rice's Center for Multimedia Communications from 1998 till 2006. He has been a Visiting Professor at IBM Federal Systems Company, Houston, Texas, the Laboratory for Communication Technology at Swiss Federal Institute of Technology (ETH), Zurich, Switzerland, the Telecommunications Laboratory at University of Oulu, Oulu, Finland, the U.S. Air Force Phillips Laboratory, Albuquerque, New Mexico, and at Nokia Mobile Phones in Irving, Texas. His research interests are in the areas of communication theory, information theory, and their applications with emphasis on multiple access communications, cellular mobile radio communications, and wireless communication networks.

Dr. Aazhang is a Fellow of IEEE, a recipient of the Alcoa Foundation Award 1993, the NSF Engineering Initiation Award 1987-1989, and the IBM Graduate Fellowship 1984-1985, and is a member of Tau Beta Pi and Eta Kappa Nu. He is a distinguished lecturer of IEEE Communication Society and also a recipient of 2004 IEEE Communication Society's Stephen O. Rice best paper award for a paper with A. Sendonaris and E. Erkip. Dr. Aazhang has been listed in the Thomson-ISI Highly Cited Researchers and has been keynote and plenary speaker of several conferences. He is serving as a guest editor for the *IEEE Journal on Selected Areas of Communication* special issue on relay and cooperative communication and the general chair of the 2006 Communication Theory Workshop, Dorado, Puerto Rico. He has served on Houston Mayor's Commission on Cellular Towers 1998-2004, as the Editor for Spread Spectrum Networks of the *IEEE Transactions on Communications* (1993-1998), the Treasurer of IEEE Information Theory Society 1995-1998, the Secretary of the Information Theory Society 1990-1993, the Publications Chairman of the 1993 IEEE International Symposium on Information Theory, San Antonio, Texas, the co-chair of the Technical Program Committee of 2001 Multi-Dimensional and Mobile Communication (MDMC) Conference in Pori, Finland, the chair of the Technical Program Committee for 2005 Asilomar Conference, Monterey, CA, and the co-chair of the Technical Program Committee of International Workshop on Convergent Technologies (IWCT), Oulu, Finland, June 6-10, 2005.