On The Delaunay Graph of a Geometric Graph

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Abstract

In this paper we study proximity structures for geometric graphs. The study of these structures was recently motivated by topology control for wireless networks [6, 7]. We obtain the following results:

(i) We prove that if G is a D_1-graph on n vertices, then it has $O(n^{3/2})$ edges.

(ii) We show that for any n there exist D_1-graphs with n vertices and $\Omega(n^{4/3})$ edges.

(iii) We prove that if G is a D_2-graph on n vertices, then it has $O(n)$ edges. This bound is worst-case asymptotically tight.

As an application of the first result, we show that:

(iv) The maximum size of a family of pairwise non-overlapping lenses in an arrangement of n unit circles in the plane is $O(n^{3/2})$.

The first two results improve the best previously known upper and lower bounds of $O(n^{5/3})$ and $\Omega(n)$ respectively (see [6]). The third result improves the best previously known upper bound of $O(n \log n)$ ([6]). Finally, our last result improves the best previously known upper bound (for the more general case of not necessarily unit circles) of $O(n^{3/2}\kappa(n))$ (see [1]), where $\kappa(n) = (\log n)^{O(\alpha^2(n))}$ and where $\alpha(n)$ is the extremely slowly growing inverse Ackermann's function.

1 Introduction

In this paper we study certain geometric-graphs which we refer to as D_k-graphs. Before defining this notion, recall that the *distance* between two vertices in a graph $G = (V, E)$, is the minimum length of a path (in the graph theoretical sense) joining these two vertices. Thus for example, two adjacent vertices are at distance 1 from each other.

Definition 1.1 Let $k \geq 1$ be some fixed integer. A geometric graph is called D_k-graph, if for every two adjacent vertices u and v in G there is a disc containing both u and v but no vertex (in G) of distance less than or equal to k neither from u nor from v (we refer to such a disc as an $N^k_{u,v}$-free disc).

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The study of such structures is motivated by the design of efficient dynamic routing protocols for *ad hoc* networks (or sensor network; see, e.g., [7, 6]).

Note that for any $k > l$ if G is a D_k-graph, then G is also a D_l-graph. Note also that if G is a Delaunay graph of some finite planar point set, then G is a D_k-graph for any k. In some sense, the notion of a D_k-graph is a generalization of that of a Delaunay graph. For example, a Delaunay graph of a finite set P of points consists of all pairs (the edges) of points that could be separated from the rest of the points by a disc. A D_1-graph has the property that any edge (pq) can be separated from the neighbors of p and of q with a disc.

In this paper we study the edge complexity of such graphs. We also show an interesting relation between the maximum edge complexity of a D_1-graph and the famous (so called) repeated distances problem of Erdős, see, e.g., [9]. We also improve the best previously known upper bound on the maximum size of a family of non-overlapping lenses in an arrangement of unit-circles. This latter problem has many applications in combinatorial and computational geometry (see, e.g., [1, 12]).

More specifically:

1. We prove that if G is a D_1-graph on n vertices, then it has $O(n^{3/2})$ edges.

2. We show that for any n there exist D_1-graphs with n vertices and $\Omega(n^{4/3})$ edges.

3. We prove that if G is a D_2-graph on n vertices, then it has at most $32n$ edges. This bound is worst-case asymptotically tight.

4. We prove that the maximum size of a family of pairwise non-overlapping lenses in an arrangement of n unit circles in the plane is $O(n^{3/2})$.

The first two results improve the best previously known upper and lower bounds of $O(n^{5/3})$ and $\Omega(n)$ respectively (see [6]). The third result improves the best previously known upper bound of $O(n \log n)$ ([6]). Finally, our last result improves the best previously known upper bound (for the more general case of not necessarily unit circles) of $O(n^{3/2}\kappa(n))$ (see [1]), where $\kappa(n) = (\log n)^{O(\alpha^2(n))}$ and where $\alpha(n)$ is the extremely slowly growing inverse Ackermann's function.

Of Particular interest is the fact (which we show here) that any unit-distance graph (i.e., a graph defined on n points in the plane such that there is an edge between two points p, q if and only if $d(p,q) = 1$, where $d(.)$ denotes the Euclidean distance function) is also a D_1-graph. Obtaining asymptotic bounds on the number of edges that a unit distance graph can have, a problem that was posed by Erdős more than 50 years ago (see, e.g., [9]), is a long-standing and one of the most famous open problems in discrete geometry.

2 D_k-graphs

Theorem 2.1 *Let $G = (V, E)$ with $|V| = n$ be a D_1-graph. Then $|E| \leq \frac{16\sqrt{2}}{\sqrt{2}-1} n^{3/2}$.*

Proof: Let Δ be a set of 8 directions, represented as points on the unit circle $\mathcal{S}^1$, with the property that for any direction u there exists a direction $u_0 \in \Delta$ such that the angle between u and u_0 is smaller than $\alpha = \pi/8$. Let G_u denote the subgraph of G consisting of all edges (p, q) such that the direction $\vec{pq}$ forms an angle of at most α with u. $\{G_u\}_{u \in \Delta}$ is a decomposition of G into 8 (not necessarily edge-disjoint) graphs. We aim to show that the number of edges in G_u is at most $\frac{2\sqrt{2}}{\sqrt{2}-1}n^{3/2}$ for each $u \in \Delta$. Assume, without loss of generality that u is the vector $(0, 1)$. We can further assume that all edges of G_u intersect a horizontal line l. Indeed, by a simple recursion it is easily seen that an upper bound of $O(n^{3/2})$ on the number of edges under the assumption that they all intersect a horizontal line, implies the same asymptotic upper bound in the general case. We formulate it more precisely in the following lemma:

Lemma 2.2 *Let $G = (V, E)$ be a geometric graph with $|V| = n$. Assume that G has the following property: Any subgraph $G' = (V, E')$, for which there exists a line that intersects all edges of E', has at most $cn^{1+\epsilon}$ edges, where $\epsilon > 0$ and c are some fixed given constants. Then $|E| \le (\frac{1}{1-\frac{1}{2^\epsilon}}) \cdot cn^{1+\epsilon}$.*

Proof: Easy and omitted

■

Lemma 2.3 *Let G_u be as above and let G'_u be a subgraph of G_u all of whose edges are intersected by a line. Then G'_u has at most $2n^{3/2}$ edges.*

Proof: We partition the edges of G'_u into two sets, according to the following rule: for each edge connecting two points (p, q) such that p is below q (i.e., the y-coordinate of p is less than the y-coordinate of q), we pick an arbitrary $N^1_{p,q}$-free disc d_{pq} (recall that by assumption, there exists such a disc). Let c denote its center. We say that the edge (p, q) is a *right edge* (resp., a *left edge*) if p, q, c is a right turn (resp., a left turn). Let G^r (resp., G^l) be the subgraph consisting of all right (resp., left) edges of G'_u. We claim that, G^r (and in a symmetric manner also G^l) cannot contain a $K_{2,2}$ as a subgraph (i.e., a complete bipartite graph with two vertices on each side). Indeed, assume to the contrary that G^r contains a $K_{2,2}$ subgraph. Without loss of generality, we can assume that all the edges of G'_u are intersected by the x-axis and that u is the vector $(0, 1)$.

In [6] it was proved that G cannot contain a self-intersecting copy of $K_{2,2}$ [1]. So the only case that remains to consider is when G^r contains a non-self-intersecting copy of $K_{2,2}$. See Figure 1 for an illustration. It is easily seen that in such a case, the four vertices x, y, z, w of this subgraph are not in convex position. Without loss of generality, assume that the point x lies in the interior of the triangle spanned by the three others points y, z, w.

Let D denote the disc which has the midpoint of the segment zy as its center and the length $|zy|$ as its diameter. Consider an $N^1_{z,y}$-free disc d (which exists by assumption) whose center lies to the right of $\vec{zy}$. It is easily seen that the point x must lie outside D (for otherwise it would belong to the interior of d contradicting the assumption on d; see, c.f., Figure 2). This implies that the angle $\measuredangle zxy$ is less than $\pi/2$, which means that the angle $\measuredangle zyx$ is at least $\pi - \pi/2 - \measuredangle yzx \ge \pi/2 - 2\alpha$. Since the edge yw does not intersect the edge zx, it must be that $\measuredangle zwy > \measuredangle zyx \ge \pi/2 - 2\alpha$ and by our choice of $\alpha = \pi/8$ implies that $zyw > \pi/4 = 2\alpha$, contradicting the assumption that every two edges have an angle of at

[1] Using this fact alone, and a result of [11], it follows that G has at most $O(n^{8/5})$ edges. Note also that G can indeed contain a non intersecting copy of $K_{2,2}$.

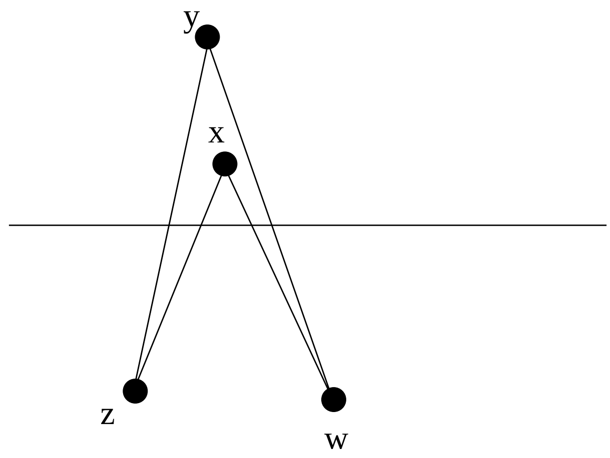

Figure 1: A non-self intersecting copy of $K_{2,2}$ in G^r, all of whose edges are intersected by a horizontal line, cannot exist.

most 2α between them in G_u. See Figure 3 for an illustration. Hence G^r does not contain a $K_{2,2}$ as a subgraph. By the Kövari-Sós-Turán Theorem in extremal graph theory (see e.g., [3]) the number of edges of G^r is at most $\frac{1}{2} \cdot (n^{3/2} + n)$. Thus, the number of edges in G'_u is at most $2n^{3/2}$ edges. This completes the proof of the lemma. ■

Applying Lemma 2.2 to G_u with $c = 2$ and $\epsilon = 1/2$ we conclude that G_u has at most $\frac{2\sqrt{2}}{\sqrt{2}-1}n^{3/2}$ edges.

We conclude that the edges of G can be decomposed into 8 subgraphs (not necessarily edge-disjoint), each containing at most $\frac{2\sqrt{2}}{\sqrt{2}-1}n^{3/2}$ edges. Hence G contains at most $\frac{16\sqrt{2}}{\sqrt{2}-1}n^{3/2}$ edges. This completes the proof of the theorem. ■

The question that naturally arises is what can we say about the maximum number of edges of D_k-graphs for $k > 1$. In the next theorem we show that such graphs have a linear number of edges already for $k = 2$.

Theorem 2.4 *Let G be a D_2-graph with n vertices. Then the number of edges of G is at most $32n$.*

Lemma 2.5 *Let G be a D_2-graph. Then G does not contain a self-intersecting copy of P_3 (i.e., a path of length three).*

Proof: Assume to the contrary that x, y, u, v are vertices of G and that $(xu), (uv), (vy)$ are three edges of a P_3, and that (vy) and (xu) cross each other.

In this case, x, y, u, v are four vertices of a convex quadrilateral. Without loss of generality assume that their clockwise order is x, y, u, v. Then, assume with out loss of generality that $\measuredangle uvx + \measuredangle xyu \geq \pi$. It follows that any disc which contains the edge (xu) will contain either v or y in its interior contradicting the assumption that G is a D_2-graph. ■

Remark: Lemma 2.5, combined with a result of [10], implies that G has at most $O(n \log n)$ edges. Using this fact alone is not enough to show that G has a linear number of edges, since the bound given in [10] is worst-case asymptotically tight.

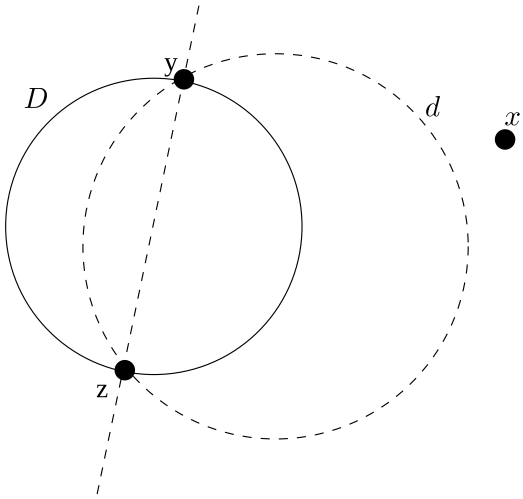

Figure 2: The half-disc inside D to the right of zy is contained in the disc d (depicted dashed).

Proof of Theorem 2.4: We start with a similar approach to that used in the proof of Theorem 2.1. We partition the edges of G into 8 (not necessarily disjoint) classes so that within each class, the orientations of any two edges differ by at most $\pi/4$.

Let G' be the subgraph spanned by one of these classes. Assume, without loss of generality that all edges in G' make an angle of at most $\pi/8$ with the y-axis. It is enough to show that the number of edges in G' is at most $4n$.

At each vertex v delete the rightmost edge and the leftmost edge that emanates upwards from v as well as the rightmost and the leftmost edges that emanates downwards from v (observe that the notion of 'to the right of an edge' is well defined since we assume that all edges form a small angle with the y-axis). Clearly, we deleted a total of at most $4n$ edges. We claim that all edges of G' are deleted. This will clearly conclude the proof of the theorem, showing that G has at most $32n$ edges.

Assume to the contrary that an edge (uv) in G' survives the deletion process. Without loss of generality, assume that v lies above u. Let d be a N^2_{uv}-free disc. Let c denote the center of d. Assume that c lies to the right of uv (i.e., that uvc is a right-turn). The case where c lies to the left of uv is treated symmetrically. Since (u, v) is not deleted, there must be an edge (ux) which is to the right of (uv) and emanates upward from u, and similarly, there is an edge (vy) which is to the right of (uv) and emanates downward from v.

Let D be the disc that has uv as its diameter. Similarly to the argument in the proof of Theorem 2.1, both x and y must lie outside D (for otherwise one of them would lie in the interior of d, contradicting the fact that d is an N^2_{uv}-free disc). Let z be the intersection point of the line l_1 spanned by (ux) and the line l_2 spanned by (vy). By Lemma 2.5, the edges (ux) and (vy) cannot intersect. Thus, z must lie outside D. This implies that $\measuredangle uzv < \pi/2$. However, by assumption on G', $\measuredangle uvz = \measuredangle uvy < \pi/4$ and $\measuredangle vuz = \measuredangle vux < \pi/4$. Hence, the sum of the angles of the triangle Δuvz is strictly less than π, a contradiction. This completes the proof of the theorem. $\square$

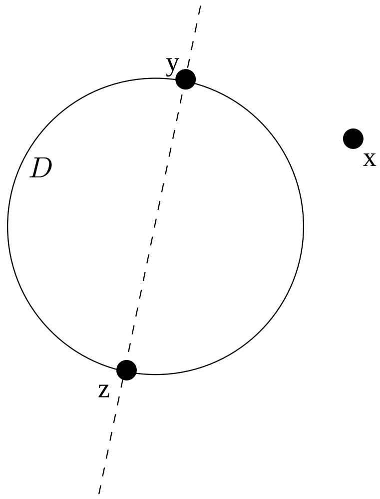

Figure 3: x must lie outside D and so the angle $\measuredangle zxy$ is less than 90^o.

3 A lower bound

In this section we construct D_1-graphs on n vertices with $\Omega(n^{4/3})$ edges. Our construction is based on a well known construction of a unit distance graph on a 3-dimensional sphere with n vertices and $\Omega(n^{4/3})$ edges (see, e.g., [8]).

We start with an old and famous construction, due to Erdős (which was rediscovered many years later by Edelsbrunner and Welzl [5]), of a configuration of n points and n lines in the plane giving rise to $cn^{4/3}$ incidences between the points and the lines (see, e.g., [9]). We then place the unit sphere $\mathcal{S}^2$ such that it touches the plane at its south pole. Centrally project the plane on the sphere, by taking the center of the sphere to be the center of projection. Then the points on the plane project to points on the (southern hemi-) sphere and the lines on the plane project to great half-circles on the sphere. After completing the half circles to full circles, we obtain a collection of n great circles and n points on the sphere, with $cn^{4/3}$ incidences. For every great circle ψ on the sphere we introduce a point P_ψ on the southern hemisphere which is equidistant from all the points on ψ. We thus obtain a set of n points on the sphere, which are the projection of the original n points in the plane, and another n points on the sphere, which rise from the n great circles. This $2n$ point set has the property that at least $cn^{4/3}$ pairs of them are at distance $\sqrt{2}$ apart. Let u and v be two such points on the sphere at distance $\sqrt{2}$ from each other. The unique circular cap C_{uv} on the sphere which has u and v diametrically opposite on its boundary has the property that every point (but u or v) on the smaller cap that it encloses on the sphere, is at distance strictly less than $\sqrt{2}$ from u and from v.

Finally, to finish the construction, we project the sphere back to the plane using the north pole as the center of projection. The $2n$ points on the sphere project to a set of $2n$ points in the plane. We define a geometric graph on this set by drawing an edge (as a straight line segment) between two points if their preimages on the sphere are at distance $\sqrt{2}$. We thus obtain a geometric graph with $2n$ vertices and at least $cn^{4/3}$ edges. We claim that this is a D_1-graph. Indeed, observe that for every u and v on the sphere at distance

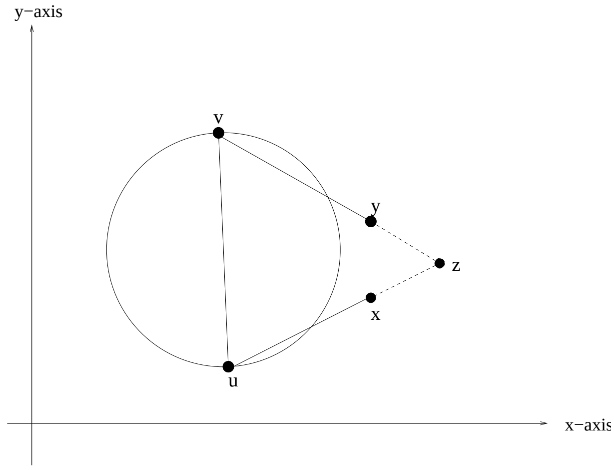

Figure 4: An illustration of Theorem 2.4

$\sqrt{2}$ from each other, the circle C_{uv} projects to a circle having the images of u and v on its boundary. Since both u and v are contained in the southern hemisphere, the smaller cap on the sphere bounded by C_{uv} does not contain the north pole, and thus, by the definition of the graph G, the disc bounded by the projection of C_{uv} is an $N^1_{u,v}$-free disc. This completes the construction.

4 Applications

In this section we present some applications of Theorem 2.1. We start with an application regarding the complexity of arrangements of unit circles in the plane.

Let $\mathcal{C}$ be a family of unit circles in the plane. For every two intersecting circles $C_1, C_2 \in \mathcal{C}$ we define the *lens* $L(C_1, C_2)$ to be the union of the arcs of C_1 and C_2 which bound the area of the intersection of the disks bounded by those circles. Two lenses are called *overlapping* if they share a common sub-arc.

The study of families of non-overlapping lenses is motivated by a problem raised by Tamaki and Tokuyama ([12]): Given a family of curves, in the plane, what is the minimum number of cuts needed in order to generate from it a family of pseudo-segments (i.e., a set of curves, every two of which intersect at most once). Tamaki and Tokuyama observed that the notion of a lens (which can be defined similarly for any family of curves) is the critical notion here. Clearly, we must cut every lens. Therefore, the minimum number of cuts is at least as large as the maximum size of a family of non-overlapping lenses. It turns out, however, that up to a constant factor this is also an upper bound.

Families of non-overlapping lenses were studied by many authors, in various settings (see, e.g., [1, 2]). Here we show how Theorem 2.1 can be used to improve the best known upper bound on the maximum size of a family of non-overlapping lenses in a family of unit circles in the plane.

in [2], Alon et al. show that the maximum size of a family of non-overlapping lenses created by n unit circles is $O(n^{3/2+\epsilon})$. This result was improved by Agarwal et al. to

$O(n^{3/2}\kappa(n))$ (see [1]). We now further improve it and show the following:

Theorem 4.1 *The maximum size of a family of non-overlapping lenses in an arrangement of n unit circles in the plane is $O(n^{3/2})$.*

Proof: Let $\mathcal{C}$ be a collection of n unit circles in the plane. Let $\mathcal{L}$ be a family of non-overlapping lenses created by circles in $\mathcal{C}$. We define a geometric graph G whose vertices are the centers of the circles in $\mathcal{C}$. We connect two vertices by an edge (drawn as a straight line segment between the two corresponding centers), if the corresponding pair of unit circles create one of the lenses in $\mathcal{L}$.

We claim that G is a D_1-graph. Once we establish this statement, Theorem 4.1 will be an immediate consequence of Theorem 2.1.

To show that G is a D_1-graph, let $e = (u, v)$ be any edge of G. The vertices u and v are, respectively, the center points of two unit circles in $\mathcal{C}$, C_u and C_v. Let D be the disc whose diameter is $\overline{uv}$ and whose center is the midpoint of the segment uv. We claim that D is an $N^1_{u,v}$-free disc. Indeed, assume to the contrary that there exists a point $w \in D$ which is a neighbor of, say u. Let $C_w \in \mathcal{C}$ be the circle whose center is w. Thus, C_w and C_u create a lens which does not overlap $L(C_u, C_v)$. We assume without loss of generality that $\overline{uv}$ is vertical and that u lies above v. Let a, b be the two intersection points of C_u and C_v, so that a is to the left of b (see Figure 5).

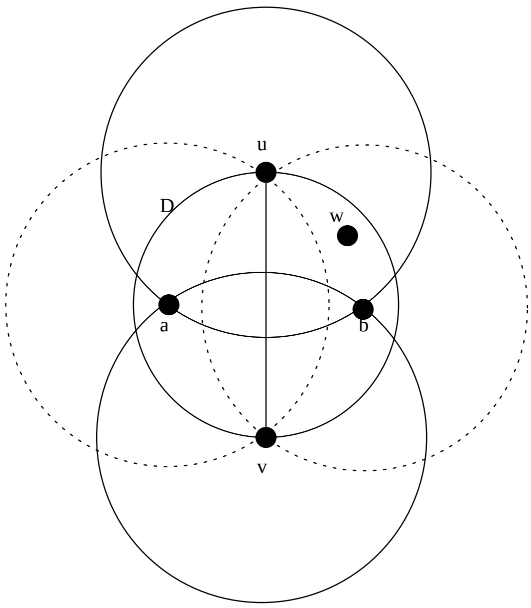

Figure 5: An illustration of the contradiction in the proof of Theorem 4.1

The unit circle centered at a intersects the boundary of D at u and v and thus encloses entirely the left half disc of D bounded by the diameter $\overline{uv}$. Similarly the unit disc centered at b encloses entirely the right half disc of D bounded by $\overline{uv}$. Since $w \in D$ it follows that the disk bounded by C_w contains one of a or b in its interior. Therefore also a portion of the arc on C_u which constitutes the lens $L(C_u, C_v)$. This implies that $L(C_w, C_u)$ and $L(C_u, C_v)$

overlap, a contradiction. Hence, the disc D cannot contain a neighbor of u in G. Thus, we have found a disc (namely D) which is $N^1_{u,v}$-free. This completes the proof of the theorem. ■

5 Open problems

- There still exists a wide gap between the lower bound of $\Omega(n^{4/3})$ and the upper bound of $O(n^{3/2})$ on the maximum number of edges of a D_1-graph, established in this paper. We leave it is a major open problem to tighten this gap.
- Can one prove a better upper bound on the size of a D_1-graph with n vertices, in the more restricted case that for each edge (p, q), we further require that the disc whose diameter is pq is an $N^1_{p,q}$-free disc. In this case the best lower bound is $\Omega(n^{1+c/\log\log n})$ and is achieved by considering n points in the plane with $\Omega(n^{1+c/\log\log n})$ pairs of them that are unit distance apart (this construction is due to Erdős; see, e.g., [9]).

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