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Discrete Quaternion Fourier Transform and Properties

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Abstract

In this paper we discuss the two-dimensional discrete quaternion Fourier transform (DQFT). We derive several properties of the DQFT which correspond to those of the (continuous) quaternion Fourier transform (QFT).

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1 Introduction

The classical Fourier transform (FT) has a long story. It is invented by the French mathematician, Joseph Fourier, in the early nineteenth century. The FT is a mathematical operation that transforms a function of a real variable in a given domain into another function in another domain. The domains differ from one application to another. In signal processing, the FT transforms the signal from the original domain to the spectral or frequency domain. In the frequency domain many characteristics of the signal are revealed.

On the other hand, the discrete Fourier transform (DFT) is widely known and used in signal and image processing. Many fundamental algorithms can be realized by DFT, such us the convolution, spectrum estimation and correlation.

Recently, the topic of generalization of the FT to the quaternion algebra called the quaternion Fourier transform (QFT) has received considerable attention. For example, Bülow [1] and Ell [2] generalized the FT by substituting the FT kernel with quaternion exponential kernel in the FT definition. Many properties of the QFT are already known such as translation, modulation, differentiation, and uncertainty principle (see [5, 7, 8]).

The purpose of this paper is to introduce the two-dimensional discrete quaternion Fourier transform (DQFT) as a generalization of the DFT using quaternion algebra. We derive its important properties which are corresponding properties of the QFT. These properties are useful to construct the discrete versions of generalized transform, such as the discrete quaternion wavelet transform, discrete quaternionic windowed Fourier transform and discrete quaternion multiplier (see [6,9]).

2 Quaternion Algebra

The first concept of quaternions [1] was formally introduced by Hamilton in 1843 and is denoted by $\mathbb{H}$. It is an associative non-commutative four dimensional algebra

$$\mathbb{H} = \{ q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3\}, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}. \tag{1}$$

The orthogonal imaginary units $\boldsymbol{i}, \boldsymbol{j},$ and $\boldsymbol{k}$ satisfy the multiplication rules:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1,\ \boldsymbol{ij} = -\boldsymbol{ji},\ \boldsymbol{ik} = -\boldsymbol{ki},\ \boldsymbol{jk} = -\boldsymbol{kj}, \text{and } \boldsymbol{ijk} = -1.$$

We may express a quaternion q as a scalar part denoted by $Sc(q) = q_0$ and a pure quaternion $\boldsymbol{q}$ denoted by $\mathrm{Vec}(q) = \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 = \boldsymbol{q}$. The conjugate of a quaternion q is obtained by changing the signs of the pure quaternion, that is,

$$\bar{q} = q_0 - \boldsymbol{i}q_1 - \boldsymbol{j}q_2 - \boldsymbol{k}q_3. \tag{2}$$

It is a linear anti-involution, that is, for every $p, q \in \mathbb{H}$ we have

$$\bar{\bar{p}} = p, \qquad \overline{p+q} = \bar{p} + \bar{q},\ \overline{pq} = \bar{q}\bar{p}. \tag{3}$$

It is not difficult to see that from equation (1) and the third term of equation (3) we obtain the norm of a quaternion q as

$$|q| = \sqrt{q\bar{q}} = \sqrt{{q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2}.$$

From equations (1), (2), and (3) we get the invers

$$q^{-1} = \frac{\bar{q}}{|q|^2}.$$

This fact shows that $\mathbb{H}$ is a normed division algebra.

For a pure quaternion $\boldsymbol{p}$ and a pure unit quaternion $\boldsymbol{q}, \boldsymbol{p}$ can be resolved into its component parallel $(\boldsymbol{p}_\parallel)$ and perpendicular $(\boldsymbol{p}_\perp)$ to $\boldsymbol{q}$, i.e.

$$\boldsymbol{p} = \boldsymbol{p}_\parallel + \boldsymbol{p}_\perp. \tag{4}$$

Here $\boldsymbol{p}_{\parallel} = \frac{1}{2}(\boldsymbol{p} - \boldsymbol{qpq})$ for $\boldsymbol{p} \parallel \boldsymbol{q}$ and $\boldsymbol{p}_{\perp} = \frac{1}{2}(\boldsymbol{p} + \boldsymbol{qpq})$ for $\boldsymbol{p} \perp \boldsymbol{q}$. Therefore, we have the following result, which will be used in the next section.

Proposition 2.1 *If* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *are two pure quaternions, then*

- $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *are parallel* $(\boldsymbol{p} \parallel \boldsymbol{q})$ *if and only if* $\boldsymbol{pq} = \boldsymbol{qp}$
- $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *are perpendicular* $(\boldsymbol{p} \perp \boldsymbol{q})$ *if and only if* $\boldsymbol{pq} = -\boldsymbol{qp}$.

Hereinafter, we will denote a finite sequence of the quaternion numbers by

$$\{f(m,n), 0 \le m,n \le M,N\} .$$

We also get the following proposition.

3 Definition of Discrete Quaternion Fourier Transform (DQFT)

Analogous to the two-dimensional discrete Fourier transform (DFT), we may define the 2D discrete quaternion Fourier transform (DQFT). Due to the non-commutative property of the quaternion multiplication, there are at least three different definitions of the DQFT. Here in defining the DQFT we adopt the type II DQFT definition proposed by Ell and Sangwine in [2, 3].

Definition 3.1. *Let* $f(m,n)$ *be a two-dimensional quaternion discrete-time sequence. The DQFT of* $f(m,n)$ *is defined by* $F_f^q(u,v) \in \mathbb{H}^{M\times N}(m = 0,1,2,\dots M-1;\ n = 0,1,2,\dots,N-1)$, *where*

$$F^q\{f\}(u,v) = F_f^q(u,v) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)e^{-\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}. \tag{5}$$

With μ *is any pure unit quaternion such that* μ^2 *=-1.*

Theorem 3.2. *The IDQFT is invertible and its inverse is given by*

$$F_f^{-q}\left[F_f^q\right](m,n) = f(m,n) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F_f^q(u,v)e^{\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}. \tag{6}$$

Proof. Substituting (5) to the right-hand side of (6) we immediately get

$$F_f^{-q}\left[F_f^q\right](m,n)$$

$$= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left[\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} f(m',n')\, e^{-\mu 2\pi(\frac{m'u}{M}+\frac{n'v}{N})}\right] e^{\mu 2\pi(\frac{mu}{M}+\frac{nv}{N})}$$

$$= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m',n') \sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} e^{\mu 2\pi\left(\frac{(m-m')u}{M}+\frac{(n-n')v}{N}\right)}$$

Notice that when $m = m'$ and $n = n'$, we have $e^{\mu 2\pi\left(\frac{(m-m')u}{M}+\frac{(n-n')v}{N}\right)} = 1$. We further obtain

$$\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} e^{\mu 2\pi\left(\frac{(m-m')u}{M}+\frac{(n-n')v}{N}\right)} = MN, \text{where } m = m' \text{ and } n = n'.$$

However, when $m \neq m'$ and $n \neq n'$, then we get

$$\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} e^{\mu 2\pi\left(\frac{(m-m')u}{M}+\frac{(n-n')v}{N}\right)} = 0.$$

It follows that

$$\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left[f(m',n') \left(\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} e^{\mu 2\pi\left(\frac{(m-m')u}{M}+\frac{(n-n')v}{N}\right)}\right)\right] = f(m,n).$$

It means that

$$\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left[\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} f(m',n')\, e^{-\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}\right] e^{\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)} = f(m,n).$$

That is $F_f^{-q}\left[\, F_f^q\right](m,n) = f(m,n)$ for $0 \leq m,n \leq (M-1),(N-1)$. This proves the theorem. ∎

4 Properties of DQFT

This section is devoted to the investigation of several properties of the DQFT. We find most of them are corresponding generalization versions of the DFT (compare to [4]).

Theorem 4.1. *The following properties hold for the DQFT*
(i) Shifts or translation. If the discrete translation of the quaternion sequence $f(m,n)$ *is defined by* $T_{m_0,n_0}f(m,n) = f(m-m_0, n-n_0)$, *then*

$$F^q_{T_{m_0,n_0}f}(u,v) = F_f^q(u,v)e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}. \tag{7}$$

Especially, if $\boldsymbol{f}(m,n)$ *is a pure quaternion sequence, then*

$$F^q_{T_{u_0,v_0}\boldsymbol{f}}(u,v) = e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}F^q_{\boldsymbol{f}_{\parallel,\mu}}(u,v) + e^{\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}F^q_{\boldsymbol{f}_{\perp,\mu}}(u,v). \tag{8}$$

(ii) Modulation. Consider the discrete modulation of the quaternion sequence $f(m,n)$ is defined by

$$M_{u_0,v_0}f(m,n) = f(m,n)e^{\mu 2\pi\left(\frac{mu_0}{M}+\frac{nv_0}{N}\right)}.$$

Then we have

$$F^q_{M_{u_0,v_0}f}(u,v) = F^q_f(u-u_0, v-v_0).$$

If $\boldsymbol{f}(m,n)$ is a pure quaternion sequence and $M_{u_0,v_0}\boldsymbol{f}(m,n) = \boldsymbol{f}(m,n)e^{\mu 2\pi\left(\frac{mu_0}{M}+\frac{nv_0}{N}\right)}$, then

$$F^q_{M_{u_0,v_0}\boldsymbol{f}}(u,v) = F^q_{\boldsymbol{f}_{\parallel,\mu}}(u-u_0, v-v_0) + F^q_{\boldsymbol{f}_{\perp,\mu}}(u+u_0, v+v_0).$$

Proof. We only prove part (*i*) and other being similar. Indeed, we have

$$F^q_{T_{m_0,n_0}f}(u,v) = \sum_{m_0=0}^{M-1}\sum_{n_0=0}^{N-1} f(m-m_0, n-n_0)\, e^{-\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}.$$

Performing the change of variables $m^{'} = m - m_0$ and $n^{'} = n - n_0$ into the above expression, we easily obtain

$$\begin{aligned} F^q_{T_{m_0,n_0}f}(u,v) &= \sum_{m^{'}=0}^{M-1}\sum_{n^{'}=0}^{N-1} f(m^{'},n^{'})\, e^{-\mu 2\pi\left(\frac{(m^{'}+m_0)u}{M}+\frac{(n^{'}+n_0)v}{N}\right)} \\ &= \sum_{m^{'}=0}^{M-1}\sum_{n^{'}=0}^{N-1} f(m^{'},n^{'})\, e^{-\mu 2\pi\left(\frac{m^{'}u}{M}+\frac{n^{'}v}{N}\right)}\, e^{-\mu 2\pi\left(\frac{m_0u}{M}+\frac{n_0v}{N}\right)}. \end{aligned}$$

Because $\boldsymbol{f}(m,n)$ is a pure quaternion sequence, then using Preposition 2.1 we may decompose $\boldsymbol{f}(m,n)$ with respect to the axis μ into $\boldsymbol{f}_{\parallel,\mu}(m,n) + \boldsymbol{f}_{\perp,\mu}(m,n)$. It means that we have

$$\begin{aligned} &F^q_{T_{m_0,n_0}\boldsymbol{f}}(u,v) \\ &= \sum_{m^{'}=0}^{M-1}\sum_{n^{'}=0}^{N-1} \boldsymbol{f}(m^{'},n^{'})\, e^{-\mu 2\pi\left(\frac{(m^{'}+m_0)u}{M}+\frac{(n^{'}+n_0)v}{N}\right)} \\ &= \sum_{m^{'}=0}^{M-1}\sum_{n^{'}=0}^{N-1} \left(\boldsymbol{f}_{\parallel,\mu}(m^{'},n^{'}) + \boldsymbol{f}_{\perp,\mu}(m^{'},n^{'})\right)\, e^{-\mu 2\pi\left(\frac{(m^{'}+m_0)u}{M}+\frac{(n^{'}+n_0)v}{N}\right)} \\ &= \sum_{m^{'}=0}^{M-1} \boldsymbol{f}_{\parallel,\mu}(m^{'},n^{'})\, e^{-\mu 2\pi\left(\frac{(m^{'}+m_0)u}{M}+\frac{(n^{'}+n_0)v}{N}\right)} \\ &\quad + \sum_{m^{'}=0}^{M-1}\sum_{n^{'}=0}^{N-1} \boldsymbol{f}_{\perp,\mu}(m^{'},n^{'})\, e^{-\mu 2\pi\left(\frac{(m^{'}+m_0)u}{M}+\frac{(n^{'}+n_0)v}{N}\right)} \end{aligned}$$

$$= \sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} \boldsymbol{f}_{\parallel,\mu}(m',n')\, e^{-\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)} e^{-\mu 2\pi\left(\frac{m_0 u}{M}+\frac{n_0 v}{N}\right)}$$

$$+ \sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} \boldsymbol{f}_{\perp,\mu}(m',n')\, e^{-\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)} e^{-\mu 2\pi\left(\frac{m_0 u}{M}+\frac{n_0 v}{N}\right)}$$

$$= e^{-\mu 2\pi\left(\frac{m_0 u}{M}+\frac{n_0 v}{N}\right)} \sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} \boldsymbol{f}_{\parallel,\mu}(m',n')\, e^{-\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}$$

$$+ e^{\mu 2\pi\left(\frac{m_0 u}{M}+\frac{n_0 v}{N}\right)} \sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} \boldsymbol{f}_{\perp,\mu}(m',n')\, e^{-\mu 2\pi\left(\frac{m'u}{M}+\frac{n'v}{N}\right)}$$

$$= e^{-\mu 2\pi\left(\frac{m_0 u}{M}+\frac{n_0 v}{N}\right)} F^q_{\boldsymbol{f}_{\parallel,\mu}}(u,v) + e^{\mu 2\pi\left(\frac{m_0 u}{M}+\frac{n_0 v}{N}\right)} F^q_{\boldsymbol{f}_{\perp,\mu}}(u,v)\,.$$

This is the desired result. ∎

The following theorem is a discrete version of the Rayleigh-Plancherel theorem.

Theorem 4.2 *For a discrete quaternion sequence* $f(m,n)$ *we get*

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} |f(m,n)|^2 = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} \left| F^q_f(u,v)\right|^2.$$

Proof. It readily follows from the definition of the DQFT (5) that

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} |f(m,n)|^2$$

$$= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)\overline{f(m,n)}$$

$$= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^q_f(u,v) e^{\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}\right)\overline{f(m,n)}$$

$$= \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^q_f(u,v) \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\overline{f(m,n)\, e^{-\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}}$$

$$= \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^q_f(u,v)\,\overline{F^q_f(u,v)}$$

$$= \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \left| F_f^q(u, v) \right|^2,$$

As desired. ∎

Theorem 4.3 *Let $f(m, n)$ be a quaternion sequence. Then we obtain*

$$F_{\bar{f}}^q(u, v) = F_{f_0}^q(u, v) - \boldsymbol{i}\, F_{f_1}^q(u, v) - \boldsymbol{j}\, F_{f_2}^q(u, v) - \boldsymbol{k}\, F_{f_3}^q(u, v).$$

Proof. Application of Definition 3.1 gives

$$F_{\bar{f}}^q(u, v)$$

$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \overline{f(m, n)}\, e^{-\mu 2\pi\left(\frac{mu}{M} + \frac{nv}{N}\right)}$$

$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (f_0(m, n) - \boldsymbol{i} f_1(m, n) - \boldsymbol{j} f_2(m, n) - \boldsymbol{k} f_3(m, n)) e^{-\mu 2\pi\left(\frac{mu}{M} + \frac{nv}{N}\right)}$$

$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_0(m, n) e^{-\mu 2\pi\left(\frac{mu}{M} + \frac{nv}{N}\right)} - \boldsymbol{i} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_1(m, n) e^{-\mu 2\pi\left(\frac{mu}{M} + \frac{nv}{N}\right)}$$

$$-\boldsymbol{j} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_1(m, n) e^{-\mu 2\pi\left(\frac{mu}{M} + \frac{nv}{N}\right)} - \boldsymbol{k} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_1(m, n) e^{-\mu 2\pi\left(\frac{mu}{M} + \frac{nv}{N}\right)}$$

$$= F_{f_0}^q(u, v) - \boldsymbol{i}\, F_{f_1}^q(u, v) - \boldsymbol{j}\, F_{f_2}^q(u, v) - \boldsymbol{k}\, F_{f_3}^q(u, v)\, .$$

This completes the proof of theorem. ∎

Theorem 4.4 *Let $f(m, n)$ be a quaternion sequence. Then we also get*

$$\overline{F_f^q(-u, -v)} = F_{f_0}^q(u, v) + \overline{F_{\boldsymbol{f}}^q(-u, -v)}\ .$$

Proof. Simple calculations yield

$$\overline{F_f^q(-u, -v)}$$

$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \overline{f(m, n)\, e^{\mu 2\pi\left(\frac{mu}{M} + \frac{nv}{N}\right)}}$$

$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \overline{(f_0(m, n) + \boldsymbol{f}(m, n))\, e^{\mu 2\pi\left(\frac{mu}{M} + \frac{nv}{N}\right)}}$$

$$= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \overline{f_0(m,n) e^{\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}} + \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \overline{\boldsymbol{f}(m,n)\, e^{\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}}$$

$$= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f_0(m,n)\, e^{-\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)} + \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \overline{\boldsymbol{f}(m,n)\, e^{\mu 2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}}$$

$$= F_{f_0}^q(u,v) + \overline{F_{\boldsymbol{f}}^q(-u,-v)}\,.$$

This finishes the proof of theorem. ■

The following corollary shows that the quaternion conjugation property of the DQFT holds if $f(m,n)$ is real sequence.

Corollary 4.5 Assume that $f(m,n)$ *is a real sequence. Then the above identity reduces to*

$$F_{\bar{f}}^q(u,v) = \overline{F_f^q(-u,-v)},$$

which resembles the analogous theorem for the DFT.

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