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Consensus problem of data-sampled networked multi-agent systems with time-varying communication delays

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Abstract

This paper investigates an observer-based control strategy for networked multi-agent systems with general linear dynamics and time-varying communication delays in a sampling setting. The communication topology is assumed to be directed fixed. Different from a traditional observer-based controller for a single system, the objects that need to be observed here are the state differences between an agent and its neighbours. Using a system transformation method, some equivalent conditions concerning the consensus of multi-agent systems are established. Moreover, we prove that both the connection weights and the communication topology play an important role in the study of multi-agent systems, and establish a linear matrix inequality based observer design method.

Keywords

Networked multi-agent systems, consensus, time-varying communication delays, sampled control, general linear dynamics, linear matrix inequalities

Introduction

In recent years, the coordination problem of multi-agent systems has attracted much attention due to its extensive applications in various areas, ranging from cooperative control of unmanned aircraft, autonomous formation flight, control of communication networks to design of sensor networks, swarm-based computing and rendezvous in space. One critical and canonical issue in the coordination of multi-agent systems is the consensus problem, which usually refers to the problem of how to reach an agreement, such as the position and velocity among a group of autonomous mobile agents in a dynamic agent system. This is a familiar phenomenon in our real life, for example, robots need to arrive at an agreement so as to accomplish some complicated tasks. Flocks of birds tend to synchronize during migration in order to resist aggression and reach their destinations. Investigations of such problems are of interest, in both theory and engineering applications.

In the past decade, consensus problems of multi-agent systems have developed very quickly and several research topics have been addressed (Ren and Cao, 2011). But, most existing consensus protocols are based on relative states between neighbouring agents. However, in a practical engineering system, it is usually impossible to directly obtain all states of systems due to economic costs or constraints on measurement. Thus, distributed estimation via observer design for multi-agent coordination attracts the attention of scholars (Hong et al., 2008; Scardovi and Sepulchre, 2009; Wang et al., 2009; Li et al., 2010). Hong et al. (2008) studied distributed observer design for leader-following control of multi-agent networks. Wang et al. (2009) constructed an observer-based dynamic output error feedback control for a general case, and proposed some sufficient conditions for achieving consensus. For multi-agent systems with time-varying topology, under the strict assumption that the matrix A in system model was Hurwitz stable or critical Hurwitz stable, Scardovi and Sepulchre (2009) studied the consensus problems with observers where it is required that the communication graph is uniformly connected. For multi-agent systems with general linear dynamics, based on relative output measurements, Li et al. (2010) proposed observer-type consensus protocols, which can be regarded as an extension of the traditional observer-based controller for a single system.

However, Scardovi and Sepulchre (2009), Wang et al. (2009) and Li et al. (2010) did not investigate the sampled-data consensus of multi-agent systems, i.e. consensus in a sampled-data setting. In fact, with the development of digital sensors and controllers, in many cases, the system dynamics are normally continuous while the synthesis of control law can only use the data sampled at the discrete sampling instants. Therefore, sampled control for continuous-time systems is more coincident with applications in our real life and has become an interesting topic (Liu et al., 2010; Zhang and Tian, 2010; Liu and Liu, 2011; Ren and Cao, 2011). On the other hand, time delay is a common phenomenon for real control systems (Jiang et al., 2010; Zhang and Tian, 2010;

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Rudy and Nejat, 2011), which may arise from the moving of vehicles, the congestion of communication channels, etc. Moreover, time delay is an important factor to cause a system to diverge or oscillate (Papachristodoulou et al., 2010).

According to the above analysis, it is necessary to investigate the observer-based consensus of networked multi-agent systems with time-varying communication delays in a sampling setting. Unfortunately, at present, few papers have studied this problem, even for linear multi-agent system with directed fixed topology. This motivated us to write this paper. Our paper can be regarded as an extension of Scardovi and Sepulchre (2009) and Wang et al. (2009). Specifically, our system model, the communication topology and communication constraints investigated in this article are quite different from Wang et al. (2009). Specifically, our system model is not a normal multi-agent systems, but is a networked multi-agent system, i.e. all the agents are connected through a communication network. Many problems such as sampled-data setting and time delay that were neglected in Wang et al. (2009) are investigated in our article. The topology in this article is directed fixed topology, which is more complicated than the undirected fixed topology in Wang et al. (2009). Compared with Scardovi and Sepulchre (2009), our consensus results can be applied to many multi-agent system since the assumption condition that the matrix A in the system model is Hurwitz stable or critical Hurwitz stable is not necessary in our article.

Our main contributions are threefold. (I) Based on matrix theory and the system transformation method, some equivalent algebraic conditions concerning the consensus of multi-agent systems are established. (II) Moreover, we prove that not only the communication topology but also the connection weights play an important role in the study of multi-agent systems. (III) Observers can be designed by solving a set of linear matrix inequalities (LMIs).

Notation. We use standard notations throughout this paper. Let M^T be the transpose of the matrix M. $M > 0$ $(M < 0)$ means that M is positive definite (negative definite). I_n represents the identity matrix of dimension n, and I denotes the identity matrix of an appropriate dimension. $\mathrm{Diag}\{A_1, \cdots, A_n\}$ represents a block-diagonal matrix with matrices $A_i, i = 1, \cdots, n$ on its diagonal. The symbol $*$ will be used to denote a symmetric structure in a matrix, that is, $\begin{bmatrix} L & N \\ * & R \end{bmatrix} = \begin{bmatrix} L & N \\ N^T & R \end{bmatrix}$. 1_n is a vector with all entries equal to 1. $\rho(\cdot)$ and $\det(\cdot)$ represent the spectral radius and determinant of a matrix, respectively. $\| x \|$ and $\| A \|$ denote the Euclidean norm of vector x and A, respectively. $A \otimes B$ denotes the Kronecker product. $A \sim B$ denotes that the matrix A is similar to the matrix B. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. $\mathbb{R}^+$ and $\mathbb{C}$ denote the sets of positive numbers and complex numbers, respectively. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ denote its real and imaginary part, respectively.

Problem formulations and preliminaries

In this section, we first introduce some graph knowledge and the networked multi-agent system model, then we formulate our problems and propose some lemmas as the preliminaries of our paper.

Graph theory

Let $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ denote a weighted graph, where $\mathscr{V} = \{1, \cdots, N\}$ is the node set, $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$ denotes the edge set and $\mathscr{A} = [a_{ij}\omega_{ij}]$ is the weighted adjacency matrix with $\omega_{ij} > 0$. Here, $\omega_{ij} > 0$ is said to be the weight between the agent i and the agent j, which reflects the dependence of the agent i on the agent j. A directed edge of $\mathscr{G}$ is denoted by $e_{ij} = (j, i)$, where j is called the parent node of i and i is called the child node of j. If the edge $e_{ij} = (j, i) \in \mathscr{E}$, then $a_{ij} = 1$, otherwise $a_{ij} = 0$. Suppose that each node has no self edge, i.e. $a_{ii} = 0$ for all i. The set of neighbours of node i is denoted by $\mathscr{N}_i = \{j \in \mathscr{V} : (j, i) \in \mathscr{E}\}$. The Laplacian matrix $L = [l_{ij}]$ of digraph $\mathscr{G}$ is defined by $l_{ij} = -a_{ij}\omega_{ij}$ for $i \neq j$ and $l_{ij} = \sum_{k=1, k \neq i}^{N} a_{ik}\omega_{ik}$ for $i = j$. A path of $\mathscr{G}$ from node i to node j is a sequence of finite-ordered edges in the form of $(i, k_1), (k_1, k_2), \cdots, (k_l, j)$. A directed graph is strongly connected if for any distinct nodes, there exists a path between them. A directed graph has or contains a directed spanning tree if there exists a node called a root such that there exists a directed path from this node to every other node. A subgraph $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1, \mathscr{A}_1)$ of $\mathscr{G}$ is a graph such that $\mathscr{V}_1 \subset \mathscr{V}$ and $\mathscr{E}_1 \subset \mathscr{E}$.

Problem formulation

Consider N agents with general linear dynamics

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t), & t \in \mathbb{R}^+ \\ y_i(t) = Cx_i(t), & i \in \mathcal{I} := \{1, \cdots, N\} \end{cases} \tag{1}$$

where $x_i(t) \in \mathbb{R}^p$ is the state, $u_i(t) \in \mathbb{R}^q$ is the control input and $y_i(t) \in \mathbb{R}^m$ is the measured output.

Remark 1. In our paper, the agents evolve in a space that possesses a measurable vector field. For example, the agents may represent vehicles travelling in an environment that possesses a temperature, chemical or magnetic field.

A model of the networked multi-agent system used in this paper is shown in Figure 1.

Throughout the paper, we make the following assumptions.

Assumption 1. In this paper, directed graphs under fixed topology are considered. Matrix A described in (1) is not Hurwitz stable, i.e. the open-loop system is not stable.

Assumption 2. For simplicity, but without loss of generality, all the time delays exist in the communication channels between the sensors and observers.

Assumption 3. Every agent is regarded as a plant. The plant output node (sensor) is assumed to be time driven, and its sampling period is h, whereas the observer is assumed to be event driven.

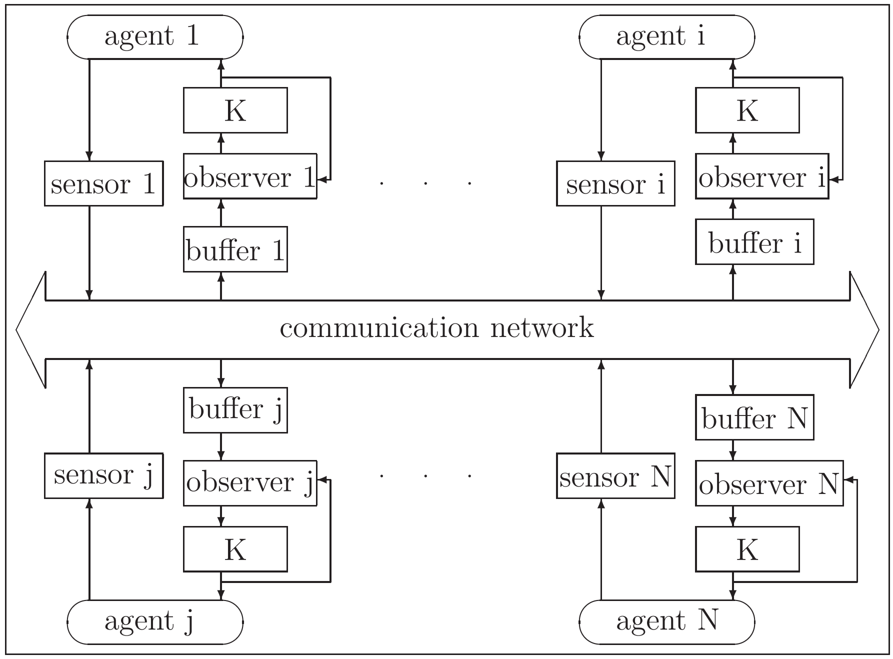

Figure 1 The structure of observer-based multi-agent systems in a communication network.

Assumption 4. The measured output of agent j at the time of kh is $y_j(kh)$. The information that agent i obtained from agent j at the time of $kh+\tau_{ij}^k$ is described by $a_{ij}\omega_{ij}[y_j(kh)-y_i(kh)]$, where a_{ij}, ω_{ij} are the adjacency relationship, the connection weight from agent j and agent i, respectively. τ_{ij}^k, $0<\tau_{ij}^k<h$, $i=1,\cdots,N, j\in N_i$ is the communication delay from agent j to agent i during the kth sampling period.

Assumption 5. Set a buffer in the receiver of every agent. Let $\tau_k=\max\limits_{i=1,\cdots,N, j\in N_i}\{\tau_{ij}^k\}$ be the maximum delay during the kth sampling period, $\tau=\max\limits_k\{\tau^k\}$ be the maximum delay of the multi-agent system. Let $kh+\tau$ be the threshold time of all the buffers during every sampling period.

For agent i, suppose the obtained information at the time of $kh+\tau$ is $\eta_i(kh)$, which is composed of the output information difference between agent i and its neighbours. Specifically

$$\eta_i(kh)=\sum_{j=1}^N a_{ij}\omega_{ij}[y_j(kh)-y_i(kh)] \tag{2}$$

Therefore, at the time of $kh+\tau$, the ith buffer releases the signal $\eta_i(kh)$, which is used to renew the input of the ith observer.

In general, most papers adopt the agreement protocol based on state feedback

$$u_i(t)=K\sum_{j=1}^N a_{ij}\omega_{ij}[x_j(kh)-x_i(kh)],\ \forall t\in[kh+\tau,(k+1)h+\tau)$$

However, in this paper, we assume that not all the states of agents can be obtained directly, and design an observer-based agreement protocol

$$\begin{cases}\dot{\hat{x}}_i(t)=A\hat{x}_i(t)+Bu_i(t)+G\{\eta_i(kh)-C\sum\limits_{j=1}^N a_{ij}\omega_{ij}[\hat{x}_j(kh)-\hat{x}_i(kh)]\}\\ u_i(t)=K\sum\limits_{j=1}^N a_{ij}\omega_{ij}[\hat{x}_j(kh)-\hat{x}_i(kh)],\ \forall t\in[kh+\tau,(k+1)h+\tau)\end{cases} \tag{3}$$

where $\hat{x}_i(t)\in\mathbb{R}^p$ is the protocol state, $i\in\mathcal{I}$, G and K are the feedback gain matrices to be designed, and a_{ij} and ω_{ij} are as defined in the subsection on graph theory. The term $C\sum\limits_{j=1}^N a_{ij}\omega_{ij}[\hat{x}_j(kh)-\hat{x}_i(kh)]$ in (3) denotes the information exchanges between agent i and those of its neighbours in the kth sampling period.

Then, by (2) and (3), system (1) can be written as

$$\dot{\bar{x}}_i(t)=\bar{A}\bar{x}_i(t)+\bar{B}\sum_{j=1}^N a_{ij}\omega_{ij}[\bar{x}_j(kh)-\bar{x}_i(kh)],\quad \forall t\in[kh+\tau,(k+1)h+\tau) \tag{4}$$

where $\bar{x}_i(t)=[\hat{x}_i^T(t),x_i^T(t)]^T$, $\bar{A}=\begin{bmatrix}A & 0\\ 0 & A\end{bmatrix}$, $\bar{B}=\begin{bmatrix}BK-GC & GC\\ BK & 0\end{bmatrix}$.

Remark 2. Compared with traditional multi-agent systems, networked multi-agent systems, such as distributed robots and mobile sensor networks, have posed a number of challenges in terms of their theoretic analysis and synthesis (Zhang and Tian, 2010). Agents in such networks are required to operate in concert with each other in order to achieve system-level objectives, for example, the consensus problem investigated in this paper.

Remark 3. As the agreement protocol describes a viable means of making networked agents achieve a common value in a decentralized manner, it makes sense to adopt decentralized observer-based control strategies since some state variables of agents may not be measured in practice (Hong et al., 2008). For example, each vehicle in a multi-vehicle system uses a digital sensor and observer to send and receive messages sampled at discrete instants, whereas the position, velocity and the direction angle are continuous physical processes. Thus, the observer-based consensus protocol designed in this paper is more suitable for multi-agent systems.

In this paper, we aim to design an observer-based control protocol to guarantee that system (1) reaches consensus.

Here, the concept of consensus is given as the following.

Definition 1. Multi-agent system (1) with strategies (2) and (3) reaches consensus if there exist gain matrices K, G and connection weights ω_{ij} such that the states of system (4) satisfy $\lim_{t\to+\infty} \| \bar{x}_i(t) - \bar{x}_j(t) \| = 0$ for arbitrary $i, j \in \mathcal{I}$.

Let $z_i(t) = \bar{x}_i(t) - \bar{x}_1(t)$, $i = 2, \cdots, N$. Define $z(t) = [z_2^T(t), \cdots, z_N^T(t)]^T$, then we can equivalently obtain a reduced system

$$\dot{z}(t) = Fz(t) + Hz(kh), \quad \forall t \in [kh + \tau, (k+1)h + \tau) \tag{5}$$

where $F = I_{N-1} \otimes \bar{A}$, $H = -\tilde{L} \otimes \bar{B}$ and $\tilde{L} = \begin{bmatrix} l_{22} - l_{12} & \cdots & l_{2N} - l_{1N} \\ \vdots & \ddots & \vdots \\ l_{N2} - l_{12} & \cdots & l_{NN} - l_{1N} \end{bmatrix}$ is defined as the reduced Laplacian matrix, where l_{ij} is the corresponding element in the Laplacian matrix L. The relationship between $\tilde{L}, L$ and the graph is given by the following lemma.

Lemma 1. In Zhang and Tian (2009), $\tilde{L}$ has no zero eigenvalue, if and only if the Laplacian matrix L has only one zero eigenvalue, if and only if the graph $\mathcal{G}$ has a directed spanning tree.

Remark 4. Obviously, $\lim_{t\to+\infty} \| \bar{x}_i(t) - \bar{x}_j(t) \| = 0$ is equivalent to $\lim_{t\to+\infty} \| z_i(t) \| = 0$, $\forall i, j \in \mathcal{I}$, i.e. the consensus problem of system (1) can be transformed into the stability problem of a reduced system (5). Hence, in the following discussion, we will focus on seeking the necessary and sufficient conditions to guarantee the stability of system (5).

Consensus analysis

In this section, we aim to establish the necessary and sufficient conditions to guarantee that system (1) reaches consensus.

Theorem 1. For a fixed topology, system (1) reaches consensus, if and only if $\rho(\Phi(\tau)) < 1$, where

$$\Phi(\tau) = \begin{bmatrix} 0 & I_{2p\times(N-1)} \\ -\tilde{L} \otimes \int_{h-\tau}^{h} e^{\bar{A}s} ds\bar{B} & I_{N-1} \otimes e^{\bar{A}h} - \tilde{L} \otimes \int_0^{h-\tau} e^{\bar{A}s} ds\bar{B} \end{bmatrix}$$

Proof. (Necessity) If system (1) reaches consensus, then system (5) is asymptotically stable.

For system (5), it is easy to obtain its corresponding discretization model

$$\begin{aligned} z((k+1)h) &= (e^{Fh} + \int_0^{h-\tau} e^{Fs} dsH)z(kh) + \int_{h-\tau}^{h} e^{Fs} dsHz((k-1)h) \\ &= (I_{N-1} \otimes e^{\bar{A}h} - \tilde{L} \otimes \int_0^{h-\tau} e^{\bar{A}s} ds\bar{B})z(kh) \\ &\quad - \tilde{L} \otimes \int_{h-\tau}^{h} e^{\bar{A}s} ds\bar{B}z((k-1)h). \end{aligned} \tag{6}$$

Defining $\bar{z}(k) = [z^T(kh), z^T((k+1)h)]^T$, then by (6), we get

$$\bar{z}(k) = \Phi(\tau)\bar{z}(k-1) \tag{7}$$

By the asymptotical stability of system (5), we get that system (7) is also asymptotically stable, i.e. $\rho(\Phi(\tau)) < 1$.

(Sufficiency) For system (7), if $\rho(\Phi(\tau)) < 1$, then we get $\| z(kh) \| \to 0, k \to \infty$. For $\forall t \in [kh, (k+1)h)$, two cases are discussed as follows.

Case 1. $\forall t \in [kh, kh + \tau)$, by system (4)

$$x_i(t) = e^{A(t-kh)} x_i(kh) + \int_0^{t-kh} e^{As} dsBK \sum_{j=1}^{N} a_{ij}\omega_{ij}[\hat{x}_j((k-1)h) - \hat{x}_i((k-1)h)]$$

Hence

$$\begin{aligned} \| x_i(t) - x_1(t) \| \leqslant{} & e^{\|A\|(t-kh)} \| x_i(kh) - x_1(kh) \| \\ & + \int_0^{t-kh} e^{\|A\|s} ds \| BK \| \\ & \times [\sum_{j=1}^{N} a_{ij}\omega_{ij} \| \hat{x}_j((k-1)h) - \hat{x}_i((k-1)h) \| \\ & + \sum_{j=1}^{N} a_{1j}\omega_{1j} \| \hat{x}_j((k-1)h) - \hat{x}_1((k-1)h) \|] \end{aligned} \tag{8}$$

Similarly

$$\begin{aligned} \hat{x}_i(t) = {} & e^{A(t-kh)} \hat{x}_i(kh) \\ & + \int_0^{t-kh} e^{As} ds\{(BK - GC) \\ & \sum_{j=1}^{N} a_{ij}\omega_{ij}[\hat{x}_j((k-1)h) - \hat{x}_i((k-1)h))] \\ & + GC \sum_{j=1}^{N} a_{ij}\omega_{ij}[x_j((k-1)h) - x_i((k-1)h)]\} \end{aligned}$$

$$\begin{aligned} \| \hat{x}_i(t) - \hat{x}_1(t) \| \leqslant{} & e^{\|A\|(t-kh)} \| \hat{x}_i(kh) - \hat{x}_1(kh) \| \\ & + \int_0^{t-kh} e^{\|A\|s} ds\{\| BK - GC \| [\sum_{j=1}^{N} a_{ij}\omega_{ij} \| \\ & \hat{x}_j((k-1)h) - \hat{x}_i((k-1)h) \| \\ & + \sum_{j=1}^{N} a_{1j}\omega_{1j} \| \hat{x}_j((k-1)h) - \hat{x}_1((k-1)h) \|] \\ & + \| GC \| [\sum_{j=1}^{N} a_{ij}\omega_{ij} \| x_j((k-1)h) - x_i((k-1)h) \| \\ & + \sum_{j=1}^{N} a_{1j}\omega_{1j} \| x_j((k-1)h) - x_1((k-1)h) \|]\} \end{aligned} \tag{9}$$

Case 2. $\forall t \in [kh+\tau,(k+1)h)$, by system (4), we have

$$
\begin{aligned}
x_i(t) = {} & e^{A(t-kh)}x_i(kh) + \int_{h-\tau}^{h} e^{As}dsBK \\
& \sum_{j=1}^{N} a_{ij}\omega_{ij}[\hat{x}_j((k-1)h) - \hat{x}_i((k-1)h)] \\
& + \int_0^{t-kh-\tau} e^{As}dsBK\sum_{j=1}^{N} a_{ij}\omega_{ij}[\hat{x}_j(kh) - \hat{x}_i(kh)] \\
\| x_i(t) - x_1(t) \| \leqslant {} & e^{\|A\|(t-kh)} \| x_i(kh) - x_1(kh) \| \\
& + \int_{h-\tau}^{h} e^{\|A\|s}ds \| BK \| [\sum_{j=1}^{N} a_{ij}\omega_{ij} \| \hat{x}_j((k-1)h) - \hat{x}_i((k-1)h) \| \\
& + \sum_{j=1}^{N} a_{1j}\omega_{1j} \| \hat{x}_j((k-1)h) - \hat{x}_1((k-1)h) \|] \\
& + \int_0^{t-kh-\tau} e^{\|A\|s}ds \| BK \| [\sum_{j=1}^{N} a_{ij}\omega_{ij} \| \hat{x}_j(kh) - \hat{x}_i(kh) \| \\
& + \sum_{j=1}^{N} a_{1j}\omega_{1j} \| \hat{x}_j(kh) - \hat{x}_1(kh) \|]
\end{aligned} \tag{10}
$$

Similarly

$$
\begin{aligned}
\hat{x}_i(t) = {} & e^{A(t-kh)}\hat{x}_i(kh) \\
& + \int_{h-\tau}^{h} e^{As}ds\{(BK - GC) \\
& \sum_{j=1}^{N} a_{ij}\omega_{ij}[\hat{x}_j((k-1)h) - \hat{x}_i((k-1)h)] \\
& + GC\sum_{j=1}^{N} a_{ij}\omega_{ij}[x_j((k-1)h) - x_i((k-1)h)]\} \\
& + \int_0^{t-kh-\tau} e^{As}ds\{(BK - GC)\sum_{j=1}^{N} a_{ij}\omega_{ij}[\hat{x}_j(kh) - \hat{x}_i(kh)] \\
& + GC\sum_{j=1}^{N} a_{ij}\omega_{ij}[x_j(kh) - x_i(kh)]\}
\end{aligned}
$$

$$
\begin{aligned}
\| \hat{x}_i(t) - \hat{x}_1(t) \| \leqslant {} & e^{\|A\|(t-kh)} \| \hat{x}_i(kh) - \hat{x}_1(kh) \| \\
& + \int_{h-\tau}^{h} e^{\|A\|s}ds\{\| BK - GC \| \\
& [\sum_{j=1}^{N} a_{ij}\omega_{ij} \| \hat{x}_j((k-1)h) - \hat{x}_i((k-1)h) \| \\
& + \sum_{j=1}^{N} a_{1j}\omega_{1j} \| \hat{x}_j((k-1)h) - \hat{x}_1((k-1)h) \|] \\
& + \| GC \| [\sum_{j=1}^{N} a_{ij}\omega_{ij} \| x_j((k-1)h) - x_i((k-1)h) \| \\
& + \sum_{j=1}^{N} a_{1j}\omega_{1j} \| x_j((k-1)h) - x_1((k-1)h) \|]\} \\
& + \int_0^{t-kh-\tau} e^{\|A\|s}ds\{\| BK - GC \| \\
& [\sum_{j=1}^{N} a_{ij}\omega_{ij} \| \hat{x}_j(kh) - \hat{x}_i(kh) \| \\
& + \sum_{j=1}^{N} a_{1j}\omega_{1j} \| \hat{x}_j(kh) - \hat{x}_1(kh) \|] + \| GC \| \\
& [\sum_{j=1}^{N} a_{ij}\omega_{ij} \| x_j(kh) - x_i(kh) \| \\
& + \sum_{j=1}^{N} a_{1j}\omega_{1j} \| x_j(kh) - x_1(kh) \|]\}
\end{aligned} \tag{11}
$$

Obviously, for $\forall t \in [kh,(k+1)h)$

$$
\begin{aligned}
& \lim_{k\to+\infty} \| z(kh) \| = 0 \quad \Leftrightarrow \\
& \lim_{k\to+\infty} \| x_i(kh) - x_1(kh) \| = 0, \lim_{k\to+\infty} ||\hat{x}_i(kh) - \hat{x}_1(kh)|| = 0, \\
& \Leftrightarrow \lim_{k\to+\infty} \| x_i(kh) - x_j(kh) \| = 0, \\
& \lim_{k\to+\infty} \| \hat{x}_i(kh) - \hat{x}_j(kh) \| = 0, \\
& i,j \in \mathcal{I}
\end{aligned} \tag{12}
$$

Furthermore, by (8), (10) and (12), we prove

$$
\begin{aligned}
& \lim_{t\to+\infty} \| x_i(t) - x_1(t) \| = 0 \Leftrightarrow \\
& \lim_{t\to+\infty} \| x_i(t) - x_j(t) \| = 0, i,j \in \mathcal{I}
\end{aligned}
$$

Similarly, by (9), (11) and (12), we get

$$
\begin{aligned}
& \lim_{t\to+\infty} \| \hat{x}_i(t) - \hat{x}_1(t) \| = 0 \Leftrightarrow \\
& \lim_{t\to+\infty} \| \hat{x}_i(t) - \hat{x}_j(t) \| = 0, i,j \in \mathcal{I}
\end{aligned}
$$

Thus, by Definition 1, multi-agent system (1) reaches consensus. □

Remark 5. Theorem 1 has established a necessary and sufficient consensus criterion, but it is difficult for us to compute $\rho(\Phi(\tau))$ for its complexity. Hence, we aim to further seek necessary and sufficient conditions to guarantee that $\rho(\Phi(\tau)) < 1$ in the following analysis.

Let $\tilde{L} = T^{-1}JT$, where J is the Jordan canonical form of $\tilde{L}$ with diagonal elements $\lambda_1, \lambda_2, \cdots, \lambda_{N-1}$. First, by the properties of the Kronecker product

$$
\begin{aligned}
\Phi(\tau) = {} & \begin{bmatrix} 0 & I_{N-1}\otimes I_{2p} \\ 0 & I_{N-1}\otimes e^{\bar{A}h} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \tilde{L}\otimes \int_{h-\tau}^{h} e^{\bar{A}s}ds\bar{B} & \tilde{L}\otimes \int_0^{h-\tau} e^{\bar{A}s}ds\bar{B} \end{bmatrix} \\
\sim {} & I_{N-1}\otimes \begin{bmatrix} 0 & I_{2p} \\ 0 & e^{\bar{A}h} \end{bmatrix} - \tilde{L}\otimes \begin{bmatrix} 0 & 0 \\ \int_{h-\tau}^{h} e^{\bar{A}s}ds\bar{B} & \int_0^{h-\tau} e^{\bar{A}s}ds\bar{B} \end{bmatrix} \\
= {} & (T^{-1}\otimes I_{4p})\tilde{\Phi}(\tau)(T\otimes I_{4p})
\end{aligned}
$$

where

$$
\tilde{\Phi}(\tau) = I_{N-1}\otimes \begin{bmatrix} 0 & I_{2p} \\ 0 & e^{\bar{A}h} \end{bmatrix} - J\otimes \begin{bmatrix} 0 & 0 \\ \int_{h-\tau}^{h} e^{\bar{A}s}ds\bar{B} & \int_0^{h-\tau} e^{\bar{A}s}ds\bar{B} \end{bmatrix}
$$

Then, we have

$$
\begin{aligned}
\det(sI_{4p(N-1)} - \Phi(\tau)) &= \det(sI_{4p(N-1)} - \tilde{\Phi}(\tau)) \\
&= \det\left(\begin{bmatrix} sI_{4p} - \tilde{\Phi}_1(\tau) & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & sI_{4p} - \tilde{\Phi}_{N-1}(\tau) \end{bmatrix}\right) \\
&= \prod_{i=1}^{N-1} \det(sI_{4p} - \tilde{\Phi}_i(\tau))
\end{aligned} \tag{13}
$$

where $\tilde{\Phi}_i(\tau) = \begin{bmatrix} 0 & I_{2p} \\ -\lambda_i \int_{h-\tau}^{h} e^{\bar{A}s} ds \bar{B} & e^{\bar{A}h} - \lambda_i \int_0^{h-\tau} e^{\bar{A}s} ds \bar{B} \end{bmatrix}$, $i = 1, \cdots, N-1$.

Remark 6. From the above analysis, it is easy to obtain that $\rho(\Phi(\tau)) < 1$ can be equivalently converted into $\rho(\Phi_i(\tau)) < 1, i = 1, 2, \cdots N-1$. It is comparatively easy to compute $\rho(\Phi_i(\tau))$.

Proposition 1. System

$$\dot{v}_i(t) = \bar{A}v_i(t) - \lambda_i \bar{B} v_i(kh), \ \forall t \in [kh + \tau, (k+1)h + \tau) \tag{14}$$

is asymptotically stable if and only if $\rho(\tilde{\Phi}_i(\tau)) < 1$, $i = 1, \cdots, N-1$, where $\bar{A}, \bar{B}$ are defined as in (4), λ_i are the eigenvalues of $\tilde{L}$ defined as in (5).

Proof. The corresponding discretization model of system (14) is given as

$$
\begin{aligned}
v_i((k+1)h) &= (e^{\bar{A}h} + \int_0^{h-\tau} e^{\bar{A}s} ds(-\lambda_i \bar{B})) v_i(kh) \\
&\quad + \int_{h-\tau}^{h} e^{\bar{A}s} ds \cdot (-\lambda_i \bar{B}) v_i((k-1)h) \\
&= (e^{\bar{A}h} - \lambda_i \int_0^{h-\tau} e^{\bar{A}s} ds \cdot \bar{B}) v_i(kh) - \lambda_i \\
&\quad \int_{h-\tau}^{h} e^{\bar{A}s} ds \cdot \bar{B} v_i((k-1)h)
\end{aligned} \tag{15}
$$

Defining $\bar{v}_i(k) = [v_i^T(kh), v_i^T((k+1)h)]^T$, then by (15), we get

$$\bar{v}_i(k) = \tilde{\Phi}_i(\tau) \bar{v}_i(k-1) \tag{16}$$

(Necessity) (14) is asymptotically stable $\Rightarrow$ (16) is asymptotically stable $\Rightarrow \rho(\tilde{\Phi}_i(\tau)) < 1$.

(Sufficiency) For $\forall t \in [kh, (k+1)h)$, two cases are discussed as follows.

Case 1. $\forall t \in [kh, kh + \tau)$

$$
\begin{aligned}
v_i(t) &= e^{\bar{A}(t-kh)} v_i(kh) + \int_0^{t-kh} e^{\bar{A}s} ds(-\lambda_i \bar{B}) v_i((k-1)h) \\
\| v_i(t) \| &\leqslant e^{\|\bar{A}\|(t-kh)} \| v_i(kh) \| \\
&\quad + \int_0^{t-kh} e^{\|\bar{A}\|s} ds \| -\lambda_i \bar{B} \| \| v_i((k-1)h) \|
\end{aligned} \tag{17}
$$

Case 2. $\forall t \in [kh + \tau, (k+1)h)$

$$
\begin{aligned}
v_i(t) &= e^{\bar{A}(t-kh)} v_i(kh) + \int_{h-\tau}^{h} e^{\bar{A}s} ds(-\lambda_i \bar{B}) v_i((k-1)h) \\
&\quad + \int_0^{t-kh-\tau} e^{\bar{A}s} ds(-\lambda_i \bar{B}) v_i(kh) \\
\| v_i(t) \| &\leqslant e^{\|\bar{A}\|(t-kh)} \| v_i(kh) \| \\
&\quad + \int_{h-\tau}^{h} e^{\|\bar{A}\|s} ds \| -\lambda_i \bar{B} \| \| v_i((k-1)h) \| \\
&\quad + \int_0^{t-kh-\tau} e^{\|\bar{A}\|s} ds \| -\lambda_i \bar{B} \| \| v_i(kh) \|
\end{aligned} \tag{18}
$$

Then, we can obtain the results as

$\rho(\tilde{\Phi}_i(\tau)) < 1 \Rightarrow \| \bar{v}_i(k) \| \to 0, \ k \to +\infty \Rightarrow \| v_i(kh) \| \to 0$, $\| v_i((k-1)h) \| \to 0, \ k \to +\infty \Rightarrow \| v_i(t) \| \to 0, \ t \to +\infty$ (by equations (17) and (18)) $\Rightarrow$ system (14) is asymptotically stable. □

For system (14), let $d(t) = t - kh$, $t \in [kh + \tau, (k+1)h + \tau)$, $d_m = \tau$, $d_M = h + \tau$, then system (14) can be equivalently converted into the following delay systems

$$
\begin{aligned}
\dot{v}_i(t) &= \bar{A}v_i(t) - \lambda_i \bar{B} v_i(t - d(t)), \ t \in [kh + \tau, \ (k+1)h + \tau), \\
& i = 1, \cdots, \ N-1
\end{aligned} \tag{19}
$$

Summarizing the above discussions, some equivalent properties about consensus problem can be obtained in the following theorem.

Theorem 2. For a fixed topology, the following propositions are equivalent.

(a) The multi-agent system (1) reaches consensus.
(b) System (5) is asymptotically stable.
(c) $\rho(\Phi(\tau)) < 1$.
(d) $\rho(\tilde{\Phi}_i(\tau)) < 1, i = 1, \cdots, N-1$.
(e) All the subsystems described in (14) are asymptotically stable.
(f) All the delay differential equations described in (19) are asymptotically stable.

Proof. By Remark 3, $(a) \Leftrightarrow (b)$. By Theorem 1, $(a) \Leftrightarrow (c)$. By equation (13), $(c) \Leftrightarrow (d)$. By Proposition 1, $(d) \Leftrightarrow (e)$. By equation (19), $(e) \Leftrightarrow (f)$. Hence, $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$. □

Remark 7. It is easy to see that the difference among all the subsystems in system (19) is caused by the different eigenvalues of $\tilde{L}$, which are affected by the connection weights and topology of multi-agent systems. Moreover, Example 1 in the section on simulations indicates that for the general linear multi-agent systems via sampled control, the connection weights cannot be chosen arbitrarily. Otherwise, there may not exist feedback gain matrices such that the multi-agent system (1) reaches consensus. Therefore, it is necessary to regard the connection weights as the parameters to be designed for improving the consentability of the multi-agent system.

Based on Theorem 2, we get the following theorem.

Theorem 3. Suppose there exist gain matrices K, G such that system

$$\dot{v}(t) = \bar{A}v(t) - \bar{B}v(t - d(t)), t \in [kh + \tau, (k+1)h + \tau) \quad (20)$$

is asymptotically stable. Then, there exist connection weights ω_{ij} such that system (1) reaches consensus if and only if the graph $\mathscr{G}$ contains a directed spanning tree.

Proof. (Necessity) Now we prove the necessity by contradiction.

Suppose there is no directed spanning tree contained in the graph $\mathscr{G}$, by Lemma 1, $\tilde{L}$ has at least one zero eigenvalue under arbitrary connection weights. Without loss of generality, suppose $\lambda_1 = 0$, we have

$$\det(sI_{4p} - \tilde{\Phi}_1(\tau)) = \det(\begin{bmatrix} sI_{2p} & -I_{2p} \\ 0 & sI_{2p} - e^{\bar{A}h} \end{bmatrix}) \\ = \det(sI_{2p}) \det(sI_{2p} - e^{\bar{A}h})$$

where $\bar{A}$ is defined as in system (4). Let $A = P_A^{-1} J_A P_A$, where J_A is the Jordan canonical form of A with diagonal elements $\mu_1, \cdots, \mu_p$. We conclude

$$\det(sI_{2p} - e^{\bar{A}h}) = \det(\begin{bmatrix} sI_p - e^{Ah} & 0 \\ 0 & sI_p - e^{Ah} \end{bmatrix}) \\ = \prod_{i=1}^{p} (s - e^{\mu_i h}) \prod_{i=1}^{p} (s - e^{\mu_i h})$$

Because A is not Hurwitz stable, there exists at least one eigenvalue μ_i that satisfies $\mathrm{Re}(\mu_i) \geqslant 0, i \in \mathcal{I}_1 := \{1, \cdots, p\}$. As a result, $|e^{\mu_i h}| \geqslant 1$ and $\rho(\tilde{\Phi}_1(\tau)) \geqslant 1$. By Theorem 2, the multi-agent system (1) cannot reach consensus. Therefore, the graph contains a directed spanning tree under the given conditions.

(Sufficiency) If the graph $\mathscr{G}$ contains a directed spanning tree, we introduce a method to choose the connection weights such that all the eigenvalues of $\tilde{L}$ are equal and not zero. Suppose $\mathscr{G}_0 = (\mathscr{V}, \mathscr{E}_0, \mathscr{A}_0)$ is a subgraph containing a directed spanning tree, then $\mathscr{E}_0 \in \mathscr{E}$. First, we renumber the agents in the following way: the number of the agent that corresponds to the root in the $\mathscr{G}_0$ is 1, whereas for the nodes corresponding to the remaining agents, the number of the child node is larger than that of its parent node. Then, let

$$\omega_{ij} = \begin{cases} 1, & \text{if } \varepsilon_{ij} \in \mathscr{E}_0 \\ 0, & \text{if } \varepsilon_{ij} \in \mathscr{E} \backslash \mathscr{E}_0 \\ \text{arbitrary}, & \text{other case} \end{cases}$$

For the given connection weights above, $\tilde{L} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \end{bmatrix}$. Obviously, $\lambda_1 = \lambda_2 = \cdots = \lambda_{N-1} = 1$. Then, all the subsystems described in system (19) become system (20). Obviously, if there exist feedback gain matrices K, G such that system (20) is asymptotically stable, by Theorem 2, system (1) reaches consensus. □

Remark 8. Theorem 3 establishes necessary and sufficient conditions on the graph $\mathscr{G}$ to guarantee the consensus of system (1). Compared with the algebraic conditions shown in Theorems 1 and 2, this theorem is easy to imply.

In the following, we establish the LMI-based stabilizability criteria for system (20).

Proposition 2. System (20) is asymptotically stable, if there exist matrices $P_1 > 0, X_1 > 0, X_2 > 0, Y_1, Y_2, S, R, Z_1, Z_2, T, G, K$ and $\alpha > 0$ satisfying the following matrix inequalities

$$\begin{bmatrix} U_0^T X U_0 - Q_0 & J_0^T(K, G) + \alpha U_0^T \\ * & Y \end{bmatrix} > 0 \quad (21)$$

$$\begin{bmatrix} U_1^T X U_1 - Q_1 & J_1^T(K, G) + \alpha U_1^T \\ * & Y \end{bmatrix} > 0 \quad (22)$$

$$\begin{bmatrix} R & T \\ * & Z_1 \end{bmatrix} > 0 \quad (23)$$

where $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} Y = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}$, $X = \alpha^2 Y^{-1}$, $Q_0 = \begin{bmatrix} \Gamma & -T^T \\ * & -S \end{bmatrix}$,

$$\Gamma = \Phi + \begin{bmatrix} 0 & P_1 \\ P_1 & 0 \end{bmatrix}, \ \Phi = \begin{bmatrix} T \\ 0 \end{bmatrix} + \begin{bmatrix} T \\ 0 \end{bmatrix}^T + \begin{bmatrix} S & 0 \\ 0 & d_M R \end{bmatrix} \\ + d_m Z_1 + (d_M - d_m) Z_2,$$

$$J_0(K, G) = \begin{bmatrix} \bar{A} & -I_{2p} & -\bar{B} \\ \bar{A} & -I_{2p} & -\bar{B} \end{bmatrix}, \ U_0 = \begin{bmatrix} I_{2p} & 0 & 0 \\ 0 & I_{2p} & 0 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} -R & 0 \\ 0 & -Z_2 \end{bmatrix}, \ J_1(K, G) = \begin{bmatrix} \bar{B} & 0 & 0 \\ \bar{B} & 0 & 0 \end{bmatrix},$$

$$U_1 = \begin{bmatrix} 0 & I_{2p} & 0 \\ 0 & 0 & I_{2p} \end{bmatrix},$$

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \bar{B} = \begin{bmatrix} BK - GC & GC \\ BK & 0 \end{bmatrix}.$$

Proof. This proposition is a special case of Theorems 2 and 3 in Naghshtabrizi and Hespanha (2005). □

Remark 9. Summarizing the whole analysis of this paper, our paper can be regarded as an extension of Scardovi and Sepulchre (2009) and Wang et al. (2009). Specifically, our system model, the communication topology and communication constraints investigated in this article are quite different from Wang et al. (2009). Specifically, our system model is not for normal multi-agent system, but networked multi-agent systems, i.e. all the agents are connected through a communication network. Many problems such as sampled-data setting and time delay that were neglected in Wang et al. (2009) are investigated in our article. The topology in this article is directed fixed topology, which is more complicated than the undirected fixed topology in Wang et al. (2009). Compared with Scardovi and Sepulchre (2009), our consensus results can be applied to many multi-agent system since the

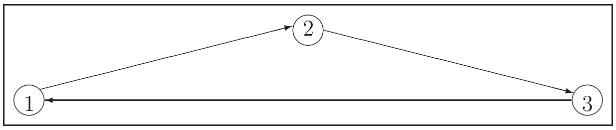

Figure 2 The topology graph of agents.

assumption condition that the matrix A in the system model is Hurwitz stable or critical Hurwitz stable is not necessary in our article.

Simulations

In this section, four examples are given to illustrate our main results in this paper.

Example 1. Consider a multi-agent supporting system (MASS) with three agents, whose interaction topology is shown in Figure 2, that obviously contains a directed spanning tree. Let $A = 1, B = 1, C = 1$, the sampling period $h = \ln 2$ and the time delay $\tau = 0$. Select the connection weights as follows: $\omega_{21} = 1$, $\omega_{32} = 5 - \sqrt{17}$, $\omega_{13} = 5 + \sqrt{17}$, whereas ω_{ij} is an arbitrary positive number for the other cases. Then, the eigenvalues of $\tilde{L} = \begin{bmatrix} 1 & 5+\sqrt{17} \\ -5+\sqrt{17} & 10 \end{bmatrix}$ are $\lambda_1 = 2$, $\lambda_2 = 9$

$$\Phi(0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2-(K-G) & -G & -(5+\sqrt{17})(K-G) & -(5+\sqrt{17})G \\ 0 & 0 & 0 & 0 & -K & 2 & -(5+\sqrt{17})K & 0 \\ 0 & 0 & 0 & 0 & -(-5+\sqrt{17})(K-G) & -(-5+\sqrt{17})G & 2-10(K-G) & -10G \\ 0 & 0 & 0 & 0 & -(-5+\sqrt{17})K & 0 & -10K & 2 \end{bmatrix}$$

$$\tilde{\Phi}_1(0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2-\lambda_1(K-G) & -\lambda_1 G \\ 0 & 0 & -\lambda_1 K & 2 \end{bmatrix},$$

$$\tilde{\Phi}_2(0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2-\lambda_2(K-G) & -\lambda_2 G \\ 0 & 0 & -\lambda_2 K & 2 \end{bmatrix}$$

It is easy to compute that

$$\det(sI_{4p} - \tilde{\Phi}_i(0)) = \begin{vmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 0 & 0 & s-(2-\lambda_i(K-G)) & \lambda_i G \\ 0 & 0 & \lambda_i K & s-2 \end{vmatrix}$$
$$= s^2[s^2 - (4-\lambda_i(K-G))s + (4 - 2\lambda_i(K-G) - \lambda_i^2 GK)] = 0, \; i = 1, 2 \tag{24}$$

By (24), $\tilde{\Phi}_i(0)$ has two nonzero eigenvalues s_{i3}, s_{i4}, satisfying $|s_{i3} + s_{i4}| = |4 - \lambda_i(K-G)|, i = 1, 2$. By Theorem 2, this multi-agent system can achieve consensus if and only if $\rho(\tilde{\Phi}_1(0)) < 1, \rho(\tilde{\Phi}_2(0)) < 1$, i.e. all the eigenvalues of $\tilde{\Phi}_1(0), \tilde{\Phi}_2(0)$ are in the unit circle. Hence, (i) $|s_{13} + s_{14}| = |4 - \lambda_1(K-G)| < |s_{13}| + |s_{14}| < 2$, i.e. $G + 1 < K < G + 3$, and (ii) $|s_{23} + s_{24}| = |4 - \lambda_2(K-G)| < |s_{23}| + |s_{24}| < 2$, i.e. $G + \frac{2}{9} < K < G + \frac{6}{9}$.

Obviously, for arbitrary G, there does not exist K satisfying (i) and (ii) simultaneously. Therefore, although the graph contains a directed spanning tree, there are no feedback gain matrices such that the multi-agent system (1) reaches consensus with the connection weights given above. Hence, the choice of connection weights has a great effect on the consensus of the multi-agent system and should be regarded as the parameter to be designed.

Example 2. This example is partly taken from Example 1 in Xi et al. (2010). For the model of a MASS given by $\ddot{\xi}_i(t) + \frac{D}{m}\dot{\xi}_i(t) + \frac{k}{m}\xi_i(t) = u_i(t)$, $i \in \{1, \cdots, N\}$, suppose that the system state cannot be obtained directly, then the observer-based control system can be given as

$$\begin{cases} \ddot{\xi}_i(t) + \frac{D}{m}\dot{\xi}_i(t) + \frac{k}{m}\xi_i(t) = u_i(t), \; i \in \{1, \cdots, N\} \\ y_i(t) = \xi_i(t) \end{cases}$$

where m is the mass of each agent, D is the damping at each agent, k is the stiffness at each agent, $\xi_i(t)$ is the height of each agent and $u_i(t)$ is the consensus protocol. Let $x_i(t) = [\xi_i(t), \dot{\xi}_i(t)]^T$, then the dynamics of the MASS with N agents can be described by (1) with $A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{D}{m} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [1 \quad 0]$. Consider a MASS with four agents, whose interaction topology is shown in Figure 3, and all the connection weights are equal to 1. Obviously, the topology graph has a directed spanning tree. The Laplacian matrix of the graph is $L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, and the eigenvalues of $\tilde{L}$ are $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Let $\alpha = 1, d_m = 0.01, d_M = 0.05$, $\tau = 0.01, m = 1000, k = 2, D = 100$. By solving LMIs

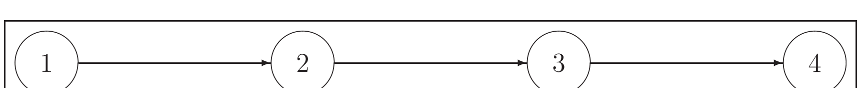

Figure 3 The topology graph with a directed spanning tree.

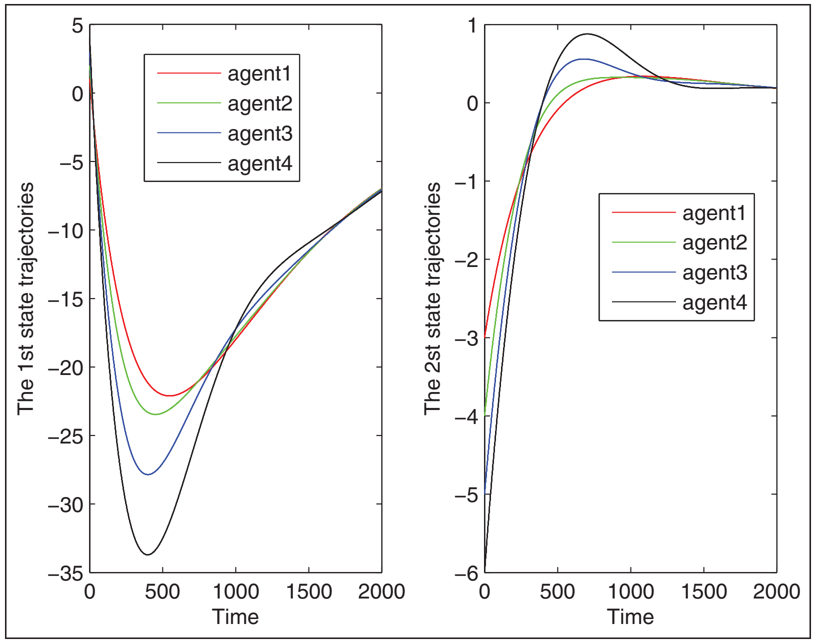

Figure 4 The first and second state trajectories of each agent.

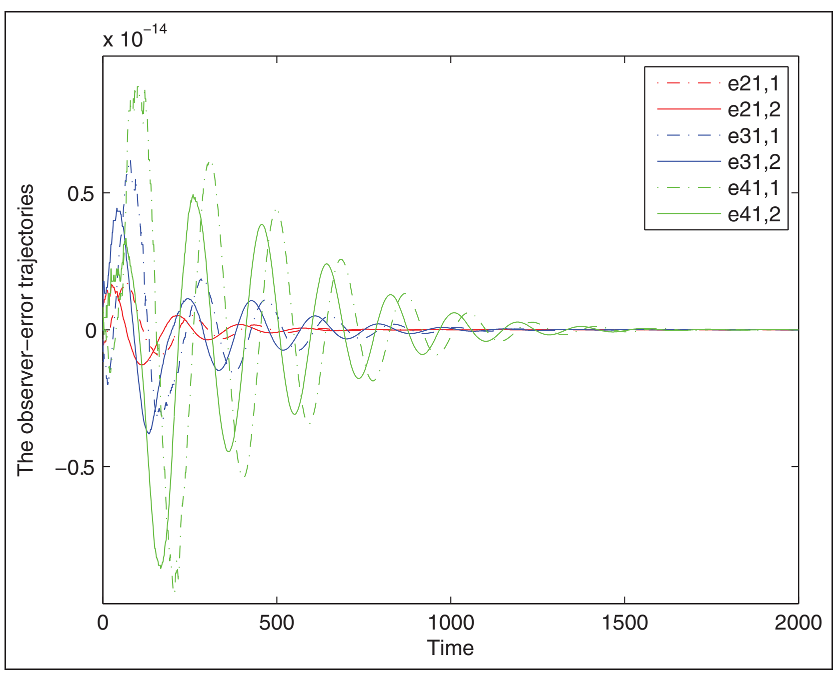

Figure 5 The observer-error trajectories.

(21)–(23), we obtain $K = [\,0.0041 \quad 5.8551\,], G = [\,-6.4916 \quad -0.0083\,]^T$. By Theorem 3, this multi-agent system can reach consensus.

As stated in Example 1 in Xi et al. (2010), our MASS can be regarded as an earthquake damage-preventing building system, where each agent in this MASS tries to keep the building horizontal when an earthquake occurs. After the earthquake, the MASS should recover to be static and horizontal, which means that the velocity and position of each agent converge to zero and a constant, respectively. Let $\xi_i(t)$, $\dot{\xi}_i(t)$ denote the absolute height/velocity of each agent in a given coordinate. Figures 4 and 5 show the simulation results with the initial values $\bar{x}_1(0) = [1, -3, 5, -11]^T$, $\bar{x}_2(0) = [2, -4, \quad 6, -12]^T$, $\bar{x}_3(0) = [3, -5, 7, -13]^T$, $\bar{x}_4(0) = [4, -6, 8, -14]^T$. One can see that this MASS can recover to be static and horizontal asymptotically.

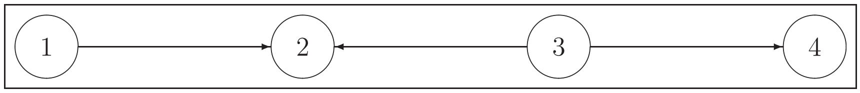

Figure 6 The topology graph with no directed spanning tree.

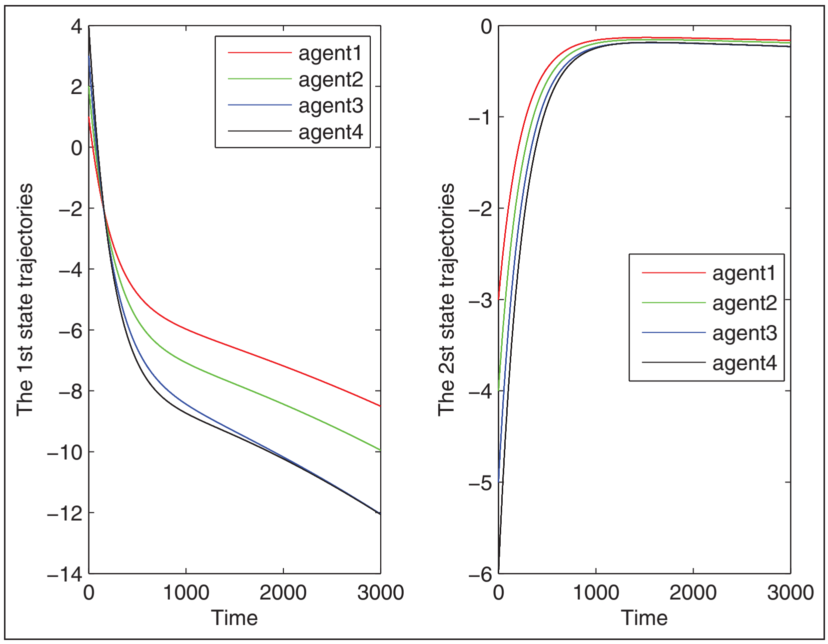

Figure 7 The state trajectories of each agent.

Example 3. Consider a multi-agent system with four agents, satisfying

$$\begin{cases} \dot{x}_i(t) = \begin{bmatrix} 0 & 0.22 \\ 0.002 & -0.1 \end{bmatrix} x_i(t) + \begin{bmatrix} 1 \\ 0.1 \end{bmatrix} u_i(t), \\ y_i(t) = [1 \quad 0] x_i(t), \quad i = 1, 2, 3, 4. \end{cases}$$

The corresponding communication topology graph given by Figure 6 does not have a directed spanning tree. By Theorem 3, the multi-agent system cannot reach consensus. Choose $\omega_{21} = 0.3, \omega_{23} = 0.2, \omega_{43} = 1$. Hence, $L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.3 & 0.5 & -0.2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. Given $K = [0.002 \quad 0.01]$, $G = [0.003 \quad 0.002]^T$. Let $\alpha = 1, d_m = 0.01, d_M = 0.05, \tau = 0.01$, then the simulation results are shown in Figures 7 and 8, where $\bar{x}_1(0) = [1, -3, 5, -11]^T$, $\bar{x}_2(0) = [2, -4, 6, -12]^T$, $\bar{x}_3(0) = [3, -5, 7, -13]^T$, $\bar{x}_4(0) = [4, -6, 8, -14]^T$.

Example 4. This example is taken from Example 6.1 in Wang et al. (2009)

$$\begin{cases} \dot{x}_i(t) = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_i(t) \\ y_i(t) = [2 \quad 1 \quad 3] x_i(t) \quad i = 1, 2, 3, 4 \end{cases}$$

Assume the adjacent graph is given by Figure 3. Let $\alpha = 1$, $d_m = 0.01$, $d_M = 0.05$, $\tau = 0.01$. By solving LMIs (21)–(23), we obtain $G = [-1.0393, -0.5135, -1.1467]^T$, $K = [-1.0960 \; -0.5956 \; 4.3218]$. By Theorem 3, this multi-agent system can reach consensus. Figures 9 and 10 show the simulation results with the initial values $\bar{x}_1(0) = [1, 7, 3, 1, 11, 1]^T$, $\bar{x}_2(0) = [-5, 2, 9, 2, -9, 2]^T$, $\bar{x}_3(0) = [-4, 6, 3, 13, 3, -3]^T$, $\bar{x}_4(0) = [7, 4, -4, 4, -14, -8]^T$. Wang et al. (2009) cannot determine whether this multi-agent system reaches consensus or not since delay phenomena were neglected. Compared with Wang et al. (2009), our simulations show that although the state trajectories in our paper change in a larger range, networked multi-agent systems with both time delays and sampled-data control can still reach consensus by using longer times.

Conclusions

This paper has studied the consensus problem of data-sampled multi-agent systems with time-varying communication delays under fixed topology. Some necessary and sufficient conditions for the consensus problem have been obtained. The study of networked multi-agent systems is still a challenging problem, and this paper can serve as a stepping stone to study more complicated networked systems with both sampling and time delays. Future work includes consensus problems in the stochastic switching topology case and

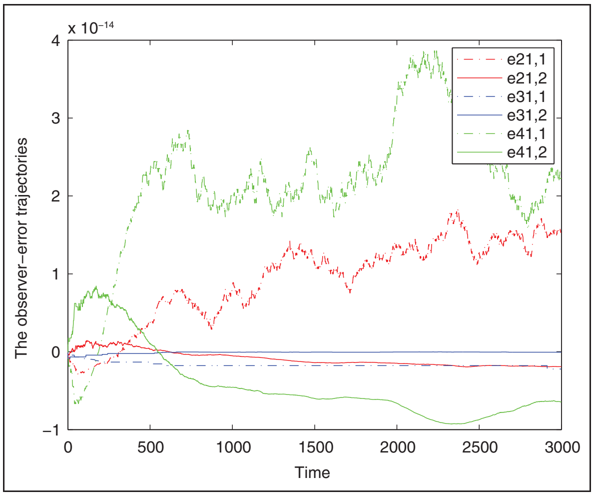

Figure 8 The observer-error trajectories.

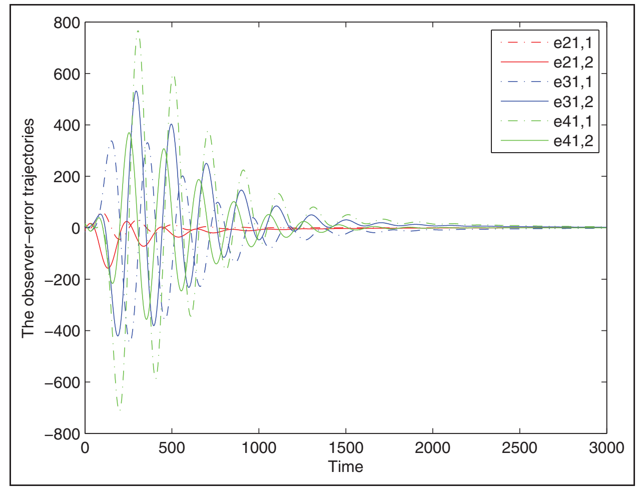

Figure 10 The observer-error trajectories.

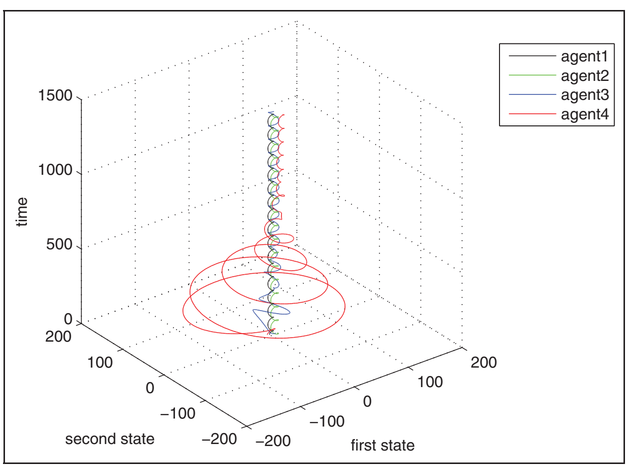

Figure 9 The state trajectories of each agent.

the case that the mobile agents communicate via a network with noise, variable delays and occasional packet losses.

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