REAL ANALYSIS with ECONOMIC APPLICATIONS

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... mathematics is very much like poetry ... what makes a good poem - a great poem - is that there is a large amount of thought expressed in very few words. In this sense formulas like

$$e^{\pi i} + 1 = 0 \quad \text{or} \quad \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

are poems.

Lipman Bers

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Preface

This is primarily a textbook on mathematical analysis for graduate students in economics. While there are a large number of excellent textbooks on this broad topic in the mathematics literature, most of these texts are overly advanced relative to the needs of a vast majority of economics students, and concentrate on various topics that are not readily helpful for studying economic theory. Moreover, it seems that most economics students lack the time and/or courage to enroll in a math course at the graduate level. Sometimes this is not even for bad reasons, for only few math departments offer classes that are designed for the particular needs of economists. Unfortunately, more often than not, the consequent lack of mathematical background creates problems for the students at a later stage of their education since an exceedingly large fraction of economic theory is impenetrable without some rigorous background in real analysis. The present text aims at providing a remedy for this inconvenient situation.

My treatment is rigorous, yet selective. I prove a good number of results here, so the reader will have plenty of opportunity to sharpen his/her understanding of the "theorem-proof" duality, and to work through a variety of "deep" theorems of mathematical analysis. However, I take many shortcuts. For instance, I avoid complex numbers at all cost, assume compactness of things when one could get away with separability, introduce topological and/or topological linear concepts only via metrics and/or norms, and so on. My objective is not to report even the main theorems in their most general form, but rather to give a good idea to the student why these are true, or even more importantly, why one should suspect that they must be true even before they are proved. But the shortcuts are not overly extensive in the sense that the main results covered here possess a good degree of applicability, especially for mainstream economics. Indeed, the purely mathematical development of the text is put to good use through several applications that provide concise introductions to a variety of topics from economic theory. Among these topics are individual decision theory, cooperative and noncooperative game theory, welfare economics, information theory, general equilibrium and finance, and intertemporal economics.

An obvious dimension that differentiates this text from various books on real analysis pertains to the choice of topics. I put much more emphasis on topics that are immediately relevant for economic theory, and omit some standard themes of real analysis that are of secondary importance for economists. In particular, unlike most treatments of mathematical analysis found in the literature, I work here quite a bit on order theory, convex analysis, optimization, linear and nonlinear correspondences, dynamic programming, and calculus of variations. Moreover, apart from direct applications to economic theory, the exposition includes quite a few fixed point theorems, along with a leisurely introduction to differential calculus in Banach spaces. (Indeed, the latter half of the text can be thought of as providing a modest introduction to geometric (non)linear analysis.) However, because they play only a minor role in modern economic theory, I do not at all discuss topics like Fourier analysis, Hilbert spaces and spectral theory in this book.

While I assume here that the student is familiar with the notion of "proof" – within the first semester of a graduate economics program, this goal must be achieved – I also spend quite a bit of time to tell the reader why things are proved the way they are, especially in the earlier part of each chapter. At various points there are (hopefully) visible attempts to help one "see" a theorem (either by discussing informally the "plan of attack," or by providing a "false-proof") in addition to confirming its validity by means of a formal proof. Moreover, whenever it was possible, I have tried to avoid the rabbit-out-of-the-hat proofs, and rather gave rigorous arguments which "explain" the situation that is being analyzed. Longer proofs are thus often accompanied by footnotes that describe the basic ideas in more heuristic terms, reminiscent of how one would "teach" the proof in the classroom.[1] This way the text is hopefully brought down to a level which would be readable for most second or third semester graduate students in economics and advanced undergraduates in mathematics, while it still preserves the aura of a serious analysis course. Having said this, however, I should note that the exposition gets less restrained towards the end of each chapter, and the analysis is presented without being overly pedantic. This goes especially for the "starred" sections which cover more advanced material than the rest of the text.

The basic approach is, of course, primarily that of a textbook rather than a reference. Yet, the reader will still find here the careful, yet unproved, statements of a good number of "difficult" theorems that fit well with the overall development. (*Examples.* Blumberg's Theorem, non-contractibility of the sphere, Rademacher's Theorem on the differentiability of Lipschitz continuous functions, Motzkin's Theorem, Reny's Theorem on the existence of Nash equilibrium, etc..) At the very least, this should hint at the student what might be expected at a higher level course. Furthermore, some of these results are widely used in economic theory, so it is desirable that the students begin at this stage developing a precursory understanding of them. To this end, I discuss a few of these "difficult" theorems at some length, talk about their applications, and at times give proofs for special cases. It is worth noting that the general exposition relies on a select few of these results.

Last, but not least, it is my sincere hope that the present treatment provides glimpses of the strength of "abstract reasoning," may it come from applied mathematical analysis or from pure analysis. I have tried hard to strike a balance in this regard. Overall, I put far more emphasis on the applicability of the main theorems relative to their generalizations and/or strongest formulations, only rarely mention if something can be achieved without invoking the Axiom of Choice, and use the method of "proof-by-contradiction" more frequently than a "purist" might like to see. On the other hand, by means of various remarks, exercises and "starred" sec-

[1] In keeping with this, I have written most of the footnotes in the first person *singular* pronoun, while using exclusively the first person *plural* pronoun in the body of the text.

tions, I try to touch upon a few topics that carry more of a pure mathematician's emphasis. (*Examples.* The characterization of metric spaces with the Banach fixed point property, the converse of Weierstrass' Theorem, various characterizations of infinite dimensional normed linear spaces, etc..) This very much reflects my full accord with the following wise words of Tom Körner:

> "... A good mathematician can look at a problem in more than one way. In particular, a good mathematician will 'think like a pure mathematician when doing pure mathematics and like an applied mathematician when doing applied mathematics'. (Great mathematicians think like themselves when doing mathematics.)"[2]

On the Structure of the Text. This book consists of four parts:

I. Set Theory (Chapters A-B)
II. Analysis on Metric Spaces (Chapters C-E)
III. Analysis on Linear Spaces (Chapters F-H)
IV. Analysis on Metric/Normed Linear Spaces (Chapters I-K)

Part I provides an elementary, yet fairly comprehensive, overview of (intuitive) set theory. Covering the fundamental notions of sets, relations, functions, real sequences, basic calculus, and countability, this part is a prerequisite for the rest of the text. It also introduces the Axiom of Choice and some of its equivalent formulations, and sketches a precursory introduction to order theory. Among the most notable theorems covered here are Tarski's Fixed Point Theorem and Sziplrajn's Extension Theorem.

Part II is (almost) a standard course on real analysis on metric spaces. It studies at length the (topological) properties of separability and compactness, and the (uniform) property of completeness, along with the theory of continuous functions and correspondences, in the context of metric spaces. I also talk about the elements of fixed point theory (in Euclidean spaces) and introduce the theories of stationary dynamic programming and Nash equilibrium. Among the most notable theorems covered here are the Contraction Mapping Principle, the Stone-Weierstrass Theorem, the Tietze Extension Theorem, Berge's Maximum Theorem, the fixed point theorems of Brouwer and Kakutani, and Michael's Selection Theorem.

Part III begins with an extensive review of some linear algebraic concepts (such as linear spaces, bases and dimension, linear operators, etc.), and then proceeds to convex analysis. A purely linear algebraic treatment of both the analytic and geometric

[2] Little is lost in translation if one adapts this quote for economic theorists. You decide:

> "... A good economic theorist can look at a problem in more than one way. In particular, a good economic theorist will 'think like a pure theorist when doing pure economic theory and like an applied theorist when doing applied theory'. (Great economic theorists think like themselves when doing economics.)"

forms of the Hahn-Banach Theorem is given here, along with several economic applications that range from individual decision theory to financial economics. Among the most notable theorems covered are the Hahn-Banach Extension Theorem, the Krein-Rutman Theorem, and the Dieudonné Separation Theorem.

Part IV can be considered as a primer on geometric linear and nonlinear analysis. Since I wish to avoid the consideration of general topology in this text, the entire discussion is couched within metric and/or normed linear spaces. The results on the extension of linear functionals and the separation by hyperplanes are sharpened in this context, an introduction to infinite dimensional convex analysis is outlined, and the fixed point theory developed earlier within Euclidean spaces is carried into the realm of normed linear spaces. The final chapter considers differential calculus and optimization in Banach spaces, and by way of an application, provides an introductory, but rigorous, approach to calculus of variations. Among the most notable theorems covered here are the Separating Hyperplane Theorem, the Uniform Boundedness Principle, the Glicksberg-Fan Fixed Point Theorem, Schauder's Fixed Point Theorems, and the Krein-Milman Theorem.

On the Exercises. As in most mathematics textbooks, the exercises provided throughout the text are integral to the present exposition, and hence appear within the main body of various sections. Many of them appear following the introduction of a particularly important concept to make the reader get better acquainted with that concept. Others are given after a major theorem in order to illustrate "how to apply" the associated result or the method of proof that yielded it.

Some of the exercises look like this:

<u>Exercise 6.</u> ...

Such exercises are "must do" ones which will be used in the material that follows them. Other exercises look either like

Exercise 6. ...

or

Exercise 6. ...

I strongly encourage the reader to work on those of the former type, for they complement the exposition at a basic level, even though skipping them would not impair one's ability to move on to subsequent material. Those of the latter kind, on the other hand, aim to provide a practice ground for the students to improve their understanding of the related topic, and at times suggest directions for further study.[3]

While most of the exercises in this book are quite "doable" – well, with a reasonable amount of suffering – some are challenging (these are "starred") and some are for the very best students (these are "double starred.") Hints and partial solutions

[3] While quite a few of these exercises are original, several of them come from the problem sets of my teachers Tosun Terzioglu, Peter Lax and Oscar Rothaus.

are provided for about one-third of them at the end of the book.[4] All in all – and this will be abundantly clear from early on – the guiding philosophy behind this text strongly subscribes to the view that there is only one way of learning mathematics, and that is *learning-by-doing.* In his preface, Chae (1995) uses the following beautiful Asian proverb to drive this point home:

> I hear, and I forget;
> I see, and I remember;
> I do, and I understand.

This recipe, I submit, should be tried also by those who wish to have some fun throughout the following five hundred some pages.

On Measure Theory and Integration. This text omits the theory of measure and Lebesgue integration in its entirety. These topics are covered in a forthcoming companion volume called *Probability Theory with Economic Applications.*

On Alternative Uses of the Text. This book is intended to serve as a textbook for a number of different courses, and also for independent study.

- *A second graduate course on mathematics-for-economists.* Such a course would use Chapter A for review, and cover the first section of Chapter B, along with pretty much all of Chapters C, D and E. This should take something like one half to two-thirds of a semester, depending on how long one wishes to spend on the applications of dynamic programming and game theory. The remaining part of the semester may then be used to go deeper into a variety of fields, such as convex analysis (Chapters F–H and parts of Chapters I and J), introductory linear analysis (Chapters F-J), or introductory nonlinear analysis and fixed point theory (parts of the Chapters F, G and I, and Chapters J-K). Alternatively, one may alter the focus, and offer a little course in probability theory whose coverage may now be accelerated. (For what's its worth, this is how I taught from the text at NYU for about 6 years with some success.)

- *A first graduate course on mathematics-for-economists.* Unless the math preparation of the class is extraordinary, this text would not serve well as a primary textbook for this sort of a course. However, it may be useful for complementary reading on a good number of topics that are traditionally covered in a first math-for-econ course, especially if the instructor wishes to touch upon infinite dimensional matters as well. (*Examples.* The earlier parts of Chapters C-E complements the standard coverage of real analysis within $\mathbb{R}^n$, Chapter C spends quite a bit of time on the Contraction Mapping Theorem and its

[4] *To the Student*: Please work on the exercises as hard as you can, before seeking out these hints. This is for your own good. Believe it or not, you'll thank me later.

applications, Chapters E provides an extensive coverage of matters related to correspondences, Chapters F-G investigate linear spaces, operators, and basic convex analysis, and include numerous separating and supporting hyperplane theorems of varying generality.)

- *An advanced (under)graduate course on real analysis for mathematics students.* While my topic selection is dictated by the needs of modern economic theory, the present text is foremost a mathematics book. It is therefore duly suitable to be used as a textbook for a course on mathematical analysis at the senior undergraduate or first year graduate level. Especially if the instructor wishes to emphasize the fixed point theory and some economic applications (regarding, say, individual decision theory), it may well help organize a full fledged math course.

- *Independent study.* One of the major objectives of this text is to provide the student with a glimpse of what lies behind the horizon of the standard mathematics that is covered in the first year of most graduate economics programs. Good portions of Chapters G-K, for instance, are relatively advanced, and hence may be deemed unsuitable for the courses mentioned above. Yet I have truly tried my best to be able make these chapters accessible to the student who needs to learn the related material but finds it difficult to follow the standard texts on linear and nonlinear functional analysis. It may eventually be necessary to study matters from more advanced treatments, but the coverage of this book may perhaps ease the pain by building a bridge between advanced texts and a standard "math-for-econ" course.

On Related Textbooks. A few words about how this text fares with other related textbooks are in order. It will become abundantly clear early on that my treatment is a good deal more advanced than that of the excellent introductory book by Simon and Blume (1994) and of the slightly more advanced text by de la Fuente (1999). While the topics of Simon and Blume (1994) are prerequisites for the present course, de la Fuente (1999) dovetails with my treatment. It is, on the other hand, for the most part equally advanced as the popular treatise by Stokey and Lucas (1989) which is sometimes taught as a second math course for economists. Most of what is assumed to be known in the latter reference is covered here. So, upon finishing the present course, the student (who wishes to take an introductory class on the theory of dynamic programming and discrete stochastic systems) would be able to read this book with a considerably accelerated pace. Similarly, after the present course, the advanced texts like Mas-Colell (1989), Duffie (1996), and Becker and Boyd III (1997) should be within reach.

Within the mathematics folklore, this book would be viewed as a continuation of a first "mathematical analysis" course, which is usually taught after or along with "advanced calculus." In that sense, it is more advanced than the expositions of Rudin (1976), Ross (1980), and Körner (2003), and is roughly at the same level with Kolmogorov and Fomin (1970), Haaser and Sullivan (1991), and Carothers (2000). The coverage of the widely popular Royden (1986) and Folland (1999) overlap quite a bit with mine as well, but the level of those books are a bit more advanced. Finally, a related text which is exceedingly more advanced than the present one is that of Aliprantis and Border (1999). That book covers an amazing plethora of topics from functional analysis, and should serve as a useful advanced reference book for any student of economic theory.

Errors. While I desperately tried to avoid them, a number of errors have surely managed to sneak past me. I can only hope that they are not substantial. The errors that I have identified after the publication of the book will be posted in my webpage http://homepages.nyu.edu/~eo1/books.html. Please do not hesitate to email me of the ones you find – my email address is efe.ok@nyu.edu.

Acknowledgments. There are many economists and mathematicians who have contributed significantly to this book. My good students Sophie Bade, Boğaçhan Çelen, Juan Dubra, Andrei Gomberg, Yusufcan Masatlioglu, Francesc Ortega, Onur Özgür, Liz Potamites, Maher Said, and Hilary Sarneski-Hayes carefully read substantial parts of the manuscript and identified a good deal of errors. All the figures in the text are drawn kindly, and with painstaking care, by Boğaçhan Çelen – I owe a lot to him.

In addition, numerous comments and corrections I received from Jose Apesteguia, Jean-Pierre Benoît, Alberto Bisin, Kim Border, Victor Klee, Peter Lax, Claude Lemaréchal, Jing Li, Massimo Marinacci, Tapan Mitra, Louise Nirenberg, Debraj Ray, Ennio Stachetti, and Srinivasa Varadhan shaped the structure of the text considerably. Especially with Jean-Pierre Benoît and Ennio Stachetti I had long discussions about the final product. Finally, I should acknowledge that my mathematical upbringing, and hence the making of this book, owes much to the very many discussions I had with Tapan Mitra at Cornell by his famous little blackboard.

At the end of the day, however, my greatest debt is to my students who have unduly suffered the preliminary stages of this text. I can only hope that I was able to teach them as much as they taught me.

Efe A. Ok
New York, 2005

PREREQUISITES

This text is intended primarily for an audience who has taken at least one "mathematics for economists" type course at the graduate level or an "advanced calculus" course *with proofs*. Consequently, it is assumed that the reader is familiar with the basic methods of calculus, linear algebra, and nonlinear (static) optimization, which would be covered in such a course. For completeness purposes, a relatively comprehensive review of the basic theory of real sequences, functions and ordinary calculus is provided in Chapter A. In fact, many facts concerning real functions are reproved later in the text in a more general context. Nevertheless, having a good understanding of real-to-real functions often helps getting intuition about things in more abstract settings. Finally, while most students come across metric spaces by the end of the first semester of their graduate education in economics, I do not assume any prior knowledge of this topic here.

To judge things for yourself, check if you have some "feeling" for the following facts:

- Every monotonic sequence of real numbers in a closed and bounded interval converges in that interval.
- Every concave function defined on an open interval is continuous and quasiconcave.
- Every differentiable function on $\mathbb{R}$ is continuous, but not conversely.
- Every continuous function defined on a closed and bounded interval attains its maximum.
- A set of vectors that spans $\mathbb{R}^n$ has at least n vectors.
- A linear function defined on $\mathbb{R}^n$ is continuous.
- The (Riemann) integral of every continuous function defined on a closed and bounded interval equals a finite number.
- The Fundamental Theorem of Calculus.
- The Mean Value Theorem.

If you have certainly seen these results before, and if you can sketch a quick (informal) argument regarding the validity of about half of them, you are well prepared to read this book. (All of these results, or substantial generalizations of them, are proved within the text.)

The economic applications covered here are foundational for the large part, so they do not require any sophisticated economic training. However, if you have taken at least one graduate course on microeconomic theory, then you would probably appreciate the importance of these applications better.

BASIC CONVENTIONS

• The frequently used phrase "if and only if" is often abbreviated in the text as "iff."

• Roughly speaking, I label a major result as a *theorem*, a result less significant than a theorem, but still of interest, as a *proposition*, a more or less immediate consequence of a theorem or proposition as a *corollary*, a result whose main utility derives from its aid in deriving a theorem or proposition as a *lemma*, and finally, certain auxiliary results as *claims*, *facts* or *observations*.

• Throughout this text n stands for an arbitrary positive integer. This symbol will correspond almost exclusively to the (algebraic) dimension of a Euclidean space, hence the notation $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, then it is understood that x_i is the real number that corresponds to the ith coordinate of x, that is, $x = (x_1, ..., x_n)$.

• I use the notation $\subset$ in the strict sense. That is, implicit in the statement $A \subset B$ is that $A \neq B$. The "subsethood" relation in the weak sense is denoted by $\subseteq$.

• Throughout this text the symbol $\square$ symbolizes the ending of a particular discussion, may it be an example, observation, or a remark. The symbol $\|$ ends a claim within a proof of a theorem, proposition, etc., while $\blacksquare$ ends the proof itself.

• For any symbols $\clubsuit$ and $\heartsuit$, the expressions $\clubsuit := \heartsuit$ and $\heartsuit =: \clubsuit$ mean that $\clubsuit$ is defined by $\heartsuit$. (This is the so-called "Pascal notation.")

• While the chapters are labeled by Latin letters, the sections and subsections of the chapters are labeled by positive integers. Consider the following sentence:

> By Proposition 4, the conclusion of Corollary B.1 would be valid here, so by using the observation noted in Example D.3.[2], we find that the solution to the problem mentioned at the end of Section J.7 exists.

Here Proposition 4 refers to the proposition numbered 4 in the chapter that this sentence is taken from. Corollary B.1 is the Corollary 1 of Chapter B, Example D.3.[2] refers to part 2 of Example 3 that is given in Chapter D, and Section J.7 is the seventh section of Chapter J. (*Theorem.* The chapter from which this sentence is taken cannot be any one of the chapters B, D and J.)

• The rest of the notation and conventions that I adopt throughout the text are explained in Chapter A.