# Mixed OLS-TLS for the estimation of Dynamic Processes with a Linear Source Term*

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**Abstract.** We present a novel technique to eliminate strong biases in parameter estimation were part of the data matrix is not corrupted by errors. Problems of this type occur in the simultaneous estimation of optical flow and the parameter of linear brightness change as well as in range flow estimation. For attaining highly accurate optical flow estimations under real world situations as required by a number of scientific applications, the standard brightness change constraint equation is violated. Very often the brightness change has to be modelled by a linear source term. In this problem as well as in range flow estimation, part of the data term consists of an exactly known constant. Total least squares (TLS) assumes the error in the data terms to be identically distributed, thus leading to strong biases in the equations at hand. The approach presented in this paper is based on a mixture of ordinary least squares (OLS) and total least squares, thus resolving the bias encountered in TLS alone. Apart from a thorough performance analysis of the novel estimator, a number of applications are presented.
**Keywords.** *parameter estimation, least squares, dynamic processes, brightness change, optical flow.*

## 1 Introduction

Many different methods to recover the optical flow exist [3]. In the context of this paper a gradient based technique for optical flow estimation is used. Here motion computations are motivated by scientific applications. As such they were extended to parameterize the underlying physical processes [6, 12, 13].

In most gradient based techniques the optical flow estimates are obtained by pooling local constraints over a small spatio-temporal neighborhood in a least squares sense. This approach does of course assume the parameters of the constraint equations to be constant throughout the region of support. This assumption can be violated at motion discontinuities, thus leading astray the estimator presented in this paper. To overcome this limitation the estimator can readily be extended to robust statistic by means of M- or LMSOD estimation [1, 6, 8, 11].

Using ordinary least squares (OLS) techniques the temporal derivatives are treated as erroneous observations and the spatial gradients as error free. This approach will

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lead to biases in the estimates, as all gradients are generally obscured by noise [14]. Under these circumstances the use of a total least squares (TLS) method [20] is the estimator of choice [16]. The local constraints of gradient based optical flow techniques do generally not incorporate brightness changes. This can of course only be a first approximation, as brightness changes due to inhomogeneous or fluctuating illumination prevail in real world scenes. Moreover, in scientific applications these brightness changes may be induced by physical processes. Hence the parameters of brightness change might be equally important as the actual optical flow [13]. A number of physically induced brightness changes as well as those caused by inhomogeneous illumination can be modelled quite accurately by a source term in the constraint equation. Additionally does the computation of surface motion from range data lead to the same type of constraints [19]. This type of equations can be thought of as multivariate intercept models. The data matrix of such a model for the TLS estimator contains a column of exactly known elements thus inducing a strong bias in the estimation. This bias can be efficiently eliminated by mixing the OLS and TLS estimator as outlined in the next section.

## 2 Mixing Ordinary Least Squares and Total Least Squares

In TLS estimates the parameter vector $\boldsymbol{p}^{\mathrm{est}}$ converges to the true vector $\boldsymbol{p}$ only for independently and identically distributed errors in the observations [5, 9, 20]. This means that all observations should have the same standard deviation $\sigma$, which can be achieved by scaling the data accordingly, an approach also known as equilibration [10, 15, 17]. However, there are instances when one column in the data matrix is known *exactly*, that is it is not subject to any errors. This is the case in intercept models of the form

$$c + a_1 x_1 + \cdots + a_m x_m = b, \tag{1}$$

which will be used in a number of applications introduced in Section 4. Such a model gives rise to an overdetermined set of equations of the form

$$(1_N; \boldsymbol{A}) \begin{pmatrix} c \\ \boldsymbol{x} \end{pmatrix} = b, \tag{2}$$

where $1_N = (1, \ldots, 1)^\top$ is the first column of the data matrix and thus exactly known.

The accuracy of the estimated parameters can be maximized by requiring that the exactly known columns in the data matrix be unperturbed [4, 20]. This can be achieved by reformulating the TLS problem in a more general form by mixing OLS and TLS:

**Definition 1** *Given a set of $n$ linear equations with $p$ unknown parameters $\boldsymbol{x}$*

$$(\boldsymbol{A}_1, \boldsymbol{A}_2)\,\boldsymbol{x} = \boldsymbol{b}, \quad \text{with} \quad \boldsymbol{A}_1 \in \mathrm{I\!R}^{n \times p_1}, \boldsymbol{A}_2 \in \mathrm{I\!R}^{n \times p_2}, \boldsymbol{x} \in \mathrm{I\!R}^{p}, \boldsymbol{b} \in \mathrm{I\!R}^{n}, \tag{3}$$

*and $p_1 + p_2 = p$. The mixed OLS-TLS problem then seeks to minimize*

$$\begin{aligned} &\min \;\; \left[(\boldsymbol{A}_2, \boldsymbol{b})\,\boldsymbol{p}_2\right]^2 \\ &\textit{subject to} \;\; (\boldsymbol{A}_1, \boldsymbol{A}_2)\,\boldsymbol{x} = \boldsymbol{A}_1\boldsymbol{x}_1 + \boldsymbol{A}_2\boldsymbol{x}_2 = \boldsymbol{b}, \end{aligned} \tag{4}$$

*where $\boldsymbol{p} = \left(\boldsymbol{x}^\top, -1\right)^\top$, $\boldsymbol{p}_2 = \left(\boldsymbol{x}_2^\top, -1\right)^\top$ and $\boldsymbol{x} = \left(\boldsymbol{x}_1^\top, \boldsymbol{x}_2^\top\right)^\top$.*

In the specific example of Equation (1) $p_1 = 1$ and $p_2 = m$. Equation (4) can thus be depicted as first finding a TLS solution on the reduced subspace of erroneous observations and than choosing from this set the one solution that solves the equations of unperturbed data exactly.

In the event of all observations $\boldsymbol{A}$ being known exactly, the OLS-TLS solution reduces to the OLS solution, while at the other extreme of only erroneous data the problem reduces to the TLS problem.

### 2.1 Implementation of Mixed OLS-TLS Estimator

The implementation of the mixed OLS-TLS estimator is straightforward. The columns of the data matrix $\boldsymbol{A}$ are permuted by a permutation matrix $\boldsymbol{P}$ in such a way, that the submatrix $\boldsymbol{A}_1$ contains the $p_1$ exactly known observations, that is

$$\boldsymbol{A}\cdot\boldsymbol{P} = (\boldsymbol{A}_1, \boldsymbol{A}_2)\,, \text{ where } \boldsymbol{A} \in \mathbb{R}^{n\times p}, \boldsymbol{A}_1 \in \mathbb{R}^{n\times p_1}, \boldsymbol{A}_2 \in \mathbb{R}^{n\times p_2}, \boldsymbol{P} \in \mathbb{R}^{p\times p}. \quad (5)$$

In a next step a QR factorization of the matrix $(\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{b})$ is performed, thus

$$(\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{b}) = \boldsymbol{Q}\begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{0} & \boldsymbol{R}_{22} \end{pmatrix}, \quad (6)$$

with $\boldsymbol{Q}$ being orthogonal and $\boldsymbol{R}_{11}$ upper triangular. The QR factorization is justified because the singular vectors and singular values of a matrix are not changed by multiplying it by an orthogonal matrix [10].

The solution for the sub system of equations $\boldsymbol{R}_{22}\boldsymbol{p}_2 = 0$ is computed in a TLS sense, which boils down to an singular value analysis of the data matrix $\boldsymbol{R}_{22}$ [20].

With the known estimate of $\boldsymbol{p}_2$ the system of equations $\boldsymbol{R}_{11}\boldsymbol{p}_1 + \boldsymbol{R}_{12}\boldsymbol{p}_2 = 0$ is solved for $\boldsymbol{p}_1$ by back-substitution. The parameter vector $\boldsymbol{p} = \left(\boldsymbol{p}_1^\top, \boldsymbol{p}_2^\top\right)^\top$ has then to be transformed back reversing the initial permutations of the columns by $\boldsymbol{p} \leftarrow \boldsymbol{P}^{-1}\boldsymbol{p}$. The step of permuting the data and parameters can of course be omitted by formulating the problem in such a way that the constant terms are in the first columns of the data matrix, as will be done in the remainder of this paper.

## 3 Comparison of OLS-TLS and TLS

In this section the properties of both the mixed OLS-TLS and standard TLS estimator shall be analyzed. We make use of the Generalized Brightness Change Constraint Equation (GBCCE) with constant linear motion and the brightness change modelled with a source term, that is

$$\begin{pmatrix} -1 & g_{x,1} & g_{y,1} & g_{t,1} \\ -1 & g_{x,2} & g_{y,2} & g_{t,2} \\ \vdots & \vdots & \vdots & \vdots \\ -1 & g_{x,n} & g_{y,n} & g_{t,n} \end{pmatrix} \cdot \begin{pmatrix} c \\ \delta x \\ \delta y \\ \delta t \end{pmatrix} = \boldsymbol{D}\cdot\boldsymbol{p} = 0, \quad \text{with} \quad \boldsymbol{D} \in \mathbb{R}^{n\times 4},\ \boldsymbol{p} \in \mathbb{R}^4. \quad (7)$$

Here $n$ represents the size of the spatio-temporal neighborhood and $g_{i,j}$ the partial derivative of the grey value $g$ with respect to the coordinate $i$ at pixel location $j$.

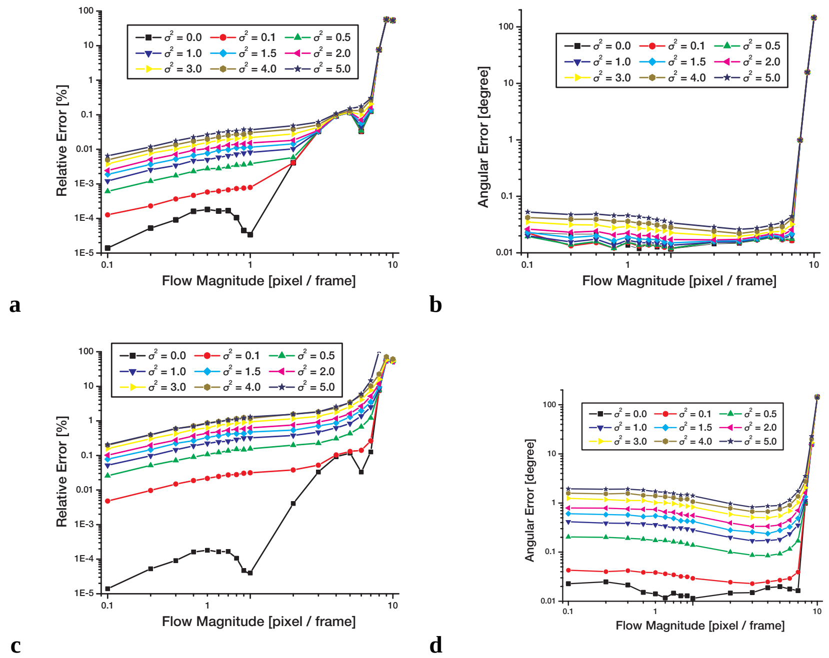


**Fig. 1.** Comparison of the relative error $E_r$ of the flow magnitude and corresponding angular errors $E_\phi$. In **a** the relative Error $E_r$ of the mixed OLS-TLS estimator is shown and in **b** the corresponding angular Error. Respectively, in **c** and **d** both $E_r$ and $E_\phi$ are shown for the TLS estimator.

Following [2] the algorithms were tested on a sinusoidal test sequence. For optical flow computation it is interesting to study the dependence of the computed optical flow $\boldsymbol{f} = (u, v)^\top = (dx/dt, dy/dt)^\top$ on the noise added to the synthetic sequence. In the present context it is of equal importance to know how accurate the intensity change present in the sequence can be detected. To address these issues first a constant intensity change was uniformly added to the sequence. The magnitude of the flow was varied from no movement ($v_{\mathrm{corr}} = 0$ pixel / frame) up to $v_{\mathrm{corr}} = 10$ pixel / frame in 20 steps, with the direction of the velocity vector along one coordinate axis. Although this is not a common situation encountered in real world situations, most gradient filters possess optimum properties along this direction [18]. Hence results presented here give a lower bound for movement along other directions. The reason for choosing this specific direction is that the performance of the optical flow computation was to be analyzed independent of the actual optimization of the gradient filter used. Along other directions the actual performance of gradient filters can vary significantly and is subject to filter optimization [18]. The results of this analysis are shown in Figure 1.

The accuracy of establishing an estimate for the parameter $c$ of brightness change in Equation (7) was examined with the three alternative techniques, namely the mixed OLS-TLS, the scaled TLS and the plain TLS estimator. Also the accuracy of detecting

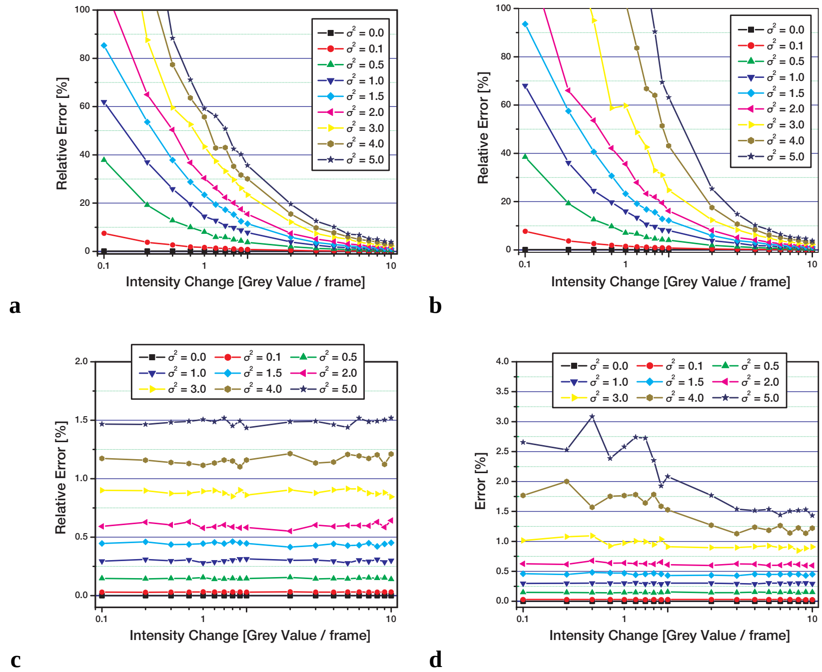


**Fig. 2.** Comparison of the relative errors in estimating an intensity change at fixed flow magnitude. In **a** the intensity change is computed from the mixed OLS-TLS estimator and in **b** with the TLS estimator. Shown in **c** and **d** are the relative errors in computing the magnitude of the optical flow $|\boldsymbol{f}_{\text{est}}|$ for an increasing level of intensity change for both the mixed (**c** ) and the TLS (**d** ) estimators.

the optical flow $f_{\text{est}} = (u_{\text{est}}, v_{\text{est}})^\top$ under different flow magnitudes and different intensity changes was inspected. Not all the resulting plots are presented in this paper. In Figure 2 the relative errors of the intensity changes are shown. It can be seen that the OLS-TLS estimator presents the most accurate results, while the scaled TLS estimate is prone to slightly larger errors. The unscaled TLS technique proves to be quite inaccurate, most notably on higher noise levels. Generally all estimators exhibit the highest accuracy on large intensity changes. The accuracy of recovering the flow magnitude proved to be independent of the intensity change in the OLS-TLS estimator and depends linearly on the noise level $\sigma$. The TLS estimate is biased towards higher intensity changes.

In Figure 3 the performance of OLS-TLS, scaled TLS and TLS are shown in fitting a line with intersect, where parameter $a$ represents the slope of the line and parameter $b$ the offset. The slope is of course equivalent to the optical flow and the offset to the source term of intensity change. The great reduction of error of the OLS-TLS estimator with respect to the TLS can be seen quite nicely. Albeit reducing the bias, the scaled TLS estimator is still less acurate than the OLS-TLS by roughly 2%.

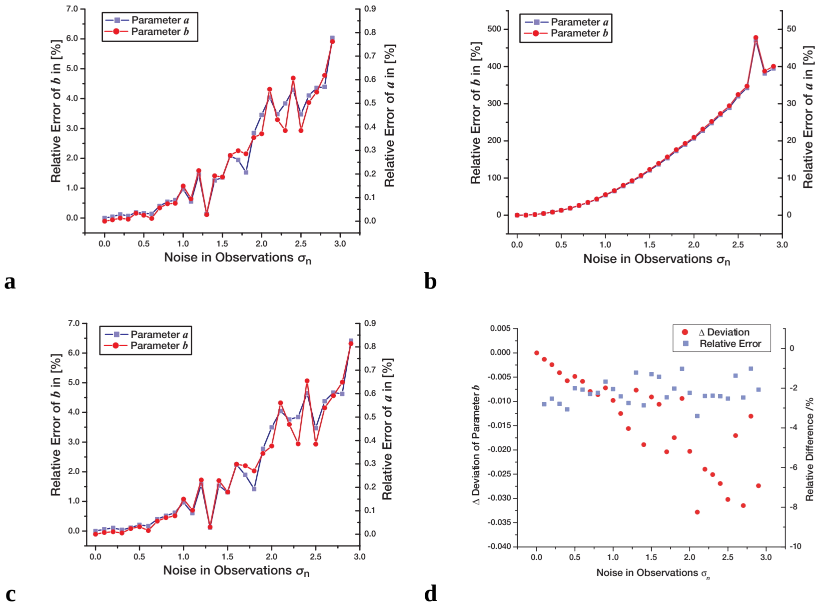


**Fig. 3.** The relative error measure $E_r$ for the OLS-TLS and unscaled TLS estimators is shown in **a** and **b** respectively. The same for the scaled TLS estimate in **c** . The relative difference between the two OLS-TLS and scaled TLS estimator is shown in **d** , indicating that the OLS-TLS estimator still has a better performance than the scaled TLS estimator by roughly $2\%$.

## 4 Applications

The mixed OLS-TLS estimator was tested on synthetic data in the previous section. In this section its performance is analyzed on real data. It is difficult to obtain ground truth data for optical flow with intensity change. To this end ground truth data was recorded with a range sensor and a structured light system. Correct movement of two objects (crumbled paper and a toy tiger, see Figure 4) was established by placing them on a system of linear positioners. Range flow can be estimated by employing the range flow constraint equation [21], were the depth velocity can be treated as a source term [19]. The relative error of the flow can thus be estimated. Comparing the mixed OLS-TLS and TLS estimators we obtain:

| method | $E_{\text{paper}}[\%]$ | density$_{\text{paper}}[\%]$ | $E_{\text{tiger}}[\%]$ | density$_{\text{tiger}}[\%]$ |
|---|---|---|---|---|
| TLS | $0.29 \pm 2.88$ | 12.3 | $6.0 \pm 6.5$ | 16.1 |
| Mixed | $0.18 \pm 0.13$ | 22.1 | $2.1 \pm 2.9$ | 10.0 |

These findings underline the superior performance of the mixed OLS-TLS estimator. It is evident that the mixed estimator can under some circumstances help to increase the density of the estimate. As it turns out, choosing correct thresholds for correct estimates is less dependent of the specific image sequences at hand, greatly simplifying

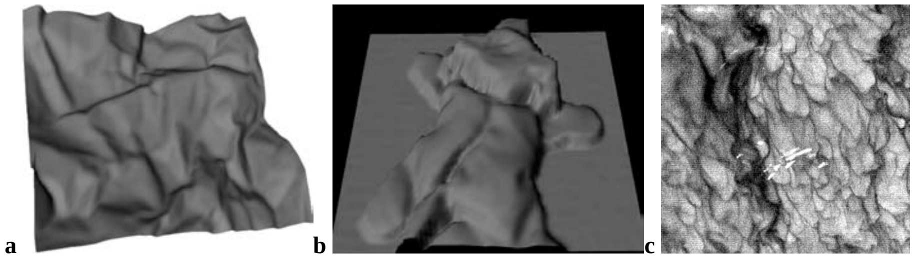

**Fig. 4.** Real test sequences: **a** crumpled sheet of paper **b** toy tiger **c** thermographic image of ocean surface.

the use of this estimator in image processing applications. The reduction of the bias, evident in the smaller relative Error $E$ is remarkable. Only with this reduced bias is it possible to employ the presented estimator on a number of scientific applications, where the source term might be an important parameter in the underlying physical processes.

One such process of interest is the heat transfer at the sea surface. It can be shown that this source term of linear intensity change is equivalent to the total derivative of the temperature structures recorded from infrared sequences[8]. Such a thermographic image is shown in Figure 4. From this total derivative with respect to time the net heat flux at the sea interface can be estimated. Due to this use of thermography coupled with the estimator presented in this paper both spatially and temporally highly resolved heat flux estimates at the sea surface could be attained for the first time[7]. Without the reduction of the bias by the mixed estimator presented in this paper a sufficiently accurate estimate of the net heat flux would not be feasible.

## 5 Conclusion

In this paper a mixed OLS-TLS estimator was presented that allows to significantly reduce the bias in the estimation of the parameters in differential equations with a linear source term. This type of equation can be used to estimate the optical flow subject to linear intensity changes in the scene or optical flow from range data. The standard TLS estimator proved to be too inaccurate for this type of problem. It was shown that it exhibits a strong bias and thus depends highly on the noise level and intensity change present in the imagery. By performing a column scaling of the data matrix $\boldsymbol{D}_{\text{noise}}$, this bias could be lessened somewhat. The virtual deviation of the exactly known first column of the data matrix has to be scaled with a variance of at least four orders of magnitude smaller than that found in the other columns. Numerically more attractive and also providing the most accurate results is the mixed OLS-TLS estimator presented in this paper, which is an unbiased estimator under iid Gaussian noise. This results was verified both on synthetic and real data. These findings emphasize the importance of this mixed estimator for accurate estimation of optical flow with linear intensity change or range flow based on range data.

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