A Bass formula for Gorenstein injective dimension

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Abstract

In this paper a generalized version of the Bass formula is proved for finitely generated modules of finite Gorenstein injective dimension over a commutative noetherian ring.

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Introduction

In 1969, M. Auslander and M. Bridger (cf. [1]) introduced and studied the "G-dimension" of a finitely generated module over a noetherian ring. This homological invariant is a refinement of the classical projective dimension and shares some of its nice properties. The dual notion of "Gorenstein injective dimension" was defined in the mid nineties by E. E. Enochs and O. M. G. Jenda [10]. It can also be considered as a generalization of the classical notion of injective dimension.

This paper deals with Gorenstein injective dimension of finitely generated modules. The main result (theorem 2.1) gives a Gorenstein injective version of the classical "Bass formula" over a commutative noetherian ring.

Theorem 2.1 *Let S be a commutative noetherian ring. If M is a finitely generated*

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S-module of finite Gorenstein injective dimension, then

$$\mathrm{Gid}_S M = \sup\{\,\mathrm{depth}\, S_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}(M)\,\}.$$

As a corollary, in 2.2 we prove that over a commutative noetherian local ring R with $\dim R - \mathrm{depth}\, R \leq 1$, the equality $\mathrm{Gid}_R M = \mathrm{depth}\, R$ holds for every finitely generated R-module M of finite Gorenstein injective dimension. This result generalizes theorem 6.2.15 of [5], where the same formula is proved over a Cohen-Macaulay local ring which admits a dualizing module.
In the last part of the paper we deal with another "Bass type" equation due to Ischebeck (cf. [14, 2.6]). Example 2.5 shows that a Gorenstein version of the formula is not true, but we prove another generalization (proposition 2.6) which gives rise to the following generalization of the classical Bass formula.

Corollary 2.7 *Let $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local homomorphism of noetherian local rings. If N is a finitely generated S-module of finite injective dimension over R then* $\mathrm{depth}\, R = \mathrm{id}_R N$.

Convention. Throughout this paper, all rings are unitary, commutative and noetherian. Furthermore, $(R, \mathfrak{m}, k)$ denotes a local ring with maximal ideal $\mathfrak{m}$ and residue field k.

1 Basic Definitions

In this section we review basic definitions and properties of Gorenstein injective dimensions. For details and proofs see [12] or [5].

Definition 1.1. *An R-module M is said to be* Gorenstein injective *if and only if there exists an exact complex of injective R-modules,*

$$I = \cdots \to I_2 \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow I_{-2} \longrightarrow \cdots$$

such that the complex $\mathrm{Hom}_R(J, I)$ is exact for every injective R-module J and M is the kernel in degree 0 of I.

It is clear that every injective module is Gorenstein injective and therefore one can construct a Gorenstein injective resolution of any module.

Definition 1.2. *Let M be an R-module. A* Gorenstein injective resolution *of M is an exact sequence*

$$0 \to M \to G_0 \to G_{-1} \to \cdots$$

such that G_i is Gorenstein injective for all $i \geq 0$.
We say that the module M has Gorenstein injective dimension *less than or equal to n,* $\mathrm{Gid}_R M \leq n$, *if M has a Gorenstein injective resolution*

$$0 \to M \to G_0 \to G_{-1} \to \cdots \to G_{-n} \to 0.$$

It is clear that one always has

$$\mathrm{Gid}_R M \leq \mathrm{id}_R M.$$

The equality holds if $\mathrm{id}_R M < \infty$ (cf. [5, 6.2.6]).

Note that if the Gorenstein injective dimension is finite, then it can be computed in terms of vanishing of the Ext functors (cf. [13, 2.22]).

Theorem 1.3. *Let M be an R-module of finite Gorenstein injective dimension. Then*

$$\mathrm{Gid}_R M = \sup\{i \,|\, \mathrm{Ext}^i_R(J, M) \neq 0 \text{ for an } R\text{−module } J \text{ with } \mathrm{id}_R J < \infty\}.$$

2 Main Results

Recall that if a finitely generated module over a local ring has finite injective dimension then its injective dimension is equal to the depth of the base ring. This is known as the Bass formula (cf. [3, 3.1.17]). In [5] Christensen has proved that over a Cohen-Macaulay local ring with a dualizing module, one can replace injective dimension with Gorenstein injective dimension. More recently, the result has been proved over a local ring which admits a dualizing complex (cf. [7, 6.4]).

The following theorem, which is the main result of this paper, gives a Gorenstein injective version of the Bass formula over an *arbitrary* commutative noetherian ring.

Theorem 2.1. *Let S be a ring and M a finitely generated S-module of finite Gorenstein injective dimension. Then*

$$\mathrm{Gid}_S M = \sup\{\, \operatorname{depth} S_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}(M)\,\}.$$

Proof. First assume that $\mathrm{Gid}_S M = 0$. To prove the theorem in this case, it is enough to show that $\operatorname{depth} S_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p} \in \operatorname{Supp}(M)$.

Suppose that $\operatorname{depth} S_{\mathfrak{p}} > 0$ for a prime ideal $\mathfrak{p} \in \operatorname{Spec}(S)$. Then $\mathfrak{p}S_{\mathfrak{p}}$ contains an $S_{\mathfrak{p}}$-regular element, say x. Since M is a Gorenstein injective S-module, truncating its complete injective resolution, we get an exact sequence

$$0 \to N \to I \to M \to 0$$

where I is an injective S-module.
Apply the functor $\operatorname{Hom}_{S_{\mathfrak{p}}}(S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}, -)$ to the localization of the short exact sequence above and use the fact that $I_{\mathfrak{p}}$ is an injective $S_{\mathfrak{p}}$-module to get the following exact sequence.

$$0 \to \operatorname{Ext}^1_{S_{\mathfrak{p}}}(S_{\mathfrak{p}}/xS_{\mathfrak{p}}, M_{\mathfrak{p}}) \to \operatorname{Ext}^2_{S_{\mathfrak{p}}}(S_{\mathfrak{p}}/xS_{\mathfrak{p}}, N_{\mathfrak{p}}) \to 0$$

But $\operatorname{Ext}^2_{S_{\mathfrak{p}}}(S_{\mathfrak{p}}/xS_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ because $\operatorname{pd}_{S_{\mathfrak{p}}} S_{\mathfrak{p}}/xS_{\mathfrak{p}} = 1$. Thus we have

$$\operatorname{Ext}^1_{S_{\mathfrak{p}}}(S_{\mathfrak{p}}/xS_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0.$$

On the other hand, the exact sequence

$$0 \to S_{\mathfrak{p}} \xrightarrow{.x} S_{\mathfrak{p}} \to S_{\mathfrak{p}}/xS_{\mathfrak{p}} \to 0$$

induces the following exact sequence.

$$M_{\mathfrak{p}} \xrightarrow{.x} M_{\mathfrak{p}} \to \operatorname{Ext}^1_{S_{\mathfrak{p}}}(S_{\mathfrak{p}}/xS_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$$

Nakayama's lemma shows that $\mathfrak{p} \notin \operatorname{Supp}(M)$, which proves the desired formula for a finitely generated Gorenstein injective module.

Now assume that $\mathrm{Gid}_S M = n > 0$. By [7, 2.14], there exists a short exact sequence

$$0 \to K \to L \to M \to 0,$$

where K is a Gorenstein injective S-module and $\mathrm{id}_S L = \mathrm{Gid}_S M = n$.

By Chouinard's equality [4, 3.1], we have

$$\mathrm{id}_S L \;=\; \sup\{\, \operatorname{depth} S_{\mathfrak p} - \operatorname{width}_{S_{\mathfrak p}} L_{\mathfrak p} \mid \mathfrak p \in \operatorname{Supp}(L)\,\},$$

where by definition

$$\operatorname{width}_{S_{\mathfrak p}} L_{\mathfrak p} \;=\; \inf\{\, i \mid \operatorname{Tor}_i^{S_{\mathfrak p}}(S_{\mathfrak p}/\mathfrak p S_{\mathfrak p}\,,\, L_{\mathfrak p}) \neq 0\}.$$

Choose a prime ideal $\mathfrak p \in \operatorname{Supp}(L)$. The sequence

$$0 \to K_{\mathfrak p} \to L_{\mathfrak p} \to M_{\mathfrak p} \to 0$$

of $S_{\mathfrak p}$-modules and $S_{\mathfrak p}$-homomorphisms is exact.

Assume that $\mathfrak p \in \operatorname{Supp}(M)$. Applying the functor $(- \otimes_{S_{\mathfrak p}} S_{\mathfrak p}/\mathfrak p S_{\mathfrak p})$ to the sequence above, we get the exact sequence $L_{\mathfrak p}/\mathfrak p L_{\mathfrak p} \to M_{\mathfrak p}/\mathfrak p M_{\mathfrak p} \to 0$. By Nakayama's lemma $M_{\mathfrak p}/\mathfrak p M_{\mathfrak p}$ and therefore $L_{\mathfrak p}/\mathfrak p L_{\mathfrak p}$ is not zero. Hence $\operatorname{width}_{S_{\mathfrak p}} L_{\mathfrak p} = 0$ for every $\mathfrak p \in \operatorname{Supp}(M)$.

If $\mathfrak p \notin \operatorname{Supp}(M)$, then $L_{\mathfrak p} \cong K_{\mathfrak p}$. Localizing the complete injective resolution of K to the prime ideal $\mathfrak p$, we get an exact sequences

$$\cdots \to I_1 \to I_0 \to L_{\mathfrak p} \to 0,$$

where I_i's are injective $S_{\mathfrak p}$-modules. Suppose that K_i is the kernel of the homomorphism $I_i \to I_{i-1}$ in this complex.

For any $S_{\mathfrak p}$-module T of finite projective dimension t and for any positive integer i, the modules $\operatorname{Ext}^i_{S_{\mathfrak p}}(T, L_{\mathfrak p})$ and $\operatorname{Ext}^{i+t}_{S_{\mathfrak p}}(T, K_t)$ are isomorphic. So they are both zero modules since $i + t > \operatorname{pd}_{S_{\mathfrak p}} T$.
Using [6, 5.3] to get the second equality below, we have

$$\begin{aligned} 0 \;&= \sup\{\, i \mid \operatorname{Ext}^i_{S_{\mathfrak p}}(T, L_{\mathfrak p}) \neq 0, \text{ for some } S_{\mathfrak p}\text{−module } T \text{ with } \operatorname{pd}_{S_{\mathfrak p}} T < \infty \,\} \\ &= \sup\{\, \operatorname{depth} S_{\mathfrak q} - \operatorname{width}_{S_{\mathfrak q}} L_{\mathfrak q} \mid \mathfrak q S_{\mathfrak p} \in \operatorname{Supp}(L_{\mathfrak p})\,\}. \end{aligned}$$

So $\operatorname{depth} S_{\mathfrak p} - \operatorname{width}_{S_{\mathfrak p}} L_{\mathfrak p} \le 0$ for every $\mathfrak p \in \operatorname{Supp}(L) \backslash \operatorname{Supp}(M)$.

Therefore,

$$\begin{aligned}\operatorname{Gid}_S M &= \operatorname{id}_S L \\ &= \sup\{\operatorname{depth} S_{\mathfrak{p}} - \operatorname{width}_{S_{\mathfrak{p}}} L_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}(L)\} \\ &= \sup\{\operatorname{depth} S_{\mathfrak{p}} - \operatorname{width}_{S_{\mathfrak{p}}} L_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}(M)\} \\ &= \sup\{\operatorname{depth} S_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}(M)\}. \qquad \square\end{aligned}$$

The following corollary is a local version of theorem 2.1 which generalizes Christensen's Gorenstein version of Bass formula [5, 6.2.15]. We say that a local ring R is *almost Cohen-Macaulay* if the inequality $\dim R - \operatorname{depth} R \leq 1$ holds.

Corollary 2.2. *Let $(R, \mathfrak{m}, k)$ be an almost Cohen-Macaulay local ring and let M be a finitely generated R-module. If $\operatorname{Gid}_R M < \infty$ then*

$$\operatorname{Gid}_R M = \operatorname{depth} R.$$

Proof. Use theorem 2.1 and the fact that over an almost Cohen-Macaulay ring, for prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$, the inequality $\operatorname{depth} R_{\mathfrak{p}} \leq \operatorname{depth} R_{\mathfrak{q}}$ holds (cf. [6, 3.1]). $\square$

The next corollary of theorem 2.1 is a change of rings result for Gorenstein injective dimension. In lemma 2 of [15], Salarian, Sather-Wagstaff and Yassemi prove that over a local ring, Gorenstein injective dimension "behaves well with respect to killing a regular element". Namely, if M is a finitely generated module over a local ring $(R, \mathfrak{m}, k)$ then $\operatorname{Gid}_R M < \infty$ implies $\operatorname{Gid}_{R/xR} M/xM < \infty$, where x is an R- and M-regular element in $\mathfrak{m}$. The following corollary is the quantitative version of that result.

Corollary 2.3. *Let $(R, \mathfrak{m}, k)$ be a local ring and M a finitely generated R-module. If $x \in \mathfrak{m}$ is an R- and M-regular element, then*

$$\operatorname{Gid}_{R/xR} M/xM \leq \operatorname{Gid}_R M - 1.$$

Furthermore, the equality holds when R is almost Cohen-Macaulay and $\operatorname{Gid}_R M$ is finite.

Proof. If $\operatorname{Gid}_R M$ is not finite then the inequality is clear. Now assume that M has finite Gorenstein injective dimension. To prove the desired inequality, it is sufficient to use theorem 2.1 and the following facts.

- $\mathrm{Supp}(M/xM) = \{\, \mathfrak{p}/xR \,|\, \mathfrak{p} \in \mathrm{Supp}(M)$ and $x \in \mathfrak{p} \,\}$.
- If $x \in \mathfrak{p}$ then $\mathrm{depth}\,(R/xR)_{\mathfrak{p}/xR} = \mathrm{depth}\, R_{\mathfrak{p}} - 1$.

The last part of the corollary is a consequence of corollary 2.2. $\square$

The following immediate corollary of 2.1 shows that finite Gorenstein injective dimension does not grow under localization.

Corollary 2.4. *Let S be a ring and M a finitely generated S-module. If $\mathfrak{p} \subseteq \mathfrak{q}$ are prime ideals and $M_{\mathfrak{p}}$ has finite Gorenstein injective dimension then*

$$\mathrm{Gid}_{S_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \mathrm{Gid}_{S_{\mathfrak{q}}} M_{\mathfrak{q}}.$$

Now we study another Bass type equality. In [14, 2.6], Ischebeck proves the following formula from which the classical Bass formula can be recovered by setting M equal to the residue field of the base ring.

Theorem *Let $(R, \mathfrak{m}, k)$ be a local ring and let M and N be finitely generated R-modules. If $\mathrm{id}_R N < \infty$ then*

$$\mathrm{depth}\, R - \mathrm{depth}_R M = \sup\{i \,|\, \mathrm{Ext}^i_R(M, N) \neq 0\}.$$

It is natural to ask whether a Gorenstein injective version of this theorem is also true. The answer is negative.

Example 2.5. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring which is not regular. Then k has finite Gorenstein injective dimension but its projective dimension is infinite. If in Ischebeck's theorem, finite injective dimension could be relaxed to finite Gorenstein injective dimension, then $\sup\{i \,|\, \mathrm{Ext}^i_R(k, k) \neq 0\}$ would have to be finite, i.e. $\mathrm{pd}_R k$ would be finite, which is not true.*

The following statement is a partial generalization of Ischebeck's result in another direction.

Proposition 2.6. *Let $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local homomorphism of local rings and let M be an R-module. For any finitely generated S-module N of finite* *injective dimension over R the following equality holds, provided that* $\operatorname{depth}_R M = 0$ *or M is a finitely generated R-module.*

$$\operatorname{depth} R - \operatorname{depth}_R M = \sup\{i \mid \operatorname{Ext}_R^i(M, N) \neq 0\}$$

Proof. Set $\operatorname{id}_R N = t$. If $\operatorname{depth}_R M = 0$ then there exists a short exact sequence

$$0 \to k \to M \to C \to 0$$

which induces a long exact sequence

$$\cdots \to \operatorname{Ext}_R^t(M, N) \to \operatorname{Ext}_R^t(k, N) \to \operatorname{Ext}_R^{t+1}(C, N) \to \cdots .$$

Since $\operatorname{Ext}_R^{t+1}(C, N) = 0$ and $\operatorname{Ext}_R^t(k, N) \neq 0$ (cf. [2, 5.5]), we have $\operatorname{Ext}_R^t(M, N) \neq 0$ and then

$$\sup\{i \mid \operatorname{Ext}_R^i(M, N) \neq 0\} \geq t.$$

The reverse inequality holds clearly. If M is finitely generated we use induction on $\operatorname{depth}_R M$ to prove the desired equality. If $\operatorname{depth}_R M > 0$ then there exists an M-regular element $x \in \mathfrak{m}$. Using the long exact sequence induced by the exact sequence $0 \to M \xrightarrow{.x} M \to M/xM \to 0$, the equality can be proved from the induction hypothesis. $\square$

The following corollary of 2.6 is another generalization of the classical Bass formula. The result has also appeared in [16, 5.2].

Corollary 2.7. *Let $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local homomorphism of local rings. For any finitely generated S-module N of finite injective dimension over R, the following equality holds.*

$$\operatorname{depth} R = \operatorname{id}_R N$$

Proof. In 2.6, set $M = R/\mathfrak{m}$ and use [2, 5.5]. $\square$

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