Achieving Network Optima Using Stackelberg Routing Strategies

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ABSTRACT

In noncooperative networks users make control decisions that optimize their individual performance objectives. Nash equilibria characterize the operating points of such networks. Nash equilibria are inherently inefficient and exhibit suboptimal network performance. Focusing on routing, a methodology is devised for overcoming this deficiency, through the intervention of the network manager. The manager controls part of the network flow, is aware of the noncooperative behavior of the users and performs its routing aiming at improving the overall system performance. The existence of *maximally efficient* strategies for the manager, i.e., strategies that drive the system into the global network optimum, is investigated. Necessary and sufficient conditions for the existence of a maximally efficient strategy are derived, and it is shown that they are met in many cases of practical interest. The maximally efficient strategy is shown to be unique and it is specified explicitly. Such a strategy does not exists when the population of users is infinite, or when the users employ suboptimal shortest-path routing. For these cases, an optimal strategy of the manager is determined.

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1. Introduction

Control decisions in large scale networks are often made by each user independently, according to its own individual performance objectives.[1] Such networks are henceforth called *noncooperative*, and game theory [MYE91, FUD92] provides the systematic framework to study and understand their behavior. The operating points of a noncooperative network are the *Nash equilibria* of the underlying game, that is, the points where unilateral deviation does not help any user to improve its performance. Game theoretic models have been employed in the context of flow control [BOV87, HSIA91, ZHA92, ALT94, KOR95], routing [ECO91, ALT93, ORD93] and virtual path bandwidth allocation [LAZ95] in modern networking. These studies mainly investigate the structure of the Nash equilibria and provide valuable insight into the nature of networking under decentralized and noncooperative control.

Nash equilibria are inherently inefficient [DUB86] and exhibit, in general, suboptimal network performance. This deficiency can be overcome with the intervention of a *network* agent, namely the network designer or manager, that architects the network so that the resulting equilibria are efficient according to some systemwide criterion. In essence, the designer/manager architects the Nash equilibria by setting the rules of the networking game. Seen under this light, the idea is related to the economic theory of implementation in Nash equilibrium [HUR85]. In the context of computer networking, various methods have been proposed for architecting Nash equilibria:

- Through pricing mechanisms. This method has been studied extensively in the context

[1] The term "user" is purposely left ambiguous. It may refer to a network user itself or, in case that the user's traffic consists of multiple connections, to individual connections that are controlled independently.

of queueing systems [KLE67, NAO69, ADI74, MEN90], where it was observed that, by levying tolls, a system can regulate the decisions made by its noncooperative users. Its applicability to the future Internet is discussed in [COC93].

- By regulating service disciplines. In [SHE94] it is shown that a proper queue scheduling discipline can guarantee an equilibrium point with desirable properties.
- Through proper network design. In [KOR94] it is shown that, by making appropriate topology design and capacity allocation decisions, the network designer can choose a systemwide efficient equilibrium.

The above approaches demand either the addition of a new component to the networking structure, such as prices, or else *a priori* design decisions on the resource configuration and/or the service disciplines of the network. In the present study, we propose a method for architecting noncooperative equilibria in the *run time phase*, i.e., during the actual operation of the network. This approach is based on the observation that, apart from the flow generated by the self-optimizing users, typically, there is also some network flow that is controlled by a central entity, that will be referred to as the "manager." Typical examples are the traffic generated by signaling and/or control mechanisms, as well as traffic of users that belong to virtual networks. The manager attempts to optimize the system performance, through the control of its portion of the flow.

The role of the manager in a noncooperative network is investigated using routing as a control paradigm. The network is shared by a set of noncooperative users, each shipping its flow in a way that optimizes its individual performance objective. The noncooperative routing scenario applies to various modern networking environments. The Internet Protocol (IPv4), for example, provides the option of source routing [ISI81], that enables the user to determine the path(s) its flow follows from source to destination. This option allows the user to choose a routing strategy that satisfies its individual perfomance objective. Similarly, the current IP Next Generation (IPv6) Specification provides for source routing with enhanced capabilities [DEE95]. Another example is the flexible routing service as specified in the Q.1211 CCITT Recommendation for the standardized capability set of Intelligent Networks (IN CS-1) [GAR93]. One of the goals of this service is to route calls over particular facilities based on the subscriber's routing preference list or distribution algorithm.[2] Flexible routing was one of the services that were successfully implemented in Ameritech's AIN 0.0 technical

[2]The target services of IN CS-1 apply to the setup/release phase of a call. It is expected, however, that these services will be extended to the active phase of a call. For example, Bellcore's AIN Release 1 target extends flexible routing services to the active phase of a call.

trial, in April 1992 [RUS93]. Also, Bell Atlantic's AIN configuration will provide complete routing control to the customer.

The manager has the following goals and capabilities: (i) it aims to optimize the overall network performance according to some systemwide efficiency criterion, and (ii) it is cognizant of the noncooperative behavior of the users and performs its routing based on this information. The first property makes the manager just another user, whose performance objective coincides with that of the network. The second property, however, enables the manager to predict the response of the noncooperative users to any routing strategy that it chooses, and hence determine a strategy that would pilot them to an operating point that optimizes the overall network performance. Instead of *reacting* to the routing strategies of the users, the manager *fixes* this optimal strategy and lets the users converge to their respective equilibrium. This is a typical scenario of a *Stackelberg game* [OWE82, MYE91], where the manager acts as a *leader*, that imposes its strategy on the self-optimizing users that behave as *followers*.[3] Stackelberg strategies have been investigated in the context of flow control in [DOU89], and routing in [ECO90]. In these references, however, the leader was a selfish user concerned about its own rather than the system's performance.

We investigate the optimal strategy of the manager. In particular, we address the following question: is there a routing strategy of the manager that drives the system into the network optimum, i.e., to the point that corresponds to the solution of a routing problem, in which the manager has full control over the *entire* flow offered to the network? Intuitively, one would expect that the manager cannot enforce the network optimum, since it controls only part of the flow, while the rest is controlled by noncooperative users. Surprisingly, this study shows that in many cases the manager does have this capability.

The methodology is developed for a system of parallel links, which, as explained in the sequel, is well-suited for modeling typical configurations in modern networking. We derive *necessary and sufficient* conditions that guarantee that the manager can enforce an equilibrium that coincides with the network optimum, and indicate that these conditions are met in many cases of practical interest. In other words, the manager is often able to achieve, through limited control, the same system performance as in the case of centralized control. Moreover, when these conditions are satisfied, we show that there exists a unique strategy of the manager that drives the system to the network optimum, specify its structure explicitly, and comment on its scalability properties.

Three different configurations of followers will be considered. The first is the case of a *single follower*, that corresponds to the simplest Stackelberg routing game, where except

[3]The terms "manager" and "leader," as well as "users" and "followers," will be used interchangeably.

for the manager there is another, self-optimizing, entity that controls its own flow. The second is the general case of *multiple followers*, that is, an arbitrary but finite number of noncooperative users. The third is the case of *simple followers*, which corresponds to two interesting scenaria, namely, an infinite population of users, and a finite population that employs simple, suboptimal, shortest-path routing.

The outline of the paper is the following. In Section 2 we present the parallel links model and formulate the problem. Section 3 gives an outline of the main results. In Section 4 we briefly describe the structure of the network optimum and Nash equilibrium. The single-follower problem is addressed in Section 5, and the multiple-follower extension is presented in Section 6. In Section 7 we address some practical issues related to the proposed management scheme. The special case of simple followers is discussed in Section 8. Finally, Section 9 summarizes the results and delineates their implications.

2. Model and Problem Formulation

We consider a set $\mathcal{I} = \{1, \ldots, I\}$ of users, that share a set $\mathcal{L} = \{1, \ldots, L\}$ of communication links, interconnecting a common source to a common destination node. The users are non-cooperative, in the sense that each user routes its flow in a way that optimizes its individual performance objective. Apart from the flow generated by the noncooperative users, there is also some flow whose routing is controlled by a central network entity, i.e., the manager. The manager is cognizant of the noncooperative behavior of the users and performs its routing based on this information, in a way that optimizes the overall network performance. For the sake of uniform notation, the manager will also be referred to as user 0. Let $\mathcal{I}_0 = \mathcal{I} \cup \{0\}$.

Let c_l be the capacity of link l, $\mathbf{c} = (c_1, \ldots, c_L)$ the capacity configuration, and $C = \sum_{l \in \mathcal{L}} c_l$ the total capacity of the system of parallel links. We assume that $c_1 \geq \ldots \geq c_L$. Each user $i \in \mathcal{I}_0$ has a throughput demand that is some process with average rate $r^i > 0$. Without loss of generality, we assume that the throughput demands of the noncooperative users satisfy: $r^1 \geq r^2 \geq \ldots \geq r^I$. Let $r = \sum_{i \in \mathcal{I}} r^i$ denote the total throughput demand of the noncooperative users, and $R = r + r^0$ the total demand offered to the network. We assume that the system of parallel links can accommodate the total demand, i.e., that $R < C$.

User $i \in \mathcal{I}_0$ ships its flow by splitting its demand r^i over the set of parallel links. Let f_l^i denote the expected flow that user i sends on link l. The user flow configuration $\mathbf{f}^i = (f_1^i, \ldots, f_L^i)$ is called a routing *strategy* of user i and the set $F^i = \{\mathbf{f}^i \in I\!\!R^L : 0 \leq f_l^i \leq c_l,\ l \in \mathcal{L};\ \sum_{l \in \mathcal{L}} f_l^i = r^i\}$ of strategies that satisfy the user's demand is called the strategy space of user i. The system flow configuration $\mathbf{f} = (\mathbf{f}^0, \mathbf{f}^1, \ldots, \mathbf{f}^I)$ is called a routing *strategy profile* and takes values in the product strategy space $F = \otimes_{i \in \mathcal{I}_0} F^i$.

The grade of service that the flow of user $i \in \mathcal{I}_0$ receives is quantified by means of a cost function $J^i : F \to I\!R$. $J^i(\mathbf{f})$ is the cost of user i under strategy profile $\mathbf{f}$; the higher $J^i(\mathbf{f})$ is, the lower the grade of service provided to the flow of the user. Various general classes of routing cost functions have been considered in [ORD93]. In the present paper, we consider cost functions that are the sum of link cost functions:

$$J^i(\mathbf{f}) = \sum_{l \in \mathcal{L}} J^i_l(\mathbf{f}_l), \quad J^i_l(\mathbf{f}_l) = f^i_l T_l(f_l), \quad l \in \mathcal{L}, \tag{2.1}$$

where $\mathbf{f}_l = (f^0_l, f^1_l, \ldots, f^I_l)$, and $T_l(f_l)$ is the average delay per unit of flow on link l and depends only on the total flow $f_l = \sum_{i \in \mathcal{I}_0} f^i_l$ on that link. In particular, we concentrate on the M/M/1 delay function:

$$T_l(f_l) = \begin{cases} \dfrac{1}{c_l - f_l}, & f_l < c_l \\ \infty, & f_l \geq c_l \end{cases}. \tag{2.2}$$

Eqs. (2.1) and (2.2) imply that $J^i(\mathbf{f})/r^i$ is the average time-delay that the flow of user i experiences under strategy profile $\mathbf{f}$. Similarly, $J(\mathbf{f})/R$, where:

$$J(\mathbf{f}) = \sum_{i \in \mathcal{I}_0} J^i(\mathbf{f}) = \sum_{l \in \mathcal{L}} \frac{f_l}{c_l - f_l}, \tag{2.3}$$

is the average time-delay experienced by the total flow offered to the network.

The total cost $J(\mathbf{f})$ of the network depends only on the link flow configuration $(f_1, \ldots, f_L)$. Since $\sum_l f_l(c_l - f_l)^{-1}$ is a convex function of $(f_1, \ldots, f_L)$, there exists a *unique* link flow configuration $(f^*_1, \ldots, f^*_L)$ – with $f^*_l \geq 0$ and $\sum_l f^*_l = R$ – that minimizes the total cost. This is the solution of the classical routing optimization problem, where the routing of all flow in the network is centrally controlled, and will be referred to as the network optimal link flow configuration, or for simplicity as the *network optimum*. The Kuhn-Tucker optimality conditions [LUE84], imply that $(f^*_1, \ldots, f^*_L)$ is the network optimum if and only if there exists a (Lagrange multiplier) λ^*, such that for every link $l \in \mathcal{L}$:

$$\lambda^* = \frac{c_l}{(c_l - f^*_l)^2}, \quad \text{if } f^*_l > 0, \tag{2.4}$$

$$\lambda^* \leq \frac{1}{c_l}, \quad \text{if } f^*_l = 0. \tag{2.5}$$

Let J^* denote the minimal total cost, that is achieved at the network optimum $(f^*_1, \ldots, f^*_L)$. Then, for any strategy profile $\mathbf{f} \in F$, we have $J(\mathbf{f}) \geq J^*$.

2.1 Validity of the Parallel Links Model

Systems of parallel links, albeit inherently simple, represent an appropriate model for seemingly unrelated networking problems. Consider, for example, a network in which resources are preallocated to various routing paths that do not interfere. Such scenaria are common in modern networking. In broadband networks bandwidth is separated among different virtual paths, resulting effectively in a system of parallel and noninterfering "links" between source/destination pairs. Another example is that of internetworking, in which each "link" models a different subnetwork.

2.2 Noncooperative Users

Each user $i \in \mathcal{I}$ aims to find a routing strategy $\mathbf{f}^i \in F^i$ that minimizes its cost J^i, or equivalently its average time-delay. This optimization problem depends on the routing decisions of the manager and the other users, described by the strategy profile $\mathbf{f}^{-i} = (\mathbf{f}^0, \mathbf{f}^1, \ldots, \mathbf{f}^{i-1}, \mathbf{f}^{i+1}, \ldots, \mathbf{f}^I)$, since J^i is a function of the system flow configuration $\mathbf{f}$.

As already explained, the routing strategy of the manager is fixed, as long as the set of noncooperative users and their throughput demands do not change. Throughout this section we assume that the manager employs strategy $\mathbf{f}^0$, according to some criterion that will be presented in the sequel. Each noncooperative user, on the other hand, adjusts its routing strategy to the actions of the other noncooperative users, in order to minimize its cost. This self-optimizing mode of operation leads to a dynamic behavior that can be modelled as a noncooperative game. Any operating point of the network is a Nash equilibrium of this game, i.e., a strategy profile $\mathbf{f}^{-0}$ of the noncooperative users, from which no user finds it beneficial to unilaterally deviate. These operating points depend on the manager's strategy $\mathbf{f}^0$. Hence, given that the manager employs strategy $\mathbf{f}^0$, strategy profile $\mathbf{f}^{-0} \in F^{-0}$ is a Nash equilibrium of the user routing game if:

$$\mathbf{f}^i \in \arg\min_{\mathbf{g}^i \in F^i} J^i(\mathbf{g}^i, \mathbf{f}^{-i}), \quad i \in \mathcal{I}. \tag{2.6}$$

From the perspective of the users, the manager merely reduces the capacity of each link l by f_l^0. Therefore, the user routing game is equivalent to the routing game in a system of parallel links with capacity configuration $\mathbf{c} - \mathbf{f}^0$. As shown in [ORD93], this routing game has a *unique* Nash equilibrium. Hence, any strategy $\mathbf{f}^0$ of the manager induces a unique Nash equilibrium $\mathbf{f}^{-0}$ of the noncooperative users, that will be denoted by $\mathcal{N}^0(\mathbf{f}^0)$.

Given a strategy profile $\mathbf{f}^{-i}$ of the other users in $\mathcal{I}_0$, the cost of user i, as defined by eqs. (2.1) and (2.2), is a convex function of its strategy $\mathbf{f}^i$. Hence, the minimization problem in (2.6) has a unique solution. The Kuhn-Tucker optimality conditions, then, imply that $\mathbf{f}^i$ is the optimal response of user i to $\mathbf{f}^{-i}$ if and only if there exists a (Lagrange multiplier) λ^i, such that, for every link $l \in \mathcal{L}$, we have:

$$\lambda^i = \frac{c_l - f_l^{-i}}{(c_l - f_l)^2}, \quad \text{if } f_l^i > 0, \tag{2.7}$$

$$\lambda^i \leq \frac{1}{c_l - f_l}, \quad \text{if } f_l^i = 0, \tag{2.8}$$

where $f_l^{-i} = \sum_{j \in \mathcal{I}_0 \setminus \{i\}} f_l^j$ is the total flow that all users except the i-th send on link l. Therefore, $\mathbf{f}^{-0} \in F^{-0}$ is the Nash equilibrium of the self-optimizing users induced by strategy $\mathbf{f}^0$ of the manager, if and only if there exist λ^i, $i \in \mathcal{I}$, such that the optimality conditions (2.7)–(2.8) are satisfied for all $i \in \mathcal{I}$.

The function $\mathcal{N}^0 : F^0 \rightarrow F^{-0}$ that assigns to each strategy of the manager the induced equilibrium of the user routing game is called the *Nash mapping*. From [KOR94, Theorem 3.3], it follows that the Nash mapping is continuous.

2.3 The Role of the Manager

The manager has knowledge of the noncooperative behavior of the users, that enables it to determine the Nash equilibrium $\mathcal{N}^0(\mathbf{f}^0)$ induced by any routing strategy $\mathbf{f}^0$ that it chooses. Being a central network entity, the manager either has the necessary information available, or can obtain it by monitoring the behavior of the users. This way, the manager can determine a routing strategy of its own flow that gives rise to a Nash equilibrium that is optimal, according to some systemwide efficiency criterion. Therefore, the manager acts as a Stackelberg leader, that imposes its strategy on the self-optimizing users that behave as followers. The presence of sophisticated users that can acquire information about the self-optimizing behavior of the other users and become Stackelberg leaders, in order to optimize their own performance, is in general undesirable [SHE94]. The manager, however, aims at optimizing the overall network performance, thus it plays a social rather than a selfish role.

The goal of the manager is to find a routing strategy of its own flow that drives the system to the network optimum, i.e., a strategy $\mathbf{f}^0$ such that if $\mathbf{f}^{-0} = \mathcal{N}^0(\mathbf{f}^0)$, then $\sum_{i \in \mathcal{I}_0} f_l^i = f_l^*$ for all $l \in \mathcal{L}$. Any such strategy of the manager achieves the minimal total cost J^* and, therefore, leads to the most efficient utilization of network resources. Using economics terminology, this is a problem of Nash implementation of *maximal efficiency* [COC93]. A general description of Nash implementation of social choice functions – a special case of which is maximal efficiency – can be found in [HUR85]. Accordingly, let us introduce the following:

Definition 2.1 *Let* $\mathbf{f}^0 \in F^0$ *be a strategy of the manager and* $\mathbf{f}^{-0} = \mathcal{N}^0(\mathbf{f}^0)$*. Strategy* $\mathbf{f}^0$ *is called* maximally efficient *if it achieves the network optimum, i.e., if* $\sum_{i \in \mathcal{I}_0} f_l^i = f_l^*$ *for all* $l \in \mathcal{L}$*.*

Continuity of the Nash mapping implies that $J(\mathbf{f}^0, \mathcal{N}^0(\mathbf{f}^0))$ is continuous in $\mathbf{f}^0 \in F^0$, thus it attains its minimum in the compact set F^0. Therefore, an *optimal* strategy of the manager always exists. Existence of a maximally efficent strategy, however, cannot be guaranteed in general. Evidently, if a maximally efficient strategy exists, then it is an optimal strategy of the manager.

In the following sections, we derive necessary and sufficient conditions that guarantee existence of a maximally efficient strategy of the manager. Moreover, provided that these conditions are met, we show that the maximally efficient strategy of the manager is unique and we specify its structure explicitly. Before we proceed with the analysis, let us present an informal summary of the main results.

3. Outline of Results

1. In the special case of a single user, the manager can always enforce the network optimum, and we specify its maximally efficient strategy.

2. In the general case of any finite number of users, the manager can enforce the network optimum if and only if its demand is higher than some threshold $\underline{r}^0$, in which case we specify the manager's maximally efficient strategy.

3. The threshold $\underline{r}^0$ is feasible, in the sense that the total demand of the users plus $\underline{r}^0$ is lower than the total capacity of the network. Thus, for every set of users (whose total demand r is less than the total capacity C) there are managers that can enforce the network optimum.

4. In heavily loaded networks it is "easy" for the manager to enforce the network optimum (i.e., the threshold $\underline{r}^0$ is small).

5. As the number of users increases, it becomes harder for the manager to enforce the network optimum (i.e., the threshold $\underline{r}^0$ increases).

6. The higher the difference in the throughput demands of any two users, the easier it becomes for the manager to enforce the network optimum. Conversely, the threshold $\underline{r}^0$ is highest when the demands of all users are equal.

7. In the case of an infinite number of users, the manager cannot, in general, enforce the network optimum. For this case, we derive the structure of an optimal strategy of the manager, and a simple algorithm to determine it.[4]

4. Preliminary Structural Results

The structure of the Nash equilibrium in a system of parallel links shared by I noncooperative users has been investigated in [ORD93, KOR94]. The results of these references can be readily applied to characterize the structure of the network optimum $(f_1^*, \ldots, f_L^*)$ and the Nash mapping $\mathcal{N}^0 : F^0 \to F^{-0}$. In this section we briefly present the related results without proofs.

Let us first consider the network optimum $(f_1^*, \ldots, f_L^*)$. The flow f_l^* on link l, is decreasing in the link number $l \in \mathcal{L}$. Therefore, there exists some link L^*, such that $f_l^* > 0$ for $l \leq L^*$ and $f_l^* = 0$ for $l > L^*$. The threshold L^* is determined by:

$$G_{L^*} < R \leq G_{L^*+1}, \tag{4.1}$$

where:

$$G_l = \sum_{n=1}^{l-1} c_n - \sqrt{c_l} \sum_{n=1}^{l-1} \sqrt{c_n}, \quad l = 2, \ldots, L, \tag{4.2}$$

$$G_1 = 0, \quad G_{L+1} = \sum_{n=1}^{L} c_n = C.$$

Note that $c_l \geq c_{l+1}$ implies that $G_l \leq G_{l+1}$ for all $l \in \mathcal{L}$.

Using the optimality conditions (2.4)–(2.5), it can be easily verified that:

$$c_l - f_l^* \geq c_{l+1} - f_{l+1}^*, \quad l = 1, \ldots, L-1, \tag{4.3}$$

with equality holding if and only if $c_l = c_{l+1}$. Moreover, writing eq. (2.4) as $\sqrt{\lambda^*}(c_l - f_l^*) = \sqrt{c_l}$, and summing over any set of links $A \subseteq \{1, \ldots, L^*\}$, we have:

$$\lambda^* = \left[\frac{\sum_{l \in A} \sqrt{c_l}}{\sum_{l \in A}(c_l - f_l^*)} \right]^2, \quad A \subseteq \{1, \ldots, L^*\}. \tag{4.4}$$

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[4]This result applies also to the case of a finite number of users that employ suboptimal shortest-path routing.

Finally, the network optimum $(f_1^*, \ldots, f_L^*)$ is given by [KOR94]:

$$f_l^* = \begin{cases} c_l - (\sum_{n=1}^{L^*} c_n - R)\frac{\sqrt{c_l}}{\sum_{n=1}^{L^*}\sqrt{c_n}} & , \quad l = 1, \ldots, L^* \\ 0 & , \quad l = L^* + 1, \ldots, L \end{cases} . \tag{4.5}$$

Let us now consider the Nash equilibrium $\mathbf{f}^{-0} = \mathcal{N}^0(\mathbf{f}^0)$ of the users that is induced by strategy $\mathbf{f}^0$ of the manager. In order to characterize the structure of $\mathbf{f}^{-0}$, it suffices to determine the best reply $\mathbf{f}^i$ of user $i \in \mathcal{I}$ to the strategies of the other users and the manager that are described by $\mathbf{f}^{-i}$. For any link l, let $c_l^i = c_l - f_l^{-i}$ denote the residual capacity of the link as seen by user i. Then, $\mathbf{f}^i$ can be determined as the network optimum for a system of parallel links with capacity configuration $\mathbf{c}^i = (c_1^i, \ldots, c_L^i)$. Therefore, *assuming* that:

$$c_l^i \geq c_{l+1}^i, \quad l = 1, \ldots, L-1, \tag{4.6}$$

the flow f_l^i is decreasing in the link number $l \in \mathcal{L}$. This implies that there exists some link L^i, such that $f_l^i > 0$ for $l \leq L^i$ and $f_l^i = 0$ for $l > L^i$. The threshold L^i is determined by:

$$G_{L^i}^i < r^i \leq G_{L^i+1}^i, \tag{4.7}$$

where, similarly to eq. (4.2):

$$G_l^i = \sum_{n=1}^{l-1} c_n^i - \sqrt{c_l^i}\sum_{n=1}^{l-1}\sqrt{c_n^i}, \quad l = 2, \ldots, L, \tag{4.8}$$

$$G_1^i = 0, \quad G_{L+1}^i = \sum_{n=1}^{L} c_n^i = C - R^{-i},$$

where $R^{-i} = R - r^i$ is the total flow offered to the network by all users in $\mathcal{I}_0$ except for user i. Note that (4.6) implies that $G_l^i \leq G_{l+1}^i$ for all $l \in \mathcal{L}$.

Similarly to eq. (4.5), the best reply $\mathbf{f}^i$ of user i to strategy profile $\mathbf{f}^{-i}$ of the other users in $\mathcal{I}_0$ is given by:

$$f_l^i = \begin{cases} c_l^i - (\sum_{m=1}^{L^i} c_m^i - r^i)\frac{\sqrt{c_l^i}}{\sum_{m=1}^{L^i}\sqrt{c_m^i}} & , \quad 1 \leq l \leq L^i \\ 0 & , \quad L^i < l \leq L \end{cases} . \tag{4.9}$$

Eqs. (4.8) and (4.9) indicate that the information user i needs to determine its best reply $\mathbf{f}^i$ to any strategy profile $\mathbf{f}^{-i}$ is the residual capacity c_l^i seen by the user on every link $l \in \mathcal{L}$, and not a detailed description of $\mathbf{f}^{-i}$. In practice, information about the residual capacities can be acquired by measuring the link delays through an appropriate estimation technique.

5. Single-Follower Stackelberg Routing Game

In this section we consider the simplest case of a Stackelberg routing game, where the network is shared by a single self-optimizing user ($I = 1$) and the manager. The simplicity of this model will allow us to elucidate both the intuition behind the structure of the manager's maximally efficient strategy and the methodology to derive it. Moreover, the results of this section provide the foundation for the analysis of the general Stackelberg routing game that will be carried out in the following section.

We start by investigating the structure of a maximally efficient strategy $\mathbf{f}^0$ of the manager, provided that one exists. Let $\mathbf{f}^1 = \mathcal{N}^0(\mathbf{f}^0)$ be the best reply[5] of the follower to $\mathbf{f}^0$. Then:

$$f_l = f_l^0 + f_l^1 = f_l^*, \quad l \in \mathcal{L}. \tag{5.1}$$

Let us first show that the flow f_l^1 the follower sends on link l is decreasing in the link number $l \in \mathcal{L}$. Assume by contradiction that, for some n, we have $0 \leq f_n^1 < f_{n+1}^1$. Then, the optimality conditions (2.7)–(2.8) imply that:

$$\frac{1}{c_{n+1} - f_{n+1}^*} + \frac{f_{n+1}^1}{(c_{n+1} - f_{n+1}^*)^2} \leq \frac{1}{c_n - f_n^*} + \frac{f_n^1}{(c_n - f_n^*)^2},$$

which is a contradiction, since $c_n - f_n^* \geq c_{n+1} - f_{n+1}^*$ (by(4.3)) and $f_{n+1}^1 > f_n^1$ (by assumption). Therefore, there exists some link L^1, such that $f_l^1 > 0$ for $l \leq L^1$ and $f_l^1 = 0$ for $l > L^1$, that is, the follower sends its flow precisely over the links in $\{1, \ldots, L^1\}$. Furthermore, (4.3) and $f_l^1 \geq f_{l+1}^1$ imply that for any link l, we have $c_l^1 = c_l - f_l^* + f_l^1 \geq c_{l+1} - f_{l+1}^* + f_{l+1}^1 = c_{l+1}^1$, that is, the residual link capacities as seen by the follower preserve the order of the link capacities themselves. Hence, the threshold L^1 is determined by (4.7), with $i = 1$, as explained in Section 4. In view of eq. (5.1), it is evident that $L^1 \leq L^*$.

The optimality conditions (2.4)–(2.5) for $(f_1^*, \ldots, f_L^*)$ and (2.7)–(2.8) for $\mathbf{f}^1$ imply:

$$\frac{c_l^1}{c_m^1} = \frac{c_l}{c_m} = \left[\frac{c_l - f_l^*}{c_m - f_m^*}\right]^2, \quad l, m \in \{1, \ldots, L^1\},$$

[5] In the single-follower case, the Nash mapping $\mathcal{N}^0 : F^0 \to F^1$ is, in fact, the best reply function of the follower.

and taking $m = 1$, we have:

$$f_l^1 = f_l^* - \frac{c_l}{c_1}(f_1^* - f_1^1), \quad l = 1, \ldots, L^1, \tag{5.2}$$

which, together with $\sum_{l=1}^{L^1} f_l^1 = r^1$, give:

$$f_l^1 = f_l^* - \frac{c_l}{\sum_{n=1}^{L^1} c_n}(\sum_{n=1}^{L^1} f_n^* - r^1), \quad l = 1, \ldots, L^1. \tag{5.3}$$

Hence, given that the follower sends its flow over the links in $\{1, \ldots, L^1\}$, the strategy of the leader is given by:

$$f_l^0 = \begin{cases} c_l \frac{\sum_{n=1}^{L^1} f_n^* - r^1}{\sum_{n=1}^{L^1} c_n} & , \quad l = 1, \ldots, L^1 \\ f_l^* & , \quad l = L^1 + 1, \ldots, L \end{cases} . \tag{5.4}$$

According to eq. (5.4), if the leader knows *a priori* the set of links $\{1, \ldots, L^1\}$, over which its strategy $\mathbf{f}^0$ will force the follower to send its flow, then: (i) on every link l that will not receive any flow from the follower, it sends flow f_l^*, and (ii) it splits the rest of its flow ($r^0 - \sum_{l=L^1+1}^{L} f_l^* = \sum_{l=1}^{L^1} f_l^* - r^1$) among the links that will receive flow from the follower, proportionally to their capacities. Condition (4.7), however, depends on the leader's strategy $\mathbf{f}^0$ and cannot provide *a priori* knowledge about the threshold L^1 to the leader. In the sequel, we derive an alternative condition to determine L^1, that is independent of the leader's strategy. To that end, let us define:

$$H_l = \sum_{n=1}^{l-1} f_n^* - \frac{f_l^*}{c_l} \sum_{n=1}^{l-1} c_n, \quad l = 2, \ldots, L, \tag{5.5}$$

$$H_1 = 0, \quad H_{L+1} = \sum_{n=1}^{L} f_n^* = R.$$

Using eqs. (2.4) and (4.2), it is easy to see that:

$$H_l = \begin{cases} G_l / \sqrt{\lambda^* c_l} & , \quad l = 1, \ldots, L^* \\ R & , \quad l = L^* + 1, \ldots, L \end{cases} , \tag{5.6}$$

thus:

$$H_l \leq H_{l+1}, \quad l = 1, \ldots, L. \tag{5.7}$$

We are now ready to prove the following:

Lemma 5.1 *Suppose that there exists a maximally efficient strategy* $\mathbf{f}^0$ *of the manager. Let*

$\mathbf{f}^1 = \mathcal{N}^0(\mathbf{f}^0)$, *and* $\{1,\ldots,L^1\}$ *be the set of links* l *for which* $f_l^1 > 0$. *Then:*

$$H_{L^1} < r^1 \leq H_{L^1+1}. \tag{5.8}$$

Proof: See Appendix A.

□

Remark: It can be easily verified that (5.8), together with (5.7), gives:

$$\frac{c_l}{\sum_{n=1}^{l} c_n}\left(\sum_{n=1}^{l} f_n^* - r^1\right) < f_l^*, \quad l = 1,\ldots,L^1, \tag{5.9}$$

$$\frac{c_{L^1+1}}{\sum_{n=1}^{L^1+1} c_n}\left(\sum_{n=1}^{L^1+1} f_n^* - r^1\right) \geq f_{L^1+1}^*. \tag{5.10}$$

The expression on the left-hand-side of (5.9) and (5.10) is the flow that the manager sends on link $l \in \{1,\ldots,L^1+1\}$, under the assumption that link l is the last link used by the follower. Therefore, L^1 is the last link l, for which that assumption leads to $f_l^1 = f_l^* - f_l^0 > 0$. Indeed, if the manager assumed that the last link used by the follower is L^1+1, then (5.10) would imply $f_{L^1+1}^1 \leq 0$, which would contradict the assumption.

Since H_l is independent from the manager's strategy $\mathbf{f}^0$, for all l, condition (5.8) is also independent of $\mathbf{f}^0$. Furthermore, in view of (5.7), it determines the threshold L^1 uniquely. Therefore, if a maximally efficient strategy of the manager exists, then it is unique and is given by eq. (5.4) and (5.8). To establish existence of the maximally efficient strategy of the manager, it suffices to show that $\mathbf{f}^0$ given by (5.4) and (5.8) is such that:

(i) $\mathbf{f}^0$ is an admissible strategy of the manager, i.e., $f_l^0 \geq 0$, $l \in \mathcal{L}$, and $\sum_{l\in\mathcal{L}} f_l^0 = r^0$, and

(ii) $\mathbf{f}^1$, with f_l^1 given by (5.2) for $l \leq L^1$, and $f_l^1 = 0$ for $l > L^1$, is the best reply of the follower to $\mathbf{f}^0$, i.e., $\mathbf{f}^1 = \mathcal{N}^0(\mathbf{f}^0)$.

The proof is presented in the following theorem that gives the main result of this section.

Theorem 5.2 *In the single-follower Stackelberg routing game, there exists a unique maximally efficient strategy* $\mathbf{f}^0$ *of the leader that is given by:*

$$f_l^0 = \begin{cases} c_l \frac{\sum_{n=1}^{L^1} f_n^* - r^1}{\sum_{n=1}^{L^1} c_n} & , \quad l = 1,\ldots,L^1 \\ f_l^* & , \quad l = L^1+1,\ldots,L \end{cases}, \tag{5.11}$$

where L^1 *is determined by:*

$$H_{L^1} < r^1 \leq H_{L^1+1}. \tag{5.12}$$

Proof: From $H_{L^1} < r^1 \leq R = H_{L^*+1}$, we conclude that $L^1 \leq L^*$. From $H_{L^1+1} \geq r^1$ and (5.5) we have $\sum_{n=1}^{L^1} f_n^* \geq r^1$. Therefore, eq. (5.11) implies that $f_l^0 \geq 0$ for $l \leq L^1$. Nonnegativity of f_l^0 for $l > L^1$ is immediate. Furthermore, it is easy to verify that $\sum_{l \in \mathcal{L}} f_l^0 = \sum_{l \in \mathcal{L}} f_l^* - r^1 = r^0$. Thus, $\mathbf{f}^0$ is an admissible strategy of the manager.

We now proceed to show part (ii) above, i.e., that $\mathbf{f}^1$ is the best reply of the follower to strategy $\mathbf{f}^0$ of the manager. For all $l \leq L^1$, (5.9) gives $f_l^0 < f_l^*$, thus $f_l^1 = f_l^* - f_l^0 > 0$. Moreover, $\sum_{l \in \mathcal{L}} f_l^1 = r^1$, by eq. (5.2). Hence, $\mathbf{f}^1 \in F^1$. Let us now show that the residual capacities seen by the follower satisfy:

$$c_l^1 \geq c_{l+1}^1, \quad l = 1, \ldots, L-1. \tag{5.13}$$

For $l > L^1$, this is immediate from (4.3). Moreover, from eq. (5.11), we have:

$$c_l^1 = c_l \frac{\sum_{n=1}^{L^1} (c_n - f_n^*) + r^1}{\sum_{n=1}^{L^1} c_n}, \quad l = 1, \ldots, L^1. \tag{5.14}$$

Since $c_l \geq c_{l+1}$, (5.13) holds for $l < L^1$. Finally, for $l = L^1$, we have $c_{L^1}^1 = c_{L^1} - f_{L^1}^0 \geq c_{L^1} - f_{L^1}^* \geq c_{L^1+1} - f_{L^1+1}^* = c_{L^1} - f_{L^1+1}^0 = c_{L^1+1}^1$, where the first inequality follows from $f_{L^1}^0 \leq f_{L^1}^*$, and the second from (4.3). Thus, inequality (5.13) holds. This implies that the best reply of the follower to $\mathbf{f}^0$ has the threshold structure of $\mathbf{f}^1$, where the respective threshold, say N^1, is determined by $G_{N^1}^1 < r^1 \leq G_{N^1+1}^1$. To show $N^1 = L^1$, it suffices to show that $G_{L^1}^1 < r^1 \leq G_{L^1+1}^1$. This is proven in Lemma A.1 in Appendix A. Therefore, to establish that $\mathbf{f}^1 = \mathcal{N}^0(\mathbf{f}^0)$, it remains to be shown that:

$$\frac{c_l^1}{(c_l^1 - f_l^1)^2} = \frac{c_m^1}{(c_m^1 - f_m^1)^2}, \quad l, m \in \{1, \ldots, L^1\}.$$

Using eqs. (5.14) and (5.2), this is equivalent to showing:

$$\frac{c_l}{(c_l - f_l^*)^2} = \frac{c_m}{(c_m - f_m^*)^2}, \quad l, m \in \{1, \ldots, L^1\},$$

which holds due to the optimality conditions (2.4)–(2.5) for $(f_1^*, \ldots, f_L^*)$, since $L^1 \leq L^*$. This concludes the proof of the theorem.

□

The above theorem indicates that, for a single follower, the leader can always enforce the network optimum, independently of the relative sizes in terms of throughput demands of the leader and the follower. In other words, it is enough for the manager to control a nonzero portion of the network flow, in order to "tame" a single selfish user. As will be seen in the following section, this might not be the case in the presence of multiple self-optimizing users.

6. Multi-Follower Stackelberg Routing Game

Let us now proceed with the general Stackelberg routing game, where an arbitrary, but finite, number I of self-optimizing users share the system of parallel links. The following lemma describes the maximally efficient strategy of the manager – provided that one exists – as well as the corresponding Nash equilibrium of the noncooperative users. Later, we will derive necessary and sufficient conditions that guarantee existence of a maximally efficient strategy of the manager.

Lemma 6.1 *In a multi-follower Stackelberg routing game, if there exists a maximally efficient strategy* $\mathbf{f}^0$ *of the leader, then it is unique and is given by:*

$$f_l^0 = c_l \sum_{i \in \mathcal{I}_l} \frac{\sum_{n=1}^{L^i} f_n^* - r^i}{\sum_{n=1}^{L^i} c_n} - (I_l - 1) f_l^*, \quad l \in \mathcal{L}, \tag{6.1}$$

where, for every user $i \in \mathcal{I}$, L^i *is determined by:*

$$H_{L^i} < r^i \leq H_{L^i+1}, \tag{6.2}$$

and for every link $l \in \mathcal{L}$, $\mathcal{I}_l = \{i \in \mathcal{I} : l \leq L^i\}$ *and* $I_l = |\mathcal{I}_l|$. *In that case, the equilibrium strategy* $\mathbf{f}^i$ *of user* $i \in \mathcal{I}$ *is described by:*

$$f_l^i = \begin{cases} f_l^* - c_l \frac{\sum_{n=1}^{L^i} f_n^* - r^i}{\sum_{n=1}^{L^i} c_n} & , \quad l = 1, \ldots, L^i \\ 0 & , \quad l = L^i + 1, \ldots, L \end{cases} . \tag{6.3}$$

Conversely, if $\mathbf{f}^0$ *described by (6.1) and (6.2) is an admissible strategy of the leader, then it is its maximally efficient strategy.*

Proof: Assume that there exists a maximally efficient strategy $\mathbf{f}^0$ of leader, and let $\mathbf{f}^{-0} = \mathcal{N}^0(\mathbf{f}^0)$. Then, following precisely the proof of eq. (5.4) in the single-follower case, one can show that for every $i \in \mathcal{I}$ we have:

$$f_l^{-i} = f_l^* - f_l^i = \begin{cases} c_l \frac{\sum_{n=1}^{L^i} f_n^* - r^i}{\sum_{n=1}^{L^i} c_n} & , \quad l = 1, \ldots, L^i \\ f_l^* & , \quad l = L^i + 1, \ldots, L \end{cases} , \tag{6.4}$$

and eq. (6.3) follows. Precisely as in the single-follower case, it can be seen that, for any $i \in \mathcal{I}$, (4.6) holds, thus the threshold L^i is determined by (4.7). Moreover, using eq. (6.4), one can show that (4.7) implies (6.2). Finally, using eq. (6.3) and $\sum_{i \in \mathcal{I}_0} f_l^i = f_l^*$, $l \in \mathcal{L}$, eq. (6.1) is immediate.

Suppose now that $\mathbf{f}^0$ given by (6.1) and (6.2) is an admissible strategy of the leader. If for all $i \in \mathcal{I}$, $\mathbf{f}^i$ is given by (6.3), it is easy to see that $\sum_{i \in \mathcal{I}_0} f_l^i = f_l^*$, $l \in \mathcal{L}$. Therefore, it suffices to show that $\mathbf{f}^{-0} = \mathcal{N}^0(\mathbf{f}^0)$, or equivalently, that $\mathbf{f}^i$ is the best reply of follower $i \in \mathcal{I}$ to the strategy profile $\mathbf{f}^{-i}$ of the other followers and the manager. It is easy to verify that for any link $l \in \mathcal{L}$ eq. (6.4) holds. Observe that this is the maximally efficient strategy of the leader in a single-follower Stackelberg game where the follower has demand r^i and the demand of the leader is R^{-i}, according to Theorem 5.2. Following precisely the proof of that theorem, one can show that $\mathbf{f}^i$ is indeed the best reply of user $i \in \mathcal{I}$ to $\mathbf{f}^{-i}$.

□

Note that, if a maximally efficient strategy of the manager exists, then the induced Nash equilibrium of the followers, as described by eq. (6.3) and (6.2), has precisely the same structure with the best reply of the follower in the single-follower case, that is given by eq. (5.3) and (5.8).

Remarks:

(i) $\mathbf{f}^0$ given by (6.1) and (6.2) might fail to be an admissible strategy of the leader; it merely decreases/increases the capacity of link $l \in \mathcal{L}$ when f_l^0 is positive/negative. From the previous proof, it follows that, even if $\mathbf{f}^0$ is nonadmissible, $\mathbf{f}^{-0}$ with $\mathbf{f}^i$ given by eq. (6.3) for $i \in \mathcal{I}$ is the induced Nash equilibrium of the followers.

(ii) Under eq. (6.3), $\{1, \ldots, L^i\}$ is the set of links that receive flow from follower $i \in \mathcal{I}$. Thus, $\mathcal{I}_l$ is precisely the set of followers that send flow on link $l \in \mathcal{L}$. Since $H_1 = 0 < r^i$, $i \in \mathcal{I}$, all users send flow on link 1, that is, $\mathcal{I}_1 = \mathcal{I}$.

(iii) For every link $l \in \mathcal{L}$ such that $\mathcal{I}_l = \emptyset$, eq. (6.1) gives $f_l^0 = f_l^*$.

(iv) Since $r^i \geq r^{i+1}$, (6.2) implies $L^i \geq L^{i+1}$ for all $i < I$, and $\mathcal{I}_{l+1} \subseteq \mathcal{I}_l$ for all $l < L$. Furthermore, since $r^i \leq R = H_{L^*+1}$, (6.2) implies that $L^i \leq L^*$, $i \in \mathcal{I}$.

Let us now investigate the admissibility of $\mathbf{f}^0$. To this end, observe that:

$$\begin{aligned} \sum_{l=1}^{L} f_l^0 &= \sum_{l=1}^{L} \sum_{i \in \mathcal{I}_l} c_l \frac{\sum_{n=1}^{L^i} f_n^* - r^i}{\sum_{n=1}^{L^i} c_n} - \sum_{l=1}^{L} I_l f_l^* + \sum_{l=1}^{L} f_l^* \\ &= \sum_{i=1}^{I} \sum_{l=1}^{L^i} c_l \frac{\sum_{n=1}^{L^i} f_n^* - r^i}{\sum_{n=1}^{L^i} c_n} - \sum_{l=1}^{L} I_l f_l^* + r^0 + \sum_{i=1}^{I} r^i \\ &= \sum_{i=1}^{I} \sum_{n=1}^{L^i} f_n^* - \sum_{l=1}^{L} I_l f_l^* + r^0 = r^0, \end{aligned} \tag{6.5}$$

since $\sum_{i=1}^{I} \sum_{n=1}^{L^i} f_n^* = \sum_{l=1}^{L} I_l f_l^*$. Thus, $\mathbf{f}^0$ is admissible if and only if $f_l^0 \geq 0$, for all $l \in \mathcal{L}$. Let us now show that this condition can be relaxed to $f_1^0 \geq 0$. It suffices to show the following:

Lemma 6.2 *Consider the (possibly nonadmissible) strategy $\mathbf{f}^0$ of the leader, that is given by eq. (6.1) and (6.2). For every link $l > 1$, we have:*

$$f_l^0 < 0 \Rightarrow f_{l-1}^0 < 0.$$

Proof: Suppose that $f_l^0 < 0$. Eq. (6.1), then, gives:

$$\sum_{i \in \mathcal{I}_l} \frac{\sum_{n=1}^{L^i} f_n^* - r^i}{\sum_{n=1}^{L^i} c_n} < (I_l - 1) \frac{f_l^*}{c_l} \leq (I_l - 1) \frac{f_{l-1}^*}{c_{l-1}}, \tag{6.6}$$

since $f_l^*/c_l \leq f_{l-1}^*/c_{l-1}$, as implied by the optimality conditions (2.4)–(2.5) for $(f_1^*, \ldots, f_L^*)$. If $\mathcal{I}_{l-1} = \mathcal{I}_l$, then $f_{l-1}^0 < 0$ is immediate from (6.6). Assume that $\mathcal{I}_{l-1} \setminus \mathcal{I}_l \neq \emptyset$. For all $i \in \mathcal{I}_{l-1} \setminus \mathcal{I}_l$, we have $L^i = l - 1$, and using inequality $H_{L^i} < r^i$, one can verify that:

$$\frac{\sum_{n=1}^{L^i} f_n^* - r^i}{\sum_{n=1}^{L^i} c_n} < \frac{f_{l-1}^*}{c_{l-1}}, \quad i \in \mathcal{I}_{l-1} \setminus \mathcal{I}_l.$$

Summing this inequality over all $i \in \mathcal{I}_{l-1} \setminus \mathcal{I}_l$, and adding it to (6.6), we obtain:

$$\sum_{i \in \mathcal{I}_{l-1}} \frac{\sum_{n=1}^{L^i} f_n^* - r^i}{\sum_{n=1}^{L^i} c_n} < (I_l - 1) \frac{f_{l-1}^*}{c_{l-1}} + (I_{l-1} - I_l) \frac{f_{l-1}^*}{c_{l-1}} = (I_{l-1} - 1) \frac{f_{l-1}^*}{c_{l-1}},$$

thus $f_{l-1}^0 < 0$.

□

The previous lemma, together with Lemma 6.1, implies that a maximally efficient strategy of the leader exists if and only if f_1^0 given by eq. (6.1) is nonnegative. The following lemma shows that f_1^0 is an increasing function of the throughput demand r^0 of the leader. This monotonicity property is used in the sequel to establish that a maximally efficient strategy of the leader exists if and only if its demand is sufficiently large.

Lemma 6.3 *Let f_1^0 be as in eq. (6.1). Then, f_1^0 is a continuous increasing function of the throughput demand $r^0 \in [0, C-r]$ of the leader.*

Proof: The proof is given in Lemmata B.1 and B.2, in Appendix B.

□

Remark: If $r^0 = C-r$, then $R = C$ and the network becomes saturated. Allowing, however, r^0 to take this value is a mere technicality that will be used in the proof of the following theorem. Note that when the network is saturated, $f_l^* = c_l$ for every link $l \in \mathcal{L}$.

We are now ready to prove the main result of this section that is given in the following:

Theorem 6.4 *There exists some $\underline{r}^0$, with $0 \leq \underline{r}^0 < C - r$, such that the leader in a multi-follower Stackelberg routing game can enforce the network optimum if and only if its throughput demand r^0 satisfies $\underline{r}^0 \leq r^0 < C - r$. Then, the maximally efficient strategy of the leader is given by eq. (6.1) and (6.2).*

Proof: Recall that even if $\mathbf{f}^0$ is nonadmissible, it satisfies the demand constraint of the leader, according to eq. (6.5). By virtue of Lemma 6.2, this implies that at $r^0 = 0$ we have $f_1^0 \leq 0$, since $f_1^0 > 0$ would imply $f_l^0 \geq 0$, for $l = 2, \ldots, L$, and the demand constraint of the leader would be violated.

Suppose now that $r^0 = C - r$. Then $f_l^* = c_l$, $l \in \mathcal{L}$, and from eq. (5.5) we have $H_l = 0$, for $l \in \mathcal{L}$, while $H_{L+1} = R$. Thus, $L^i = L$ for every follower $i \in \mathcal{I}$. Therefore:

$$f_1^0 = c_1 \frac{I \sum_{n=1}^{L} f_l^* - r}{C} - (I-1) f_1^* = c_1(1 - r/C) > 0,$$

where positivity of f_1^0 follows from $r < C$.

Since f_1^0 is continuous increasing in $[0, C-r]$, nonpositive at $r^0 = 0$ and positive at $r^0 = C - r$, there exists a unique $\underline{r}^0 \in [0, C-r)$, such that $f_1^0 = 0$ at $r^0 = \underline{r}^0$. Thus, $f_1^0 \geq 0$ if and only if $r^0 \in [\underline{r}^0, C-r)$, and the result follows.

□

As seen by the previous proof, the threshold $\underline{r}^0$ of the leader is the unique solution of the equation "$f_1^0(r^0) = 0$" in $r^0 \in [0, C-r)$. Since f_1^0 is an increasing function of r^0, this equation can be easily solved using standard numerical techniques.

The above theorem implies that, for any finite set of followers with total demand r that does not exceed the total capacity C of the system, there is always a (feasible) leader, with $\underline{r}^0 \leq r^0 < C - r$, that can enforce the network optimum. Moreover, when $r \to C$, we have $\underline{r}^0 \to 0$, meaning that in heavily loaded networks it suffices to control just a small portion of the flow in order to drive the system into the network optimum. Even though this behavior might seem surprising, it has a rather intuitive explanation. In the heavy load region, the average delay increases rapidly to infinity, thus small changes in the flow configuration result in drastic changes of the average delay. Therefore, although the leader controls only a small part of the total flow, it has the power to steer the network to the desired network optimum. This result is quite encouraging, because it is in heavily loaded networks where the presence of a manager/leader is particularly important.

The threshold $\underline{r}^0$ on the leader's throughput demand depends on the number and the throughput demands of the followers. This dependence is investigated in the following section.

6.1 Properties of the Leader Threshold $\underline{r}^0$

Let us first examine the dependence of $\underline{r}^0$ on the number of followers, when their throughput demand r is fixed. To simplify the formulation of the problem, we concentrate on followers with identical throughput demands, i.e., with $r^i = r^j$ for all $i, j \in \mathcal{I}$. This class of followers will be referred to as *identical followers*, and the special structure of their Nash equilibrium has been investigated in [ORD93]. The following proposition shows that as the number of followers increases, it becomes harder for the leader to enforce the network optimum.

Proposition 6.5 *Suppose that the followers are identical and their total throughput demand r is fixed. Then, the minimum throughput demand $\underline{r}^0$ that enables the leader to enforce the network optimum $(f_1^*, \ldots, f_L^*)$ is nondecreasing in the number of followers.*

Proof: By the definition of $\underline{r}^0$, it suffices to show that, with the demand r^0 of the leader fixed, f_1^0 is nonincreasing in the number of followers. Let $\mathbf{f}^0$ and $\hat{\mathbf{f}}^0$ be the strategy of the leader, given by eq. (6.1) and (6.2), when there are I and $I+1$ followers, respectively. Note that in both cases the network optimum $(f_1^*, \ldots, f_L^*)$ is the same, since it depends on the total throughput demand $R = r^0 + r$, and not on the number of followers. Therefore, H_l is the same in both cases, for all $l \in \mathcal{L}$.

Since the followers are identical, their associated thresholds are equal, according to (6.2). Let L^1 and $\hat{L}^1$ be the thresholds when there are I and $I+1$ followers, respectively. In the former case, the demand of each follower is r/I and in the latter $r/(I+1)$. Therefore, (6.2) implies that $L^* \geq L^1 \geq \hat{L}^1$. From eq. (6.1), we have:

$$\frac{f_1^0}{c_1} = \frac{I\sum_{n=1}^{L^1} f_n^* - r}{\sum_{n=1}^{L^1} c_n} - (I-1)\frac{f_1^*}{c_1}, \quad \frac{\hat{f}_1^0}{c_1} = \frac{(I+1)\sum_{n=1}^{\hat{L}^1} f_n^* - r}{\sum_{n=1}^{\hat{L}^1} c_n} - I\frac{f_1^*}{c_1}.$$

Hence, to prove $f_1^0 \geq \hat{f}_1^0$, we have to show:

$$\frac{I\sum_{n=1}^{L^1} f_n^* - r}{\sum_{n=1}^{L^1} c_n} - \frac{I\sum_{n=1}^{\hat{L}^1} f_n^* - r}{\sum_{n=1}^{\hat{L}^1} c_n} \geq \frac{\sum_{n=1}^{\hat{L}^1} f_n^*}{\sum_{n=1}^{\hat{L}^1} c_n} - \frac{f_1^*}{c_1} \tag{6.7}$$

The expression on the right-hand-side of (6.7) is nonpositive, since $f_1^*/c_1 \geq f_l^*/c_l$, for all $l \leq L^*$, as implied by the optimality conditions for $(f_1^*, \ldots, f_L^*)$. Therefore, it suffices to show that:

$$\frac{I\sum_{n=1}^{L^1} f_n^* - r}{\sum_{n=1}^{L^1} c_n} \geq \frac{I\sum_{n=1}^{\hat{L}^1} f_n^* - r}{\sum_{n=1}^{\hat{L}^1} c_n}. \tag{6.8}$$

Since (6.8) holds trivially for $L^1 = \hat{L}^1$, we only need to consider the case $L^1 > \hat{L}^1$. Without loss of generality, assume that $\hat{L}^1 = L^1 - 1$. Then, (6.8) is equivalent to:

$$\frac{r}{I} \geq \sum_{n=1}^{L^1-1} f_n^* - \frac{f_{L^1}^*}{c_{L^1}} \sum_{n=1}^{L^1-1} c_n = H_{L^1},$$

which is true, by the definition of the threshold L^1. This concludes the proof.

□

Let us now concentrate on the dependence of $\underline{r}^0$ on differences of the demands of the followers, when their total throughput demand r is fixed. The following proposition shows that the higher the difference in the throughput demand of any two followers, the easier it becomes for the leader to enforce the network optimum.

Proposition 6.6 *Suppose that the total throughput demand r of the followers is fixed. Then, for any two followers j and k, the minimum throughput demand $\underline{r}^0$ that enables the leader to enforce the network optimum $(f_1^*, \ldots, f_L^*)$ is nonincreasing in $|r^j - r^k|$. Therefore, $\underline{r}^0$ attains its maximum value when all followers are identical.*

Proof: Suppose that $r^j \geq r^k$, and let $\mathbf{f}^0$ be the strategy of the leader given by eq. (6.1). It suffices to show that if the demands of the followers become $r^j + \varepsilon$ and $r^k - \varepsilon$, $0 \leq \varepsilon \leq r^k$, and $\hat{\mathbf{f}}^0$ is the resulting strategy of the leader – according to eq. (6.1) – then $\hat{f}_1^0 \geq f_1^0$. Since the total demand of the followers is fixed, the network optimum $(f_1^*, \ldots, f_L^*)$ and the threshold

L^i of every follower $i \in \mathcal{I} \setminus \{j,k\}$ remain the same. Therefore, it suffices to show that:

$$\phi(\varepsilon) \equiv \frac{\sum_{n=1}^{L^j} f_n^* - r^j - \varepsilon}{\sum_{n=1}^{L^j} c_n} + \frac{\sum_{n=1}^{L^k} f_n^* - r^k + \varepsilon}{\sum_{n=1}^{L^k} c_n}, \tag{6.9}$$

is an nondecreasing function of $\varepsilon \in [0, r^k]$.

Note that L^j and L^k in eq. (6.9) are also functions of ε. In particular, (6.2) implies that L^j is nondecreasing and L^k nonicreasing in ε. Then, it is easy to see that there exists a finite number of points $\alpha_1 < \ldots < \alpha_M$ in $(0, r^k)$, such that:

(i) for all ε in the same interval $[0, \alpha_1]$, $(\alpha_m, \alpha_{m+1}]$, $m = 1, \ldots, M-1$, $(\alpha_M, r^k]$, both the thresholds L^j and L^k remain the same, and

(ii) at any point α_m, either L^j is increased, or L^k is decreased.

Clearly, ϕ is continuous in every interval $[0, \alpha_1]$, $(\alpha_m, \alpha_{m+1}]$, $m = 1, \ldots, M-1$, $(\alpha_M, r^k]$. Using precisely the same technique as in the proof of Lemma B.1 in Appendix B, it is easy to see that ϕ is also continuous at every point α_m, $m = 1, \ldots, M$. Therefore, it is a continuous function of $\varepsilon \in [0, r^k]$. Hence, to show that it is also nondecreasing, it suffices to show that it is nondecreasing in every $(\alpha_m, \alpha_{m+1}]$ interval, where the thresholds L^j and L^k are fixed. But this is immediate from eq. (6.9), since $L^j \geq L^k$ implies that $\varepsilon(1/\sum_{n=1}^{L^k} c_n - 1/\sum_{n=1}^{L^j} c_n)$ is nondecreasing in ε. This completes the proof.

□

Let us now demonstrate the properties of $\underline{r}^0$, established in the previous propositions, by means of a numerical example. We consider a system of parallel links with capacity configuration $\mathbf{c} = (12, 7, 5, 3, 2, 1)$, shared by I identical followers with total demand r. The threshold $\underline{r}^0$ of the leader is depicted in Figure 1 as a function of r, for various values of I. In the same figure, we also show the saturation line "$\underline{r}^0 + r = C$". From the figure, one can see that $\underline{r}^0$ always lies below the saturation line, in accordance with Theorem 6.4. Furthermore, $\underline{r}^0$ increases with the number of users.

From the same figure, we observe that in the light load region (i.e., when the total demand r of the followers is low compared to the total capacity C) $\underline{r}^0$ increases with r, that is, the higher the demand of the followers, the more difficult it becomes for the leader to drive the system to the network optimum. In the moderate and heavy load regions, on the other hand, $\underline{r}^0$ is decreasing in r. This behavior has been explained in the discussion following Theorem 6.4.

Note that in the light load region, the curves for the various values of I have a common part. This behavior can be explained as follows. In the corresponding load region, r is such

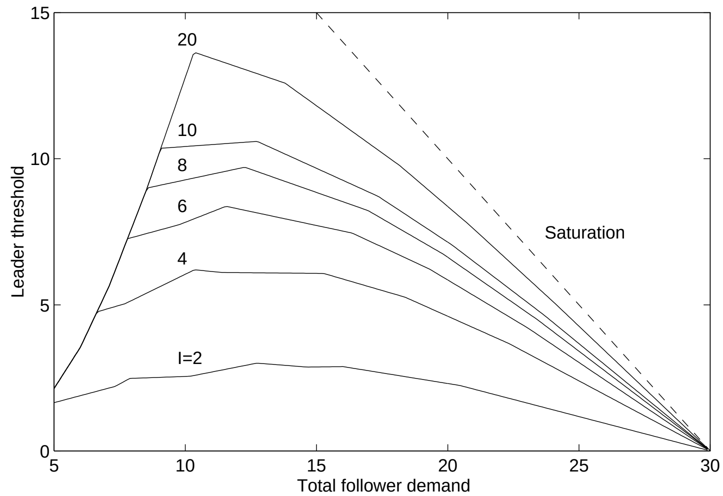

Figure 1: Leader threshold as a function of total follower demand.

that the followers send their flow only on the link with the highest capacity, i.e., on link 1. Therefore, $f_1^0 = f_1^* - r$ and it is independent of the number of followers I, thus so is $\underline{r}^0$.

7. Practical Considerations

In this section we discuss some practical issues regarding the proposed mechanism of enforcing the network optimum by means of the manager's routing strategy.

7.1 Scalability

Assume that the manager can enforce the network optimum, i.e., that its throughput demand satisfies $r^0 \geq \underline{r}^0$. According to Lemma 6.1, the manager can determine its maximally efficient strategy $\mathbf{f}^0$ by eq. (6.1) and (6.2). In order to do so, it must have information about the throughput demand r^i of every user $i \in \mathcal{I}$, and about the network optimum $(f_1^*, \ldots, f_L^*)$. The network optimum can be readily computed from eq. (4.5) and (4.1), given the total load R offered to the network. Hence, the manager needs only information about the throughput demand of every user. If user flows are accepted by means of some admission control mechanism, this information can be readily available to the manager.[6] Each time a user arrives to or departs from the network, the manager can simply adjust its strategy to the maximally efficient one, using the information about the throughput demand of that user. In that sense, the proposed mechanism of enforcing the network optimum by means of the manager's routing strategy is scalable.

[6]Otherwise, it has to obtain estimates of the user loads through measurements.

7.2 Achieving the Threshold $\underline{r}^0$

An important question that arises from the present work is whether and how the manager can satisfy the necessary and sufficient condition that allows it to drive the system to the network optimum. As indicated by Proposition 6.5, the minimum demand $\underline{r}^0$ that enables the manager to enforce the network optimum decreases with the number of noncooperative users. Therefore, one way to achieve this threshold is to provide incentives to "small" users to join "larger" (but still self-optimizing) network entities, such as Virtual Networks. It is worth noting that, while bifurcated routing might seem impractical in the single (small) user case, a VN control entity can implement (optimal) bifurcation by routing the flow of different VN users over its various paths.

An alternative way to achieve the loading threshold $\underline{r}^0$ is to provide incentives to the (noncooperative) users to join a "social" entity (e.g., a "social" VN), that is, one whose flow is directly controlled by the manager. This way, not only the number of noncooperative users is reduced, but also the total flow controlled by the manager is increased.

A key question is, then, what are the possible incentives that would persuade a user to join such larger network entities. One way to achieve this is through appropriate pricing mechanisms. A user may decide to join a VN controlled by the manager, for example, provided that lower prices would compensate for loosing control of its flow. Moreover, the manager has the flexibility to provide different grade of service to the various VNs (or users) it controls, by routing their flow over different paths, while still implementing the maximally efficient strategy for the total flow it controls. The manager can, then, charge a VN (or user) according to the grade of service that it receives. Since pricing is one of the key factors for the deployment of future broadband/multimedia networks, investigating such mechanisms is a challenging problem for future research.

8. Simple-Follower Stackelberg Routing Game

In the previous sections we have assumed that the behavior of each user is mandated by the desire to minimize an individual cost function, namely, its average time-delay. In practice, however, users often employ simpler, suboptimal routing strategies, due to complexity considerations. Many typical routing schemes, for example, send flows through shortest paths (paths of minimal delay), without accounting for delay derivatives or bifurcating flows. This motivates the following:

Definition 8.1 *A user is said to be* simple *if it routes its flow through links (or paths) of minimal delay.*

In this section, we consider a system of parallel links shared by a set of simple users and the manager. The Nash equilibrium of a set of simple users sharing a system of parallel links with capacity configuration $\mathbf{c}$ is unique with respect to the *total* link flows [ORD93]. Therefore, the set of simple followers can be viewed as a single follower that routes its flow over links of minimal delay. Based on this observation and for the sake of simplicity, throughout this section we adopt the notation of the single simple follower case, keeping in mind that the analysis applies for any number of simple followers. For instance, the total flow sent by the simple followers on link l will be denoted by f_l^1.

Suppose that the manager employs a strategy $\mathbf{f}^0 \in F^0$. Then, the simple followers are presented with a system of parallel links with capacity configuration $\mathbf{c} - \mathbf{f}^0$. Therefore, their induced Nash equilibrium[7] $\mathbf{f}^1 = \mathcal{N}^0(\mathbf{f}^0)$ is unique – in total link flows – and the corresponding necessary and sufficient conditions require the existence of some λ^1, such that [KOR94]:

$$\lambda^1 = T_l = \frac{1}{c_l - f_l^0 - f_l^1}, \quad \text{if } f_l^1 > 0, \tag{8.1}$$

$$\lambda^1 \leq T_l = \frac{1}{c_l - f_l^0}, \quad \text{if } f_l^1 = 0, \tag{8.2}$$

for all $l \in \mathcal{L}$. From (8.1)–(8.2), it is easy to see that users that route according to the optimality conditions (2.7)–(2.8) become simple as their population grows to infinity and their individual demands become infinitesimally small, while their total demand remains r. This is the typical scenario in a transportation network.

Recall that, when the followers are identical, as their number increases, it becomes more difficult for the manager of enforce the network optimum (Proposition 6.5). Since the case of simple followers is equivalent to that of an infinite number of identical self-optimizing followers, one would expect that it corresponds to the worst case scenario for the manager. Indeed, let us now explain that in the simple-follower case the manager cannot force, in general, the network optimum $(f_1^*, \ldots, f_L^*)$, independently of its throughput demand. To see this, assume that $c_l > c_{l+1}$ for all links l, and that the throughput demand of the simple followers is $r > f_1^*$. Suppose that a maximally efficient strategy $\mathbf{f}^0$ of the manager exists, and let $\mathbf{f}^1 = \mathcal{N}^0(\mathbf{f}^0)$. Since $r > f_1^* \geq f_l^*$, $l \in \mathcal{L}$, it is evident that the simple followers ship their flow over at least two links, say m and n, in $\mathcal{L}$. Then, (8.1) implies $c_m - f_m^* = c_n - f_n^*$, while the optimality conditions for $(f_1^*, \ldots, f_L^*)$ give $c_m - f_m^* > c_n - f_n^*$. Therefore, for any $r^0 \in [0, C - r)$, the manager cannot force the network optimum, i.e., a maximally efficient strategy of the manager does not exist. Intuitively, the simple followers, by equating the

[7] By abuse of notation, throughout this section $\mathcal{N}^0$ denotes the Nash mapping in the simple-follower case.

delay on all links they send their flow to, do not allow the manager to drive the system to the network optimum.

In view of this negative result, in the remaining of this section, we concentrate on the problem of determining an *optimal* strategy of the manager, that is, a strategy $\mathbf{f}^0 \in F^0$ that minimizes the total cost $J(\mathbf{f}^0, \mathcal{N}^0(\mathbf{f}^0))$.[8] Let us start with the following:

Lemma 8.2 *There exists an optimal strategy of* $\mathbf{f}^0$ *of the leader, such that, if* $\mathbf{f}^1 = \mathcal{N}^0(\mathbf{f}^0)$, *then, for every link* $l \in \mathcal{L}$, *we have:*

$$f_{l+1}^1 > 0 \;\Rightarrow\; f_l^1 > 0. \tag{8.3}$$

Proof: Let $\hat{\mathbf{f}}^0$ be an optimal strategy of the leader and $\hat{\mathbf{f}}^1 = \mathcal{N}^0(\mathbf{f}^0)$. Assume that $\hat{\mathbf{f}}^0$ is such that (8.3) does not hold. Based on $\hat{\mathbf{f}}^0$, we will construct another optimal strategy $\mathbf{f}^0$ of the leader that satisfies (8.3).

Let us first assume that there exists exactly one link n, such that $\hat{f}_n^1 = 0$ and $\hat{f}_{n+1}^1 > 0$. Then (8.1) and (8.2) give:

$$c_n - \hat{f}_n^0 \leq c_{n+1} - \hat{f}_{n+1}^0 - \hat{f}_{n+1}^1 < c_{n+1} - \hat{f}_{n+1}^0. \tag{8.4}$$

Consider now a strategy $\mathbf{f}^0$ of the leader, such that:

$$f_n^0 = c_n - c_{n+1} + \hat{f}_{n+1}^0, \quad f_{n+1}^0 = c_{n+1} - c_n + \hat{f}_n^0, \tag{8.5}$$

$$f_l^0 = \hat{f}_l^0, \quad l \in \mathcal{L} \setminus \{n, n+1\}.$$

Using (8.4) and (8.5), it is easy to verify that $0 \leq f_n^0 < c_n$, $0 < f_{n+1}^0 < c_{n+1}$, and $\sum_{l \in \mathcal{L}} f_l^0 = \sum_{l \in \mathcal{L}} \hat{f}_l^0 = r^0$, i.e., $\mathbf{f}^0$ is an admissible strategy of the leader. Throughout this proof, "hat" values will refer to strategy $\hat{\mathbf{f}}^0$ of the leader, while "non-hat" values to strategy $\mathbf{f}^0$. For example, $\hat{c}_l^1$ denotes the residual capacity of link l as seen by the simple users when the leader employs strategy $\hat{\mathbf{f}}^0$. Eq. (8.5) implies that:

$$c_n^1 = \hat{c}_{n+1}^1, \quad c_{n+1}^1 = \hat{c}_n^1, \tag{8.6}$$

$$c_l^1 = \hat{c}_l^1, \quad l \in \mathcal{L} \setminus \{n, n+1\}.$$

[8] Using the methodology developed in [KOR94], it is easy to show that the Nash mapping is continuous in the case of simple followers. Hence, the total cost is a continuous function of the manager's strategy and, therefore, a minimizing strategy $\mathbf{f}^0 \in F^0$ exists.

In other words, the followers are presented with exactly the same residual link capacities under both $\mathbf{f}^0$ and $\hat{\mathbf{f}}^0$, but with the roles of links n and $n+1$ interchanged. Therefore, if $\mathbf{f}^1 = \mathcal{N}^0(\mathbf{f}^0)$, we have:

$$f_n^1 = \hat{f}_{n+1}^1 > 0, \quad f_{n+1}^1 = \hat{f}_n^1 = 0, \tag{8.7}$$
$$f_l^1 = \hat{f}_l^1, \quad l \in \mathcal{L} \setminus \{n, n+1\}.$$

Eqs. (8.6) and (8.7) imply that $J_n^1 = \hat{J}_{n+1}^1$, $J_{n+1}^1 = \hat{J}_n^1 = 0$ and, therefore, $J^1 = \hat{J}^1$. Let us now consider the cost of the leader. Note that eqs. (8.6) and (8.7), together with the optimality conditions (8.1)–(8.2), imply $T_n = \hat{T}_{n+1} \leq \hat{T}_n = T_{n+1}$, while the delays on all other links are the same under both $\mathbf{f}^0$ and $\hat{\mathbf{f}}^0$. Thus, using eq. (8.4), we get:

$$J^0 - \hat{J}^0 = (c_n - c_{n+1})(\hat{T}_{n+1} - \hat{T}_n) \leq 0.$$

Therefore, $J \leq \hat{J}$ and since $\hat{\mathbf{f}}^0$ is an optimal strategy of the leader, we conclude that $J = \hat{J}$, i.e., $\mathbf{f}^0$ is also an optimal strategy. Note that $\mathbf{f}^0$ satisfies (8.3).

Similarly, if $\hat{\mathbf{f}}^0$ is such that $\hat{\mathbf{f}}^1 = \mathcal{N}^0(\hat{\mathbf{f}}^0)$ violates (8.3) in more than one link, we can construct inductively an optimal strategy $\mathbf{f}^0$ of the leader so that (8.3) is satisfied.

□

Let $\mathbf{f}^0$ be an optimal strategy of the leader, such that the Nash equilibrium $\mathbf{f}^1 = \mathcal{N}^0(\mathbf{f}^0)$ of the simple followers satisfies (8.3). Then, there exists some link $L^1 \in \mathcal{L}$, such that $f_l^1 > 0$ for all links $l \leq L^1$, and $f_l^1 = 0$ for all $l > L^1$. The previous lemma implies that we can restrict our attention to strategies of the leader, such that the Nash equilibrium of the followers has precisely this threshold structure. In the sequel, we derive an algorithm to determine such an optimal strategy of the leader.

Suppose that $\mathbf{f}^0$ is an optimal strategy of the leader, $\mathbf{f}^1$ is the Nash equilibrium of the simple followers and $\{1, \ldots, L^1\}$ the set of links l with $f_l^1 > 0$. Then, eq. (8.1) gives:

$$\lambda^1 = \frac{L^1}{\sum_{n=1}^{L^1} c_n - \sum_{n=1}^{L^1} (f_n^1 + f_n^0)}. \tag{8.8}$$

Therefore, $(\mathbf{f}^0, \mathbf{f}^1)$ is a solution to the following optimization problem:

$$\min_{(\mathbf{f}^0, \mathbf{f}^1)} \sum_{l=1}^{L} \frac{f_l^0 + f_l^1}{c_l - (f_l^0 + f_l^1)}, \tag{8.9}$$

subject to:

$$c_l - (f_l^0 + f_l^1) = \frac{\sum_{n=1}^{L^1} c_n - \sum_{n=1}^{L^1} (f_n^1 + f_n^0)}{L^1}, \quad l = 1, \ldots, L^1, \tag{8.10}$$

$$c_l - (f_l^0 + f_l^1) \leq \frac{\sum_{n=1}^{L^1} c_n - \sum_{n=1}^{L^1}(f_n^1 + f_n^0)}{L^1}, \quad l = L^1+1, \ldots, L, \tag{8.11}$$

$$\sum_{l=1}^{L^1} f_l^1 = r \tag{8.12}$$

$$\sum_{l=1}^{L} f_l^0 = r^0, \tag{8.13}$$

$$f_l^1 > 0, \ l = 1, \ldots, L^1, \quad f_l^1 = 0, \ l = L^1+1, \ldots, L, \tag{8.14}$$

$$f_l^0 \geq 0, \quad l = 1, \ldots, L. \tag{8.15}$$

According to eq. (8.8), constraints (8.10)–(8.11) are precisely (8.1)–(8.2), which guarantee that $\mathbf{f}^1 = \mathcal{N}^0(\mathbf{f}^0)$.

Consider now the following optimization problem, with respect to the total link flow configuration $(f_1, \ldots, f_L)$:

$$\min_{(f_1,\ldots,f_L)} \sum_{l=1}^{L} \frac{f_l}{c_l - f_l}, \tag{8.16}$$

subject to:

$$c_l - f_l = \frac{\sum_{n=1}^{L^1} c_n - \sum_{n=1}^{L^1} f_n}{L^1}, \quad l = 1, \ldots, L^1, \tag{8.17}$$

$$c_l - f_l \leq \frac{\sum_{n=1}^{L^1} c_n - \sum_{n=1}^{L^1} f_n}{L^1}, \quad l = L^1+1, \ldots, L, \tag{8.18}$$

$$\sum_{l=1}^{L^1} f_l \geq r, \tag{8.19}$$

$$\sum_{l=1}^{L} f_l = R, \tag{8.20}$$

$$f_l \geq 0, \quad l = 1, \ldots, L. \tag{8.21}$$

Let $P1(L^1)$ denote the problem described by (8.16)–(8.21), and $J^*(L^1)$ the cost at its optimal solution, provided that one exists. If $P1(L^1)$ is infeasible, define $J^*(L^1) = \infty$. The cost to be minimized in $P1(L^1)$ is a convex function of $(f_1, \ldots, f_L)$, therefore, if an optimal solution exists, it is unique. Let $(f_1, \ldots, f_L)$ be the optimal solution of $P1(L^1)$. For any strategy $\mathbf{f}^0$ of the leader, such that:

$$\sum_{l=1}^{L^1} f_l^0 = \sum_{l=1}^{L^1} f_l - r, \tag{8.22}$$

$$0 \leq f_l^0 < f_l, \quad l = 1, \ldots, L^1, \tag{8.23}$$

$$f_l^0 = f_l, \quad l = L^1+1, \ldots, L, \tag{8.24}$$

it is easy to verify that, if $\mathbf{f}^1$ is such that $f_l^1 = f_l - f_l^0$ for all $l \in \mathcal{L}$, then $(\mathbf{f}^0, \mathbf{f}^1)$ is a solution to the optimization problem (8.9)–(8.15). Moreover, for any solution $(\mathbf{f}^0, \mathbf{f}^1)$ of problem (8.9)–(8.15), $(f_1, \ldots, f_L)$ with $f_l = f_l^0 + f_l^1$ for all $l \in \mathcal{L}$ is a solution of problem $P1(L^1)$.

If $\mathbf{f}^0$ is an optimal strategy of the leader, such that $\mathbf{f}^1 = \mathcal{N}^0(\mathbf{f}^0)$ satisfies (8.3), then the above analysis shows that the link flow configuration $(f_1, \ldots, f_L)$ with $f_l = f_l^0 + f_l^1$ for all $l \in \mathcal{L}$ is the solution of $P1(L^1)$, for some $L^1 \in \mathcal{L}$. Then, for any $L^1 \in \mathcal{L}$, such that:

$$L^1 \in \arg\min_{N \in \mathcal{L}} J^*(N), \tag{8.25}$$

the optimal solution of $P1(L^1)$ is an optimal link flow configuration and any $\mathbf{f}^0$ satisfying (8.22)–(8.24) is an optimal strategy of the leader. Therefore, an algorithm to determine an optimal strategy $\mathbf{f}^0$ of the leader is the following:

1. For every $N \in \mathcal{L}$, solve problem $P1(N)$.
2. Find an L^1 that satisfies (8.25).
3. Let $(f_1^*, \ldots, f_L^*)$ be the optimal solution of $P1(L^1)$ and choose any $\mathbf{f}^0$ according to (8.22)–(8.24).

For any $N \in \mathcal{L}$, problem $P1(N)$ can be solved using standard convex programming techniques [LUE84]. In Appendix C, we present a simple iterative algorithm to solve $P1(N)$ – or determine that it is infeasible – that is based on the explicit solution of the general single-user routing optimization problem (see Section 4).

9. Conclusions

The practical inability to achieve global cooperation in many modern networking environments, typically results in an inefficient use of the network resources. This situation might be prohibitive for future broadband networks that are expected to support numerous resource consuming applications, such as multimedia. In recent years, a number of methods have been proposed to overcome this problem. These methods improve the network performance either through proper design of the resource configuration and/or the service disciplines of the network, or by introducing some "external" component such as prices.

We proposed a new method for improving the performance of noncooperative networks. This approach calls for the intervention of a social agent, namely the network manager, that tries to optimize the network performance, through the limited control that it routinely employs during the run time phase of the network. Specifically, we considered a network manager that acts as a Stackelberg leader. The manager controls only part of the network flow, and is cognizant of the presence of noncooperative users. Considering a system of parallel links, we showed that, by controlling just a small portion of the network flow, the operating point of the system can often be driven into the network optimum. In particular, we demonstrated that a maximally efficient strategy always exists in heavily loaded networks, i.e., the manager can enforce maximal efficiency when it is most needed. When the users employ suboptimal shortest-path routing, or when their population is infinite, the manager cannot, in general, drive the system to the network optimum. For this class of users, we derived the structure of an optimal strategy of the manager and proposed a simple algorithm to determine it.

It should be noted, though, that our analysis depends on the specific structure of the model. The extent to which these results can be generalized is an important subject for further research. Nonetheless, the ability to obtain efficient strategies for simple networking models has, per se, important implications. We indicated, for example, that systems of parallel links appropriately model scenaria that become common in modern networking. Indeed, current practices tend to decrease the degrees of freedom in networks, as is the case, for example, when bandwidth is separated among virtual paths. The present work indicates that such practices make the network less vulnerable to the deficiencies of noncooperation. This is yet a further indication of the potential benefit of decoupling complex structures in a network.

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APPENDIX

A. Single-Follower Stackelberg Routing Game

In this appendix we give the proofs of some of the results in Section 5. Recall that, if the manager employs the strategy $\mathbf{f}^0$ given by eq. (5.4), then the residual capacity seen by the follower on any link $l \leq L^1$ is:

$$c_l^1 = c_l \frac{\sum_{n=1}^{L^1}(c_n - f_n^*) + r^1}{\sum_{n=1}^{L^1} c_n}, \quad l = 1, \ldots, L^1. \tag{A.1}$$

Proof of Lemma 5.1: Since the follower sends its flow precisely over the links in $\{1, \ldots, L^1\}$, we have $G_{L^1}^1 < r^1 \leq G_{L^1+1}^1$.

Let us first show that $H_{L^1} < r^1$. Using eqs. (4.8) and (A.1), $r^1 > G_{L^1}^1$ is equivalent to:

$$r^1 > \left[\sum_{n=1}^{L^1-1} c_n - \sqrt{c_{L^1}} \sum_{n=1}^{L^1-1} \sqrt{c_n}\right] \frac{\sum_{n=1}^{L^1}(c_n - f_n^*) + r^1}{\sum_{n=1}^{L^1} c_n},$$

or:

$$r^1 \sqrt{c_{L^1}} \sum_{n=1}^{L^1} \sqrt{c_n} > \left[\sum_{n=1}^{L^1-1} c_n - \sqrt{c_{L^1}} \sum_{n=1}^{L^1-1} \sqrt{c_n}\right] \sum_{n=1}^{L^1}(c_n - f_n^*). \tag{A.2}$$

Since $L^1 \leq L^*$, taking $A = \{1, \ldots, L^1\}$ in eq. (4.4), we get:

$$\sqrt{\lambda^*} = \frac{\sum_{n=1}^{L^1} \sqrt{c_n}}{\sum_{n=1}^{L^1}(c_n - f_n^*)}. \tag{A.3}$$

Thus, (A.2) is equivalent to:

$$r^1 > \frac{1}{\sqrt{\lambda^* c_{L^1}}} \left[\sum_{n=1}^{L^1-1} c_n - \sqrt{c_{L^1}} \sum_{n=1}^{L^1-1} \sqrt{c_n}\right] = \frac{G_{L^1}}{\sqrt{\lambda^* c_{L^1}}} = H_{L^1}.$$

Let us now proceed to show that $r^1 \leq H_{L^1+1}$. If $f_{L^1+1}^* = 0$, then $L^* = L^1$ and $H_{L^1+1} = R > r^1$, by (5.6). Therefore, we concentrate on the case where $f_{L^1+1}^* > 0$. Using eqs. (4.8) and (A.1), $r^1 \leq G_{L^1+1}^1$ is equivalent to:

$$r^1 \leq \sum_{l=1}^{L^1} c_l \frac{\sum_{n=1}^{L^1}(c_n - f_n^*) + r^1}{\sum_{n=1}^{L^1} c_n} - \sqrt{c_{L^1+1} - f_{L^1+1}^*} \sum_{l=1}^{L^1} \sqrt{c_l} \frac{\sqrt{\sum_{n=1}^{L^1}(c_n - f_n^*) + r^1}}{\sqrt{\sum_{n=1}^{L^1} c_n}},$$

or, after some algebraic manipulation, to:

$$r^1 \leq \frac{1}{c_{L^1+1} - f^*_{L^1+1}} \left[\frac{\sum_{n=1}^{L^1} (c_n - f^*_n)}{\sum_{n=1}^{L^1} \sqrt{c_n}} \right]^2 \sum_{n=1}^{L^1} c_n - \sum_{n=1}^{L^1} (c_n - f^*_n),$$

and, using eq. (A.3), equivalent to:

$$r^1 \leq \frac{1}{c_{L^1+1} - f^*_{L^1+1}} \frac{1}{\lambda^*} \sum_{n=1}^{L^1} c_n - \frac{\sum_{n=1}^{L^1} \sqrt{c_n}}{\sqrt{\lambda^*}}. \tag{A.4}$$

Since $f^*_{L^1+1} > 0$, eq. (2.7) gives $c_{L^1+1} - f^*_{L^1+1} = \sqrt{c_{L^1+1}/\lambda^*}$, and (A.4) is equivalent to:

$$r^1 \leq \frac{1}{\sqrt{\lambda^* c_{L^1+1}}} \left[\sum_{n=1}^{L^1} c_n - \sqrt{c_{L^1+1}} \sum_{n=1}^{L^1} \sqrt{c_n} \right] = \frac{G_{L^1+1}}{\sqrt{\lambda^* c_{L^1+1}}} = H_{L^1+1}. \tag{A.5}$$

□

Let us now proceed with the following lemma that is used in the proof of Theorem 5.2.

Lemma A.1 *Consider the strategy* $\mathbf{f}^0$ *of the manager that is given by (5.11) and (5.12). Then, we have* $G^1_{L^1} < r^1 \leq G^1_{L^1+1}$.

Proof: From (5.12), we have $H_{L^1} < r^1 \leq H_{L^1+1}$. As explained in the proof of Theorem 5.2, $L^1 \leq L^*$. Then, as shown in the proof of Lemma 5.1, $H_{L^1} < r^1$ is equivalent to $G^1_{L^1} < r^1$. Let us now show that:

$$r^1 \leq G^1_{L^1+1}. \tag{A.6}$$

As seen in the proof of Lemma 5.1, if $f^*_{L^1+1} > 0$, then (A.6) is equivalent to $r^1 \leq H_{L^1+1}$. Therefore, we only need to establish (A.6) in the case where $f^*_{L^1+1} > 0$. In that case, (A.6) is equivalent to (A.4). Furthermore, (2.8) implies that $c_{L^1+1} - f^*_{L^1+1} = c_{L^1+1} \leq \sqrt{c_{L^1+1}/\lambda^*}$. Thus, to show (A.4) it suffices to show (A.5), which holds true.

□

B. Multi-Follower Stackelberg Routing Game

In this appendix we present the proof of Lemma 6.3. The proof is given in the following two lemmata. The first establishes that $\mathbf{f}^0$, given by eq. (6.1) and (6.2), is a continuous function of the leader's demand r^0, while the second shows that f^0_1 is an increasing function of r^0.

Lemma B.1 *The (possibly non-admissible) strategy* $\mathbf{f}^0$ *of the leader, given by eq. (6.1) and (6.2), is a continuous function of* $r^0 \in [0, C - r]$.

Proof: Following the methodology developed in [KOR94], it can be shown that the network optimum $(f_1^*, \ldots, f_L^*)$ is a continuous function of the total throughput demand $R \in [0, C)$ and, therefore, of the demand $r^0 \in [0, C-r)$ of the leader. Furthermore, it can be easily seen that $(f_1^*, \ldots, f_L^*)$ is continuous at $r^0 = C - r$, i.e., at the point where the network becomes saturated, where $f_l^* = c_l$, for all links $l \in \mathcal{L}$. Then, eq. (5.5) implies that, for every $l \in \mathcal{L}$, H_l is a continuous function of $r^0 \in [0, C-r]$.

Taking $A = \{1, \ldots, L^*\}$ in eq. (4.4), we get $\sqrt{\lambda^*} = \sum_{l=1}^{L^*} \sqrt{c_l} / (\sum_{l=1}^{L^*} c_l - R)$. Thus, λ^* is increasing in r^0,[9] and eq. (5.6) implies that H_l is a decreasing function of $r^0 \in [0, C-r]$, for all $l = 1, \ldots, L^*$. Then, from (6.2), the threshold L^i is a nondecreasing (integer-valued) function of r^0, for all $i \in \mathcal{I}$. Since there is a finite number of followers i and L^i takes values in a finite set, this implies that there exists a finite number of points $\alpha_1 < \ldots < \alpha_M$ in $(0, C-r)$, such that:

(i) for all r^0 in the same interval $[0, \alpha_1]$, $(\alpha_m, \alpha_{m+1}]$, $m = 1, \ldots, M-1$, $(\alpha_M, C-r]$ the threshold L^i of every follower $i \in \mathcal{I}$ remains constant, and

(ii) at any point α_m, there exists at least one follower $j \in \mathcal{I}$, for which the threshold changes from L^j to $L^j + 1$,[10] i.e., according to (6.2):

$$r^j = H_{L^j+1}(\alpha_m) = \sum_{n=1}^{L^j} f_n^*(\alpha_m) - \frac{f_{L^j+1}^*(\alpha_m)}{c_{L^j+1}} \sum_{n=1}^{L^j} c_n. \tag{B.1}$$

The strategy $\mathbf{f}^0$ is continuous in every interval $[0, \alpha_1]$, $(\alpha_m, \alpha_{m+1}]$, $m = 1, \ldots, M-1$, $(\alpha_m, C-r]$, since $(f_1^*, \ldots, f_L^*)$ is continuous in r^0 and all thresholds L^i are constant in each such interval. Therefore, we have to show that it is also continuous at every point α_m, $m = 1, \ldots, M$.

Let j be a follower for which the threshold L^j changes to $L^j + 1$ at $r^0 = \alpha_m$. Without loss of generality, assume that j is the only user for which the threshold changes at this point. By its definition in (6.2), L^j is left-continuous at α_m and so is $\mathbf{f}^0$:

$$\lim_{r^0 \uparrow \alpha_m} f_l^0(r^0) = f_l^0(\alpha_m) = c_l \sum_{i \in \mathcal{I}_l} \frac{\sum_{n=1}^{L^i} f_n^*(\alpha_m) - r^i}{\sum_{n=1}^{L^i} c_n} - (I_l - 1) f_l^*(\alpha_m), \quad l \in \mathcal{L}.$$

If $\mathcal{I}_l$ is the set of followers that send flow on link l when $r^0 \in (\alpha_{m-1}, \alpha_m]$, then for $r^0 \in (\alpha_m, \alpha_{m+1}]$ the set of followers that send flow on link $l \in \mathcal{L} \setminus \{L^j + 1\}$ is the same, while

---

[9] Note that in view of (4.1), L^* is nondecreasing with r^0.

[10] To simplify the analysis, we assume that $c_1 > \ldots > c_L$, so that $H_l < H_{l+1}$, for all $l = 1, \ldots, L^*$; cases where $c_l = c_{l+1}$, for some link l, can be handled based on elementary reasoning.

for link L^j+1 it is $\mathcal{I}_{L^j+1} \cup \{j\}$. By continuity of $(f_1^*, \ldots, f_L^*)$, for every link $l \in \mathcal{L} \setminus \{L^j+1\}$, we have:

$$\lim_{r^0 \downarrow \alpha_m} f_l^0(r^0) = c_l \sum_{i \in \mathcal{I}_l \setminus \{j\}} \frac{\sum_{n=1}^{L^i} f_n^*(\alpha_m) - r^i}{\sum_{n=1}^{L^i} c_n} + c_l \frac{\sum_{n=1}^{L^j+1} f_n^*(\alpha_m) - r^j}{\sum_{n=1}^{L^j+1} c_n} - (I_l - 1) f_l^*(\alpha_m),$$

while for link L^j+1:

$$\lim_{r^0 \downarrow \alpha_m} f_{L^j+1}^0(r^0) = c_{L^j+1} \sum_{i \in \mathcal{I}_{L^j+1}} \frac{\sum_{n=1}^{L^i} f_n^*(\alpha_m) - r^i}{\sum_{n=1}^{L^i} c_n} + c_{L^j+1} \frac{\sum_{n=1}^{L^j+1} f_n^*(\alpha_m) - r^j}{\sum_{n=1}^{L^j+1} c_n} - I_{L^j+1} f_{L^j+1}^*(\alpha_m).$$

Therefore, to establish continuity at $r^0 = \alpha_m$, we need to show that:

$$\frac{\sum_{n=1}^{L^j} f_n^*(\alpha_m) - r^j}{\sum_{n=1}^{L^j} c_n} = \frac{\sum_{n=1}^{L^j+1} f_n^*(\alpha_m) - r^j}{\sum_{n=1}^{L^j+1} c_n},$$

$$c_{L^j+1} \frac{\sum_{n=1}^{L^j+1} f_n^*(\alpha_m) - r^j}{\sum_{n=1}^{L^j+1} c_n} = f_{L^j+1}^*(\alpha_m).$$

It can be easily verified that both the above equations are equivalent to eq. (B.1). Thus, $\mathbf{f}^0$ is also continuous at every point α_m, $m = 1, \ldots, M$, and this concludes the proof of the lemma.

□

Lemma B.2 *Let f_1^0 be as in eq. (6.1). Then, f_1^0 is an increasing function of the throughput demand $r^0 \in [0, C-r]$ of the leader.*

Proof: Let α_m, $m = 1, \ldots, M$, be as in the proof of the previous lemma. Since f_1^0 is continuous in $r^0 \in [0, C-r]$, in order to show that it is an increasing function, it suffices to show that it is increasing in every interval $[0, \alpha_1]$, $(\alpha_m, \alpha_{m+1}]$, $m = 1, \ldots, M-1$, $(\alpha_M, C-r]$, where the threshold L^i of every follower i is constant. Let us concentrate on the case $r^0 \in (\alpha_m, \alpha_{m+1}]$. From eq. (6.1), f_1^0 can be written as:

$$f_1^0 = c_1 \sum_{i=1}^{I-1} \left[\frac{\sum_{n=1}^{L^i} f_n^*}{\sum_{n=1}^{L^i} c_n} - \frac{f_1^*}{c_1} \right] + c_1 \left[\frac{\sum_{n=1}^{L^I} f_n^*}{\sum_{n=1}^{L^I} c_n} - \sum_{i=1}^{I} \frac{r^i}{\sum_{n=1}^{L^i} c_n} \right]. \tag{B.2}$$

Since, as shown in the proof of Lemma B.1, λ^* increases with r^0, eq. (2.4) implies that f_l^*, $l = 1, \ldots, L^*$, is increasing in r^0. Therefore, the second term in eq. (B.2) is increasing in $r^0 \in (\alpha_m, \alpha_{m+1}]$. Solving eq. (2.4) with respect to f_l^*, after some algebraic manipulation one can verify that:

$$\frac{\sum_{n=1}^{L^i} f_n^*}{\sum_{n=1}^{L^i} c_n} - \frac{f_1^*}{c_1} = -\frac{1}{\sqrt{\lambda^*}} \frac{\sum_{n=1}^{L^i} \sqrt{c_n}(\sqrt{c_1} - \sqrt{c_n})}{\sqrt{c_1} \sum_{n=1}^{L^i} c_n} \leq 0, \quad i \in \mathcal{I}.$$

Since λ^* is increasing in r^0, this implies that the first term in eq. (B.2) is nondecreasing in $r^0 \in (\alpha_m, \alpha_{m+1}]$. Therefore, f_1^0 is increasing in $r^0 \in (\alpha_m, \alpha_{m+1}]$, and this concludes the proof.

□

C. Simple-Follower Stackelberg Routing Game

In Section 8, it was shown that an optimal strategy of the leader can be determined by solving the optimization problem $P1(L^1)$ for all $L^1 \in \mathcal{L}$. In this appendix we develop an iterative algorithm to solve $P1(L^1)$, that is based on the explicit solution of the general single-user routing optimization problem, that is given by eq. (4.5) and (4.1).

In view of eq. (8.17), (8.16) can be written as:

$$\min_{(f_1,\dots,f_L)} \left\{ L^1 \frac{\sum_{n=1}^{L^1} f_n}{\sum_{n=1}^{L^1} c_n - \sum_{n=1}^{L^1} f_n} + \sum_{l=L^1+1}^{L} \frac{f_l}{c_l - f_l} \right\}. \tag{C.1}$$

Note also that (8.21) for $l = 1, \dots, L^1$ is equivalent to $c_l \geq (\sum_{n=1}^{L^1} c_n - \sum_{n=1}^{L^1} f_n)/L^1$. Since $c_{L^1} \leq c_l$, for $l \leq L^1$, these L^1 nonnegativity constraints can be replaced by:

$$\frac{\sum_{n=1}^{L^1} f_n}{L^1} \geq \frac{\sum_{n=1}^{L^1} c_n}{L^1} - c_{L^1}. \tag{C.2}$$

Define now $\mathbf{d}$, with:

$$d_l = \begin{cases} \sum_{n=1}^{L^1} c_n / L^1 & , \quad 1 \leq l \leq L^1 \\ c_l & , \quad L^1 < l \leq L \end{cases},$$

and

$$\beta_{L^1} = \max\left\{ \frac{r^1}{L^1}, \frac{\sum_{n=1}^{L^1} c_n}{L^1} - c_{L^1} \right\} > 0,$$

and consider the following optimization problem in $\mathbf{g} = (g_1, \dots, g_L)$, that will be denoted by $P2(L^1)$:

$$\min_{\mathbf{g}} \sum_{l=1}^{L} \frac{g_l}{d_l - g_l}, \tag{C.3}$$

subject to:

$$d_l - g_l \leq d_1 - g_1, \quad l = L^1 + 1, \dots, L, \tag{C.4}$$

$$\sum_{l=1}^{L} g_l = R, \tag{C.5}$$

$$g_l \geq \beta_{L^1}, \quad l = 1, \dots, L^1, \tag{C.6}$$

$$g_l \geq 0, \quad l = L^1 + 1, \dots, L. \tag{C.7}$$

The cost function in (C.3) is convex, therefore, if an optimal solution exists, then it is unique. Let $\mathbf{g}$ be the optimal solution to $P2(L^1)$. Then, it is easy to verify that $d_1 = \ldots = d_{L^1}$ implies $g_1 = \ldots = g_{L^1}$. Adding the latter as a constraint to $P2(L^1)$, (C.3) can be replaced by:

$$\min_{\mathbf{g}} \left\{ L^1 \frac{g_1}{d_1 - g_1} + \sum_{l=L^1+1}^{L} \frac{g_l}{d_l - g_l} \right\}. \tag{C.8}$$

Consider now $(f_1, \dots, f_L)$, with:

$$f_l = \begin{cases} c_l - (\sum_{n=1}^{L^1} c_n - g_1)/L^1 & , \quad 1 \leq l \leq L^1 \\ g_l & , \quad L^1 < l \leq L \end{cases}. \tag{C.9}$$

Then, it is easy to verify that:

$$\sum_{l=1}^{L^1} f_l = g_1, \tag{C.10}$$

therefore, $(f_1, \dots, f_L)$ satisfies (8.17). In view of (C.6), it also satisfies (8.19) and (C.2), which implies the nonnegativity constraints (8.21) for $l \leq L^1$. The nonnegativity constraints (8.21) for $l > L^1$ follow from (C.7). Finally, constraints (8.18) and (8.20) coincide with (C.4) and (C.5), respectively. Therefore, $(f_1, \dots, f_L)$ satisfies the constraints of problem $P1(L^1)$. Since the cost functions in (C.1) and (C.8) are equal (according to (C.10)), we conclude that $(f_1, \dots, f_L)$ is the optimal solution of $P1(L^1)$. Similarly, if $(f_1, \dots, f_L)$ is the optimal solution of $P1(L^1)$, then $\mathbf{g}$ with $g_l = \sum_{n=1}^{L^1} f_n / L^1$ for $l \leq L^1$ and $g_l = f_l$ for $l > L^1$ is the optimal solution of $P2(L^1)$.

Note that problem $P2(L^1)$ is a modified version of the problem of optimally routing a throughput demand of R over a system of parallel links with capacity configuration $\mathbf{d}$. In $P2(L^1)$, the flow at each link $l \leq L^1$ is required to be higher than a positive constant (by (C.6)), and the residual capacity of each link $l > L^1$ is required to be less than the residual capacity of link 1 (by (C.4)). In the sequel, we develop an algorithm to find the optimal solution of $P2(L^1)$, provided that one exists, based on the explicit solution of the general routing optimization problem, that has been presented in Section 4.

Let $\mathbf{g}$ be the optimal solution of $P2(L^1)$. Let us start by showing that:

$$d_l - g_l \geq d_{l+1} - g_{l+1}, \quad l = L^1 + 1, \ldots, L - 1. \tag{C.11}$$

To see this, assume that there is some $n \geq L^1 + 1$, with $d_n - g_n < d_{n+1} - g_{n+1} \leq d_1 - g_1$. Then $d_n > d_{n+1}$, since $d_n = d_{n+1}$ would contradict the first inequality. Consider now $\hat{\mathbf{g}}$, with $\hat{g}_n = g_{n+1} + d_n - d_{n+1}$, $\hat{g}_{n+1} = g_n - d_n + d_{n+1}$, and $\hat{g}_l = g_l$, for all $l \neq n, n+1$. It is easy to verify that $\hat{\mathbf{g}}$ satisfies the constraints of $P2(L^1)$. If J and $\hat{J}$ are the costs under $\mathbf{g}$ and $\hat{\mathbf{g}}$, respectively, then $\hat{J} - J = (d_n - d_{n+1})[(d_{n+1} - g_{n+1})^{-1} - (d_n - g_n)^{-1}] < 0$, which contradicts the optimality of $\mathbf{g}$.

Denote the optimal solution of the problem of optimally routing demand R over $\mathbf{d}$ by $\mathbf{g}^*$. As shown in [ORD93], $\mathbf{g}^*$ satisfies (C.4) for all $l \in \mathcal{L}$. Thus, if $g_1^* \geq \beta_{L^1}$, $\mathbf{g}^*$ is the optimal solution of $P2(L^1)$, i.e., $\mathbf{g} = \mathbf{g}^*$. Suppose now that $g_1^* < \beta_{L^1}$. Then, $g_1 = \beta_{L^1}$, since it is easy to verify that $g_1 > \beta_{L^1}$ would imply $\mathbf{g} = \mathbf{g}^*$, and thus $\beta_{L^1} < g_1 = g_1^* < \beta_{L^1}$. In this case, the cost at each "link" $l \leq L^1$ is fixed – recall that $g_1 = \ldots = g_{L^1}$ – and $(g_{L^1+1}, \ldots, g_L)$ minimizes the total cost over $\{L^1 + 1, \ldots, L\}$, subject to the constraints. Let $\mathbf{g^{(1)}}$ be the solution to the problem of optimally routing demand $R - L^1\beta_{L^1}$ over $(d_{L^1+1}, \ldots, d_L)$. Suppose that $d_{L^1+1} - g_1^{(1)} \leq d_1 - \beta_{L^1}$. From (C.11), $\mathbf{g}$ with $g_l = \beta_{L^1}$ for $l \leq L^1$ and $g_l = g_{l-L^1}^{(1)}$ for $l > L^1$ satisfies the constraints of $P2(L^1)$, thus it is its optimal solution. Similarly, if $d_{L^1+1} - g_1^{(1)} > d_1 - \beta_{L^1}$, then $d_{L^1+1} - g_{L^1+1} = d_1 - \beta_{L^1}$,[11] i.e., g_{L^1+1} is fixed and $(g_{L^1+2}, \ldots, g_L)$ minimizes the total cost over $\{L^1 + 2, \ldots, L\}$, subject to the constraints. Proceeding inductively (in view of (C.11)), we either determine the optimal solution $\mathbf{g}$, or conclude that $P2(L^1)$ is infeasible.

The above discussion shows that an iterative algorithm to solve $P2(L^1)$ is the following:

Step 0: Find the solution $\mathbf{g}^*$ of the problem of optimally routing demand R over $\mathbf{d}$. If $g_1^* > \beta_{L^1}$, then set $\mathbf{g} = \mathbf{g}^*$ and go to **Final Step**. Otherwise, set:

$$g_1 = \beta_{L^1}, \quad R^{(1)} = R - L^1\beta_{L^1},$$

and proceed to **Step 1**.

Step n: If $R^{(n)} < 0$, stop; $P2(L^1)$ is infeasible. If $R^{(n)} = 0$, set $g_{L^1+l} = 0$ for $l = n, \ldots, L - L^1$ and go to **Final Step**. If $R^{(n)} > 0$, find the solution $\mathbf{g^{(n)}}$ of the problem of optimally routing demand $R^{(n)}$ over $(d_{L^1+n}, \ldots, d_L)$.

If $d_{L^1+n} - g_1^{(n)} \leq d_1 - \beta_{L^1}$, then set $g_{L^1+l} = g_{n+1-l}^{(n)}$, $l = n, \ldots, L - L^1$ and go to **Final**

[11] It is easy to see that $d_{L^1+1} - g_{L^1+1} < d_1 - \beta_{L^1}$ implies that $\mathbf{g^{(1)}} = (g_{L^1+1}, \ldots, g_L)$, which contradicts $d_{L^1+1} - g_1^{(1)} > d_1 - \beta_{L^1}$.

Step. Otherwise, set:

$$g_{L^1+n} = d_{L^1+n} - (d_1 - \beta_{L^1}), \quad R^{(n+1)} = R^{(n)} - g_{L^1+n}.$$

If $g_{L^1+n} < 0$, stop; $P2(L^1)$ is infeasible. Otherwise, proceed to **Step n+1** if $n < L - L^1$, or to **Final Step** if $n = L - L^1$.

Final Step: If $\sum_{l=1}^{L} g_l = R$, then $\mathbf{g}$ is the optimal solution of $P2(L^1)$. Otherwise, the problem is infeasible.

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