Int. J. Pure Appl. Sci. Technol., 4(1) (2011), pp. 1-9

International Journal of Pure and Applied Sciences and Technology

ISSN 2229 - 6107

Available online at [***www.ijopaasat.in***](www.ijopaasat.in)

Research Paper

Fixed Point Theorem for Mappings with Cyclic Contraction in Menger Spaces

Binayak S. Choudhury [1], Krishnapada Das [2,*] and Samir Kumar Bhandari [3]

[1,2,3]Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah-711103, India.

* Corresponding author, e-mail: (kestapm@yahoo.co.in)

(Received: 31-01-2011; Accepted: 26-03-2011)

Abstract: *In this paper we introduce the concept of cyclic contraction in probabilistic metric spaces. We show that such contractions necessarily have unique fixed points in a complete Menger space. An illustrative example is given.*

Keywords: Menger space, Cauchy sequence, fixed point, Φ-function.

---

1. Introduction

Banach's contraction is one of the most important results in modern mathematics. This principle was generalized in various directions. A probabilistic generalization was proposed by Sehgal and Bharucha-Reid in 1972 [22]. They proved the result in the context of probabilistic metric spaces. This generalization is called Sehgal contraction or B-contraction. The inherent flexibility of the probabilistic metric spaces allows us to extend the contraction mapping principle in more than one inequivalent ways. Thus another extension of contraction mapping was established in probabilistic metric spaces by Hicks in 1983 [9]. The notion introduced by Hicks was known as C-contraction. Subsequently, fixed point theory in probabilistic metric spaces has developed in a large way. A comprehensive survey of this development upto 2001 described in [8] by Hadzic and Pap. Some more recent references may be noted in [1, 2, 3, 5, 6, 13, 14, 16, 17, 18] and [23].

In metric fixed point theory, altering distance functions have been used by many authors in a number of works. An "altering distance function" is a control function which alters the distances between two points in a metric space. This concept was introduced by Khan, Swaleh and Sessa [12]. Some of the other works in this line of research are noted in [15, 19] and [20]. Recently "altering distance functions" have been extended in the context of Menger space by Choudhury and Das in [1]. This idea of control function in Menger space has opened the possibility of proving new probabilistic fixed point results. Some recent results on fixed point and coincidence point problems using this control function are obtained in works like [2, 3, 5, 6] and [14].

In recent years cyclic contraction and cyclic contractive type mapping have appeared in several works. This line of research was initiated by Kirk, Srinivasan and Veeramani [11], where they, amongst other results, established the following generalization of the contraction mapping principle.

Theorem 1.1 [11] Let A and B be two non-empty closed subsets of a complete metric space X and suppose $f : X \to X$ satisfies:

(1) $fA \subseteq B$ and $fB \subseteq A$,

(2) $d(fx, fy) \le kd(x, y)$ for all $x \in A$ and $y \in B$ where $k \in (0,1)$.

Then f has a unique fixed point in $A \cap B$.

The problems of cyclic contractions have been strongly associated with proximity point problems. Some other results dealing with cyclic contractions and proximity point problems may be noted in [4, 7, 10, 24, 25] and [26].

Definition 1.1 [8, 21] A mapping $F : R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$, where R is the set of all real numbers and R^+ denotes the set of all non-negative real numbers.

Definition 1.2 t-norm [8, 21]

A t-norm is a function $\Delta : [0,1]\times[0,1] \to [0,1]$ which satisfies the following conditions for all $a, b, c, d \in [0,1]$

(i) $\Delta(1, a) = a$,

(ii) $\Delta(a,b) = \Delta(b,a)$,

(iii) $\Delta(c,d) \ge \Delta(a,b)$ whenever $c \ge a$ and $d \ge b$,

(iv) $\Delta(\Delta(a,b),c) = \Delta(a,\Delta(b,c))$.

Definition 1.3 Menger space [8, 21]

A Menger space is a triplet (X, F, Δ), where X is a non empty set, F is a function defined on $X \times X$ to the set of distribution functions and Δ is a t-norm, such that the following are satisfied:

(i) $F_{x,y}(0) = 0$ for all $x, y \in X$,

(ii) $F_{x,y}(s) = 1$ for all $s > 0$ and $x, y \in X$ if and only if $x = y$,

(iii) $F_{x,y}(s) = F_{y,x}(s)$ for all $x, y \in X$, $s > 0$ and

(iv) $F_{x,y}(u+v) \geq \Delta(F_{x,z}(u), F_{z,y}(v))$ for all $u, v \geq 0$ and $x, y, z \in X$.

Definition 1.4 A sequence $\{x_n\} \subset X$ is said to converge to some point $x \in X$ if given $\varepsilon > 0, \lambda > 0$ we can find a positive integer $N_{\varepsilon,\lambda}$ such that for all $n > N_{\varepsilon,\lambda}$

$$F_{x_n,x}(\varepsilon) \geq 1 - \lambda. \tag{1.1}$$

Definition 1.5 A sequence $\{x_n\}$ is said to be a Cauchy sequence in X if given $\varepsilon > 0, \lambda > 0$ there exists a positive integer $N_{\varepsilon,\lambda}$ such that

$$F_{x_n,x_m}(\varepsilon) \geq 1 - \lambda \text{ for all } m, n > N_{\varepsilon,\lambda}. \tag{1.2}$$

Definition 1.6 A Menger space (X, F, Δ) is said to be complete if every Cauchy sequence is convergent in X .

Definition 1.7 Φ-function [1]
A function $\phi : R \to R^+$ is said to be a Φ-function if it satisfies the following conditions:
(i) $\phi(t) = 0$ if and only if $t = 0$,
(ii) $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,
(iii) ϕ is left continuous in $(0, \infty)$,
(iv) ϕ is continuous at 0.

Definition 1.8 [1] Let (X, F, Δ) be a Menger space. A self map $f : X \to X$ is said to be ϕ-contractive if

$$F_{fx,fy}(\phi(t)) \geq F_{x,y}(\phi(\frac{t}{c})) \tag{1.3}$$

where $0 < c < 1, x, y \in X$ and *t>0* and the function ϕ is a Φ-function.

2. Main Result

Theorem 2.1 Let (X, F, Δ) be a complete Menger space where Δ is the minimum t-norm and let there exist two closed subsets A and B of X where $A \cap B$ is nonempty such that the mapping $T : A \cup B \to A \cup B$ satisfies the following conditions:

(i) $TA \subseteq B$ and $TB \subseteq A$, (2.1)

(ii) $F_{Tx,Ty}(\phi(ct)) \geq F_{x,y}(\phi(t))$ (2.2)

for all $x \in A$ and $y \in B$ where $c \in (0,1)$ and $t > 0$. Then T has a unique fixed point in $A \cap B$.

Proof: Let x be any arbitrary point in A . Now we define the sequence $\{x_n\}_{n=1}^{\infty}$ in X by

$$x_n = T^n x, n \geq 1.$$

As $x\in A, Tx\in B, T^2x\in A, T^3x\in B$ and so on, in general we obtain

$$x_{2n}=T^{2n}x\in A,\ x_{2n+1}=T^{2n+1}x\in B \text{ for all } n\geq 0. \tag{2.3}$$

For any $t>0, n\geq 1$ we have,

$$F_{T^{2n+1}x,T^{2n+2}x}(\phi(t))=F_{TT^{2n}x,TT^{2n+1}x}(\phi(t))\geq F_{T^{2n}x,T^{2n+1}x}(\phi(\frac{t}{c})). \tag{2.4}$$

Again for any $t>0, n\geq 1$ we have,

$$\begin{aligned} F_{T^{2n}x,T^{2n+1}x}(\phi(t)) &= F_{TT^{2n-1}x,TT^{2n}x}(\phi(t)) \\ &= F_{TT^{2n}x,TT^{2n-1}x}(\phi(t)) \\ &\geq F_{T^{2n}x,T^{2n-1}x}(\phi(\frac{t}{c})) \\ &= F_{T^{2n-1}x,T^{2n}x}(\phi(\frac{t}{c})). \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), for all $n\geq 1$ and $t>0$ we have,

$$F_{x_n,x_{n+1}}(\phi(t))\geq F_{x_{n-1},x_n}(\phi(\frac{t}{c})). \tag{2.6}$$

By successive application of the above inequality for all $n\geq 1$ and $t>0$, we have

$$F_{x_n,x_{n+1}}(\phi(t))\geq F_{x_{n-1},x_n}(\phi(\frac{t}{c}))\geq ....\geq F_{x_0,x_1}(\phi(\frac{t}{c^n})).$$

Taking $n\to\infty$ in the above inequality we have $\lim_{n\to\infty} F_{x_n,x_{n+1}}(\phi(t))=1$. Again, by virtue of a property of ϕ, given $s>0$ we can find $t>0$ such that $s>\phi(t)$.

Thus the above limit implies that for all $s>0$,

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(s)=1. \tag{2.7}$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exist $t_1>0$ and $0<\lambda<1$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k)>m(k)>k$ such that

$$F_{x_{m(k)},x_{n(k)}}(t_1)<1-\lambda. \tag{2.8}$$

We take $n(k)$ corresponding to $m(k)$ to be the smallest integer satisfying (2.8), so that

$$F_{x_{m(k)},x_{n(k)-1}}(t_1)\geq 1-\lambda. \tag{2.9}$$

If $t'<t_1$ then we have

$$F_{x_{m(k)},x_{n(k)}}(t')\leq F_{x_{m(k)},x_{n(k)}}(t_1).$$

We conclude that it is possible to construct $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ with $m(k)<n(k)$ and satisfying (2.8) and (2.9) whenever t_1 is replaced by a smaller positive value. As ϕ is

continuous at 0 and strictly monotone increasing with $\phi(0)=0$, it is possible to obtain $t>0$ such that $\phi(t)<t_1$.

Then, by the above argument, it is possible to obtain an increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k)>m(k)>k$ such that

$$F_{x_{m(k)},x_{n(k)}}(\phi(t))<1-\lambda \tag{2.10}$$

and

$$F_{x_{m(k)},x_{n(k)-1}}(\phi(t))\geq 1-\lambda. \tag{2.11}$$

We have the following four possible cases.

Case I: The integers $m(k)$ is odd and $n(k)$ is even for an infinite number of values of k, that is, there exist $\{m(k(p))\}\subset\{m(k)\},\{n(k(p))\}\subset\{n(k)\}$ where $\{n(k(p))\}$ is even and $\{m(k(p))\}$ is odd for all p. Then, by (2.3), we have $x_{m(k(p))-1}\in A$ and $x_{n(k(p))-1}\in B$.

Therefore,

$$\begin{aligned} 1-\lambda &> F_{x_{m(k(p))},x_{n(k(p))}}(\phi(t)) \\ &= F_{Tx_{m(k(p))-1},Tx_{n(k(p))-1}}(\phi(t)) \\ &\geq F_{x_{m(k(p))-1},x_{n(k(p))-1}}(\phi(\frac{t}{c})). \end{aligned} \tag{2.12}$$

Since ϕ is strictly increasing we can choose $\eta>0$ such that $\phi(\frac{t}{c})-\eta>\phi(t)$. Also by (2.7) we can choose a $\lambda_1>0$ with $\lambda_1<\lambda$ such that

$$F_{x_{m(k(p))-1},x_{m(k(p))}}(\eta)\geq 1-\lambda_1 \tag{2.13}$$

for all $p\geq N_1$, where N_1 is a suitable large positive integer.

Now using (2.12) and (2.13), for all $p\geq N_1$ we have,

$$\begin{aligned} 1-\lambda &> F_{x_{m(k(p))-1},x_{n(k(p))-1}}(\phi(\frac{t}{c})) \\ &\geq \Delta(F_{x_{m(k(p))-1},x_{m(k(p))}}(\eta),F_{x_{m(k(p))},x_{n(k(p))-1}}(\phi(\frac{t}{c})-\eta)) \\ &\geq \Delta(F_{x_{m(k(p))-1},x_{m(k(p))}}(\eta),F_{x_{m(k(p))},x_{n(k(p))-1}}(\phi(t))) \\ &\geq \Delta(1-\lambda_1,1-\lambda) \\ &\geq 1-\lambda, \text{ (since } \lambda_1<\lambda \text{ by our choice)} \end{aligned}$$

which is a contradiction.

Case II: The integers $m(k)$ is even and $n(k)$ is odd for an infinite number of values of k. Here we arrive at a contradiction exactly as in the Case-I above.

Case III: The integers $m(k)$ and $n(k)$ both are even for an infinite number of values of k, that is, there exist $\{m(k(p))\} \subset \{m(k)\}, \{n(k(p))\} \subset \{n(k)\}$ where $\{m(k(p))\}$ and $\{n(k(p))\}$ are even for all p. As ϕ is strictly increasing we can choose $\eta_1 > 0$ such that

$$\phi(t) - \eta_1 \geq \phi(ct).$$

Therefore from (2.10) we have,

$$\begin{aligned} 1-\lambda &> F_{x_{m(k(p))}, x_{n(k(p))}}(\phi(t)) \\ &\geq \Delta(F_{x_{m(k(p))}, x_{m(k(p))+1}}(\eta_1), F_{x_{m(k(p))+1}, x_{n(k(p))}}(\phi(t) - \eta_1)) \\ &\geq \Delta(F_{x_{m(k(p))}, x_{m(k(p))+1}}(\eta_1), F_{x_{m(k(p))+1}, x_{n(k(p))}}(\phi(ct))) \\ &\geq \Delta(F_{x_{m(k(p))}, x_{m(k(p))+1}}(\eta_1), F_{x_{m(k(p))}, x_{n(k(p))-1}}(\phi(t))). \quad \text{(by (2.2))} \end{aligned} \tag{2.14}$$

By (2.7), we can choose a $0 < \lambda_2 < \lambda$ such that

$$F_{x_{m(k(p))}, x_{m(k(p))+1}}(\eta_1) > 1 - \lambda_2$$

for all $p > N_2$ where N_2 is a suitably large positive integer. Using this in (2.14), for all $p > N_2$,

$$\begin{aligned} 1-\lambda &> \Delta(F_{x_{m(k(p))}, x_{m(k(p))+1}}(\eta_1), F_{x_{m(k(p))}, x_{n(k(p))-1}}(\phi(t))) \\ &\geq \Delta(1-\lambda_2, 1-\lambda) \\ &\geq 1-\lambda, \text{ (since } \lambda_2 < \lambda \text{ by our choice)} \end{aligned}$$

which is a contradiction.

Case IV: The integers $m(k)$ and $n(k)$ both are odd for an infinite number of values of k. Here we arrive at a contradiction exactly as in the Case-III above.

Considering the above four cases we conclude that $\{x_n\}$ be a Cauchy sequence. Since X is complete, we have $x_n \to z$ in X for $n \to \infty$. The subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ of $\{x_n\}$ also converges to z. Now $\{x_{2n}\} \subset A$ and A is closed. Therefore $z \in A$. Similarly, $\{x_{2n-1}\} \subset B$ and B is closed. Therefore $z \in B$. Thus we have $z \in A \cap B$.

Now we prove that $Tz = z$.

For this we have, for all $t > 0$,

$$F_{z,Tz}(\phi(t)) \geq \Delta(F_{z,x_{2n+1}}(\eta), F_{x_{2n+1},Tz}(\phi(t) - \eta)) \text{ for some } \eta > 0. \tag{2.15}$$

By the properties of ϕ we can choose $s > 0$ such that $\phi(t) - \eta \geq \phi(s)$ .Therefore we have from (2.15),

$$\begin{aligned} F_{z,Tz}(\phi(t)) &\geq \Delta(F_{z,x_{2n+1}}(\eta), F_{x_{2n+1},Tz}(\phi(t) - \eta)) \\ &\geq \Delta(F_{z,x_{2n+1}}(\eta), F_{x_{2n+1},Tz}(\phi(s))) \\ &\geq \Delta(F_{z,x_{2n+1}}(\eta), F_{x_{2n},z}(\phi(\tfrac{s}{c}))). \quad \text{(using (2.2))} \end{aligned} \tag{2.16}$$

Taking limit on both sides of (2.16) as $n \to \infty$ we have,

$$\begin{aligned} F_{z,Tz}(\phi(t)) &\geq \Delta(F_{z,z}(\eta), F_{z,z}(\phi(\frac{s}{c}))) \\ &= \Delta(1,1) \\ &= 1. \end{aligned}$$

As ϕ is continuous at 0 and strictly monotone increasing with $\phi(0) = 0$, it is possible to obtain $t > 0$ such that $\phi(t) < s$ and we can conclude that $z = Tz$.

To prove the uniqueness of the fixed point, let us take $v \in A \cap B$ be another fixed point of T, that is, $Tv = v$.

$$\begin{aligned} \text{Now, } F_{z,v}(\phi(t)) &= F_{Tz,Tv}(\phi(t)) \\ &\geq F_{z,v}(\phi(\frac{t}{c})) \\ &\geq F_{z,v}(\phi(\frac{t}{c^2})) \\ &\ldots \ldots \ldots \ldots \\ &\geq F_{z,v}(\phi(\frac{t}{c^n})) \to 1 \text{ as } n \to \infty. \end{aligned}$$

Again by the properties of ϕ we can conclude that $z = v$.

Therefore $z \in A \cap B$ is a unique fixed point of T.

Now we give the following example to validate our result.

Example 2.1 Let $X = \{x_1, x_2, x_3, x_4\}$, $A = \{x_1, x_2\}$ and $B = \{x_2, x_3, x_4\}$. Here the t-norm $\Delta(a,b) = \min(a,b)$ and $F_{x,y}(t)$ be defined as

$$F_{x_1,x_2}(t) = F_{x_1,x_3}(t) = F_{x_1,x_4}(t) = F_{x_2,x_4}(t) = F_{x_3,x_4}(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ 0.40, & \text{if} \quad 0 < t < 4, \\ 1, & \text{if} \quad t \geq 4, \end{cases}$$

$$F_{x_2,x_3}(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ 0.60, & \text{if} \quad 0 < t \leq 7, \\ 1, & \text{if} \quad t > 7. \end{cases}$$

It is easy to verify that (X, F, Δ) be a complete Menger space. If we define $T : A \cup B \to A \cup B$ as follows : $Tx_1 = x_3, Tx_2 = x_2, Tx_3 = x_2, Tx_4 = x_1$ then it satisfies all the conditions of the Theorem 2.1 where $\phi(t) = t$ and x_2 is the unique fixed point of T. If we take $t = 3$ and $c = \frac{1}{2}$ then T is not a ϕ-contraction as $F_{Tx_3,Tx_4}(\phi(3)) < F_{x_3,x_4}(\phi(6))$.

Note: In this paper the property (iii) of Definition 1.7 has not been used.

Acknowledgements

The work is partially supported by CSIR, Govt. of India, Research Project No – 25(0168)/09/EMR-II. The support is gratefully acknowledged.

References

[1] B.S. Choudhury and K.P. Das, A new contraction principle in Menger spaces, *Acta Mathematica Sinica*, *English Series*, 24 (2008), 1379-1386.

[2] B.S. Choudhury, P.N. Dutta and K.P. Das, A fixed point results in Menger spaces using a real function, *Acta Math. Hungar.*, 122 (2008), 203-216.

[3] B.S. Choudhury and K.P. Das, A coincidence point result in Menger spaces using a control function, *Chaos*, *Solitons and Fractals*, 42 (2009), 3058-3063.

[4] C. Di Baria, T. Suzuki and C. Vetro, Best proximity points for cyclic Meir–Keeler contractions, *Nonlinear Analysis*, 69 (2008), 3790-3794.

[5] P.N. Dutta, B.S. Choudhury and K.P. Das, Some fixed point results in Menger spaces using a control function, *Surveys in Mathematics and its Applications*, 4 (2009), 41-52.

[6] P.N. Dutta and B.S. Choudhury, A generalized contraction principle in Menger spaces using control function, *Anal. Theory Appl.*, 26 (2010), 110-121.

[7] A. Fernandez-Leon, Existence and uniqueness of best proximity points in geodesic metric spaces, *Nonlinear Analysis*, 73 (2010), 915-921.

[8] O. Hadzic and E. Pap, Fixed Point Theory in Probabilistic Metric Spaces, Kluwer Academic Publishers, 2001.

[9] T.L. Hicks, Fixed point theory in probabilistic metric spaces, *Zb. Rad. Priod. Mat. Fak. Ser. Mat.*, 13 (1983), 63-72.

[10] S. Karpagam and S. Agrawal, Best proximity point theorems for cyclic orbital Meir–Keeler contraction maps, *Nonlinear Analysis*, 74 (2011), 1040-1046.

[11] W.A. Kirk, P.S. Srinivasan and P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, *Fixed Point Theory*, 4 (2003), 79-89.

[12] M.S. Khan, M. Swaleh and S. Sessa, Fixed point theorems by altering distances between the points, *Bull. Austral. Math. Soc.*, 30 (1984), 1-9.

[13] Yicheng Liu and Zhixing Li, Coincidence point theorem in probabilistic and fuzzy metric spaces, *Fuzzy Sets and Systems*, 158 (2007), 58-70.

[14] D. Mihet, Altering distances in probabilistic Menger spaces, *Nonlinear Analysis*, 71 (2009), 2734-2738.

[15] S.V.R. Naidu, Some fixed point theorems in metric spaces by altering distances, *Czechoslovak Mathematical Journal*, 53 (2003), 205-212.

[16] D. O'Regan and R. Saadati, Nonlinear contraction theorems in probabilistic spaces, *Appl. Math. Comput.*, 195 (2008), 86-93.

[17] V. Radu, Sehgal contractions on Menger spaces, *Fixed Point Theory*, 7 (2006), 315-322.

[18] A. Razani and K. Fouladgar, Extension of contractive maps in the Menger probabilistic metric space, *Chaos, Solitons and Fractals*, 34 (2007), 1724-1731.

[19] K.P.R. Sastry and G.V.R. Babu, Some fixed point theorems by altering distances between the points, *Ind. J. Pure. Appl. Math.*, 30 (1999), 641-647.

[20] K.P.R. Sastry, S.V.R. Naidu, G.V.R. Babu and G.A. Naidu, Generalisation of common fixed point theorems for weakly commuting maps by altering distances, *Tamkang Journal of Mathematics*, 31 (2000), 243-250.

[21] B. Schweizer and A. Sklar, Probabilistic Metric Spaces, Elsevier, North-Holland, (1983).

[22] V.M. Sehgal and A.T. Bharucha-Reid, Fixed point of contraction mappings on PM space, *Math. Sys. Theory*, 6 (1972), 97-100.

[23] S. Shakeri, A contraction theorem in Menger probabilistic metric spaces, *The Journal of Nonlinear Science and Applications*, 1 (2008), 189-193.

[24] C. Vetro, Best proximity points: Convergence and existence theorems for p-cyclic mappings, *Nonlinear Analysis*, 73 (2010), 2283-2291.

[25] K. Wlodarczyk, R. Plebaniak and A. Banach, Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces, *Nonlinear Analysis*, 70 (2009), 3332–3341.

[26] K. Wlodarczyk, R. Plebaniak and C. Obczy«ski, Convergence theorems, best approximation and best proximity for set-valued dynamic systems of relatively quasi-asymptotic contractions in cone uniform spaces, *Nonlinear Analysis*, 72 (2010), 794-805.