ON SEMILOCAL MODULES AND RINGS

CHRISTIAN LOMP
HEINRICH HEINE UNIVERSITÄT
D-40225 DÜSSELDORF, GERMANY
LOMPMATH.UNI-DUESSELDORF.DE

Abstract. It is well-known that a ring R is semiperfect if and only if ${}_RR$ (or R_R) is a supplemented module. Considering *weak supplements* instead of supplements we show that *weakly supplemented* modules M are *semilocal* (i.e., $M/Rad(M)$ is semisimple) and that R is a semilocal ring if and only if ${}_RR$ (or R_R) is weakly supplemented. In this context the notion of *finite hollow dimension* (or *finite dual Goldie dimension*) of modules is of interest and yields a natural interpretation of the Camps-Dicks characterization of semilocal rings. Finitely generated modules are weakly supplemented if and only if they have finite hollow dimension (or are semilocal).

1. Preliminaries

Let R be an associative ring with unit and throughout the paper M will be a left unital R-module. By $N \trianglelefteq M$ we denote an essential submodule $N \subset M$. M is *uniform* if $M \neq 0$ and every non-zero submodule is essential in M, and M has *finite uniform dimension* (or *finite Goldie dimension*) if there exists a sequence

$$0 \longrightarrow \bigoplus_{i=1}^n U_i \xrightarrow{\ f\ } M,$$

where all the U_i are uniform and the image of f is essential in M. Then n is called the uniform dimension of M and we write $udim(M) = n$. It is well known that this is equivalent to M having no infinite

I would like to thank Nyguen Viet Dung for bringing the R. Camps and W. Dicks paper to my attention. Moreover I want to express my thanks to Patrick F. Smith and Robert Wisbauer for their interest and helpful suggestions.

independent family of non-zero submodules (there is a maximal finite independent family of uniform submodules).

We denote a small submodule N of M by $N \ll M$. A module M is said to be *hollow* if $M \neq 0$ and every proper submodule is small in M. M is said to have *finite hollow dimension* (or *finite dual Goldie dimension*) if there exists an exact sequence

$$M \xrightarrow{\ g\ } \bigoplus_{i=1}^{n} H_i \longrightarrow 0$$

where all the H_i are hollow and the kernel of g is small in M. Then n is called the hollow dimension of M and we write $hdim(M) = n$.

B. Sarath and K. Varadarajan showed in [SV, Theorem 1.8] that in this case M does not allow an epimorphism to a direct sum with more than n summands. Dual to the notion of an independent family of submodules we have:

Definition 1.1. *Let M be an R-module and $\{K_\lambda\}_\Lambda$ a family of proper submodules of M. $\{K_\lambda\}_\Lambda$ is called coindependent (see* [T76]*) if for every $\lambda \in \Lambda$ and finite subset $J \subseteq \Lambda \setminus \{\lambda\}$*

$$K_\lambda + \bigcap_{j \in J} K_j = M$$

holds(convention: if J is the empty set, then set $\bigcap_J K_j := M$).

For finitely generated modules it usually suffices to consider coindependent families of finitely generated submodules as our next observation shows.

Lemma 1.2. *Let M be a finitely generated R-module and $\{N_1, \ldots, N_m\}$ a coindependent family of submodules. Then there exists a coindependent family of finitely generated submodules $L_i \subseteq N_i$, $1 \leq i \leq m$.*

Proof. Since M is finitely generated, for each $1 \leq i \leq m$, there exist finitely generated submodules $X_i \subseteq N_i$ and $Y_i \subseteq \bigcap_{j \neq i} N_j$ such that $X_i + Y_i = M$. Let $L_i := X_i + \sum_{j \neq i} Y_j \subseteq N_i$. As $L_i + \bigcap_{j \neq i} L_j \supseteq X_i + Y_i = M$ holds the result follows. □

Theorem 1.3 (Grezeszcuk, Puczyłowski, Reiter, Takeuchi, Varadarajan)**.** *For an R-module M the following statements are equivalent:*

(a) *M has finite hollow dimension.*
(b) *M does not contain an infinite coindpendent family of submodules.*
(c) *There exists a unique number n and a coindependent family $\{K_1, \dots, K_n\}$ of proper submodules, such that $M/K_1, \dots, M/K_n$ are hollow modules and $K_1 \cap \dots \cap K_n \ll M$.*
(d) *For every descending chain $K_1 \supset K_2 \supset K_3 \supset \cdots$ of submodules of M, there exits a number n such that $K_n/K_m \ll M/K_m$, for all $m \geq n$.*

Proof. The equivalence of $(b), (c), (d)$ can be found in [GP]. The equivalence of (a) and (c) is given by the chinese remainder theorem (see [W, 9.12]). $\square$

Remark 1.4. Let M be an R-module and N, L submodules of M. Then the following properties hold:
(1) $hdim(M/N) \leq hdim(M)$;
(2) $N \ll M \Rightarrow hdim(M) = hdim(M/N)$;
(3) $hdim(N \oplus L) = \ hdim(N) + hdim(L)$.
Moreover if M is self-projective and has finite hollow dimension, then every surjective endomorphism is an isomorphism.

We refer to [GP], [HaS], [HeS],[Lo], [Re], [T76] and [V] for more information on dual Goldie dimension.

The following theorem can be seen as an attempt to transfer R. Camps and W. Dicks characterization of semilocal rings [CD] to arbitrary modules with finite hollow dimension. Denote by $\mathcal{L}(M)$ the lattice of submodules of a module M.

Theorem 1.5. *For M the following statements are equivalent:*
(a) *M has finite hollow dimension.*
(b) *There exists an $n \in I\!N$ and a mapping $d : \mathcal{L}(M) \to \{0, 1, \dots, n\}$ such that for all $N, L \in \mathcal{L}(M)$:*
   (i) *If $d(N) = 0$, then $N = M$.*
   (ii) *If $N + L = M$, then $d(N \cap L) = d(N) + d(L)$.*
(c) *There exists a partial ordering $(\mathcal{L}(M), \leq)$ such that*
   (i) *$(\mathcal{L}(M), \leq)$ is an artinian poset;*

(ii) *for all* $N, L \in \mathcal{L}(M)$ *with* $N + L = M$*: if* $L \neq M$*, then* $N > N \cap L$.

Proof. $(a) \Rightarrow (b)$ Let $d(N) := hdim(M/N)$; then the conditions (i) and (ii) are easily checked.

$(b) \Rightarrow (c)$ Let $N < L :\Leftrightarrow d(N) < d(L)$ and $N = L :\Leftrightarrow d(N) = d(L)$ then $(\mathcal{L}(M), \leq)$ is artinian. Let $N, L \in \mathcal{L}(M)$ with $N + L = M$ and $L \neq M$. By (i) and (ii) we have $d(N \cap L) = d(N) + d(L) < d(N)$. Hence $N > N \cap L$.

$(c) \Rightarrow (a)$ Assume that $\{K_i\}_{\mathbb{N}}$ is an infinite coindependent family of submodules of M. Then we have for all $i \in \mathbb{N}$: $K_1 \cap \cdots \cap K_i + K_{i+1} = M$ and $K_{i+1} \neq M$. Hence by (ii) we get the infinite descending chain

$$K_1 > K_1 \cap K_2 > \cdots > K_1 \cap \cdots \cap K_i > \cdots$$

contradicting property (i). Hence M does not contain an infinite coindependent family of submodules. □

2. Weakly Supplemented Modules

Dual to a complement of a submodule N of M the *supplement* of N is defined as a submodule L of M minimal with respect to $N + L = M$. This is equivalent to $N + L = M$ and $N \cap L \ll L$. Recall that M is *supplemented* if every submodule has a supplement in M.

More generally, a submodule N of M has a *weak supplement* L in M if $N + L = M$ and $N \cap L \ll M$, and M is called *weakly supplemented* if every submodule N of M has a weak supplement (see Zöschinger [Z78a]). Examples for weakly supplemented modules are supplemented, artinian, linearly compact, uniserial or hollow modules. For supplemented modules over commutative local noetherian rings we refer to [Z78a], [Z78b], [Z86] and [Ru].

Before we give a summarizing list of properties of weakly supplemented modules, we will state a general result:

Proposition 2.1. *For a proper submodule* $N \subset M$*, the following are equivalent:*

(a) M/N *is semisimple;*

(b) *for every* $L \subseteq M$ *there exists a submodule* $K \subseteq M$ *such that* $L + K = M$ *and* $L \cap K \subseteq N$*;*

(c) *there exists a decomposition* $M = M_1 \oplus M_2$ *such that* M_1 *is semisimple,* $N \trianglelefteq M_2$ *and* M_2/N *is semisimple.*

Proof. (a) ⇒(c) Let M_1 be a complement of N. $M_1 \simeq (M_1 \oplus N)/N$ is a direct summand in M/N, hence semisimple and there is a semisimple submodule M_2/N such that $(M_1 \oplus N)/N \oplus M_2/N = M/N$. Thus $M = M_1 + M_2$ and $M_1 \cap M_2 \subseteq N \cap M_1 = 0$ implies $M = M_1 \oplus M_2$. Since M_1 is a complement we have by the natural isomorphisms $N \simeq (M_1 \oplus N)/M_1 \trianglelefteq M/M_1 \simeq M_2$ that $N \trianglelefteq M_2$.

(c) ⇒(a) ⇒ (b) clear.

(b) ⇒(a) Let $L/N \subseteq M/N$; then there exists a submodule $K \subseteq M$ such that $L+K = M$ and $L \cap K \subseteq N$. Thus $L/N \oplus (K+N)/N = M/N$. Hence every submodule of M/N is a direct summand. □

Let $\text{Rad}(M)$ denote the radical of M. We call M a *semilocal module* if $M/\text{Rad}\,(M)$ is semisimple. Any semilocal module M is a *good* module, i.e., for every homomorphism $f : M \to N$, $f(\text{Rad}\,(M)) = \text{Rad}\,(f(M))$ (see [W]).

We call N a *small cover* of a module M if there exists an epimorphism $f : N \to M$ such that $\text{Ker}\,(f) \ll M$. Then f is called a *small epimorphism.* N is called a *flat cover*, *projective cover* resp. *free cover* of M if N is a small cover of M and N is a flat, projective resp. free module. Note that this definition of a flat cover is different from Enochs' definition.

Proposition 2.2. *Assume* M *to be weakly supplemented. Then:*

(1) M *is semilocal;*

(2) $M = M_1 \oplus M_2$ *with* M_1 *semisimple,* M_2 *semilocal and* $\text{Rad}\,(M) \trianglelefteq M_2$*;*

(3) *every factor module of* M *is weakly supplemented;*

(4) *any small cover of* M *is weakly supplemented;*

(5) *every supplement in* M *and every direct summand of* M *is weakly supplemented.*

Proof. (1) and (2) follow from Proposition 2.1 since for every $L \subseteq M$ there exists a weak supplement $K \subseteq M$ such that $L + K = M$ and $L \cap K \subseteq \mathrm{Rad}\,(M)$.

(3) Let $f : M \to N$ be an epimorphism and $K \subset N$, then $f^{-1}(K)$ has a weak supplement L in M and it is straightforward to prove that $f(L)$ is a weak supplement of K in N.

(4) Let N be a small cover of M and $f : N \to M$ be a small epimorphism. First note that $f^{-1}(K) \ll N$ for every $K \ll M$ holds since $\mathrm{Ker}\,(f) \ll N$. Let $L \subset N$. Then $f(L)$ has a weak supplement X in M. Again it is easy to check that $f^{-1}(X)$ is a weak supplement of L in N.

(5) If $N \subseteq M$ is a supplement of M, then $N + K = M$ for some $K \subseteq M$ and $K \cap N \ll N$. By (3), $M/K \simeq N/(N \cap K)$ is weakly supplemented and by (4), N is weakly supplemented. Direct summands are supplements and hence weakly supplemented. □

Let *length*(M) denote the length of the module M.

Corollary 2.3. *An R-module M with* $\mathrm{Rad}\,(M) = 0$ *is weakly supplemented if and only if M is semisimple. In this case hdim(M) = length(M) holds.*

Proof. This follows by Proposition 2.2(1). □

We need the following technical lemma to show that every finite sum of weakly supplemented modules is weakly supplemented.

Lemma 2.4. *Let M be an R-module with submodules K and M_1. Assume M_1 is weakly supplemented and $M_1 + K$ has a weak supplement in M. Then K has a weak supplement in M.*

Proof. By assumption $M_1 + K$ has a weak supplement $N \subseteq M$, such that $M_1 + K + N = M$ and $(M_1 + K) \cap N \ll M$. Because M_1 is weakly supplemented, $(K + N) \cap M_1$ has a weak supplement $L \subseteq M_1$. So

$M = M_1 + K + N = L + ((K + N) \cap M_1) + K + N = K + (L + N)$ and

$K \cap (L+N) \subseteq ((K+L) \cap N) + ((K+N) \cap L \subseteq ((K+M_1) \cap N) + ((K+N) \cap L) \ll M.$

Hence $N + L$ is a weak supplement of K in M. □

Proposition 2.5. *Let $M = M_1 + M_2$, where M_1 and M_2 are weakly supplemented, then M is weakly supplemented.*

Proof. For every submodule $N \subseteq M$, $M_1 + (M_2 + N)$ has the trivial weak supplement 0 and by the Lemma above $M_2 + N$ has a weak supplement in M as well. Applying the Lemma again we get a weak supplement for N. $\square$

Corollary 2.6. *Every finite sum of weakly supplemented modules is weakly supplemented.*

The relationship between the concepts 'hollow dimension' and 'weakly supplemented' is expressed in the following theorem.

Theorem 2.7. *Consider the following properties:*

(i) *M has finite hollow dimension;*
(ii) *M is weakly supplemented;*
(iii) *M is semilocal.*

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $hdim(M) \geq length(M/Rad(M))$ holds.
If $Rad(M) \ll M$ then $(iii) \Rightarrow (ii)$ holds.
If M is finitely generated then $(iii) \Rightarrow (i)$ and $hdim(M) = length(M/Rad(M))$ holds.

Proof. $(i) \Rightarrow (ii)$ There is a small epimorphism $f : M \to \bigoplus_{i=1}^{n} H_i$ with hollow modules H_i. Since hollow modules are (weakly) supplemented we get by Corollary 2.6 that $\bigoplus_{i=1}^{n} H_i$ is weakly supplemented. Since f is a small epimorphism we get by Proposition 2.2(4) that M is weakly supplemented.

$(ii) \Rightarrow (iii)$ by Propositon 2.2(1).

If $Rad(M) \ll M$, then $(iii) \Rightarrow (ii)$ follows by Proposition 2.2(4).

If M is finitely generated and (iii) holds, then M is a small cover of $M/\mathrm{Rad}(M)$. By Corollary 2.3, $hdim(M/\mathrm{Rad}(M)) = length(M/\mathrm{Rad}(M))$, and by remark 1.4(2), $hdim(M) = length(M/\mathrm{Rad}\,(M))$. $\square$

3. Semilocal Modules and Rings

Let $Gen(M)$ denote the class of M-generated modules.

Theorem 3.1. *The following statements about M are equivalent:*

(a) *M is semilocal;*
(b) *any $N \in Gen(M)$ is semilocal;*
(c) *any $N \in Gen(M)$ is a direct sum of a semisimple module and a semilocal module with essential radical;*
(d) *any $N \in Gen(M)$ with small radical is weakly supplemented;*
(e) *any finitely generated $N \in Gen(M)$ has finite hollow dimension.*

Proof. $(a) \Rightarrow (b)$ For every $N \in Gen(M)$ there exists a set Λ and an epimorphism $f : M^{(\Lambda)} \to N$. Since $f(\text{Rad}\,(M^{(\Lambda)})) \subseteq \text{Rad}\,(N)$ and $M^{(\Lambda)}/\text{Rad}\,(M^{(\Lambda)}) \simeq (M/\text{Rad}\,(M))^{(\Lambda)}$ always holds we get an epimorphism $\bar{f} : (M/\text{Rad}\,(M))^{(\Lambda)} \to N/\text{Rad}\,(N)$. Hence N is semilocal.

$(b) \Rightarrow (a)$ trivial.

$(b) \Leftrightarrow (c)$ by Proposition 2.1.

$(b) \Leftrightarrow (d) \Leftrightarrow (e)$ by Theorem 2.7 $\square$

Recall that the ring R is *semilocal* if ${}_RR$ (or R_R) is a semilocal R-module.

Corollary 3.2. *For a ring R the following statements are equivalent:*

(a) *${}_RR$ is weakly supplemented;*
(b) *${}_RR$ has finite hollow dimension;*
(c) *R is semilocal;*
(d) *R_R has finite hollow dimension;*
(e) *R_R is weakly supplememted.*

In this case $hdim({}_RR) = length(R/\text{Jac}\,(R)) = hdim(R_R)$.

Proof. Apply Theorem 2.7 and use that 'semilocal' is a left-right symmetric property. $\square$

Remark 3.3. Consider the ring

$$R := \mathbb{Z}_{p,q} := \left\{ \frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0, p \nmid b \text{ and } q \nmid b \right\},$$

where p and q are primes. Then R is a commutative uniform semilocal noetherian domain with two maximal ideals. Since R is uniform, the decomposition of $R/\mathrm{Jac}\,(R)$ cannot be lifted to R. Moreover the maximal ideals pR and qR are weak supplements but not supplements of each other. So R is a semilocal ring which is not semiperfect.

For our next result we need the following:

Lemma 3.4. *Let R be a ring, $r, a \in R$ and $b := 1 - ra$. Then $Ra \cap Rb = Rab$.*

Proof. $x \in Ra \cap Rb$, then $x = ta = sb = s(1 - ra) \Rightarrow s = (t + sr)a \in Ra$. Hence $Ra \cap Rb \subseteq Rab$. Conversely $Rab = Ra(1 - ra) = R(1 - ar)a \subseteq Ra \cap Rb$. □

We are now ready to give characterizations of semilocal rings in terms of finite hollow dimension and to prove results from Camps-Dicks (see [CD, Theorem 1]) in a module-theoretic way.

Note that for a semilocal ring R, ${}_RR$ is a good module, and so for any left R-module N we have $\mathrm{Rad}\,(M) = \mathrm{Jac}\,(R)\,M$ (see [W, 23.7]).

Theorem 3.5. *For any ring R the following statements are equivalent:*

(a) *R is semilocal;*
(b) *every left R-module is semilocal;*
(c) *every left R-module is the direct sum of a semisimple module and a semilocal module with essential radical;*
(d) *every left R-module with small radical is weakly supplemented;*
(e) *every finitely generated left R-module has finite hollow dimension;*
(f) *every product of semisimple left R-modules is semisimple;*
(g) *there exists an $n \in \mathbb{N}$ and a map $d : R \to \{0, 1, \ldots, n\}$ such that for all $a, b \in R$ the following holds:*
   (i) *$d(a) = 0 \Rightarrow a$ is a unit;*
   (ii) *$d(a(1 - ba)) = d(a) + d(1 - ba)$;*
(h) *there exists a partial ordering $(R, \leq)$ such that:*
   (i) *$(R, \leq)$ is an artinian poset;*
   (ii) *for all $a, b \in R$ such that $1 - ba$ is not a unit, we have $a > a(1 - ba)$.*

In this case $hdim(R) \leq n$ *holds.*

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$ follow from Theorem 3.1.

$(b) \Rightarrow (f)$ By the remark above, semilocal rings are good rings and hence $\text{Rad}\,(M) = \text{Jac}\,(R)M$ holds for every left R-module. Let M be a product of semisimple modules. Since for all $m \in M$ $\text{Jac}\,(R)Rm = 0$ holds as Rm is semisimple, we have $\text{Rad}\,(M) = \text{Jac}\,(R)M = 0$. By (b) M is semisimple.

$(f) \Rightarrow (a)$ $R/\text{Jac}\,(R)$ is a submodule of a product of simple modules. By (f) this product is semisimple and so is $R/\text{Jac}\,(R)$.

$(a) \Rightarrow (g)$ By Corollary 3.2 ${}_RR$ has finite hollow dimension. By Theorem 1.5 there is a map d'. Set $d(a) := d'(Ra)$ for all $a \in R$ and (i) and (ii) follow easily from the properties of d'.

$(g) \Rightarrow (h)$ Let $a < b :\Leftrightarrow d(a) < d(b)$ and $a = b :\Leftrightarrow d(a) = d(b)$ for all $a, b \in R$. If $1 - ba$ is not a unit, then $d(1 - ba) > 0$ implies $d(a) < d(a(1 - ba))$ and hence $a > a(1 - ba)$.

$(h) \Rightarrow (a)$ Assume that there exists a left ideal $I \subset R$ that has no weak supplement. Then we can construct an infinite descending chain of elements

$$1 > b_1 > b_2 > \cdots > b_n > \cdots$$

such that for all $n \in \mathbb{N}$ we have $I + Rb_n = R$. Since $(R, \leq)$ is artinian - this is a contradiction, hence I must have a weak supplement in R. By Corollary 3.2 R is semilocal.

We can construct the chain as follows: Let $n = 1$. Since $I \not\ll R$ there is an $a \in I$ such that $1 - a$ is not a unit in R. Hence $1 > 1 - a =: b_1$ and $I + Rb_1 = R$ holds.

Now assume that we constructed a chain $1 > b_1 > b_2 > \cdots > b_n$ for $n \geq 1$ with $I + Rb_n = R$. By assumption $I \cap Rb_n \not\ll R$ implies that there is an $r \in R$, such that $rb_n \in I$ and $x := 1 - rb_n$ is not a unit in R. Hence

$$b_n > b_n(1 - rb_n) = b_n x =: b_{n+1}.$$

Moreover, by the modularity law, we have $Rb_n = (I \cap Rb_n) + (Rb_n \cap Rx)$. Together with Lemma 3.4, $R = I + Rb_n = I + (Rb_n \cap Rx) = I + Rb_{n+1}$ holds. $\square$

Remark 3.6. Theorem 3.5 generalizes the well-known fact that a ring R is semiperfect if and only if every finitely generated R-module is supplemented.

Recall that every finitely generated R-module over a semiperfect ring R has a projective cover.

Corollary 3.7. *Every finitely generated R-module over a semilocal ring R is a direct summand of a module having a finitely generated free cover.*

Proof. Let M be a finitely generated R-module. Then there exists a number k and an epimorphism $f : R^k \to M$. Since R is semilocal, R^k is weakly supplemented. Hence $K := \text{Ker}\,(f)$ has a weak supplement $L \subseteq R^k$. Thus the natural projection $R^k \to M \oplus (R^k/L)$ with kernel $K \cap L \ll R^k$ implies that R^k is a projective cover for $M \oplus (R^k/L)$. $\square$

Comparing semiperfect and semilocal rings the following fact is of interest:

Theorem 3.8. *For a ring R the following statements are equivalent.*

(a) *R is semiperfect;*
(b) *R is semilocal and every simple R-module has a flat cover;*
(c) *R is semilocal and every finitely generated R-module has a flat cover.*

Proof. $(a) \Rightarrow (c)$ holds since projective modules are flat.

$(c) \Rightarrow (b)$ is trivial.

$(b) \Rightarrow (a)$ Assume R is semilocal and consider $R/\text{Jac}(R) = E_1 \oplus \cdots \oplus E_n$ with E_i simple R-modules. Every simple R-module is isomorphic to one of the $E_i's$. By hypothesis every E_i has a flat cover L_i. Thus $L := L_1 \oplus \cdots \oplus L_n$ is a flat cover of $R/\text{Jac}\,(R)$. Hence we obtain the following diagram:

$$\begin{array}{ccccc} & & R & & \\ & & \downarrow & & \\ L & \xrightarrow{f} & R/\text{Jac}\,(R) & \longrightarrow & 0 \end{array}$$

that can be extended by a homomorphism $g : R \to L$. Since f is a small epimorphism and gf is epimorph, g must be epimorph with $\text{Ker}\,(g) \subseteq \text{Ker}\,(gf) = \text{Jac}\,(R)$. Hence R is a projective cover of the flat module L. By [W, 36.4], $L \simeq R$ and hence all L_i must be projective. Thus each simple R-module has a projective cover and so R is semiperfect (see [W, 42.6]). □

Remark 3.9. It follows also from Theorem [W, 36.4] that a ring R is semisimple if and only if R is semilocal and every simple R-module is flat. Since in this case R is a projective cover of the flat module $R/\text{Jac}\,(R)$ and hence $R \simeq R/\text{Jac}\,(R)$ holds.

The following result was first proved by T.Takeuchi in [T94]. We will give a new proof of his result.

Theorem 3.10 (Takeuchi)**.** *Let M be a self-projective R-module. Then M has finite hollow dimension if and only if $S := \text{End}(M)$ is semilocal. Moreover we have $hdim({}_RM) = hdim(S)$.*

Proof. $\Rightarrow$: Let $\{I_1, \ldots, I_n\}$ be a coindependent family of proper left ideals of ${}_SS$. By Lemma 1.2, we may assume that the $I_k{}'s$ are finitely generated. Consider the epimorphism

$$S \longrightarrow \bigoplus_{k=1}^n S/I_k \longrightarrow 0.$$

Applying $M \otimes_S -$ we get the exact sequence

$$M \longrightarrow \bigoplus_{k=1}^n M/MI_k \longrightarrow 0,$$

since $M \otimes_S S/I_k \simeq M/MI_k$. We have $I_k = \text{Hom}\,(M, MI_k)$ and hence $MI_k \neq M$. Thus $hdim(S) \leq hdim({}_RM)$ and so S is semilocal by Corollary 3.2.

$\Leftarrow$: Consider an epimorphism (with $N_i \neq M$)

$$M \longrightarrow \bigoplus_{i=1}^n M/N_i \longrightarrow 0.$$

Since M is self-projective, $\text{Hom}\,(M, -)$ yields an exact sequence

$$S \longrightarrow \bigoplus_{i=1}^n \text{Hom}\,(M, M/N_i) \longrightarrow 0,$$

showing that $hdim({}_RM) \leq hdim(S)$. □

Remark 3.11. More generally, if P is an M-projective module that generates M, then one can apply $\text{Hom}(P, -)$ in the same way as in Theorem 3.10 to obtain $hdim({}_R M) \leq hdim({}_S \text{Hom}(P, M))$, where $S := \text{End}(P)$.

The following Corollaries are immediate consequences from Takeuchi's result.

Corollary 3.12. *A ring R is semilocal if and only if every finitely generated, self-projective left (or right) R-module has a semilocal endomorphism ring.*

Proof. The assertion follows from Theorem 3.5 and Theorem 3.10. □

Corollary 3.13. *Let M be a self-projective R-module with semilocal endomorphism ring. Then* $\text{End}(M/N)$ *is semilocal for any fully invariant submodule N of M.*

Proof. Since M is self-projective and N fully invariant we get by [W, 18.2] that M/N is self-projective. By Theorem 3.10 we have $hdim(End(M)) = hdim(M) \geq hdim(M/N) = hdim(End(M/N))$. □

Analogous to the fact that a projective module has a semiperfect endomorphism ring if and only if it is finitely generated and supplemented (see [W, 42.12]) we get the following corollary:

Corollary 3.14. *Let M be a self-projective R-module. M is finitely generated and weakly supplemented if and only if* $\text{End}(M)$ *is semilocal and* $\text{Rad}(M) \ll M$.

Proof. This follows from Theorem 2.7, Theorem 3.10 and the fact that a module with finite hollow dimension and small radical is finitely generated. □

The author does not know if the hypothesis of a small radical of M is necessary. He raises the following

Question: Is every (self-)projective R-module with semilocal endomorphism ring finitely generated ?

Remark 3.15. This question is closely related to an old problem of D. Lazard. He considered rings with the property that all projective modules P with $P/\mathrm{Rad}\,(P)$ finitely generated are already finitely generated. Following H. Zöschinger, rings with this property are called L-rings. He proved in [Z81] that this property is left-right symmetric, i.e. R is a left L-ring if and only if it is a right L-ring. Moreover he showed that a ring R is an L-ring if and only if every supplement in R is a direct summand ([Z81, Satz 2.3]). Hence semiperfect and semiprimitive rings, i.e rings with zero Jacobson radical, are L-rings. In [J], S.Jøndrup showed that every PI-ring is an L-ring. A good resource for some characterizations of L-rings is [MS].

Corollary 3.16. *Let R be an L-ring and P a projective R-module. Then P is finitely generated and weakly supplemented if and only if* $\mathrm{End}\,(P)$ *is semilocal.*

Proof. Assume $\mathrm{End}\,(P)$ to be semilocal. By Takeuchi's result (Theorem 3.10) P has finite hollow dimension and hence $P/\mathrm{Rad}(P)$ is finitely generated. As R is an L-ring, P is finitely generated. □

Remark 3.17. In [GS] V.N. Gerasimov and I.I. Sakhaev constructed a non - commutative semilocal ring that is not an L-ring (see also [S91], [S93]). Hence a negative answer to the question above might be more likely, but the condition of a semilocal endomorphism ring $\mathrm{End}\,(P)$ is stronger than $P/\mathrm{Rad}\,(P)$ being finitely generated.

A ring R is left *f-semiperfect* or *semiregular* if every finitely generated left ideal has a supplement in ${}_RR$, equivalently, $R/\mathrm{Jac}\,(R)$ is von Neumann regular and idempotents in $R/\mathrm{Jac}\,(R)$ can be lifted to ${}_RR$ (see [W, 42.11]). Analogous to that we have:

Proposition 3.18. *For any ring R the following statements are equivalent:*

(a) *every principal left ideal of R has a weak supplement in ${}_RR$;*
(b) $R/\mathrm{Jac}\,(R)$ *is von Neumann regular;*
(c) *every principal right ideal of R has a weak supplement in R_R;*

Proof. $(a) \Rightarrow (b)$ Let $a \in R$. By assumption there exists a weak supplement $I \subset R$ of Ra. Hence there exist $b \in R$ and $x \in I$ such that $x = 1 - ba$. Moreover, by Lemma 3.4, $Rax = Ra \cap Rx \subseteq Ra \cap I \ll R$ implies $ax = a - aba \in \mathrm{Jac}\,(R)$. Thus $R/\mathrm{Jac}\,(R)$ is von Neumann regular.

$(b) \Rightarrow (a)$ For any $a \in R \setminus \mathrm{Jac}\,(R)$ we get an element $b \in R \setminus \mathrm{Jac}\,(R)$ such that $a - aba \in \mathrm{Jac}\,(R)$. Hence $R(1 - ba)$ is a weak supplement of Ra in ${}_RR$ by Lemma 3.4.

$(b) \Leftrightarrow (c)$ analogous. □

Remark 3.19. It is not difficult to see that Ra has a weak supplement in ${}_RR$ if and only if aR has a weak supplement in R_R for all $a \in R$. The situation for supplements is not that clear. Indeed H.Zöschinger proved that the property: Ra has a supplement in ${}_RR$ implies aR has a supplement in R_R is equivalent with R being an L-ring (see [Z81]).

The following proposition, due to S. Page, relates uniform and hollow dimension.

Proposition 3.20 (Page)**.** *Let ${}_RQ$ be an injective cogenerator in R-Mod, $T = \mathrm{End}(Q)$ and M any R-module. Then $hdim({}_RM) = udim(\mathrm{Hom}(M,Q)_T)$.*

Proof. See [P, Proposition 1]. □

Remark 3.21. More generally this result can be extended to any injective cogenerator ${}_RQ$ in $\sigma[{}_RM]$ - the full subcategory of R-Mod that contains all M-subgenerated left R-modules. Hence for any ${}_RN \in \sigma[M]$ the formula $hdim({}_RN) = udim(\mathrm{Hom}\,(N,Q)_T)$ holds where $T := \mathrm{End}\,(Q)$ (see [Lo] for details).

Using S. Page's result we get another characterization of semilocal rings in terms of hollow and uniform dimension.

Theorem 3.22. *The following statements are equivalent for a ring R.*

(a) *R is semilocal;*

(b) *there exists a generator ${}_RG$ in R-Mod such that $G_{\mathrm{End}\,(G)}$ has finite hollow dimension;*

(b') *for any generator* $_RG$ *in R-Mod,* $G_{\mathrm{End}\,(G)}$ *has finite hollow dimension;*

(c) *there exists an injective cogenerator* $_RQ$ *in R-Mod such that* $Q_{\mathrm{End}\,(Q)}$ *has finite uniform dimension;*

(c') *for any injective cogenerator* $_RQ$ *in R-Mod,* $Q_{\mathrm{End}\,(Q)}$ *has finite uniform dimension.*

In this case $hdim(G_{\mathrm{End}\,(G)}) = length(R/\mathrm{Jac}\,(R)) = udim(Q_{\mathrm{End}\,(Q)})$ *holds.*

Proof. Let $_RG \in R$-Mod and $S = \mathrm{End}\,(G)$. From [W, 18.8] we know that $_RG$ is a generator in R-Mod if and only if G_S is finitely generated, projective in S-Mod and $R \simeq End_S(G_S)$. Hence by Theorem 3.10, we get $hdim(G_S) = hdim(End_S(G_S)) = hdim(R)$. This proves $(a) \Leftrightarrow (b) \Leftrightarrow (b')$.

By Page's Proposition 3.20 we have $hdim(R) = udim(Q_T)$ where $T = \mathrm{End}\,(Q)$ for any injective cogenerator $_RQ$. This proves $(a) \Leftrightarrow (c) \Leftrightarrow (c')$. □

References

[CD] R.Camps and W.Dicks, *On semilocal rings*, Israel J. Math. 81, 203 - 221 (1993)

[GP] P.Grezeszcuk and E.R.Puczyłowski, *On Goldie and dual Goldie dimension*, J. Pure and Applied Algebra 31, 47-54 (1984)

[GS] V.N. Gerasmiov and I.I. Sakhaev, A counterexample to two hypotheses on projective and flat modules, Siberian Math. J. 25, 855 - 859 (1984)

[HaS] A.Hanna and A.Shamsuddin, *Dual Goldie dimension*, Rendiconti dell' Istitutio di Mathematica dell' Universita di Trieste 24, No.1-2, 25-38 (1992)

[HeS] D.Herbera and A.Shamsuddin, *Modules with semi-local endomorphism ring*, Proceedings American Math. Soc. 123, 3593-3600 (1995)

[J] S. Jøndrup, *Projective modules*, Proceedings American Math. Soc. 59, 217 - 221 (1976)

[La] D. Lazard, *Liberte des gros modules projectifs*, J. Algebra 31, 437 - 451 (1974)

[Lo] C.Lomp, *On dual Goldie dimension*, Diplomarbeit, HHU Düsseldorf (1996)

[MS] A. Mohammed and F.L. Sandomierski, *Complements in projective modules* J. Algebra 127, 206 - 217 (1989)

[P] S.Page *Relatively semiperfect rings and corank of modules* Communication Algebra, 21, 975 - 990 (1993)

[Re] E.Reiter, *A dual to the Goldie ascending chain condition on direct sums of submodules*, Bulletin Calcutta Math. Soc. 73, 55-63 (1981)

[Ru] P.Rudlof, *On the structure of couniform and complemented modules*, J. Pure and Applied Algebra 74, 281 - 305 (1991)

[SV] B.Sarath and K.Varadarajan, *Dual Goldie dimension II*, Communication Algebra, 7, 1885 - 1899 (1979)

[S91] I.I. Sakhaev, *On lifting finite generacy of a projective module modulo its radical*, Math. Notes 49, 293 - 301 (1991)

[S93] I.I. Sakhaev, *The finite generation of projective modules*, Algebra(Proc. Int. Conf. Krasnoyarsk 1993), edited by Y.L. Ershov et al., 209-216, de Gruyter, Berlin 1996

[T76] T.Takeuchi, *On cofinite-dimensional modules*, Hokkaido Math. J. 5, 1-43 (1976)

[T94] T.Takeuchi, *Coranks of a quasi-projective module and its endomorphism ring*, Glasgow Math. J. 36, 381 - 383 (1994)

[V] K.Varadarajan, *Dual Goldie dimension*, Communication Algebra, 7, 565-610 (1979)

[W] R.Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading (1991)

[Z78a] H.Zöschinger, *Invarianten wesentlicher Überdeckungen*, Math. Annalen 237, 193 - 202 (1978)

[Z78b] H.Zöschinger, *Über Torsions- und κ-Elemente von $Ext(C, A)$*, J. Algebra 50, 295 - 336 (1978)

[Z81] H.Zöschinger, *Projektive Moduln mit endlich erzeugtem Radikalfaktormodul*, Math. Annalen 255, 199 - 206 (1981)

[Z86] H.Zöschinger, *Minimax-Moduln*, J. Algebra 102, 1-32 (1986)