Achievable Rates of Physical Layer Network Coding Schemes on the Exchange Channel

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***Abstract*— Network coding, where relay nodes combine the information received from multiple links rather than simply replicating and forwarding the received packets, has shown the promise of significantly improving system performance. In very recent works, multiple researchers have presented methods for increasing system throughput by employing network coding inspired methods to mix packets at the physical layer: physical-layer network coding (PNC). A common example used to validate much of this work is that of two sources exchanging information through a single intervening relay - a situation that we denote the "exchange channel". In this paper, achievable rates of various schemes on the exchange channel are considered. Achievable rates for traditional multi-hop routing approaches, network coding approaches, and various PNC approaches are considered. A new method of PNC inspired by Tomlinson-Harashima precoding (THP), where a modulo operation is used to control the power at the relay, is introduced.**

I. Introduction

In the past decade, network coding has been a research topic of significant interest [1]-[3]. Network coding, where information streams are coded within the network rather than only at the network edges, provides the possibility to improve network performance. The extension of network coding to wireless networks has naturally also attracted a lot of research interest [4]-[6]. In fact, since network coding may also be able to simplify the difficult wireless network routing problem, wireless networks are a natural setting. Numerous researchers have studied the performance of network coding in wireless networks to take into account the properties of omnidirectional transmissions, half-duplex communications, etc.

An important property of wireless networks is the broadcast nature of the medium. Whereas most wireless network research has traditionally considered interference due to simultaneous transmissions as having a negative effect, [10] proposes a new strategy that views this broadcast property as a capacity-boosting advantage. Instead of treating the concurrent transmission as interference, this pioneering method exploits the fact that the relay in the exchange channel needs only obtain the exclusive-or ("XOR") of the two sources' data, and uses such an observation to map the sum of the simultaneously received electromagnetic waves at the relay into a decision on the XOR of the two sources' data bits (termed "Physical Layer Network Coding"). However, one major challenge to the method of [10] is that the gains and phases of the signals sent from the two sources must be set precisely in order to have successful reception at the relay, and this phase-synchronization is difficult to achieve in practical realizations. If not synchronized, the PNC suffers up to a 6dB SNR loss when two-dimensional constellations are employed. This limits the application of the PNC described in [10].

In [8], another scheme where a type of network coding is applied at the physical layer is introduced to the research community. Instead of attempting any decoding, the relay in this scheme just amplifies and forwards (AM/FW) the mixed signal to the destinations. At the destination, the desired packets can be extracted since the destination node knows its own contribution to the mixed signal (see also [9]). The main contributions of [8] is to provide a clever method of extracting the desired signal (that from the other source) noncoherently. However, one disadvantage of this scheme versus that of [10] is that the signal transmitted by the relay is a sum over the reals of the two received signals. Thus, from the perspective of a given receiver, a portion of the transmit power budget of the relay is wasted in transmitting already known information.

Inspired by Tomlinson-Harashima precoding (THP), where a modulo operator is put on the transmitter to eliminate the power amplification effect caused by the pre-coder, a THP-based wireless network coding scheme is proposed in this paper. It is similar to the AM/FW scheme except that a modulo operation is applied at the relay and spectral shaping is performed to support such. In particular, instead of forwarding the mixed signal directly, the relay in this THP scheme carries an extra modulo operation on the mixed analog signal before amplifying and transmitting. At a given destination receiver, a modulo operator helps to recover the desired signal. Similar to the Tomlinson-Harashima precoding, this reduces the necessary power consumption at the relay with only a slight impact on performance.

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The purpose of this paper is twofold. First, and foremost, we desire to put the recently introduced PNC schemes into perspective. In particular, much has been made of the ability of such schemes to support transmission across the exchange channel in only two transmissions. However, this can also be performed in other classical ways, such as a standard multiple-access channel (MAC) from the sources to the relays, followed by a broadcast transmission of the XOR of the two sources' information. Second, we introduce the THP-based PNC method described above, and compare it to the previous schemes. It will be observed that it shows a slight gain over the scheme of [8] that can be further enhanced with transmit signal shaping.

The paper is organized as follows. In Section II, the system model and the various transmission schemes are described. In Section III, the achievable rates are derived for each of these schemes. In Section IV, numerical results are shown and discussed, and conclusions are drawn in Section V.

II. System Model and Different Transmission Schemes

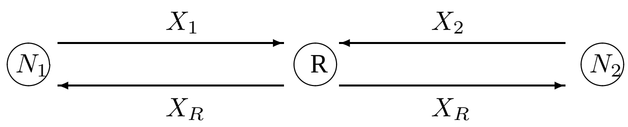

Fig. 1. An Exchange Channel. Sources N_1 and N_2, which are not in the radio range of each other, send information to each other through a common relay R that is in the radio range of both N_1 and N_2.

In this paper, we study a simple model termed the exchange channel. Figure 1 shows a three-node network realization of this channel. We make the assumption that all three nodes are half-duplex; in other words, due to the difficulty of transmit/receive isolation, each of them cannot transmit and receive at the same time [7]. Sources N_1 and N_2 are nodes that exchange information packets X_1 and X_2, but are out of each other's transmission range (which differs from [7]). Node R is the relay node between them and is assumed to be within the transmission ranges of both sources. Thus, relay R helps finish the exchange by transmitting packet X_R. For simplicity, all channels are assumed to be real additive white Gaussian noise (AWGN) channels.

This three-node model plays an important role and is considered a basic unit in the implementation of multi-hop wireless network transmission. In the traditional approach, Source N_1 sends its packet to relay R and R forwards this packet to N_2. Then N_2 sends its packet to relay R and R forwards this packet to N_1. Thus, four time slots are required to exchange two packets. Table I illustrates this scheme. Recently, this straightforward approach has been shown to be inefficient, and several alternate transmission strategies have been introduced. They will be described before the introduction of the THP scheme proposed in this paper.

Node	Stage 1	Stage 2	Stage 3	Stage 4
N_1	Transmit	Idle	Idle	Receive
N_2	Idle	Receive	Transmit	Idle
R	Receive	Transmit	Receive	Transmit

TABLE I
TRADITIONAL SCHEDULING SCHEME.

A. Multiple Access Channel (MAC) Followed by Broadcast Channel (MAC/BC) Scheme

The timing schedule of this scheme is illustrated in Table II. During the first stage, source nodes N_1 and N_2 share the wireless channel and send packets X_1 and X_2 to the relay R simultaneously. Relay R estimates X_1 and X_2 separately. Since the independent information X_1 and X_2 need to be recovered at the relay, the two senders contend not only with the noise at the relay, but with interference from each other as well. Thus, this exchange channel under this scenario can be considered as the traditional multiple-access channel, in which two (or more) senders send information to a common receiver. During stage 2, R sends X_R to both N_1 and N_2, where

$$X_R = XOR(X_1, X_2) \quad (1)$$

Since the relay is sending the common information X_R to each of the destinations, this does not fit the traditional "broadcast channel" of information theory, where independent information is sent to each of two receivers. However, with this caveat, we still will term this the "broadcast channel" phase and the whole scheme as MAC/BC. At node N_1, since it knows the information X_1 that it sent out during the first stage, it can recover X_2 by $XOR(X_1, X_R)$ as follows:

$$X_2 = XOR(X_1, X_R) = XOR(X_1, XOR(X_1, X_2)) \quad (2)$$

The same reception procedure is applied at node N_2. Thus, this scheme requires two stages.

Node	Stage 1	Stage 2
N_1	Transmit	Receive
N_2	Transmit	Receive
R	Receive	Transmit

TABLE II
MAC/BC SCHEME SCHEDULE.

B. Digital Network Coding (DNC) Scheme

Another solution for the exchange channel is the commonly-considered example of digital network coding (DNC) [4], which applies the standard network coding strategy in this three-node wireless system. In this scheme, two sources transmit their packets separately. At the first stage, N_1 sends X_1

to R. At second stage, N_2 sends X_2 to R. After receiving X_1 and X_2, R broadcasts $X_R = XOR(X_1, X_2)$ to N_1 and N_2. Similar to the MAC/BC scheme, when N_1 receives X_R, it extracts X_2 from X_R since it knows X_1. Table III shows the timing of DNC scheme. This scheme requires three stages.

Observe that the MAC/BC scheme also employs a similar form of network coding, but, that the two sources share the channel when they send packets to the relay.

Node	Stage 1	Stage 2	Stage 3
N_1	Transmit	Idle	Receive
N_2	Idle	Transmit	Receive
R	Receive	Receive	Transmit

TABLE III
DNC SCHEME SCHEDULE.

C. Physical Layer Network Coding (PNC) Scheme

In [10], network coding is applied at the physical layer at the relay. During stage 1, sources N_1 and N_2 send out packets to the relay R simultaneously. In contrast to the MAC/BC scheme, the relay doesn't decode X_1 and X_2 separately. If the time and phase of the signals from both nodes are well synchronized at the relay, the relay estimates the XOR of X_1 and X_2 directly for certain modulation schemes. The following example with BPSK modulation helps explain how the modulation/demodulation mapping of PNC works.

For BPSK modulation, the signal send from $S_i,\ i = 1, 2$ is either 1 ($X = 1$) or -1 ($X = 0$), i.e., $S_i \in \{-1, 1\}$. Assuming both phase and amplitude synchronization of the signals from the two senders is obtained at relay, the received signal is: $Y_R \in \{-2, 0, 2\}$, where the noise has been omitted for simplicity. Table IV illustrates how the relay demodulates. With reference to Table IV, the relay R obtains the desired XOR information $X_R = XOR(X_1, X_2)$. In other words, the XOR operation in the straightforward digital network coding scheme can be realized through PNC mapping.

Modulation Mapping at Source Nodes				Demodulation Mapping at Relay Node	
Packet		Signal		Signal	Output
X_1	X_2	S_1	S_2	Y_R	X_R
0	0	-1	-1	-2	0
0	1	-1	1	0	1
1	0	1	-1	0	1
1	1	1	1	2	0

TABLE IV
MODULATION/DEMODULATION MAPPING OF PNC SCHEME.

During the second stage, the XOR packet is sent out to both N_1 and N_2. Finally, the destination nodes extract the desired packet from this XOR associated with the packet it sent out. This scheme needs two stages and is illustrated per Table V.

The synchronization at the relay of signals from the two isolated source nodes N_1 and N_2 is hard or costly. When they are not well synchronized, the performance of such a scheme rapidly deteriorates. For a modulation with a one-dimensional constellation (e.g. BPSK), the performance loss is relatively small. But for a modulation employing a two-dimensional constellations (e.g. QPSK), the performance loss is significant. For example, the SNR loss is up to 6dB when QPSK modulation is employed in an uncoded system.

Node	Stage 1	Stage 2
N_1	Transmit	Receive
N_2	Transmit	Receive
R	Receive	Transmit

TABLE V
PNC SCHEME SCHEDULE.

D. AM/FW Scheme

The fourth scheme considered in this paper is the AM/FW scheme [8]. In this scheme, sources N_1 and N_2 send out packets to the relay R simultaneously during the first stage. Suppose the signals sent from N_1 and N_2 are S_1 and S_2 respectively, then the signal received at R is

$$Y_R = S_1 + S_2 + n_R \tag{3}$$

where n_R is the Gaussian noise added at the relay R. Unlike previous schemes, here the relay does not try to decode, either $\{X_1,\ X_2\}$ or $XOR(X_1, X_2)$. Instead, it forwards the signal it received directly to nodes N_1 and N_2 during stage 2. To satisfy the power constraints, it scales the signal as:

$$S_R = \beta Y_R = \beta(S_1 + S_2 + n_R) \tag{4}$$

where $\beta = \sqrt{P_R/(P_1 + P_2 + \sigma^2_{n_R})}$, P_R, P_1 and P_2 are the power of nodes R, N_1 and N_2 respectively, that are constrained by P. $\sigma^2_{n_R}$ is the variance of the noise. The received signal at N_1 is then $Y_1 = S_R + n_1 = \beta(S_1 + S_2 + n_R) + n_1$, where n_1 is the Gaussian noise with variance $\sigma^2_{n_1}$. Since node N_1 knows the signal it sent out, S_1, it can subtract this part from the received signal, and thus get

$$Z_1 = \beta S_2 + \beta n_R + n_1 \tag{5}$$

It then estimates S_2 from Z_1. Notice here that the received signal-to-noise ratio (SNR) is no longer P_2/σ_{n_1} but $\beta^2 P_2/(\sigma^2_{n_1} + \beta^2 \sigma_{n_R})$. This scheme requires two stages. Table VI illustrates the schedule of the AM/FW scheme.

Node	Stage 1	Stage 2
N_1	Transmit	Receive
N_2	Transmit	Receive
R	Receive	Transmit

TABLE VI
AM/FW SCHEME SCHEDULE.

E. THP Scheme

In this subsection, we introduce the THP scheme. The schedule for this scheme is identical to that of the AM/FW scheme. Sources N_1 and N_2 send out signals S_1 and S_2 to the relay R simultaneously during the first stage. But the signal sent from the relay differs. Notice that in the AM/FW scheme, the amplitude of the transmitted signal must be adjusted by β (4). Per (5), one may see an SNR loss at node N_1 when it decodes X_2. Suppose the variances of the noises n_1 and n_2 are the same, $\sigma_{n_1}^2 = \sigma_{n_R}^2 = \sigma^2$, then the SNR loss is roughly 4.7dB. If this SNR loss coming from the amplitude adjustment could be reduced, performance gains could be obtained.

Inspired by the Tomlinson-Harashima precoding (THP) strategy [11], a novel scheme on the exchange channel is proposed here. Recall in THP that a modulo-t operator Γ_t is put on the transmitter precoding equalizer to reduce the amplitude of the transmitted signal. The same strategy can be applied to the signal sent from the relay. The procedure at the

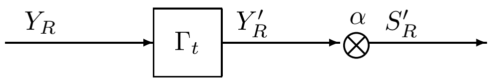

Fig. 2. The processor at the relay. Γ_t is the modulo-t operator.

relay is illustrated in Figure 2. The received signal at the relay is the same as that in the AM/FW scheme, $Y_R = S_1+S_2+n_R$. But before the signal is forwarded, a modulo-t operator is applied to the received signal, i.e.,

$$S'_R = \alpha Y'_R = \alpha \Gamma_t(Y_R) \tag{6}$$

The modulo-t operator Γ_t maps the real number onto $(-t/2, t/2]$, where t is a positive number. The modulo-t operation can be viewed as the signal-dependent addition, i.e., $Y'_R = \Gamma_t(Y_R) = Y_R + l \times t$, where l is the integer for which $Y'_R \in (-t/2, t/2]$. α is the amplitude adjustment factor for which the power of the signal sent from node R, S'_R, satisfies the power constraint. But due to the modulo-t mapping, the power of the signal Y'_R is lower than that of Y_R; thus, α is smaller than the corresponding β in the AM/FW scheme. In particular, as shown in the following Section (III), if S_1 and S_2 are i.i.d. uniformly distributed on $(-t/2, t/2]$, the power of Y'_R is P, and thus α is 1.

During the second stage, signal S'_R is sent to both N_1 and N_2. Suppose that S_1 and S_2 are i.i.d. uniformly distributed on $(-t/2, t/2]$, then the received signal at N_1 is,

$$Y'_1 = S'_R + n_1 = \Gamma_t(S_2 + S_1 + n_R) + n_1 \tag{7}$$

Before estimating X_2, Y'_1 is processed through another modulo-t operation, as shown in Figure 3.

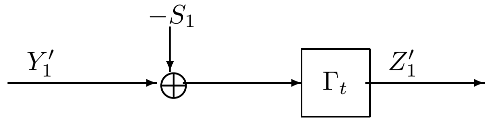

Fig. 3. The processor at the destination N_1. Γ_t is the modulo-t operator.

Then node N_1 estimates S_2 from the resulting Z'_1, where,

$$\begin{aligned} Z'_1 &= \Gamma_t(S'_R - S_1 + n_1) \\ &= \Gamma_t(\Gamma_t(S_2 + S_1 + n_R) - S_1 + n_1) \\ &= \Gamma_t(S_2 + S_1 + n_R + l \times t - S_1 + n_1) \\ &= \Gamma_t(S_2 + n_R + n_1) \\ &\approx S_2 + n_R + n_1 \end{aligned} \tag{8}$$

The approximation holds when the SNR is high.

Per Eq. (8), it can be seen that the SNR loss is roughly 3dB, 1.7dB less than that of the AM/FW scheme. The timing scheme of THP scheme is the same as that of the AM/FW, which demands two stages.

III. ACHIEVABLE RATES ANALYSIS

The exchange channel shown in Figure 1 seems simple, but the total throughput capacity of such a channel is still an open research topic. In this section, the achievable overall symmetric-rate throughput of the schemes described in the previous Section (II) are analyzed. Under a total power constraint, the overall throughput is maximized.

A. MAC/BC Scheme

The channel capacities of both multiple access channel and broadcast channel have been well studied [12]. The capacity of the multiple access channel can be illustrated by the *capacity region*, which is the closure of the set of achievable rate pairs $(R_{N_1\to R}, R_{N_2\to R})$, where $R_{N_i\to R}$ is the rate from N_i to R, $i = 1, 2$. For Additive White Gaussian Noise (AWGN) channel, the pair of rate is limited by ([12]:

$$\begin{aligned} R_{N_i\to R} &\le 1/2\log(1 + P_i/\sigma^2),\ i = 1, 2 \\ R_{N_1\to R} + R_{N_2\to R} &\le 1/2\log(1 + (P_1 + P_2)/\sigma^2) \end{aligned} \tag{9}$$

An example of the capacity region of MAC is shown per Figure 4. For the broadcast channel, the capacity region is given by:

$$\begin{aligned} R_{R\to N_1} &\le 1/2\log_2(1 + \alpha P_R/\sigma^2) \\ R_{R\to N_2} &\le 1/2\log_2(1 + (1-\alpha)P_R/(\alpha P_R + \sigma^2)), \end{aligned} \tag{10}$$

where α may be arbitrarily chosen $(0 \le \alpha \le 1)$. The capacity of broadcast channel is the convex hull of all $(R_{R\to N_1}, R_{R\to N_2})$ satisfying 10.

Using the model shown in Figure 1, the main concern is how to maximize the overall throughput $R_{N_1\to N_2} + R_{N_2\to N_1}$ subject to the power constrain P, where $R_{N_1\to N_2}$ and $R_{N_2\to N_1}$

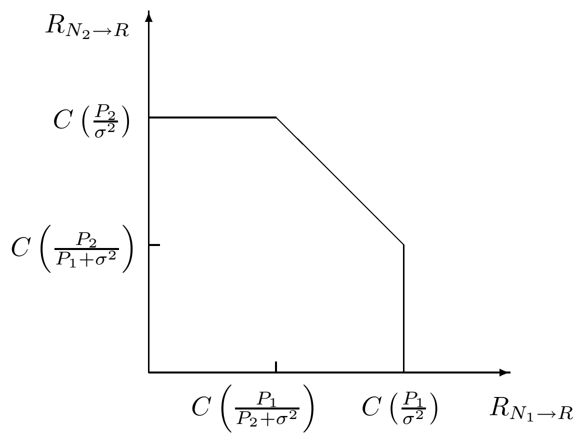

Fig. 4. An example of the capacity region of MAC. $C(x) = \frac{1}{2}\log(1+x)$.

are the rate from N_1 to N_2 and that from N_2 to N_1 respectively. For the multiple access channel (first stage), the achievable rates are limited by the sum rate line (Figure 10). Since $R_{N_1\to R}$ and $R_{N_2\to R}$ are to be set equal,

$$\begin{aligned} R_{N_1\to R} &= R_{N_2\to R} \leq 1/2(1/2\log_2(1+2P_1/\sigma^2)) \\ &= 1/4\log_2(1+2P_1/\sigma^2)\ bits/channeluse \end{aligned} \tag{11}$$

Recalling that this broadcast channel is sending common information (in contrast to the traditional broadcast channel), the achievable rates on the two directions are the same and limited by the capacity:

$$R_{R\to N_1} = R_{R\to N_2} \leq 1/2\log_2(1+P_R/\sigma^2) \tag{12}$$

To maximize total throughput, one can adapt the power and timeshare these two channels. Suppose the fractional time used for the first stage is ρ, and for the second stage $1-\rho$. Thus the information (in bits) sent from N_1 to R (or from N_2 to R) is $\rho/4\log_2(1+2P_1/\sigma^2)$ and the the information (in bits) sent from R to N_1 (or from R to N_2) is $(1-\rho)/2\log(1+P_R/\sigma^2)$. Setting them equal,

$$\rho/4\log(1+2P_1/\sigma^2) = (1-\rho)/2\log(1+P_R/\sigma^2) \tag{13}$$

and, to satisfy the power constraint:

$$2\rho P_1 + (1-\rho)P_R \leq P \tag{14}$$

thus $P_1 = (P-(1-\rho)P_R/(2\rho)$. P_R is a free parameter. ρ is adjustable to achieve the equality in (13) and to achieve the maximum throughput in the MAC/BC Scheme. Numerical results are shown in Section IV.

B. DNC Scheme

In this scheme, the channels in the first and second stages are standard AWGN channels without multi-user interference. Thus, the maximum rates that can be achieved through the first and second stages are:

$$\begin{aligned} R_{N_1\to R} &\leq 1/2\log_2(1+P_1/\sigma^2) \quad bits/\ channel\ use \\ R_{N_2\to R} &\leq 1/2\log_2(1+P_1/\sigma^2) \quad bits/\ channel\ use \end{aligned}$$

and the maximum rates that can be achieved through the third stage are:

$$\begin{aligned} R_{R\to N_1} &= R_{R\to N_2} \\ &\leq 1/2\log_2(1+P_R/\sigma^2)\ bits/channel\ use \end{aligned}$$

Applying the same time sharing strategy employed by the MAC/BC scheme, the maximum overall throughput can be achieved by making

$$\rho 1/2\log_2(1+P_1/\sigma^2) = (1-2\rho)/2\log_2(1+P_R/\sigma^2)$$

subject to the power constraint: $2\rho P_1 + (1-2\rho)P_R \leq P$.

C. PNC Scheme

The analysis of capacity or the achievable rate of such scheme is a challenging task since it depends on the precise modulation method. From the perspective of the source and destination, PNC induces a binary symmetric channel (BSC) between the transmitter and the receiver. The crossover probability of the BSC is obtained for the simple example of BPSK as follows. When node N_1 sends to N_2, an error occurs when either the relay mis-decodes the XOR of the two source bits (X_1 and X_2), or N_2 mis-decodes the XOR from the relay. The probability of such an error (P_e) is easily calculated if the SNR is known in the system, and, given this error probability P_e, the system is equivalent to the BSC shown in Figure 5.

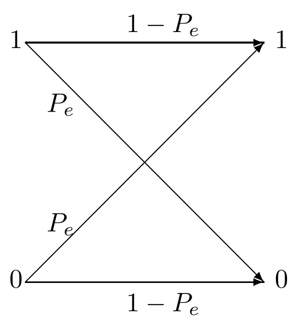

Fig. 5. The BSC model corresponding to the PNC scheme with BPSK modulation.

The maximum rate of such BSC model can be calculated as: $R \leq 1/2(1-H(P_e))$, where P_e is a function of SNR and the factor $1/2$ indicates that it is realized through two transmissions. Finally, this equivalent view of PNC extends to a discrete symmetric channel, corresponding to M-ary Amplitude Shift Keying (M-ary ASK). Each of the different modulations (BPSK, 4-ASK, 8-ASK, etc) provides a plot on the rate versus SNR domain. The envelope of these plots is the achievable rate of the PNC scheme.

D. Amplify and Forward (AM/FW) Scheme

In this scheme, the signal through which N_1 estimates the packet sent from N_2 is shown per Eq. (5). The corresponding received SNR is then,

$$SNR_{AM/FW} = \beta^2 P_1/(\sigma^2 + \beta^2\sigma) \tag{15}$$

Thus the maximum achievable rate for such a scheme is

$$\begin{aligned} R &\leq 1/4\log_2(1 + SNR_{AM/FW}) \\ &= 1/4\log_2(1 + \beta^2 P/(\sigma^2 + \beta^2\sigma^2)) \end{aligned} \tag{16}$$

E. THP Scheme

Per Eq. (8), destination node N_1 estimates X_2 through the signal $Z_1' = \Gamma_t(S_2 + n_R + n_1) \triangleq \Gamma_t(S_2 + n)$, where $n \triangleq n_R + n_1$ is a additive white Gaussian noise with variance $2\sigma^2$. The mutual information for this channel is:

$$\begin{aligned} I(Z_1'; S_2) &= h(Z_1') - h(Z_1'|S_2) \\ &= h(\Gamma_t(S_2 + n)) - h(\Gamma_t(n)) \quad (17) \\ &\leq \log_2(t) - h(\Gamma_t(n)) \quad (18) \end{aligned}$$

where $h(.)$ denotes differential entropy. The upper bound of (18) follows from the fact that a random variable with constrained support t achieves maximum differential entropy when it follows i.i.d. uniform distribution. When S_1 and S_2 follow i.i.d. uniform distribution, it can be shown that Z_1', Z_2' and Y_R' are all uniformly distributed on $(-t/2, t/2]$:

Theorem 3.1: If S_1 and S_2 are i.i.d. uniformly distributed random variables with support $(-t/2, t/2]$ and n is a zero-mean Gaussian random variable with variance τ^2, then 1) $\Gamma_t(S_i + n), i = 1, 2$ is uniformly distributed on $(-t/2, t/2]$, and 2) $\Gamma_t(S_1 + S_2 + n)$ is uniformly distributed on $(-t/2, t/2]$.

Proof: First, consider the first part. Suppose f_{S_1} and f_n are the probability density functions (pdf) of S_1 and n, respectively. then

$$f_{S_i}(s_i) = 1/t, \quad -t/2 \leq s_i < t/2, \ i = 1, 2$$

and 0 otherwise,

$$f_n(n) = \frac{1}{\sqrt{2\pi}\tau}\exp(-\frac{n^2}{2\tau^2}) \tag{19}$$

Then the pdf of $S_2 + n \triangleq S$ is:

$$\begin{aligned} f_S(s) &= f_{S_2}(s) * f_n(n) \\ &= \int_{-\infty}^{\infty} f_{S_2}(u) f_n(s-u) du \\ &= \frac{1}{t}\int_{-t/2}^{t/2} f_n(s-u) du \\ &= \frac{1}{t}[F_\tau(s + t/2) - F_\tau(s - t/2)] \end{aligned} \tag{20}$$

where $F_\tau(s)$ is the cumulative distribution function (CDF) of a zero-mean Gaussian random variable with variance τ^2. Let V denote $\Gamma_t(S)$, then the pdf of V is the folded version of that of the S onto $(-t/2, t/2]$, i.e., for $\forall v \in (-t/2, t/2]$

$$\begin{aligned} f_V(v) &= \sum_{k=-\infty}^{\infty} f_S(v + kt) \\ &= \lim_{k\to\infty} \sum_{k=-K}^{k=K} f_S(v + kt) \\ &= \frac{1}{t}\lim_{k\to\infty}[F_\tau(v - Kt + t/2) - F_\tau(v - Kt - t/2) \\ &\quad + F_\tau(v - Kt + t + t/2) - F_\tau(v - Kt + t - t/2) \\ &\quad + \ldots + F_\tau(v + Kt + t/2) - F_\tau(v + Kt - t/2)] \\ &= \frac{1}{t}\lim_{k\to\infty}[F_\tau(v + Kt + t/2) - F_\tau(v - Kt - t/2)] \\ &= \frac{1}{t}[F_\tau(\infty) - F_\tau(-\infty)] \\ &= \frac{1}{t} \end{aligned} \tag{21}$$

Thus, V follows a uniform distribution with support $(-t/2, t/2]$. This completes the proof of the first part. Next, move to the second part of the theorem. Let W denote $\Gamma_t(S_1 + S_2 + n)$. Notice $\Gamma_t(S_1 + S_2 + n) = \Gamma_t(\Gamma_t(S_1 + S_2) + n)$. $S_1 + S_2$ follows a triangle distribution with support $(-t, t]$, thus $\Gamma_t(S_1 + S_2)$ follows a uniform distribution on the support of $(-t/2, t/2]$. So, W is also uniformly distributed on $(-t/2, t/2]$. This finishes the proof of Theorem 3.1.

Per Theorem 3.1, the bound in (18) can be achieved by choosing S_1 and S_2 to be i.i.d. uniform over the interval $(-t/2, t/2]$. In such a scenario, the transmitted powers of the source nodes are all $t^2/12$. At the relay node, the power of the signal output from the modulo-t operator, Y_R', is also $t^2/12$. This explains the reason that α in Eq. (6) can be set to unity under such a scenario.

IV. Numerical Results

In this section, the achievable rates of the schemes described preciously are presented and analyzed. Figure 6 shows the numerical results achieved through the analysis in Section III. The uppermost plot is the capacity of the standard AWGN channel, $C = 1/4\log_2(1 + SNR)$. An extra $1/2$ indicates the time-sharing for the two sources. This capacity, which corresponds to the case where each of the two messages does not affect the transmission of the other, provides an upper bound on the achievable rate of the three-node exchange channel model shown in Figure 1. Per the plot, the best of the considered schemes is the PNC scheme. The dotted lines are the achievable rates of the discrete symmetric channel corresponding to the PNC schemes with modulations of BPSK, 4-ASK, 8-ASK and 16-ASK, respectively. The envelope of these curves is superior to the rest of schemes. But, unlike the rest of the schemes, the PNC scheme is based on the assumption that the signals from two isolated transmitter are perfectly synchronized (phase, amplitude) at the relay. If this synchronization is absent, a 6 dB SNR loss is anticipated.

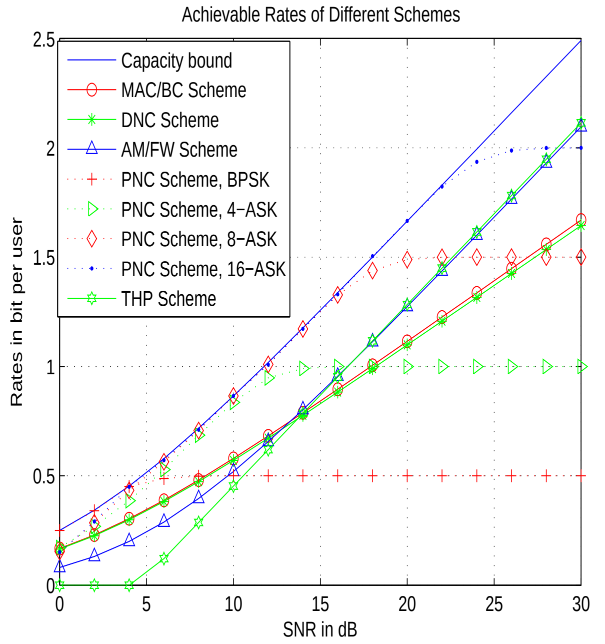

Fig. 6. Numerical results: the achievable rates of different network coding schemes on the exchange channel.

As expected, the MAC/BC scheme and DNC scheme have identical performances. In particular, since the broadcast transmission from the relay in each case is of common information, the limiting link is that from the sources to the relay. The DNC scheme employs a time-division multiple access (TDMA) approach for this portion, which is known to achieve the boundary of the MAC scheme on the equal rate line considered here [12]. Thus, they are identical from the perspective of information theory.

The AM/FW scheme and THP scheme are superior to the other non-PNC schemes - the MAC/BC scheme and DNC scheme. This reinforces the motivation of improving the throughput by applying network coding at the physical layer. In addition, an SNR gain of 1-2dB is anticipated compared to the PNC scheme when synchronization is absent. Finally, the AM/FW scheme and THP schemes are compared. The performance gain is (perhaps unexpectedly) negligible at the medium to high SNRs of interest. The reason for this is as follows. Recall that a anticipated 1.7dB SNR gain over the AM/FW scheme should be obtained by the THP scheme because of the modulo-t operation at the relay, but this is not observed. In this example of the THP scheme, a uniform distribution is applied at the source nodes. Relative to the optimal Gaussian distribution, this induces a 1.5 dB "shaping loss", thus nearly negating the SNR gain provided by the modulo operation.

Studying this more carefully, two issues should be taken into account here:

1) The results for the AM/FW scheme are the *maximum* rates that it will achieve. On the contrary, the THP results are *achievable* rates and certainly not maximum ones. Notice that in (18) the bound follows that a finite-support random variable achieves the maximum differential entropy if it is i.i.d. uniformly distributed given the *support*. Instead the maximization considered here is subject to the the *power* constraint. Thus, the shaping loss can be reduced. An initial foray in this direction is shown below.
2) More important, the results of AM/FW is achieved by choosing a Gaussian codebook. On the other hand, the THP scheme uses a uniform codebook, which is closer to a practical realization. Thus, in a practical implementation, it is anticipated that the THP scheme will outperform the AM/FW scheme; in particular, we expect the anticipated 1.7dB SNR gain to be realized.

As indicated previously, the "shaping loss" negates the SNR gain provided by the modulo operation when the input is chosen from a uniform distribution. This loss is anticipated to be reduced when other distributions are employed. A good starting example is the truncated Gaussian distribution. The pdf of truncated Gaussian distribution is as follows:

$$f_{TG} = \frac{1}{B}\frac{1}{\sqrt{2\pi}\lambda}\exp(-\frac{x^2}{2\lambda^2}) \quad x \in (-b/2, b/2] \tag{22}$$

where

$$B = erf(\frac{b}{2\sqrt{2}\lambda})$$

$$erf(b) = \frac{2}{\sqrt{\pi}}\int_0^b \exp(-x^2)dx$$

and $b = \gamma\lambda$. γ is a free parameter. The power of the truncated Guassian random variable is :

$$\eta^2 = \lambda^2 - \frac{\lambda^2\gamma\exp(-\gamma^2/8)}{\sqrt{2\pi}erf(\gamma/\sqrt{8})} \tag{23}$$

γ plays an important role here. On one hand, it decides the "shape" of the pdf. When γ is large, the pdf of the truncated Gaussian is close to that of a Gaussian random variable; thus a larger value of the first term in (17) is expected. When γ is small, the pdf is close to that of a uniform distribution, for which "shaping loss" is expected. On the other hand, one may check that the larger γ is, the bigger α in (6) is, and thus more power is demanded. Therefore, γ decides the *tradeoff* between the power consumption and the mutual information. To achieve the maximum rates with the truncated Gaussian input, the optimal value of γ, $\hat{\gamma}$, needs to be calculated.

The rates of the THP scheme with a truncated Gaussian input are shown in Figure 7. At low SNRs, the THP scheme is identical to the AM/FW scheme. At medium to high SNRs, the THP scheme outperforms the AM/FW scheme, with an SNR gain of 0.4-0.6dB. The truncated Gaussian distribution sets a lower bound on the achievable rate of the THP scheme. Other distributions are under consideration.

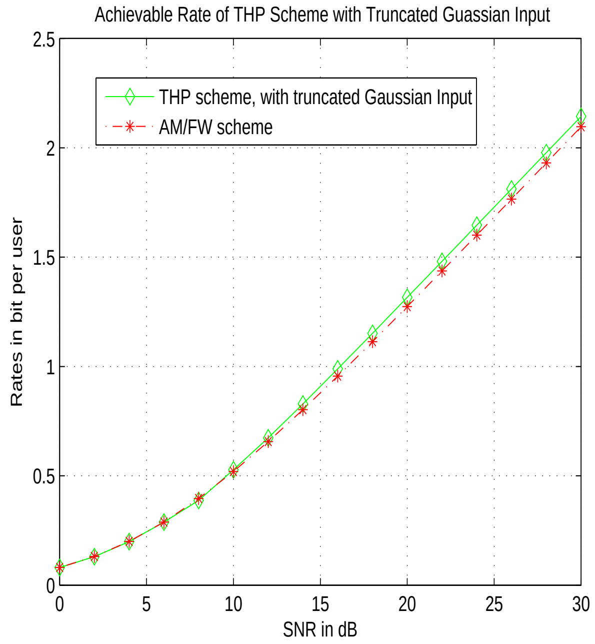

Fig. 7. Numerical results: the achievable rates of the THP scheme with truncated Gaussian input.

V. Conclusions

In this paper, several transmission schemes that have been proposed for the exchange channel have been considered. These include network coding above the physical layer, and also two recently-introduced physical layer network coding (PNC) schemes. In addition, a novel physical layer network coding inspired by Tomlinson-Harashima precoding (THP) is introduced. The achievable throughput rates of each of these schemes is analyzed to provide perspective on this emerging research area. As expected, applying network coding type approaches at the physical layer provide performance improvement. If some implementation issues are ignored, the approach of [10] is clearly favorable to the other considered schemes. Under more realistic assumptions, the THP scheme presented in this paper is slightly preferable to the PNC of [8] and [9] from the theoretical perspective (by 0.4-0.6dB gain). If the shaping loss of the proposed THP scheme is recoverable (a topic currently under consideration), it will demonstrate more significant gains that we also anticipate observing in practical systems.

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