Structural Properties of Complete Problems for Exponential Time[1]

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ABSTRACT The properties and structure of complete sets for exponential-time classes are surveyed. Strong reductions, those implying many-one completeness, are considered as strengthenings of the usual completeness notions. From the results on strong reductions, immunity properties of complete sets are derived. Differences are shown between complete sets arising from the various polynomial-time reductions. These include most of the "weak" reduction between $\leq_m^p$ and $\leq_T^p$. Finally we consider complete sets for some other classes such as r.e. sets along with structural properties of these sets.

1 Introduction

Complete problems play a central and defining role in complexity theory. They are the canonical sets within a complexity class, are almost always the only sets which arise naturally within a class and as such are the most thoroughly studied. One would like to understand the inherent properties which make a set complete and in particular capture the core of what makes them natural and yet hard to compute. As they do occur naturally we would like to understand the difficulties involved in efficiently solving them, or at least solving as many instances as possible of the problem. In this paper we survey the properties of the various types of complete sets which arise in deterministic and nondeterministic exponential time. In this setting natural problems are not as ubiquitous as in smaller classes such as P or NP. However we have the advantage that superpolynomial time affords us, and we are able to perform constructions and analyses which are not possible with smaller time bounds. As such, complete problems in these classes are both interesting in their own right and provide possible insights into the structure of complete sets for other, central complexity classes.

In recent years there has been a great deal of activity and considerable success in studying the structural properties of sets within E, the deterministic exponential time computable problems. E is a simply defined subrecursive class which contains a number of interesting complete problems. It provides a useful platform for studying polynomial-time reductions, polynomial-time completeness notions and the structure of intractable problems. Moreover, there are several advantages to studying problems in this class as opposed to some other more difficult classes. Most importantly these include the fact that E, being deterministic, is closed under complement and as well that there is a universal predicate for polynomial-time functions within E. These facts allow for a good understanding of the structure of polynomial time reductions on problems in E. This has resulted in progress on several areas of study concerning the structure of exponential time sets, particularly complete sets.

Studying nondeterministic time classes is more difficult. In particular, proving almost anything absolute about NP is difficult, at least in part because the two properties of E mentioned above do not seem valid here. In this paper we concentrate on the properties of nondeterministic exponential time, NE. While NE is large enough to allow for the enumeration of all polynomial-time computable functions, it is (probably) not closed under complement. For this reason, NE seems more similar to NP than does any deterministic class, and results concerning NE perhaps give a better indication of structural properties of nondeterministic classes. In studying this class we have to come to grips with some of the problems intrinsic to nondeterminism.

There is no unique notion of a complete set in a complexity class as each different reduction gives rise to a corresponding notion of completeness. For a complexity class $\mathcal{C}$ and a reducibility $\leq$ we say that A is $\leq$-complete for $\mathcal{C}$ if $A \in \mathcal{C}$ and $\forall B \in \mathcal{C}(B \leq A)$. All of the reducibilities considered in this paper are polynomial-time bounded. As is customary, we call A $\mathcal{C}$-complete if A is $\leq_m^p$-complete for $\mathcal{C}$. (Here $\leq_m^p$ denotes polynomial-time many-one reducibility.) We call reductions which imply $\leq_m^p$ stronger reductions, and reduction implied by $\leq_m^p$ weaker reductions. So stronger reduction imply weaker reductions.

We assume familiarity with the standard polynomial-time reductions and the standard polynomial and exponential time (deterministic and nondeterministic) complexity classes. Let E = DTIME $\left(2^{linear}\right)$, NE = NTIME $\left(2^{linear}\right)$, EXP= DTIME $\left(2^{poly}\right)$, and NEXP= NTIME $\left(2^{poly}\right)$. All of the problems considered here will be thought of as coded by finite strings over a fixed finite alphabet Σ. We make use of a polynomial time computable and invertible pairing function (. , .).

In Section 2 we introduce the basic complete sets for E, NE, EXP and NEXP . We examine efficient reductions to complete sets and show how to strengthen these reductions in certain respects. One approach to the question of what makes a complete problem hard is to ask about the frequency of the hard instances. Conversely, one can ask what large subsets of these sets can be found which are easier than the complete set to compute. These question have consequence for approximations to hard problems, as we would like to know if we can solve these problems efficiently on large sets of instances. These considerations lead to the concept of immunity.

Definition 1.1 Let $\mathcal{C}$ be a collection of sets. A set A is $\mathcal{C}$-immune if A is infinite and does not have an infinite subset in $\mathcal{C}$.

We would like to know if intractable sets, particularly many-one complete sets, can be P-immune. This would tell us that not only is membership in the sets themselves hard to decide, but they have no infinite tractable subsets. All natural complete sets can easily be seen to have many easily computable subsets and so are not P-immune. What is more difficult is determining if this property is a consequence of completeness.

It has long been known that every set polynomial-time many-one-complete for E is not P-immune (L. Berman [Be76]). Recently, Nicholas Tran [Tran] has proved this same fact for NE-complete problems (and their complements). These non-P-immunity results are witnessed by sparse sets, and it is not known if there are dense P subsets of these complete sets for E and NE. We do know that NE-complete sets contain infinite, dense subsets in the classes E and UP. Results such as these are discussed in Section 3 of this paper.

In Section 4 a comparison between complete sets for the most common polynomial-time reductions is presented. With a few unexpected exceptions we show that these completeness concepts differ on both E and NE. More precisely we differentiate between completeness with respect to many-one reductions, bounded and unbounded truth-table reductions, and Turing reductions. The exceptions are that many-one and 1-truth-table completeness turn out to coincide for the classes considered.

In Section 5 we consider a few other properties of complete sets, particularly those which have consequences for the weaker truth-table and Turing complete problems. We also expand our discussion to include complexity classes larger than exponential time, particularly the class of r.e. sets. We conclude with a list of open problems.

Two earlier surveys on parts of this subject can be found in the proceedings of past Structure in Complexity Theory conferences. The starting point for this article is the paper I wrote for Structures '90 [H90]. A more recent survey with a somewhat different focus was written by Burhman and Torenvliet [BT94] and can be found in Structures '94.

All of these exponential time classes have easy-to-define canonical complete problems. In fact, they have 1-1, length increasing, invertible and paddable complete sets.

For E the canonical complete set is defined by:
$\{(e, x, t)|$ the e^{th} deterministic exponential time Turing machine accepts input x in $\leq t$ steps$\}$. (Here t is a binary integer, and we are making use of a canonical and efficient enumeration of the exponential time sets.) A similar set, based on an enumeration of NE machines, is 1-1, length increasing, invertible complete for NE.

Most of the results in this paper are stated for the "linear exponential" classes E and NE. However, they all hold for the larger classes EXP and NEXP (as well as for any reasonable, larger deterministic and nondeterministic time classes). That EXP and NEXP have these same properties follows from straightforward padding arguments. One of the more fundamental padding arguments is the proof that any E-complete (NE-complete) set is also EXP-complete (NEXP-complete). As we later refer to padding arguments, we present a brief proof of this fact here.

Theorem 1.1 *Any E-complete set is also EXP-complete.*

Proof. Let C be E-complete and let A be a set in EXP, say $A \in \mathrm{DTIME}\left(2^{n^k}\right)$. Define $B = \{(x, O^{|x|^k}) \mid x \in A\}$. So $A \leq_m^p B$ via the functions $f(x) = (x, O^{|x|^k})$. It is straightforward to check that $B \in E$ and so $B \leq_m^p C$ implying $A \leq_m^p C$ and proving C is EXP-complete. □

For the definitions and basic facts concerning the most common and important polynomial-time bounded reductions we refer the reader to Ladner, Lynch and Selman [LLS75] . A more recent exposition of efficient reductions can be found in Buhrman [Bu93]. As usual we let $\leq_T^p$ denote polynomial Turing reductions. Being less widely used, the definitions for the truth-table reductions are included here.

Definition 1.2 A *reduction* is an oracle Turing machine computation. A *truth-table* (or *tt* or *non-adaptive*) reduction is an oracle computation where all oracle queries are made before any of the answers are read by the Turing machine computation. That is, the queries to the oracle depend only on the input and the Turing machine program and not on the answers to previous oracle queries. A reduction is $k - tt$ of the number of oracle queries, on any input, is bounded by k. A reduction is *b-tt* if it is *k-tt* for some integer $k > 0$.

We can now define several polynomial time truth-table reductions.

Definition 1.3 A set A_1 is polynomial time truth-table reducible to A_2 ($A_1 \leq_{tt}^p A_2$) iff there exists a polynomial-time-bounded *tt*-reduction M^{A_2}, with queries to the oracle set A_2, such that $x \in A_1 \iff M^{A_2}(x)$ accepts. Polynomial-time bounded k-truth-table reducibility ($\leq_{k-tt}^p$) and bounded-truth-table reducibility ($\leq_{btt}^p$) are defined similarly using polynomial time *k-tt* and *btt* reductions.

2 Strong Reductions to Complete Sets

Many properties of complete problems follow from strengthening the reductions to the problem. In particular, the (many-one) complete sets for E and NE are actually complete for somewhat stronger reductions. We use these facts to prove several of the immunity properties which appear in the next section. Furthermore the existence of stong reductions to complete sets is of independent interest. They provide evidence for the validity of the isomorphism conjecture (see Berman and Hartmanis [BH77]) which in the exponential time setting says that all E-complete sets are polynomial-time isomorphic. If we could strengthen the reductions to complete sets sufficiently then the polynomial-time isomorphism of these sets would follow.

We present the proofs of two such strengthened reductions in this section. They are used in the next section to derive some interesting immunity properties of complete sets. Every $\leq_m^p$-complete set for E is complete with respect to 1-1 length-increasing polynomial-time reductions. Every $\leq_m^p$-complete set for NE is complete with respect to 1-1 exponentially-honest polynomial-time reductions. The first theorem, concerning E-complete sets, is due to Len Berman and can be found in his thesis [Ber77]. The proof given here is from Ganesan and Homer [GH89].

Theorem 2.1 *[Ber77] All $\leq_m^p$-complete sets for E are 1-1 length-increasing equivalent.*

Proof.

Let A be any arbitrary m-complete set in E and let K be any complete set in E. It is enough to show that $K \leq^p_{1-li} A$.

Let $f_1, f_2, \ldots$ be an enumeration of all polynomial time computable machines, such that $f_i(x)$ can be computed from i and x in time $2^{\mathcal{O}(|i|+|x|)}$. (See [GH89] for a proof that such enumerations exist.) We construct a set M in such that the reduction, say, f_j from M to A is 1-1-li on $\{j\} \times N$. In addition the set M is constructed so that the function, $g(x) = (j, x)$ will be a reduction from K to M. The required 1-1-li reduction from K to A, then would be $f(x) = f_j(g(x))$.

The following program describes the set M.

1. input (i, x)
2. if $|f_i(i, x)| \leq |(i, x)|$
3. then accept (i, x) iff $f_i(i, x) \notin A$.
4. else if $\exists\, y < x$ such that $f_i(i, x) = f_i(i, y)$
5. then accept (i, x) iff $y \notin K$.
6. else accept (i, x) iff $x \in K$.

Lemma 2.1 *M is in E.*

Proof. Let us compute the time required for M on input (i, x). Note that computing $f_i(i, x)$ takes time $2^{\mathcal{O}(|i|+|x|)}$. Since, there are only $2^{\mathcal{O}(|(i,x)|)}$ strings of the form (i, y) less than (i, x) all of them can be computed in time $2^{\mathcal{O}(|(i,x)|)}$. Hence the condition on line 5 of the algorithm can be performed in time $2^{\mathcal{O}(|(i,x)|)}$. There are only three cases where the decision to accept (i, x) is made.

Case 1 $|f_i(i, x)| \leq |(i, x)|$. In this case we accept (i, x) iff $f_i(i, x)$ is not in A. This can obviously be done in time $2^{\mathcal{O}(|(i,x)|)}$.

Case 2 Condition on line 4 holds. Since $|y| < |(i, x)|$ membership of y in K can be decided in time $2^{\mathcal{O}(|(i,x)|)}$.

Case 3 M accepts (i, x) iff $x \in K$. This can be done directly in time $2^{\mathcal{O}(|(i,x)|)}$.

It is clear from the above discussion that M is in E.

Lemma 2.2 *If f_j is a reduction from M to A, then f_j is 1-1-li on $\{j\} \times N$. Moreover, $g(x) = (j, x)$ is a reduction from K to M.*

Proof. If f_j is not length increasing, it is not a reduction from M to A because of line 3 of the construction. So, f_j has to be length increasing. Suppose, f_j is not 1-1. Let x_2 be the least element such that for some $x_1 < x_2$, $f_j(j, x_2) = f_j(j, x_1)$. By definition of M, $(j, x_1) \in M \Leftrightarrow x_1 \in K$ and $(j, x_2) \in M \Leftrightarrow x_1 \notin K$. So, f_j can not be a reduction from M to A, contradiction. Hence f_j is 1-1-li on $\{j\} \times N$. Note that $(j, x) \in M$ iff $x \in K$ from the way M is defined. The elements of the form (j, x) will always fall in Case 3 of the algorithm. Hence, $g(x) = (j, x)$ is a reduction from K to M.

Lemma 2.3 $K \leq^p_{1-li} A$.

Proof. Define $f(x) = f_j(g(x))$. Clearly g is 1-1-li. Since f_j is 1-1-li on the range of g, f is 1-1-li. f is also computable in polynomial time. It is easy to check that f is a reduction from K to A.

For the next corollary we need the following worst-case definition of a one-way function. It is quite different than the probabilistic notion of one-way function which is found in modern cryptography.

Definition 2.1 A function f is *polynomially honest* if there is a polynomial p such that $\forall x(|x| \leq p(|f(x)|))$.
A *one-way function* is a one-to-one, polynomially honest polynomial time computable function whose inverse in not polynomial time computable.

Corollary 2.1 *If there are no one-way functions then all E-complete sets are P-isomorphic.*

Proof. Let C be E-complete and K the canonical E-complete set. By the above theorem, $K \leq^p_{1-1,l.i.} C$. Furthermore the reduction is P-time invertible by the assumption of the nonexistence of one-way functions. Similarly, the canonical properties of K yield the existence of a reduction from C to K which is 1-1, length-increasing and P-invertible. Now the main theorem of Berman and Hartmanis [BH77] tells us that two sets which are interreducible by 1-1, length increasing and P-invertible reductions are P-isomorphic.

We now turn to the nondeterministic complexity class NE. We would like to prove the same results for for NE as we have for E. But the nondeterminism presents added difficulty which we have not been able to completely overcome. We are able to prove the 1-1 completeness of NE-complete sets, but not the length increasing completeness. These results hold for other larger nondeterministic classes as well. First a definition.

Definition 2.2 A function f is *exponentially honest* if $\forall x(2^{|f(x)|} \geq |x|)$.

Theorem 2.2 ([GH88]) *All $\leq^p_m$-complete sets for NE are 1-1 exponentially honest equivalent.*

Proof. Let A be any arbitrary $\leq^p_m$-complete set in NE and let K be any 1-1, length-increasing complete set for NE. The canonical NE-complete set has this property. It is sufficient to show that K is 1-1 exponentially honest reducible to A.

Let $f_1, f_2, \ldots$ be an enumeration of all polynomial time computable functions such that $f_i(x)$ can be computed from i and x in time $2^{O((|i|+\log(|x|))^2)}$. (See [GH88] for a proof of the existence of such an enumeration.) Let M be the set of pairs accepted by the following algorithm.

1. input (i, x)
2. if $2^{|f_i(i,x)|} < |(i, x)|$
3. then accept (i, x) iff $f_i(i, x) \notin A$.
4. if $\exists y < x$ such that $f_i(i, y) = f_i(i, x)$
5. then if $2^{|y|} \leq |x|$ then accept (i, x) iff $y \notin K$,
6. else reject (i, x).
7. else accept (i, x) if either (1) or (2) holds.
8. (1) $\exists\, y > x$ such that $|y| \leq 2^{|x|}$ and $f_i(i, x) = f_i(i, y)$.
9. (2) $x \in K$.

Proposition 2.1 *M is in NEXP.*

Proof. Let c and d be constants such that the running of time of K is bounded by $2^{c\cdot|x|}$ and the running time of A is bounded by $2^{d\cdot|x|}$. Let us compute the time required for M on input (i, x). Note that computing $f_i(i, x)$ takes time at most $2^{O((|i|+\log(|x|))^2)}$. If $2^{|f_i(i,x)|} < |(i, x)|$ the membership of $f_i(i, x)$ in A can be decided deterministically in time $2^{O(|i|+|x|)}$. Since, there are only $2^{O(|(i,x)|)}$ strings of the form (i, y) less than (i, x), the condition on line 4 of the algorithm can be computed in time $2^{O(|i|+|x|)}$. If $2^{|y|} \leq |x|$, we can decide if $y \in K$ deterministically in time $2^{2^{c|y|}} < 2^{|x|^c}$. Thus steps 5 and 6 can be done in time $2^{O(|x|^c)}$. In step 8, we can guess a string y which is of length at most $2^{|x|}$ and compute $f_i(i, y)$ in time $2^{O((|i|+|x|)^2)}$. Thus the set M is in NEXP . □

Any m-complete set for NE is also m-complete for NEXP . So there is a reduction from M to A.

Proposition 2.2 *Let f_j be any reduction from M to A. Then $f(x) = (j, x)$ is a 1-1 exponentially honest reduction from K to A.*

Proof. It is clear from the definition that the reduction has to be exponentially honest, otherwise f_j can not be a reduction due to the diagonalization in steps 2 and 3. Assume f is not 1-1. Let x_2 be the least elements such that for some $x_1 < x_2$, $f_j(j, x_1) = f_j(j, x_2)$. There are two cases.
Case 1. $2^{|x_1|} \leq |x_2|$. In this case, $(j, x_1) \in M$ iff $x_1 \in K$. $(j, x_2) \in M$ iff $x_1 \notin K$. Hence f_j cannot be a reduction. Contradiction.
Case 2. $2^{|x_1|} > |x_2|$. In this case, $(j, x_1) \in M$ and $(j, x_2) \notin M$. So again f_j cannot be a reduction. Contradiction.

Hence, f is 1-1. Having proved f is 1-1, it is easy to verify that f is a reduction from K to A. □

A stubborn open question remains here. Namely, can we improve Theorem 2.2 to get the 1-1 reductions to be length increasing, as we can for E-complete sets ? This technical question turns out to be both difficult and most challenging.

3 Immunity for Complete Problems

In this section the question of whether complete problems always have easy-to-compute subsets is investigated. Furthermore, we explore the density of such easy subsets. Motivating these questions is the idea that as E- and NE-complete problems are intrinsically difficult, it is desirable to find large subsets of these problems which are more tractable. We require that our subsets be infinite, and if possible dense. The subsets should certainly be easier to compute than the original complete problem. Ideally they would be polynomially time computable but, as we shall see, this goal is not always attained.

The first strong results in this direction were given by L. Berman [Be76, Be77]. He proved that every E-complete set has an infinite polynomial time subset. The key element of this proof is Berman's theorem of the last section that E-complete sets are actually one-one *length increasing* complete. Since the complement of an E-complete set is also E-complete, the complement cannot be P-immune either. Hence the situation here is as good as could be hoped.

Let $A^{\leq n}$ denote the set of strings in A of length not greater than n. A is (polynomially) sparse if there is a polynomial p such that for all n, $||A^{\leq n}|| \leq p(n)$. A is *dense* if it is not sparse.

Theorem 3.1 *No E-complete set or complement of an E-complete set is P-immune.*

Proof. Let A be E-complete. So A is 1-1 length increasing complete. Let $S = \{0^n : n > 0\}$. Then there is a total, 1-1, length increasing, and polynomial time computable function f such that $S \leq_m^p A$ via f. It is easy to see that $f(S)$ is an infinite polynomial-time subset of A. □

The same proof works for the complement of A by considering a reduction of $\Sigma^* - \{0\}^*$ to A.

It is an interesting open problem whether E-complete sets have dense P-subsets. This question was considered in Homer and Wang [HW] where several more small results are proved.

The theory for NE is more difficult and not as well-understood. Again the first results are Berman's [Be76]. In his 1976 FOCS paper he proved,

Theorem 3.2 ([Be76]) *Every NE-complete set has an infinite E subset.*

The E subset constructed by Theorem 3.2 is sparse. Using the 1-1 completeness of NE-complete sets this result can be improved in several ways. First, we can insist that the infinite E subsets be dense and second, that it be in UP as well. The use of Theorem 2.2 in the next proof is not necessary, but does simplify the proof. In particular, the methods developed in the last part of Berman's thesis [Be77] are sufficient for this theorem, although the results stated there are weaker.

Theorem 3.3 *Let C be an NE-complete set. Then there is a dense set $B \subseteq C$ with $B \in E \cap UP$.*

Proof. By Theorem 2.2 C is 1-1, exponentially honest complete. So there is a 1-1 function f which reduces Σ^* to C.
Define $B = \{f(x) \mid 2|f(x)| \geq |x|\}$
(1) $B \in UP$
$z \in B \iff \exists x(z = f(x) \wedge 2|f(x)| \geq |x|)$ (*)
Hence B is in NP . Since f is 1-1 the x is unique, and so B is in UP.
(2) $B \in E$
This too follows from (*) above. Given z, a search for the corresponding x can be done by considering the exponentially many $f(x)$'s with $|x| \leq 2|z|$ and testing for equality with z.
(3) B is dense
Assume f is computable in DTIME(n^k). Since f is 1-1, $f(\Sigma^n)$ yields 2^n elements in C of length $\leq n^k$. Of these 2^n elements, at most $2^{(n/2)+1}$ of them have length $\leq n/2$. Thus more than 2^{n-1} of the elements of $f(\Sigma^n)$ are in B. Hence $||B \cap \Sigma^{\leq n^k}|| \geq 2^{n-1}$ and so B is not sparse. □

In the above proof we could have just as easily used a 1-1 reduction from $\emptyset$ to C. Doing this enables us to obtain the same result for the complement of any NE-complete set. The argument given here yields a small result for NP . Namely, any 1-1 complete for NP set has an infinite dense UP subset.

The question of the non-P-immunity of NE-complete sets seems more difficult. Here our knowledge of the structure of nondeterministic complete sets seems inadequate to obtain an answer. The method which works successfully for E does not quite apply. As the following simple theorem shows, what is missing is the sufficient honesty of reductions to NE-complete sets.

Proposition 3.1 *Any NE-complete set which is complete with respect to polynomial-honest functions has an infinite P-subset.*

Proof. Let C be NE-complete and let f be a polynomial honest reduction from Σ^* to C with $|x| \leq p(|f(x)|)$ for some polynomial p and all x. Define $B = \{f(0^n)|n \in \mathcal{N}\}$. Then $B \subseteq C$ and $B \in P$ since to check if $z \in B$ it is sufficient to check if $f(0^n) = z$ for $n \leq p(|z|)$. □

Note that the P-subset B in the above proof is sparse. The hypothesis of Proposition 3.1 is stronger than needed. It is sufficient that the reduction be polynomial-honest on some infinite P-printable set. For some further structural conditions concerning the immunity of NE-complete sets see Wang [Wan90]. Some related results can be found in Hartmanis, Li and Yesha [HLY86] and in Buhrman [Bu93] where properties of tt-complete sets for NEXP are considered.

Quite recently, Nicholas Tran [Tran] has shown that every NE-complete set and its complement does contain an infinite P-subset. Tran's proof contains a clever use of the hierarchy theorem for nondeterministic time. It makes use of the general method of the proof of Theorem 2.1 in the last section. We still do not know if the of Proposition 3.1 is sufficient to obtain this result. That is, we do not know if every NE-complete set is complete with respect to polynomially honest functions.

Theorem 3.4 *No NE-complete set or complement of an NE-complete set is P-immune.*

Proof. Let C be an NE-complete set, $\langle f_i \rangle$ be an efficient enumeration of the polynomial time computable functions, and let T be a tally set which is in NTIME(2^{n^2}) - NTIME(2^n). Such a T

exists by the standard nondeterministic time hierarchy theorem. Define
$A = \{0^{(i,x)} | \ |f_i(0^{(i,x)})| > (i,x)/i^2$ or $0^x \in T\}$.
$A \in$ NTIME(2^{n^2}) and hence, since NE-complete sets are also NEXP-complete, is $\leq_m^p$ reducible to C via some polynomial time function f_j.

Now there are infinitely many strings $0^{(j,x)}$ such that $|f_j(0^{(j,x)})| > (j,x)/j^2$. Otherwise, for large enough x, we would have $0^{(j,x)} \in A$ iff $0^x \in T$. In this case, for large enough x, $0^x \in T$ iff $f_j(0^{(j,x)}) \in C$, putting T in NTIME(2^n) and contradicting its defining property.

Now define $B = \{y \mid y = f_j(0^{(j,x)})$ and for some $j, (j,x) < j^2|y|\}$. Clearly $B \in P$ and by the above claim B is infinite. Finally,
$y \in B \rightarrow$ for some (j,x), $(f_j(0^{(j,x)}) = y$ and $|0^{(j,x)}| < j^2|y|) \rightarrow 0^{(j,x)} \in A \rightarrow y \in C$.
And so B is the desired subset of C. A similar argument shows the existence of an infinite P-subset of the complement of C. □

Note that as every NE-complete set is also NEXP-complete, this same fact holds for NEXP-complete sets.

Few absolute results are known concerning immunity for NP sets. The above proof for NE does work in a more restricted sense, namely for sets which are polynomially-honest complete for NP. The proof of Proposition 3.1 can be used to show that every such NP-complete set has an infinite P-subset. Relative to an oracle it is known that one may have every NP-complete set having an infinite polynomial subset (Homer and Maass [HM83]). However no oracle is known relative to which there is an NP-complete set which is P-immune.

4 Differences Between Complete Sets

Corresponding to the various efficient reductions are different complete sets. In this section we take the same approach as in Section 2 and try to relate exponential time complete problems with respect to the various reducibilities. In particular, we examine how to differentiate between complete problems for the weaker reduction, those between $\leq_m^p$ and $\leq_T^p$, on exponential time classes. The main focus is on NE, as a representative nondeterministic time complexity class, although our results apply to NEXP and to any larger nondeterministic time classes. As a guide we begin with the results for E and other deterministic classes. While the results for E and NE here coincide, the proof methods do not. In part the proofs for E depend upon closure under complement while the NE results must avoid this dependence.

The systematic study of the different polynomial time reductions was begun by Ladner, Lynch and Selman in [LLS75]. In that paper they defined all of the various truth-table reductions and showed that these reductions were all distinct, and distinct from $\leq_m^p$ and $\leq_T^p$ on sets in E. They mentioned but did not approach the question of whether the complete sets with respect to these reductions were different. This question was taken up by Watanabe [Wat87b] who showed that most of these reductions yield different E-complete sets. Among the results proved there are,

Theorem 4.1 ([Wat87b]) *(i). There is a set which is $\leq_{2-tt}^p$-complete for E but not $\leq_m^p$-complete for E.*
(ii). For any integer $k \geq 1$, there exists a set which is $\leq_{k+1-tt}^p$ for E but not $\leq_{k-tt}^p$-complete for E.
(iii). There is a set which is $\leq_T^p$-complete for E but not $\leq_{tt}^p$-complete for E.

The proof of Theorem 4.1 is omitted here. A proof can be given by directly using the ideas of the proof of Theorem 4.4 below. Theorem 4.1, together with other results in [Wat87b], answer most of the questions concerning the strength of complete sets with respect to the different reductions.

One which is not answered is the comparison between many-one and 1-truth-table complete sets. Somewhat surprisingly, these two types of complete sets coincide. The proof first appears in [HKR-90].

Theorem 4.2 ([HKR90]) *Any set which is $\leq^p_{1-tt}$-complete for E is also $\leq^p_m$-complete for E.*

Proof.
Let L be 1-tt complete for EXP , let E be m-complete for EXP , and let $(t_i)_{i\in\Sigma^*}$ be an efficient enumeration of the $1-tt$ reductions. We show that E is m-reducible to L by constructing an intermediate set which does the untwisting of the $1-tt$ reductions for us.

Define A by

$$A((i,x)) \quad = \quad \begin{cases} E(x), & \text{if (i) } t_i((i,x)) = (y_x \in L)\text{; and} \\ \overline{E}(x), & \text{if (ii) } t_i((i,x)) = (y_x \notin L). \end{cases} \tag{4.1}$$

This A is easily seen to be computable in exponential-time. By the 1-tt completeness of L, there is a 1-tt reduction f of A to L. Let j be a t-index for f, i.e., $f = t_j$.

If (i) holds for (j,x), then $x \in E \iff (j,x) \in A$ and $(j,x) \in A \iff y_x \in L$; hence, $x \in E \iff y_x \in L$. If (ii) holds for (j,x), then $x \in E \iff (j,x) \notin A$ and $(j,x) \in A \iff y_x \notin L$; hence, $x \in E \iff y_x \in L$.

Therefore, $x \mapsto y_x$ is an m-reduction for E to L as required. □

Combining Theorem 4.2 with Berman's [Ber77] theorem that the m-complete languages for EXP are 1-li complete yields:

Corollary 4.1 *The 1-tt complete for EXP languages are 1-li complete.*

The proof above depends on the fact the E is closed under complement, a fact not known to be true for NE. Nonetheless, a slightly more complicated proof yields the same result for the nondeterministic class. The result is due to Buhrman, Spaan and Torenvliet [BST]. Their proof is sketched here, noting the differences with the proof of Theorem 4.2.

Theorem 4.3 ([BST]) *Any set which is $\leq^p_{1-tt}$-complete for NE is also $\leq^p_m$-complete for NE.*

Proof. First, the following weaker claim is made.
Claim: If L is $\leq^p_{1-tt}$-complete for NE and $B \in NE \cap co-NE$, then $B \leq^p_m L$.

This claim follows from the proof of Theorem 4.2 as an intermediate set can be constructed between B and L as above. We use the fact that $B \in NE \cap co-NE$ to show that the intermediate set is in NE.

Now let D be a set in NE, L a 1-truth-table complete set in NE and let M_j witness the $1-tt$ reduction from D to L. On any input x, $M_j(x)$ can end up in one of the following four situations:

1. $M_j(x)$ queries a string z and accepts iff $z \in L$
2. $M_j(x)$ queries a string z and accepts iff $z \notin L$
3. $M_j(x)$ accepts
4. $M_j(x)$ rejects

We now split set D in two subsets E and F.

$$E = \{x \mid x \in D \text{ and machine } M_j(x) \text{ is not in case } 2\}$$
$$F = \{x \mid x \in D \text{ and machine } M_j(x) \text{ is in case } 2\}$$

Claim: F is in $NE \cap co - NE$.
Proof. We need to show that there is a NE predicate for F and for the complement of F.

$$x \in F \text{ iff the computation } M_j(x) \text{ is in case 2 and } x \in D$$
$$x \notin F \text{ iff the computation } M_j(x) \text{ is not in case 2 or } z \in L$$

It is clear that both predicates are NE.

Now we can construct the many-one reduction from D to L: On input x simulate machine M_j on input x. If M_j is in case 1, then output z. If M_j in case 2, then x is in D iff x is in F. Since F is in NE $\cap$ co-NE there is, by the Claim, a many-one reduction from F to L, say g. Now output $g(x)$. If M_j is in case 3, output a fixed element $t_0 \in L$, and if M_j is in case 4, output a fixed element $t_1 \notin L$. The entire construction can be carried out in polynomial time.

For other weak reductions, truth-table and Turing, the NE-complete sets can be shown to differ. These next results are presented in detail in Buhrman, Homer and Torenvliet [BHT89]. Some related results can be found in Watanabe [Wat87a].

Theorem 4.4 ([BHT89]) *(i). There is a set which is $\leq^p_{2-tt}$-complete for NE but not $\leq^p_m$-complete for NE.*
(ii). For any integer $k \geq 1$, there exists a set which is $\leq^p_{k+1-tt}$ for NE but not $\leq^p_{k-tt}$-complete for NE.
(iii). There is a set which is $\leq^p_T$-complete for NE but not $\leq^p_{tt}$-complete for NE.

We give the proof of part (i). Part (ii) is a generalization of the ideas there. The proof of part (iii) is somewhat more complicated and can be found in [BHT89].
Proof. We first show this result for the class NEXP and then use a simple padding argument to obtain it for NE.

Let K be a polynomial-time paddable $\leq^p_m$-complete set for NE. It is easy to see (for example see Balcázar, Díaz, and Gabarró [BDG88]) that K is $\leq^p_m$-complete for NEXP as well. The set B will be constructed so that its only elements are of the form (e, x, l, i), $i = 0$ or $i = 1$.

B will be complete via the $\leq^p_{2-tt}$ reduction:

$$(e, x, l) \in K \leftrightarrow [(e, x, l, 0) \in B \vee (e, x, l, 1) \in B]$$

To ensure that B is not $\leq^p_m$-complete we diagonalize against all possible $\leq^p_m$ reductions from Σ^* to B. Let f_i be the i^{th} polynomial-time computable function in some fixed enumeration of all such functions. We may assume that f_i runs in DTIME (n^i). We need a set of elements on which to diagonalize. To this end we define a sequence of integers $\{u_n\}_n$ by $u_0 = u_1 = 1$, $u_m = 2^{(u_{m-1})^{m-1}} + 1$, for $m > 1$.

Let $H = \{0^{u_k}\}_{k \in I\!N}$. It is easy to verify that $H \in$ P. We use the sequence H to diagonalize against $\leq^p_m$ reductions

We can now describe the construction of B. The set B is constructed in stages. At stage $k = 1, 2, ...$ we determine all elements in B of length $\leq (u_k)^k$. At stage 1 we put all strings s, $|s| \leq 1$ into $\overline{B}$. Now assume we have constructed B through stage $n-1$ and describe stage $n > 1$.

stage n:

Compute $f_n(0^{u_n})$. Let s be any string of the form $(e, x, l, i), (i \in \{0, 1\})$ with $(u_{n-1})^{n-1} < |s| \leq (u_n)^n$. Then we put $s \in B$ iff $s \neq f_n(0^{u_n})$ and $(e, x, l) \in K$.

end of stage n

First note that $K \le_{2-tt}^{p} B$ via the reduction defined above. Since for any (e,x,l) if $(e,x,l) \in K$ then at least one of $(e,x,l,0),(e,x,l,1)$ is put into B (without loss of generality $|(e,x,l,0)| = |(e,x,l,1)|$) and if $(e,x,l) \notin K$ then neither of the two strings is in B.

Claim 4.1 *$B \in NEXP$*

Proof. Given a string s, $s \in B$ iff:

1. $s = (e,x,l,i)$ for some $e,x,l \in \Sigma^*, i \in \{0,1\}$,
2. $(e,x,l) \in K$, and
3. $s \neq f_k(0^{u_k})$ where u_k is the least element in the sequence $\{u_n\}_n$ with $(u_k)^k \geq s$.

Condition 1 can be tested for in linear time. Consider condition 3. By definition of u_k, $|s| > (u_{k-1})^{k-1}$ and hence $(u_k)^k \leq 2^{k|s|}$. Now since $f_k \in \mathrm{DTIME}\left(n^k\right)$ and $H \in \mathrm{P}$, the u_k as in 3 can be found and the condition in 3 checked in time $(u_k)^k \leq 2^{k|s|} \leq 2^{O\left(|s|^2\right)}$. As $K \in \mathrm{NTIME}\left(2^n\right)$ the claim follows and in fact $B \in \mathrm{NTIME}\left(2^{n^2}\right)$. □

Thus we have that B is $\le_{2-tt}^{p}$-complete for NEXP .

Claim 4.2 *B is not $\le_m^p$-complete for NEXP.*

Proof. Assume B is $\le_m^p$-complete. Then by Theorem 1.1 there is a polynomial time computable f_n which reduces Σ^* to B and which is exponentially honest.

At stage n of the construction of B we computed $f_n(0^{u_n})$. By the exponential honesty of f_n, $2^{|f_n(0^{u_n})|} \geq |0^{u_n}| = u_n = 2^{(u_{n-1})^{n-1}} + 1$, and so $|f_n(0^{u_n})| > (u_{n-1})^{n-1}$. Hence at stage n we put $f_n(0^{u_n})$ into $\overline{B}$. This contradicts the assumption that f_n is a reduction of Σ^* to B □
This completes the proof of Theorem 2.3(i) for the class NEXP . □

A standard padding argument now yields the desired result for NE.

Claim 4.3 *There is a C which is $\le_{2-tt}^{p}$-complete for NE but not $\le_m^p$-complete for NE.*

Proof. Let B be as in the previous theorem. Then, as noted above, $B \in \mathrm{NTIME}\left(2^{n^2}\right)$. Define $C = \{x10^{|x|^2} | x \in B\}$. Then

1. $C \in$ NE.
2. $B \le_m^p C$ and hence C is $\le_{2-tt}^{p}$ complete.
3. C is not $\le_m^p$-complete for NE.

Hence C has the desired properties. □

Almost nothing is known regarding these questions for NP-complete sets, even assuming that P $\neq$ NP .

5 Other Properties and Open Problems

5.1 *Properties of "weak" complete sets*

In the last section we considered reductions weaker than $\le_m^p$ and showed that such reductions generally lead to different notions of completeness. We have not mentioned any of the structural properties which such "weakly-complete" sets possess. Here we discuss one recent development concerning autoreducibility for $\leq_T^p$ sets in some detail and briefly mention a few other properties of interest.

Recently, there has been a new study of the autoreducibility of polynomial time Turing-complete sets for exponential time (and larger) classes which has interesting consequences for the relationships between important complexity classes. An autoreduction is a Turing reduction from a set to itself where the only restriction is that one cannot query the oracle on the input string itself. More formally,

Definition 5.1 A set A is (P-time) autoreducible if there is a polynomial-time oracle Turing machine M such that $A = M^A$ and for all x, $M^A(x)$ does not query its oracle A about x.

Buhrman, Fortnow and Torenvliet [BFT] proved the following results.

Theorem 5.1 *Every set which is $\leq_T^p$-complete for EXP or for EXPSPACE is autoreducible.*

Theorem 5.2 *There is a set which is $\leq_T^p$-complete for EEXPSPACE (= DSPACE $(2^{2^{poly}})$) but not autoreducible.*

From these results they noted that it follows that settling the question of the autoreducibility of EEXP-time (= DTIME $(2^{2^{poly}})$) complete sets has important consequences. In particular, if every set which is $\leq_T^p$-complete for EEXP-time is autoreducible then P is not equal to PSPACE. On the other hand if this is not the case then P is not LOGSPACE.

There are a number of other relevant and interesting concepts and results which we have not had time to discuss here and now briefly mention.

The non-uniform complexity of exponential time sets, particularly complete sets, has been extensively studied. Generally these results address the question of whether circuits families of limited size can be used to solve (many or all) exponential time problems. One major open problem is whether all of EXP can be solved with polynomial size circuits. There are numerous partial results in this study, many of which follow from measure theoretic techniques and can be found in Lutz [L92] and Lutz and Mayordomo [LM93]. Others, using more direct approaches, can be found in the Ph.D. theses of Mocas [M93] and Buhrman [Bu93] and the paper by Bin Fu [Fu93].

Tang, Fu and Liu [FLT91] and Buhrman, Hoene and Torenvliet [BHT] study the rubustness of EXP-hard sets by considering when and whether subsets of hard EXP-time sets remain hard. Fu, Li and Zhong [FLZ92] continue this study and also generalize exponential time problems to consider the notion of EXP-low sets in this regard. These ideas are expanded in the work of Buhrman [Bu93] and Buhrman, Hoene and Torenvliet [BHT] where splittings of EXP- and NEXP-complete sets are defined and explored.

5.2 Polynomial time complete recursively enumerable sets

The results presented here all hold for all reasonable (deterministic and nondeterministic) time complexity classes with time bounds larger than 2^n. The recursively enumerable sets form an interesting and somewhat different platform in which to study some of the problems discussed in this paper. The r.e. sets in many respects resemble a non-deterministic class where there are no time bounds to worry about. As with non-deterministic classes there is no closure under complement. And the very powerful methods of recursion theory can be brought to bear. So it is a tempting setting within which to try to first approach the problems discussed here. Yet even here much is not known about the properties of complete sets. We do know that any problem which is $\leq_m^p$-complete for the r.e. sets is also 1-1 complete. (For a proof of this see Ganesan and Homer [GH89].) So here the situation is the same as for NEXP -complete sets. However we do not know if complete sets are 1-1, length-increasing complete, nor do we know if every r.e.-complete set has an infinite P-subset. It seems reasonable to hope to settle one or both of these questions, but a new idea will be necessary.

5.3 A short list of open problems

We end with a short list of open problems raised in this paper. Some are easier than others.

1. Is every 1-1-complete set for NE length-increasing complete ? (Note: In this case length-increasing completeness is equivalent to polynomial-honest completeness, so it is sufficient to prove the polynomial honesty of the reductions.)
2. Does the proof of 1-1 completeness for NE-complete sets extend to nondeterministic complete sets for classes with smaller, but still superpolynomial, time bounds? For example, can we show that every NTIME(n^{logn})-complete set is 1-1 complete ?
3. Is every $1-tt$-complete set for NP m-complete ?
4. Is there a reasonable complexity-theoretic assumption which implies that the various polynomial time reducibilities differ on NP sets ? [LLS75] For one answer to this question refer to the paper by Jack Lutz in this volume.
5. Does every E-complete set contain a *dense* infinite P-subset ? Tran [Tran] has shown this property to be oracle dependent.
6. Does every $\leq^p_{2-tt}$-complete (or $\leq^p_T$-complete) set for E contain an infinite NP (or UP) subset? Note that a negative answer to this question would imply that NP $\neq$ EXP . So one approach might be to assume NP $\neq$ EXP and try to prove the negation. It is straightforward to show that such complete sets may be P-immune. (See Kurtz, Mahaney and Royer [KMR86].)
7. Does every E-complete set contain an infinite dense R subset ?
8. Is there an oracle relative to which there is a P-immune NP complete set ?
9. Is every problem which is $\leq^p_T$-complete set for EEXP autoreducible ? Is every $\leq^p_T$-complete set for NE autoreducible ?
10. Does every $\leq^p_m$ -complete r.e. set contain an infinite P-subset ? Can the complement of such sets be P-immune ?
11. Is every $\leq^p_m$ -complete r.e. set also 1-1, length-increasing complete ?
12. One can define the non-adaptive tt-autoreducibility in a manner completely analogous to (Turing-)autoreducibility. Using this notion, one can prove that all tt-complete sets for PSPACE are tt-autoreducible and that there are sets tt-complete for EXPSPACE which are not tt-autoreducible. The question remains whether all sets which are tt-complete for for EXP are tt-autoreducible. Settling this question would separate PSPACE either from P or from EXP . For precise definitions and more details see [BFT].

Pointers to other papers in this volume:
Several papers in this volume contain results relevant to this survey.

1. The paper by Jack Lutz considers a notion of measure-theoretic completeness (weak completeness) for exponential time sets which relates to the completeness notions considered here and leads to additional structural properties.

2. Jie Wang, in Average Case Computational Complexity Theory, looks at concepts and properties of average-case completeness. While focusing on NP problems, some of this work is relevant to exponential time as well.

3. This paper has focused on efficient reductions to complete sets and the properties of complete sets which follow from such reductions. A complementary and very active area of computational complexity studies which types of problems complete sets can be reduced to (i.e. " hard sets"), and what follows from the existence of such reductions. The paper of Cai and Ogihara in this volume discusses some aspects of this question.

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