Stability analysis of two photorefractive ring resonator circuits: the flip-flop and the feature extractor

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We analyze the stability of two recently demonstrated photorefractive resonator circuits. The analysis is based on single-mode models of the multimode circuits. The flip-flop, which consists of two competitively coupled rings, is considered in the two limits where the rings share or have separate gain volumes. Both configurations are found to be stable for typical experimental conditions. The feature extractor consists of two rings with a shared gain volume. It is found to be unconditionally stable. The results are discussed in the context of the experimental demonstrations.

1. INTRODUCTION

The photorefractive ring oscillator was first demonstrated in 1982.[1] In its most basic configuration, with a single gain medium, the photorefractive ring resonator has been used for real-time beam cleanup[2] and has been proposed as a bistable device.[3] Including additional nonlinear elements or multiple photorefractive interactions in a single ring, or allowing multiple rings to couple with one another, has led to increasingly sophisticated devices and dynamics. Generalizations of the basic ring resonator have been used to demonstrate associative memories,[4–6] bistability,[7] flip-flop operation,[8] controlled competitive dynamics,[9] self-organized feature extraction,[10,11] and topology-preserving feature mappings.[12]

These more complicated resonator configurations are in some ways reminiscent of analog electronic circuits. Beams of light couple with one another in nonlinear elements, transferring energy and phase. As the circuits become more complex, predicting their steady-state operating points, and their dynamics, becomes more difficult. As is the case with electronic circuits it is desirable to have a set of analytical tools for predicting the behavior of a given circuit. This paper is devoted to the development of such a set of tools and their application to two representative circuits: the flip-flop[8] and the feature extractor.[10,11]

The photorefractive flip-flop was demonstrated with the configuration shown schematically in Fig. 1(a). Two multimode rings are photorefractively pumped by the input signal. The rings are competitively coupled in additional photorefractive loss media: part of the energy in each ring serves as a loss pump for the other ring. The result is that there are two equivalent asymmetric states corresponding to flip-flop operation: ring 1 on with ring 2 off, or ring 2 on with ring 1 off. There are also four other possible states: both rings on with equal intensities, both rings on with unequal intensities (ring 1 or ring 2 at higher intensity), and both rings off. The on state refers to the presence of a high-intensity steady-state oscillation, and the off state refers to the intensity in that ring being zero. Because the two rings are equivalent, either the left or the right ring may be assumed to be ring number 1. All six possible states may or may not be temporally stable. Below we analyze the stability of this circuit, using a plane-wave model of the photorefractive interactions.[13] The flip-flop constructed with optical resonators is more complicated than its electronic counterpart for several reasons. First, the steady states of the flip-flop correspond to parameter regimes where photorefractive two-beam coupling is nonlinear. It is necessary to account explicitly for the nonlinearity when analyzing the optical circuit. Second, because the optical circuit is much larger than the wavelength of the light there is a round-trip resonance condition $L/\lambda = m$, with m an arbitrary integer. To avoid sensitivity to L/λ the resonators discussed here incorporate multimode fiber so that they support a large number of transverse modes (~10,000). The effective cavity length depends on the transverse mode that is due to the modal dispersion of the fiber. Because the field in the resonator is free to choose, it will choose a transverse mode that puts it on resonance with the cavity. We will therefore assume the cavity to be on resonance and will describe the corresponding resonator mode as a single, albeit spatially complicated, mode. The assumption of on-resonance oscillation allows us to model the photorefractive coupling coefficient as purely real in the analysis that follows.

Two flip-flop configurations are studied below. The case of separate gain media is shown in Fig. 1(a), and the case of shared gain media is shown in Fig. 1(b). It turns out that the stability of these two configurations is considerably different. Sharing of the gain medium provides additional competitive coupling between the rings that serves to stabilize the flip-flop. This is true provided that there is no direct coupling between the spatially uncorrelated beams in the two rings.

The photorefractive feature extractor was demonstrated by use of the configuration shown schematically in Fig. 2(a).[10] Two multimode rings with a shared gain medium are photorefractively pumped by the input signals. In this case the input is not a constant beam but

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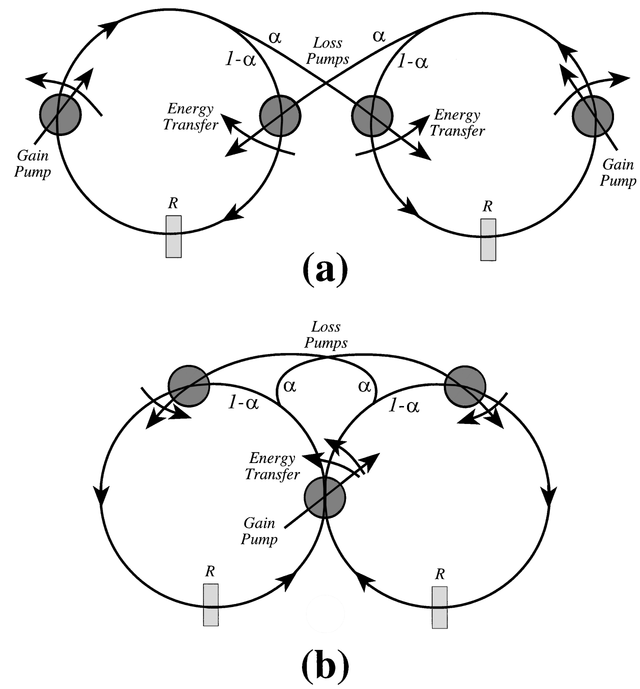

Fig. 1. Photorefractive flip-flop: (a) separate gain media and (b) shared gain media. R is the cavity reflectivity and α is the fraction of the oscillating energy that is diverted into the loss pumps.

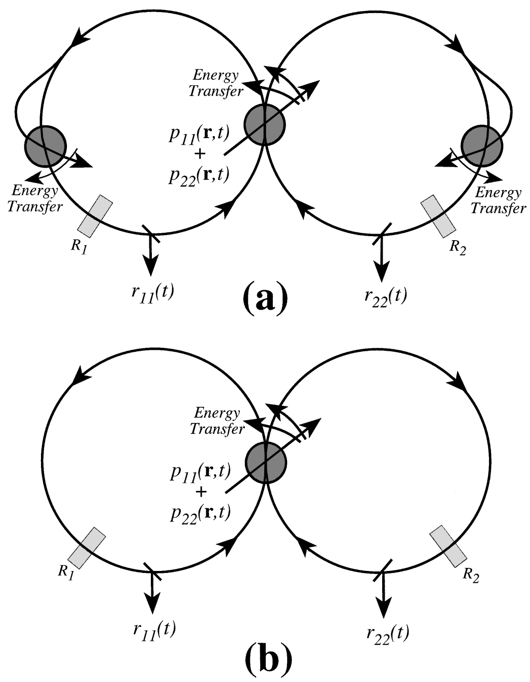

Fig. 2. Feature extractor: (a) experimental geometry with reflexive coupling in each ring and (b) simplified model.

rather a superposition of spatially and temporally orthogonal features. For example the input signals may be spatial patterns with different optical carrier frequencies[10] or spatial patterns that are presented at different times.[11] For convenience, different temporal modes will be referred to below simply as different frequencies. The resonator dynamics result in each of the input features' becoming temporarily correlated with spatially orthogonal resonator modes. As with the flip-flop, multimode rings are used to avoid sensitivity to the cavity resonance condition. However, the input signals are free to choose orthogonal resonator modes in the same ring. A useful mapping of input signals onto resonator modes is obtained only when the resonator modes are intensity, not just field, orthogonal. The additional photorefractive couplings in Fig. 2(a) prevent two input signals from choosing the same ring. A portion of the oscillating energy is coupled out of each ring and then reflexively coupled[14] back into the same ring. The excess loss that is due to the reflexive coupling is minimized when there is only one temporal signal in each ring. The result is that the circuit is driven toward a state where each signal chooses a different ring. We will therefore model the feature extractor by using the simplified configuration shown in Fig. 2(b). Each multimode ring with reflexive coupling is represented as a single-mode ring, with no additional coupling. Analysis of this configuration shows that the feature extractor is unconditionally stable.

Analysis of these circuits will be based on perturbation theory. First the possible steady states are categorized, and dispersion relations for the evolution of perturbations about the steady states are derived. The dispersion relations are valid for arbitrary circuit parameters, but they will be analyzed in the limit of large passive cavity losses and large small-signal gain and loss. These limits correspond to the experimental demonstrations of the flip-flop[8] and the feature extractor.[10] In some cases the steady-state operating points correspond to nonzero values of both pump and resonator fields. This means that the fields vary with position in the photorefractive medium. A transfer function analysis based on the assumption of undepleted pumps[15] is therefore not applicable. It is necessary to account for the propagation of fluctuations on both interacting beams through the medium. The method used is based on one developed previously for the analysis of various four-wave mixing geometries.[16] In the case of two-beam coupling it is convenient to systematize the algebra by deriving transfer matrices for the perturbations. This is done in Section 2. The transfer matrices are applied to the flip-flop with separate gain medium in Section 3 and to the flip-flop with shared gain media in Section 4. In the case of the feature extractor the analysis is tedious because of the large number of interacting fields. It turns out to be more convenient to analyze the possible steady states separately, instead of deriving general transfer matrices. The case of a single ring pumped by two input signals is analyzed in Section 5, and two signals pumping two rings is analyzed in Section 6. The results of the analysis are discussed in Section 7.

2. DERIVATION OF TRANSMISSION MATRICES FOR PERTURBATIONS

The dynamics of two-beam coupling in a photorefractive medium are described by the following set of equations[13]:

$$\frac{\partial r}{\partial x} = \nu p\,, \tag{1a}$$

$$\frac{\partial p}{\partial x} = -\nu^* r\,, \tag{1b}$$

$$\left(\tau \frac{\partial}{\partial t} + 1\right)\nu = \frac{\Gamma}{2}\,\frac{rp^*}{I_T}\,, \tag{1c}$$

where r and p are amplitudes of resonator and pumping beams, respectively, ν is the amplitude of the grating, Γ is the coupling constant, $\tau \propto 1/I_T$ is the characteristic relaxation time of the medium, and $I_T = |r|^2 + |p|^2$ is the sum of intensities of the interacting beams.

In the case of a purely real coupling constant Γ the stationary solution may be assumed real without loss of generality. Output intensities of the resonator and the pump beams are connected to their input intensities by the relations[17]

$$|r_0|^2_{\mathrm{out}} = |r_0|^2_{\mathrm{in}} M\,, \tag{2a}$$

$$|p_0|^2_{\mathrm{out}} = |p_0|^2_{\mathrm{in}} M \exp(-\Gamma)\,, \tag{2b}$$

$$M = \frac{1 + y}{y + \exp(-\Gamma)}\,, \tag{2c}$$

where $y = |r_0/p_0|^2_{\mathrm{in}}$ is the input beam intensity ratio. For convenience the length of the medium has been set equal to 1 in this and all subsequent formulas. We may dimensionalize all the results below by making the conversion $\Gamma \to \Gamma l$, where l is the length of the medium. Equations (2) allow for any sign of Γ. Below we use the notation $\Gamma \equiv \Gamma_G$, $M \equiv G$ for $\Gamma > 0$ (when a resonator beam experiences gain) and $\Gamma \equiv -\Gamma_L$, $M \equiv L$ for $\Gamma < 0$ (when it experiences loss).

Arbitrary complex amplitude perturbations about this stationary state can be separated into purely real and purely imaginary ones, and their evolution can be described independently.[18] Below we restrict ourselves to consideration of purely real perturbations:

$$r(x, t) = r_0(x) + \mathrm{Re}[\delta r(x)\exp(ft)]\,, \tag{3a}$$

$$p(x, t) = p_0(x) + \mathrm{Re}[\delta p(x)\exp(ft)]\,, \tag{3b}$$

where f is the complex frequency.

Linearizing system (1) with respect to δr and δp around stationary solution (2) and solving it allows one to obtain a transmission matrix that describes propagation of perturbations through the medium

$$\begin{bmatrix} \delta r \\ \delta p \end{bmatrix}_{\mathrm{out}} = \mathbf{T}^{(2)} \begin{bmatrix} \delta r \\ \delta p \end{bmatrix}_{\mathrm{in}}\,, \tag{4}$$

where the matrix elements are given by the relations

$$T_{11}^{(2)} = \frac{\exp(\Gamma/2)}{[1 + y \exp(\Gamma)]\sqrt{M}} \times \left\{ y \exp(\Gamma/2) + \exp\left[\frac{1}{1 + \tau f}\left(\ln M - \frac{\Gamma}{2} \right)\right]\right\}, \tag{5a}$$

$$T_{12}^{(2)} = \frac{\exp(\Gamma/2)}{[1 + y \exp(\Gamma)]\sqrt{M}} \sqrt{y} \times \left\{ \exp(\Gamma/2) - \exp\left[\frac{1}{1 + \tau f}\left(\ln M - \frac{\Gamma}{2} \right)\right]\right\}, \tag{5b}$$

$$T_{21}^{(2)} = \frac{\exp(\Gamma/2)}{[1 + y \exp(\Gamma)]\sqrt{M}} \sqrt{y} \times \left\{ 1 - \exp\left[\frac{\Gamma}{2} + \frac{1}{1 + \tau f}\left(\ln M - \frac{\Gamma}{2} \right)\right]\right\}, \tag{5c}$$

$$T_{22}^{(2)} = \frac{\exp(\Gamma/2)}{[1 + y \exp(\Gamma)]\sqrt{M}} \times \left\{ 1 + y \exp\left[\frac{\Gamma}{2} + \frac{1}{1 + \tau f}\left(\ln M - \frac{\Gamma}{2} \right)\right]\right\}. \tag{5d}$$

An analogous 3×3 matrix describes the propagation of perturbations in the case of a pumping beam p coupling with two resonator beams r_1 and r_2 that do not directly couple with each other and have spatially uncorrelated amplitudes. The equations of motion are an evident generalization of Eqs. (1):

$$\frac{\partial r_1}{\partial x} = \nu_1 p\,, \tag{6a}$$

$$\frac{\partial r_2}{\partial x} = \nu_2 p\,, \tag{6b}$$

$$\frac{\partial p}{\partial x} = \nu_1^* r_1 - \nu_2^* r_2\,, \tag{6c}$$

$$\left(\tau \frac{\partial}{\partial t} + 1\right)\nu_1 = \frac{\Gamma}{2}\,\frac{r_1 p^*}{I_T}\,, \tag{6d}$$

$$\left(\tau \frac{\partial}{\partial t} + 1\right)\nu_2 = \frac{\Gamma}{2}\,\frac{r_2 p^*}{I_T}\,. \tag{6e}$$

Here as in Eqs. (1) r_1, r_2 and p are amplitudes of resonator and pumping beams, respectively, ν_1 and ν_2 are amplitudes of the gratings, Γ is the coupling constant, $\tau \propto 1/I_T$ is the characteristic relaxation time of the medium, and $I_T = |r_1|^2 + |r_2|^2 + |p|^2$. The stationary solution of Eqs. (6) is

$$|r_{10}|^2_{\mathrm{out}} = |r_{10}|^2_{\mathrm{in}} M\,, \tag{7a}$$

$$|r_{20}|^2_{\mathrm{out}} = |r_{20}|^2_{\mathrm{in}} M\,, \tag{7b}$$

$$|p_0|^2_{\mathrm{out}} = |p_0|^2_{\mathrm{in}} M \exp(-\Gamma)\,, \tag{7c}$$

$$M = \frac{1 + y}{y + \exp(-\Gamma)}\,, \tag{7d}$$

$$y = y_1 + y_2\,, \tag{7e}$$

where $y_1 = |r_{10}/p_0|^2_{\mathrm{in}}$, $y_2 = |r_{20}/p_0|^2_{\mathrm{in}}$.

Linearizing Eqs. (6) around this stationary state gives, after some algebra, the following 3×3 transmission matrix for the perturbations:

$$\begin{bmatrix} \delta r_1 \\ \delta r_2 \\ \delta p \end{bmatrix}_{\mathrm{out}} = \mathbf{T}^{(3)} \begin{bmatrix} \delta r_1 \\ \delta r_2 \\ \delta p \end{bmatrix}_{\mathrm{in}}\,, \tag{8}$$

$$T_{11}^{(3)} = \frac{y_1}{y} T_{11}^{(2)}(y) + \frac{y_2}{y} \exp\left[\frac{1}{2(1+\tau f)} \ln M\right], \tag{9a}$$

$$T_{22}^{(3)} = \frac{y_2}{y} T_{11}^{(2)}(y) + \frac{y_1}{y} \exp\left[\frac{1}{2(1+\tau f)} \ln M\right], \tag{9b}$$

$$T_{33}^{(3)} = T_{22}^{(2)}(y), \tag{9c}$$

$$T_{12}^{(3)} = \frac{\sqrt{y_1 y_2}}{y} \left\{T_{11}^{(2)}(y) - \exp\left[\frac{1}{2(1+\tau f)} \ln M\right]\right\}, \tag{9d}$$

$$T_{21}^{(3)} = T_{12}^{(3)}, \tag{9e}$$

$$T_{23}^{(3)} = \sqrt{y_1 y_2}\, T_{12}^{(2)}(y), \tag{9f}$$

$$T_{31}^{(3)} = \sqrt{(y_1/y)}\, T_{21}^{(2)}(y), \tag{9g}$$

$$T_{32}^{(3)} = \sqrt{(y_2/y)}\, T_{21}^{(2)}(y). \tag{9h}$$

Here both the elements of the 2×2 transmission matrix $\mathbf{T}^{(2)}$, given by Eqs. (5) and the parameter M, described by Eq. (7d) depend on the argument $y = y_1 + y_2$.

3. FLIP-FLOP WITH SEPARATE GAIN VOLUMES

Consider the photorefractive flip-flop circuit depicted schematically in Fig. 1(a). It consists of two rings, which will be referred to subsequently as ring 1 and ring 2. The rings are equivalent, so either the left or the right ring may be considered to be ring 1. Each of the rings contains a gain crystal, whose properties in the stationary state are characterized by gain coefficient $G(1)$ or $G(2)$, respectively, and also a loss crystal characterized by loss coefficient $L(1)$ or $L(2)$, where both the gain and the loss coefficients are described by Eqs. (2). Specifying resonator y_{G2} at the entrance to the gain crystal in both rings determines both their normalized output intensities $G(1)y_{G1}$ and $G(2)y_{G2}$ and also the resonator-to-pump ratios at the entrance to the loss crystals:

$$y_{L1} = \frac{(1-\alpha)G(1)y_{G1}}{\alpha G(2)y_{G2}}, \tag{10a}$$

$$y_{L2} = \frac{(1-\alpha)G(2)y_{G2}}{\alpha G(1)y_{G1}}. \tag{10b}$$

Here α is the fraction of the resonator beam intensity that is diverted. Stationary states are found from the nonlinear relations

$$y_{G1} = G(1)L(1)R(1-\alpha)y_{G1}, \tag{11a}$$

$$y_{G2} = G(2)L(2)R(1-\alpha)y_{G2}, \tag{11b}$$

where R is the total accumulated passive losses in the rings ($0 < R < 1$). For convenience we have taken α, Γ_G, Γ_L, and R to be the same in both rings. In the experimental realization of this device[8] the inequalities $R \ll 1$(large passive losses)$\exp(\Gamma_G) \gg 1$ and $\exp(\Gamma_L) \gg 1$(large small-signal gain and loss) were satisfied. These inequalities will be assumed be low in order to simplify the algebra. Equations (11) mean physically that the net round-trip gain in each ring is unity in steady state.

Solution of Eqs. (10) and (11) shows that the geometry has six stationary states. The first state $y_{G1} = y_{G2} = 0$ corresponds to the situation when both modes are off, which is always a solution. The next couple of solutions correspond to asymmetric on–off states in which one mode is on and the other is off. For the solution when ring 1 is oscillating

$$y_{G1} = R(1-\alpha) - \exp(-\Gamma_G), \tag{12a}$$

$$y_{G2} = 0, \tag{12b}$$

and vice versa for ring 2 oscillating. Solution (12) exists, provided that the net small-signal gain in the ring is greater than unity, i.e., if $R(1-\alpha)\exp(\Gamma_G) > 1$.

Another solution corresponds to a symmetric on state in which both rings oscillate with the same intensity:

$$y_{G1} = y_{G2} = \frac{R(1-\alpha)}{\alpha} \exp(-\Gamma_L) - \exp(-\Gamma_G). \tag{13}$$

This solution exists, provided that $R(1-\alpha)\exp(\Gamma_G - \Gamma_L) > \alpha$.

Finally there exist two solutions in which both modes are on but their intensities are different:

$$y_G = \exp(-\Gamma_G)\frac{1-z}{z}, \tag{14a}$$

$$z = \frac{1}{2(1-\alpha)q} \times \left(q - 1 \pm \left\{\frac{[1+(1-2\alpha)q](q-3+2\alpha)}{1-2\alpha}\right\}^{1/2}\right), \tag{14b}$$

$$q = R\,\frac{(1-\alpha)}{\alpha} \exp(\Gamma_G - \Gamma_L), \tag{14c}$$

where $+$ and $-$ correspond to the pair y_{G1}, y_{G2} or vice versa, depending on in which ring the intensity of the oscillating mode is larger. Solution (14) exists in the range

$$\min[1, \alpha(3-2\alpha)] < R(1-\alpha)\exp(\Gamma_G - \Gamma_L) < \max[1, \alpha(3-2\alpha)]. \tag{15}$$

The dispersion equation for the perturbations around stationary states is readily obtained with the help of transmission matrices (4). The derivation proceeds as follows: consider perturbations δr_1 and δr_2 to the resonator beams r_1 and r_2 oscillating in rings 1 and 2, respectively. Let the amplitudes of these perturbations immediately before the respective beam splitters α be equal to δr_1(before α) and δr_2(before α). We start the derivation by following the perturbation δr_1 around ring 1. After passage through the beam splitter the amplitude of the perturbation remaining in ring 1 is δr_1(after α) $= \sqrt{1-\alpha}\,\delta r_1$(before α). In the loss crystal $(L, 1)$ (capital letters G and L denote gain and loss crystals and numerials 1 and 2 denote rings 1 and 2, respectively) this perturbation serves as a perturbation of the signal. The pump beam for the loss crystal $(L, 1)$ is supplied from ring 2, and the input amplitude of the pump beam perturbation is $\sqrt{\alpha}\,\delta r_2$(before α). Applying transmission matrices (4) to the propagation through loss crystal $(L, 1)$, one gets $\delta r_1[\text{after } (L, 1)] = T_{11}^{(2)}(L, 1)\sqrt{1-\alpha}\,\delta r_1$(before

$\alpha) + T_{12}^{(2)}(L, 1)\sqrt{\alpha}\,\delta r_2(\text{before } \alpha)$. Immediately before the gain crystal $(G, 1)$ the amplitude of the perturbation is $\delta r_1[\text{before } (G, 1)] = \sqrt{R}\,\delta r_1[\text{after } (L, 1)]$. The pump beam for gain crystal $(G, 1)$ is supplied externally, and the input amplitude of its perturbation is zero. The amplitude of the perturbation of resonator beam 1 after gain crystal $(G, 1)$ is then coupled to its input value by the relation [see Eq. (4)] $\delta r_1[\text{after } (G, 1)] = T_{11}^{(2)}(G, 1)\delta r_2[\text{before } (G, 1)]$. On the other hand, $\delta r_1[\text{after } (G, 1)] = \delta r_1(\text{before } \alpha)$, so we get the relation

$$\delta r_1(\text{before } \alpha) = T_{11}^{(2)}(G, 1)\sqrt{R} \times [T_{11}^{(2)}(L, 1)\sqrt{1-\alpha}\,\delta r_1(\text{before } \alpha) + T_{12}^{(2)}(L, 1)\sqrt{\alpha}\,\delta r_2(\text{before } \alpha)]. \tag{16}$$

Repeating the same procedure for the second ring, we arrive at the identical relation

$$\delta r_2(\text{before } \alpha) = T_{11}^{(2)}(G, 2)\sqrt{R} \times [T_{11}^{(2)}(L, 2)\sqrt{1-\alpha}\,\delta r_2(\text{before } \alpha) + T_{12}^{(2)}(L, 2)\sqrt{\alpha}\,\delta r_1(\text{before } \alpha)]. \tag{17}$$

The condition that Eqs. (16) and (17) have nonzero solutions δr_1 and δr_2 results in the dispersion equation

$$[1 - \sqrt{R(1-\alpha)}\,T_{11}^{(2)}(G, 1)T_{11}^{(2)}(L, 1)] \times [1 - \sqrt{R(1-\alpha)}\,T_{11}^{(2)}(G, 2)T_{11}^{(2)}(L, 2)] = R\alpha T_{11}^{(2)}(G, 1)T_{11}^{(2)}(G, 2)\,T_{12}^{(2)}(L, 1)T_{12}^{(2)}(L, 2). \tag{18}$$

In the case of symmetric on or off stationary states

$$T_{11}^{(2)}(G, 1) = T_{11}^{(2)}(G, 2) \equiv T_{11}^{(2)}(G), \tag{19a}$$

$$T_{11}^{(2)}(L, 1) = T_{11}^{(2)}(L, 2) \equiv T_{11}^{(2)}(L), \tag{19b}$$

$$T_{12}^{(2)}(L, 1) = T_{12}^{(2)}(L, 2) \equiv T_{12}^{(2)}(L), \tag{19c}$$

and Eq. (18) reduces to

$$1 - \sqrt{R(1-\alpha)}\,T_{11}^{(2)}(G)T_{11}^{(2)}(L) = \pm\sqrt{R\alpha}\,T_{11}^{(2)}(G)T_{12}^{(2)}(L), \tag{20}$$

where plus or minus corresponds to symmetric or antisymmetric perturbations, respectively. In the first case perturbations in both rings have the same sign, and the intensity in both rings either increases or decreases simultaneously. In the second case these perturbations have opposite sign and so, while the intensity in one ring increases, it decreases in the second ring.

For the symmetric off-stationary state Eq. (20) is further simplified to

$$R(1-\alpha)\exp\left(\frac{\Gamma_G}{1+\tau_G f}\right) = 1. \tag{21}$$

There is no difference here between symmetric and antisymmetric perturbations because the ground state is exactly zero and the relaxation time $\tau_L \propto 1/I_T(L)$ in the loss crystals is equal to infinity. Solutions of Eq. (21) have positive real parts of the complex frequency f corresponding to instability of the stationary off state, provided that $R(1-\alpha)\exp(\Gamma_G) > 1$. In other words, provided that there is sufficient gain for the circuit to turn on, the off state is unstable.

For the asymmetric on–off state Eq. (18) splits into two independent equations for the on and the off rings because the product $T_{12}^{2}(L, 1)T_{12}^{(2)}(L, 2)$ turns out to be equal to zero. The dispersion equation for the on ring yields [in accordance with Eqs. (12) we assume that ring 1 is on and ring 2 is off]

$$f\tau_G(1) = -\Gamma_G - \ln[R(1-\alpha)]. \tag{22}$$

Note that this solution contains neither the loss coefficient nor the relaxation time of the loss crystal because loss in the on ring is determined by the off ring, and there is no radiation there. Because the on–off stationary state exists, provided that $R(1-\alpha)\exp(\Gamma_G) > 1$, the on ring is always stable. Simplification of the dispersion equation for the off ring results in the relation

$$\sqrt{R(1-\alpha)}\,\exp\left\{\frac{\Gamma_G}{2[1+\tau_G(2)f]} - \frac{\Gamma_L}{2[1+\tau_L(2)f]}\right\} = 1. \tag{23}$$

Stability of the off ring (negative real part of f) requires fulfillment of the following inequalities:

$$\Gamma_L + C > \Gamma_G, \tag{24a}$$

$$\frac{\tau_G}{\tau_L} > \frac{\Gamma_G - C}{\Gamma_L + C}, \tag{24b}$$

where $C = -\ln[R(1-\alpha)]$. The first inequality is not restrictive, but the second one turns out to be much more stringent when the passive losses in the ring are large ($R \ll 1$), and that is usually the case. The reason is that the characteristic relaxation time in a photorefractive medium is inversely proportional to its illumination. In the bad-cavity limit discussed here ($R \ll 1$), there is essentially no buildup in the on ring, and so the intensity at the output of the gain crystal in the on ring is at most equal to the pump intensity. The intensity of the pumping beam for the loss crystal in the off ring is therefore at most equal to the intensity of the signal at the output of the gain crystal in the on ring divided by passive losses on its passage from the gain to the loss crystal. Usually these losses constitute a considerable part of the total passive losses R, and hence

$$\Gamma_L > \frac{\tau_L}{\tau_G}(\Gamma_G - C) - C \tag{25}$$

in order for the flip-flop with separate gain volumes to be stable.

Inequalities (24) are formally identical to the conditions that determine the stability of the off state of a single bistable ring with gain and loss,[19] despite the fact that the physical situation is considerably different. In the case of the bistable ring with gain and loss one cannot satisfy Eq. (24b) by simply increasing Γ_L, as doing so would imply that the on state no longer existed. Rather, the bistable ring can be stabilized only if the loss is made sufficiently fast. In the case of the flip-flop Γ_L may be increased as much as desired without affecting the existence of the on–off asymmetric state.

Inequalities (24) allow for a simple physical interpretation in the case of large passive losses, corresponding to the presence of two different time scales τ_G and τ_L in the system. The first of inequalities (24) corresponds to the stability of the system with respect to slow perturbations, which change at times of the order of τ_L, and is obtained from Eq. (23) in the limit $f = 0$. The second one corresponds to fast perturbations, which change at times of the order of

$$\sqrt{R(1-\alpha)}\,\exp\left[\frac{\Gamma_G}{2(1+\tau_G f)}\right] = 1\,. \tag{26}$$

Equation (26) immediately gives $\Gamma_G - C \leq 0$ as the necessary condition of stability. The above dispersion equation shows that perturbations changing at times faster than the characteristic relaxation time of the loss crystal do not feel any losses. This fact is well known and has found application, e.g., in photorefractive novelty filters.[15]

For the symmetric on state [Eqs. (13)] dispersion equation (20) for the symmetric perturbations (plus) gives a stable solution, whereas for the antisymmetric perturbations it reduces to

$$q\,\exp\left[\frac{\tau_G f}{1+\tau_G f}\left(\frac{\Gamma_G}{2} - \ln q\right)\right] = (1-2\alpha) + 2\,\exp\left[\frac{\tau_L f}{1+\tau_L f}\left(\frac{\Gamma_L}{2} + \ln \alpha\right)\right], \tag{27}$$

where $q = R(1-\alpha)\alpha^{-1}\exp(\Gamma_G - \Gamma_L)$ (the stationary state exists for $q > 1$). In the experimental limit of $\tau_G \ll \tau_L$ there are no stable solutions of Eq. (27). Thus the symmetric on state is unstable. It should be reemphasized that this analysis, and the conclusion that the symmetric on state is unstable, is based on the assumption of large small-signal loss [$\exp(\Gamma_L) \gg 1$]. It is intuitively clear that in the opposite limit, where $\Gamma_L \to 0$, there is no coupling between the rings, and the symmetric on state is stable.

Finally, investigation of the stability of the asymmetric on state [Eqs. (14)] shows that the state is also always unstable for $\tau_G \ll \tau_L$.

To summarize the results of this section, when the two rings have separate gain volumes all the stationary states, including the asymmetric on–off state that corresponds to flip-flop operation, turn out to be unstable, provided that the relaxation times in the loss crystals are considerably larger than those in the gain crystals. The asymmetric on–off state can be stabilized by increasing the loss coupling until inequality (25) is satisfied.

4. FLIP-FLOP WITH SHARED GAIN VOLUME

In the case when the rings share a common gain medium [Fig. 1(b)], the resonator-to-pump-beam ratios at the entrance to the loss crystals are given by

$$y_{L1} = \frac{(1-\alpha)y_{G1}}{\alpha y_{G2}}\,, \tag{28a}$$

$$y_{L2} = \frac{(1-\alpha)y_{G2}}{\alpha y_{G1}}\,, \tag{28b}$$

because both signals receive the same gain G. Stationary states of this geometry are again given by Eqs. (11), where $G(1) = G(2) = G$. Solution of Eqs. (11) and (28) shows that there are four stationary states that are remarkably similar to those discussed in Section 3. Thus there exists the symmetric off state $y_{G1} = y_{G2} = 0$. There also exist two asymmetric on–off solutions, in which one ring is on and the other is off. They are exactly analogous to those discussed in Section 3 and are described by Eqs. (12). The final solution is the symmetric on state, where both rings are on with the same intensity. This solution is given by

$$y_{G1} = y_{G2} = \frac{1}{2}\left[\frac{R(1-\alpha)}{\alpha}\exp(-\Gamma_L) - \exp(-\Gamma_G)\right], \tag{29}$$

which differs from Eqs. (13) only by a factor of 2. There are no asymmetric on solutions similar to Eqs. (14).

The dispersion relation for the perturbations is obtained with the help of transmission matrices (4) for the loss crystals and of transmission matrix (8) for the gain crystal and takes the form

$$\begin{aligned}&[1 - \sqrt{R(1-\alpha)}\,T_{11}^{(3)}T_{11}^{(2)}(L,1) - \sqrt{R\alpha}\,T_{21}^{(3)}T_{12}^{(2)}(L,1)]\\ &\quad\times[1 - \sqrt{R(1-\alpha)}\,T_{22}^{(3)}T_{11}^{(2)}(L,2) - \sqrt{R\alpha}\,T_{12}^{(3)}T_{12}^{(2)}(L,2)]\\ &= [\sqrt{R(1-\alpha)}\,T_{12}^{(3)}T_{11}^{(2)}(L,1) + \sqrt{R\alpha}\,T_{22}^{(3)}T_{12}^{(2)}(L,1)]\\ &\quad\times[\sqrt{R(1-\alpha)}\,T_{21}^{(3)}T_{11}^{(2)}(L,2) - \sqrt{R\alpha}\,T_{11}^{(3)}T_{12}^{(2)}(L,2)].\end{aligned} \tag{30}$$

In the case of symmetric on or off states Eq. (30) reduces to

$$1 = \sqrt{R}\,T_{11}^{(2)}(G)[\sqrt{1-\alpha}\,T_{11}^{(2)}(L) + \sqrt{\alpha}\,T_{12}^{(2)}(L)] \tag{31}$$

for symmetric perturbations and to

$$1 = \sqrt{R}\,\exp\left[\frac{1}{2(1+\tau_G f)}\ln G\right] \times [\sqrt{1-\alpha}\,T_{11}^{(2)}(L) - \sqrt{\alpha}\,T_{12}^{(2)}(L)] \tag{32}$$

for antisymmetric perturbations.

Stability analysis of the symmetric off state gives the same result as in Section 3; namely, Eq. (30) is further simplified to Eq. (21), which means that the symmetric off state is unstable, provided that $R(1-\alpha)\exp(\Gamma_G) > 1$.

For the symmetric on state [Eq. (29)], dispersion equation (31) for symmetric perturbations gives only stable solutions, whereas Eq. (32) for antisymmetric perturbations simplifies to

$$\exp\left[\frac{\tau_G f}{1+\tau_G f}\left(\frac{\Gamma_G}{2} - \ln\sqrt{q}\right)\right] = (1-2\alpha) + 2\,\exp\left[\frac{\tau_L f}{1+\tau_L f}\left(\frac{\Gamma_L}{2} - \ln \alpha\right)\right], \tag{33}$$

where $q = R(1-\alpha)\alpha^{-1}\exp(\Gamma_G - \Gamma_L)$. This equation is only slightly different from Eq. (27) and analogously has no stable solutions for $\tau_G \ll \tau_L$. The symmetric on stationary state is therefore unstable. As was emphasized in the case of the flip-flop with separate gain volumes, the analysis has been based on the assumption of large

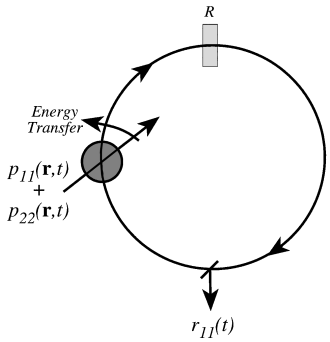

Fig. 3. Single ring pumped by two signals.

small-signal loss. In the limit where $\Gamma_L \rightarrow 0$ the symmetric on state is stable, provided that the two rings have the same passive losses. When the passive losses are unequal, only the ring with lower loss will oscillate.

Finally, consider the asymmetric on–off state. In this case, as in Section 3, general dispersion equation (30) splits into two independent equations for the on and the off rings. Solution of the dispersion equation for the on ring yields only stable solutions with a decay rate given by Eq. (22). The dispersion equation for the off ring can be reduced to

$$\exp\left[\frac{\tau_G f}{1+\tau_G f}\left(\frac{\Gamma_G}{2} - \ln\sqrt{q}\right) + \frac{1}{1+\tau_L f}\,\frac{\Gamma_L}{2}\right] = 1\,, \quad (34)$$

where $q = R(1-\alpha)\exp(\Gamma_G)$ (the on–off state exists for $q > 1$). Equation (34) has no solutions corresponding to a positive real part of f. Hence, in contrast to the results of Section 3, the on–off solution is always stable in the flip-flop with shared gain volumes.

To summarize, when the two rings have a shared gain volume and $\tau_L \gg \tau_G$, only the asymmetric on–off state corresponding to flip-flop operation is stable. The symmetric off state is unstable, provided that there is sufficient gain for the circuit to turn on, and the symmetric on state is unstable, in the investigated limit of large small-signal loss.

5. FEATURE EXTRACTOR: ONE RING

Before analyzing the feature extractor it is useful to consider the one-ring circuit shown in Fig. 3. The ring is pumped by two beams p_{11} and p_{22} having different frequencies (temporal modes) and spatially uncorrelated transverse amplitude distributions. Each of these pumps can in principle excite oscillating signals in the ring at its frequency (r_{11} or r_{12}). The first index refers to the spatial mode, and the second index refers to the frequency. We assume that the ring can support oscillations in only a single spatial mode, so these signals have the same transverse amplitude distributions. This means that, e.g., resonator mode r_{12} will scatter off a grating written by pumping beam p_{11} and resonator mode r_{11}. This results in the appearance of pumping beam p_{12} that has the frequency of pumping beam p_{22} but the spatial amplitude distribution of pumping beam p_{11}. The input amplitude of this beam is zero, but it is generated inside the photorefractive medium. Analogously, readout of the grating written by pump p_{22} with resonator mode r_{12} by resonator mode r_{11} results in the generation of pumping beam p_{21}, which has the frequency of pumping beam p_{11} but the spatial amplitude distribution of pumping beam p_{22}.

The system of equations describing the evolution of these fields inside the photorefractive medium is

$$\frac{\partial r_{11}}{\partial x} = \nu_{11} p_{11} + \nu_{12} p_{21}\,, \quad (35a)$$

$$\frac{\partial r_{12}}{\partial x} = \nu_{11} p_{12} + \nu_{12} p_{22}\,, \quad (35b)$$

$$\frac{\partial p_{11}}{\partial x} = -\nu_{11}^* r_{11}\,, \quad (35c)$$

$$\frac{\partial p_{12}}{\partial x} = -\nu_{11}^* r_{12}\,, \quad (35d)$$

$$\frac{\partial p_{21}}{\partial x} = -\nu_{12}^* r_{11}\,, \quad (35e)$$

$$\frac{\partial p_{22}}{\partial x} = -\nu_{12}^* r_{12}\,, \quad (35f)$$

$$\left(\tau\frac{\partial}{\partial t} + 1\right)\nu_{11} = \frac{\Gamma}{2I_T}\,(r_{11}p_{11}^* + r_{12}p_{12}^*)\,, \quad (35g)$$

$$\left(\tau\frac{\partial}{\partial t} + 1\right)\nu_{12} = \frac{\Gamma}{2I_T}\,(r_{11}p_{21}^* + r_{12}p_{22}^*)\,, \quad (35h)$$

where ν_{ij} is the amplitude of the refractive-index grating that couples resonator spatial mode i with pump spatial mode j. In the one-ring geometry there is only one resonator spatial mode ($i = 1$) and two pump spatial modes ($j = 1, 2$). Γ and $\tau \propto I_T^{-1}$ are the coupling coefficient and the relaxation time, respectively, where $I_T = \sum_{ij} |r_{ij}|^2 + |p_{ij}|^2$ is the total illumination of the crystal. The boundary conditions for system (35) are

$$p_{11,\mathrm{in}} = p_{11,\mathrm{in}}^{(0)}\,, \quad (36a)$$

$$p_{22,\mathrm{in}} = p_{22,\mathrm{in}}^{(0)}\,, \quad (36b)$$

$$p_{12,\mathrm{in}} = p_{21,\mathrm{in}} = 0\,, \quad (36c)$$

$$r_{11,\mathrm{out}}\sqrt{R} = r_{11,\mathrm{in}}\,, \quad (36d)$$

$$r_{12,\mathrm{out}}\sqrt{R} = r_{12,\mathrm{in}}\,, \quad (36e)$$

where R is the total passive loss in the ring ($0 < R < 1$). It should be emphasized that Eqs. (36d) and (36e) are mutually consistent only when the frequency difference between the pump signals is much less than c/L. When this is not the case, the boundary conditions should in principle include a round-trip phase for one of the signals. Strictly speaking, neglecting this phase means that we are considering signals with the same frequency but presented at different times. As inclusion of the round-trip phase would serve only to strengthen the competition between signals, the analysis presented below corresponds to a worst-case situation.

The stationary solution of interest of Eqs. (35) and (36) corresponds to the situation when only one frequency (say, r_{11}) oscillates in the ring. In this case $p_{12} = p_{21} = 0$, $p_{22} = p_{22,\mathrm{in}}^{(0)} = \text{constant}$, whereas for p_{11} and r_{11} one gets

$$|r_{11}|^2_{\rm out} = |r_{11}|^2_{\rm in} G\,, \tag{37a}$$

$$|p_{11}|^2_{\rm out} = |p_{11}|^2_{\rm in} G \exp(\Gamma_n)\,, \tag{37b}$$

$$G = \frac{1+y}{y+\exp(-\Gamma_n)}\,, \tag{37c}$$

$$\Gamma_n = \Gamma\,\frac{1+y}{1+y+m}\,, \tag{37d}$$

where Γ_n is the normalized coupling constant, $y = |r_{11}/p_{11}|^2_{\rm in}$, $m = |p_{22}/p_{11}|^2_{\rm in}$, and the value of y is found from the stationary solution to be equal to

$$y = R - \exp(-\Gamma_n)\,. \tag{38}$$

The stationary solution exists for $R > \exp(-\Gamma_n)$.

To investigate stability we proceed as before and represent all fields $a = \{r_{ij},\, p_{ij}\}$ in the form

$$a(x, t) = a^{(0)}(x) + \mathrm{Re}[\delta a(x)\exp(ft)]\,. \tag{39}$$

When Eqs. (35) are linearized about the stationary solution they split into two independent sets. The first set describes perturbations to fields that have nonzero values in the stationary state:

$$\frac{\mathrm{d}\delta r_{11}}{\mathrm{d}x} = \delta\nu_{11} p_{11}^{(0)} + \nu_{11}^{(0)}\delta p_{11}\,, \tag{40a}$$

$$\frac{\mathrm{d}\delta p_{11}}{\mathrm{d}x} = -\delta\nu_{11} r_{11}^{(0)} - \nu_{11}^{(0)}\delta r_{11}\,, \tag{40b}$$

$$\delta\nu_{11} = \frac{\Gamma}{2I_T^{(0)}(1+\tau f)} \times \left[r_{11}^{(0)}\delta p_{11} + p_{11}^{(0)}\delta r_{11} - r_{11}^{(0)} p_{11}^{(0)} \frac{\delta I_T}{I_T^{(0)}} \right], \tag{40c}$$

$$\delta p_{11,\rm in} = 0\,, \tag{40d}$$

$$\delta r_{11,\rm out}\sqrt{R} = \delta r_{11,\rm in}\,, \tag{40e}$$

and the second set describes evolution of the fields that were equal to zero:

$$\frac{\mathrm{d}\delta r_{12}}{\mathrm{d}x} = \delta\nu_{12} p_{22}^{(0)} + \nu_{11}^{(0)}\delta p_{12}\,, \tag{41a}$$

$$\frac{\mathrm{d}\delta p_{12}}{\mathrm{d}x} = -\nu_{11}^{(0)}\delta r_{12}\,, \tag{41b}$$

$$\frac{\mathrm{d}\delta p_{21}}{\mathrm{d}x} = -\delta\nu_{12} r_{11}^{(0)}\,, \tag{41c}$$

$$\delta\nu_{12} = \frac{\Gamma}{2I_T^{(0)}(1+\tau f)}[\delta r_{12} p_{22}^{(0)} + r_{11}^{(0)}\delta p_{21}], \tag{41d}$$

$$\delta p_{12,\rm in} = \delta p_{21,\rm in} = 0\,, \tag{41e}$$

$$\delta r_{12,\rm out}\sqrt{R} = \delta r_{12,\rm in}\,. \tag{41f}$$

Note that, because contributions to δp_{22} first appear at second order in the perturbations, $p_{22} = p_{22}^{(0)}$ both in Eqs. (40) and in Eqs. (41).

Solution of Eqs. (40) yields

$$\delta r_{11,\rm out} = T\delta r_{11,\rm in}\,, \tag{42a}$$

$$T = \frac{\exp(\Gamma_n/2)}{\sqrt{G}[1+y\exp(\Gamma_n)]}[y\exp(\Gamma_n/2) + B] + \frac{2m}{1+m} yBI\,, \tag{42b}$$

$$B = \exp\left[\frac{1}{1+\tau f}(\ln G - \Gamma_n/2)\right], \tag{42c}$$

$$I = \frac{\Gamma_n \exp(-\Gamma_n/2)}{2(1+\tau f)}\sqrt{G}\int_0^1 \mathrm{d}x \exp\left(-\frac{\tau f}{1+\tau f} \times \left\{\frac{\Gamma_n}{2}x + \ln[y+\exp(-\Gamma_n x)]\right\}\right). \tag{42d}$$

Applying the boundary condition for δr_{11} to Eqs. (42), one gets $T\sqrt{R} = 1$. This equation has no solutions with a positive real part of f. Indeed, let us assume that such solutions exist. For $\mathrm{Re}\, f > 0$ integral I is small compared with the first term in the expression for T. The remaining part yields

$$\tau f = -\frac{2}{\Gamma_n}(\Gamma_n + \ln R)\,. \tag{43}$$

Because in all the range of existence of the stationary solution the right-hand side of Eq. (43) is less than zero, we come to a contradiction. Hence the dispersion equation (43) has only stable roots. General formulas for the damping rate are slightly cumbersome, but Eq. (43) is valid for values of τf such that $|\tau f| \leq 1$.

By introducing the functions

$$r = \delta r_{12}/r_{11}^{(0)}\,, \tag{44a}$$

$$z_1 = \frac{1}{1+\tau f}[r p_{22}^{(0)^2} + \delta p_{21} p_{22}^{(0)}], \tag{44b}$$

$$z_2 = r p_{11}^{(0)^2} - \delta p_{12} p_{11}^{(0)}\,, \tag{44c}$$

we can put Eqs. (41) into the form

$$\frac{\mathrm{d}r}{\mathrm{d}x} = \frac{\Gamma}{2I_T^{(0)}}(z_1 - z_2)\,, \tag{45a}$$

$$\frac{\mathrm{d}z_1}{\mathrm{d}x} = \frac{\Gamma}{2I_T^{(0)}(1+\tau f)}\{[r_{11}^{(0)^2} - p_{22}^{(0)^2}]z_1 + p_{22}^{(0)^2} z_2\}, \tag{45b}$$

$$\frac{\mathrm{d}z_2}{\mathrm{d}x} = \frac{\Gamma}{2I_T^{(0)}}\{p_{11}^{(0)^2} z_1 - [p_{11}^{(0)^2} + r_{11}^{(0)^2}]z_2\}. \tag{45c}$$

The boundary conditions for Eqs. (45) are $r_{\rm out} = r_{\rm in}$, $z_{1,\rm in} = r_{\rm in} p_{22,\rm in}^2$, and $z_{2,\rm in} = r_{\rm in} p_{11,\rm in}^2$.

At the instability threshold ($f = 0$) Eqs. (45) yield

$$r_{\rm out} - r_{\rm in} = \int_0^1 \mathrm{d}x(z_1 - z_2)\,, \tag{46a}$$

$$(z_1 - z_2) = (z_1 - z_2)_{\rm in} \exp\left(-\frac{\Gamma}{2}\,\frac{1+y-m}{1+y+m}\,x\right). \tag{46b}$$

The boundary condition for r, $r_{\rm out} = r_{\rm in}$ can be satisfied only if $(z_1 - z_2)_{\text{in}} = 0$, which in turn implies that $p_{11,\text{in}} = p_{22,\text{in}}$. For small growth rates f the solution of Eqs. (45) yields $\tau f \propto (p_{22,\text{in}}^2 - p_{11,\text{in}}^2)$. Hence the stationary solution corresponding to the oscillation of only one signal in the ring turns out to be stable if the input intensity of the corresponding pump is larger than that of the second pump and to be unstable in the opposite case. Physically this means that, even though both input signals may be above threshold for driving oscillation in the ring, only the temporal mode corresponding to the stronger of the two signals will oscillate. The weaker signal is completely suppressed.

6. FEATURE EXTRACTOR: TWO RINGS

The feature extractor shown in Fig. 2(b) is described by

$$\frac{\partial r_{ij}}{\partial x} = \sum_{k=1,2} \nu_{ik} p_{kj} \qquad (i, j = 1, 2), \quad (47a)$$

$$\frac{\partial p_{ij}}{\partial x} = -\sum_{k=1,2} \nu_{ki}^* r_{kj} \qquad (i, j = 1, 2), \quad (47b)$$

$$\left(\tau \frac{\partial}{\partial t} + 1\right) \nu_{ij} = \frac{\Gamma}{2I_T} \sum_{k=1,2} r_{ik} p_{jk}^* \qquad (i, j = 1, 2), \quad (47c)$$

where p_{ij} and r_{ij} are pumping and resonator beams, respectively, the first index denotes the spatial mode, and the second index denotes the temporal mode. The boundary conditions for Eqs. (47) are

$$p_{11,\text{in}} = p_{11,\text{in}}^{(0)}, \quad (48a)$$

$$p_{22,\text{in}} = p_{22,\text{in}}^{(0)}, \quad (48b)$$

$$p_{12,\text{in}} = p_{21,\text{in}} = 0, \quad (48c)$$

$$r_{ij,\text{out}} \sqrt{R_i} = r_{ij,\text{in}} \qquad (i, j = 1, 2), \quad (48d)$$

where $R_i (i = 1, 2)$ are the passive losses in the two rings. In the stationary state Eqs. (47) have the following integrals of motion:

$$c_1 = r_{11}^2 + r_{21}^2 + p_{11}^2 + p_{21}^2, \quad (49a)$$

$$c_2 = r_{12}^2 + r_{22}^2 + p_{12}^2 + p_{22}^2, \quad (49b)$$

$$c_3 = r_{11} r_{12} + r_{21} r_{22} + p_{11} p_{12} + p_{21} p_{22}, \quad (49c)$$

where, without loss of generality, the stationary solution has been taken to be purely real.

Consider the possible situation in which modes at only one frequency (say, 1) oscillate in both rings. This means that $r_{12} = r_{22} = 0$ and $p_{12} = p_{21} = 0$. The remaining equations in the stationary state are

$$\frac{\mathrm{d}r_{11}}{\mathrm{d}x} = \frac{\Gamma}{2I_T} r_{11} p_{11}^2, \quad (50a)$$

$$\frac{\mathrm{d}r_{21}}{\mathrm{d}x} = \frac{\Gamma}{2I_T} r_{21} p_{11}^2, \quad (50b)$$

$$\frac{\mathrm{d}p_{11}}{\mathrm{d}x} = -\frac{\Gamma}{2I_T} (r_{11}^2 + r_{21}^2) p_{11}. \quad (50c)$$

Solution of these equations yields

$$r_{11,\text{out}} = r_{11,\text{in}} \sqrt{G}, \quad (51a)$$

$$r_{21,\text{out}} = r_{21,\text{in}} \sqrt{G}, \quad (51b)$$

$$G = \frac{1}{y_{11} + y_{21} + \exp(-\Gamma_n)}, \quad (51c)$$

$$y_{ij} = (r_{ij}/p_{11})_{\text{in}}^2, \quad (51d)$$

$$\Gamma_n = \Gamma \frac{1}{1 + m}, \quad (51e)$$

$$m = (p_{22}/p_{11})_{\text{in}}^2. \quad (51f)$$

The boundary conditions for Eqs. (51) are $GR_1 = 1$ for r_{11} and $GR_2 = 1$ for r_{22}. These conditions can be satisfied simultaneously only if $R_1 = R_2$. This means that the solution corresponding to both rings oscillating at the same frequency has zero region of existence (i.e., it does not exist from the practical point of view). The only possible one-frequency solution corresponds to the oscillation at one frequency in one ring and to the absence of oscillation in the other ring. Suppose that the oscillating signal is r_{11}. Steady-state solutions for this case are given by Eqs. (51) with $y_{21} = 0$.

To investigate the stability of the stationary solution in which one frequency oscillates in only one ring we represent all fields $a = \{r_{ij}, p_{ij}\}$ in the form of a stationary solution plus real perturbations, as in Eq. (39), and linearize Eqs. (47). We find that the linearized Eqs. (47) split into four independent sets. The first two sets describe perturbations to fields that have nonzero values in the stationary state and the evolution of perturbations at the complimentary frequency in the oscillating ring. These sets of equations are identical to Eqs. (40) and (41) that were considered in the one-ring case in Section 5. The same conclusions that were derived there are equally valid here, namely, that the on ring is stable and that perturbations at the complimentary frequency in the on ring have positive growth rates only when the intensity of the pump corresponding to the complimentary frequency is larger than that of the pump corresponding to the stationary solution.

The final two sets of equations for the perturbations are

$$\frac{\mathrm{d}\delta r_{21}}{\mathrm{d}x} = \delta\nu_{21} p_{11}^{(0)}, \quad (52a)$$

$$\delta\nu_{21} = \frac{\Gamma}{2I_T^{(0)}(1 + \tau f)} \delta r_{21} p_{11}^{(0)}, \quad (52b)$$

$$\delta r_{21,\text{out}} \sqrt{R_2} = \delta r_{21,\text{in}}, \quad (52c)$$

corresponding to the evolution of perturbation at the already existing frequency in the off ring, and

$$\frac{\mathrm{d}\delta r_{22}}{\mathrm{d}x} = \delta\nu_{22} p_{22}^{(0)}, \quad (53a)$$

$$\delta\nu_{22} = \frac{\Gamma}{2I_T^{(0)}(1 + \tau f)} \delta r_{22} p_{22}^{(0)}, \quad (53b)$$

$$\delta r_{22,\text{out}} \sqrt{R_2} = \delta r_{22,\text{in}}, \quad (53c)$$

which describe the evolution of perturbations at the complimentary frequency in the off ring.

Solution of Eqs. (52), which describe the evolution of perturbations at the already existing frequency in the off ring, yields

$$\tau f = -\frac{\ln(1/R_2)}{\ln(R_1/R_2)}\,. \tag{54}$$

When the losses in the already oscillating ring are less than in the off ring ($R_1 > R_2$), the off ring is stable against perturbations at the already existing frequency.

Solution of Eqs. (53) describing the evolution of perturbations at the complimentary frequency in the off ring yields

$$\tau f = \frac{m\Gamma_n - \ln(1/R_2)}{\ln(1/R_2)}\,. \tag{55}$$

Recall that the threshold of existence of the stationary state [Eqs. (51)] is given by the relation $\Gamma_n = -\ln R_1$. If the passive losses R_1 and R_2 in the rings are of the same order, and the pumping beam ratio m is not abnormally large or small, then Eq. (55) yields $f > 0$ for values of the nonlinear coupling constant Γ_n slightly exceeding the threshold of excitation of the stationary solution. Hence the second ring is unstable with respect to the growth of perturbations at the complimentary frequency.

To recap, the analysis thus far shows that the stationary solution corresponding to oscillation at one frequency in one of the rings may be unstable with respect to several kinds of perturbation. The instabilities described in Eqs. (41) and (52) simply mean that the system chooses the stationary state such that the frequency of the stronger pump oscillates in the ring with the lowest losses. These instabilities do not change the character of this stationary state. The instability described by Eqs. (53) corresponding to the excitation of oscillations at the complimentary frequency in the second ring is more interesting. The resulting new stationary solution describes the situation in which each of the rings oscillates at a different frequency, i.e., the observed behavior of the feature extractor. Below we analyze the stability of this solution.

Let pump p_{11} excite resonator mode r_{11} oscillating in the first ring and pump p_{22} excite resonator mode r_{22} oscillating in the second ring. All other pump and resonator modes are not excited ($p_{12} = p_{21} = r_{12} = r_{21} = 0$). The stationary intensities of the pump and resonator beams are described by (there is no summation over the index j in the following formulas)

$$r_{jj,\text{out}}^2 = r_{jj,\text{in}}^2 G_j \qquad (j = 1, 2), \tag{56a}$$

$$p_{jj,\text{out}}^2 = p_{jj,\text{in}}^2 G_j \exp(-\Gamma_j) \qquad (j = 1, 2), \tag{56b}$$

$$G_j = \frac{1}{y_j + \exp(-\Gamma_j)} \qquad (j = 1, 2), \tag{56c}$$

$$\Gamma_1 = \Gamma\,\frac{1}{1+m}\,, \tag{56d}$$

$$\Gamma_2 = \Gamma\,\frac{m}{1+m}\,, \tag{56e}$$

where Γ_j is the normalized coupling constant for each ring, $y_j = (r_{jj}/p_{jj})_{\text{in}}^2$, $m = (p_{22}/p_{11})_{\text{in}}^2$, and the values of y_j ($j = 1, 2$) are found from the stationary solution to be equal to

$$y_j = R_j - \exp(-\Gamma_j)\,. \tag{57}$$

To investigate the stability of this stationary solution we again represent all fields $a = \{r_{ij}, p_{ij}\}$ in the form of Eq. (39) and linearize Eqs. (47). The linearized equations split into two independent sets. The first set is analogous to Eqs. (40) and describes perturbations to fields that have nonzero values in the stationary state ($j = 1, 2$):

$$\frac{\mathrm{d}\delta r_{jj}}{\mathrm{d}x} = \delta\nu_{jj} p_{jj}^{(0)} + \nu_{jj}^{(0)}\delta p_{jj}\,, \tag{58a}$$

$$\frac{\mathrm{d}\delta p_{jj}}{\mathrm{d}x} = -\delta\nu_{jj} r_{jj}^{(0)} - \nu_{jj}^{(0)}\delta r_{jj}\,, \tag{58b}$$

$$\delta\nu_{jj} = \frac{\Gamma}{2I_T^{(0)}(1+\tau f)} \times \left[r_{jj}^{(0)}\delta p_{jj} + p_{jj}^{(0)}\delta r_{jj} - r_{jj}^{(0)} p_{jj}^{(0)}\frac{\delta I_T}{I_T^{(0)}} \right]. \tag{58c}$$

Equations (58) describe the evolution of perturbations in the first ring ($j = 1$) and in the second ring ($j = 2$) that are coupled by means of perturbation to the total illumination of the crystal $8I_T = 2\delta r_{11,\text{in}} r_{11,\text{in}}^{(0)} + 2\delta r_{22,\text{in}} r_{22,\text{in}}^{(0)}$.

The second set of equations describes the evolution of fields that were equal to zero in the stationary state:

$$\frac{\mathrm{d}\delta r_{12}}{\mathrm{d}x} = \delta\nu_{12} p_{22}^{(0)} + \nu_{11}^{(0)}\delta p_{12}\,, \tag{59a}$$

$$\frac{\mathrm{d}\delta r_{21}}{\mathrm{d}x} = \delta\nu_{21} p_{11}^{(0)} + \nu_{22}^{(0)}\delta p_{21}\,, \tag{59b}$$

$$\frac{\mathrm{d}\delta p_{12}}{\mathrm{d}x} = -\delta\nu_{21} r_{22}^{(0)} - \nu_{11}^{(0)}\delta r_{12}\,, \tag{59c}$$

$$\frac{\mathrm{d}\delta p_{21}}{\mathrm{d}x} = -\delta\nu_{12} r_{11}^{(0)} - \nu_{22}^{(0)}\delta r_{21}\,, \tag{59d}$$

$$\delta\nu_{ij} = \frac{\Gamma}{2(1+\tau f)}\left[r_{ii}^{(0)}\delta p_{ji} + \delta r_{ij} p_{jj}^{(0)} \right]. \tag{59e}$$

Solution of Eqs. (58) yields

$$\begin{bmatrix} \delta r_{11} \\ \delta r_{22} \end{bmatrix}_{\text{out}} = \mathbf{T}\begin{bmatrix} \delta r_{11} \\ \delta r_{22} \end{bmatrix}_{\text{in}}, \tag{60}$$

where the elements of the 2×2 matrix $\mathbf{T}$ are

$$T_{jj} = \frac{\exp(\Gamma_j/2)}{[1 + y_j\exp(\Gamma_j)]G_j}\,[y_j \exp(\Gamma_j/2) + B_j] + 2\,\frac{m^{2-j}}{1+m}\,y_j B_j I_j\,, \tag{61a}$$

$$T_{ij(i\neq j)} = -2\,\frac{\sqrt{m}}{1+m}\,\sqrt{y_1 y_2}\,B_i I_i\,, \tag{61b}$$

$$B_j = \exp\left[\frac{1}{1+\tau f}\left(\ln G_j - \frac{\Gamma_j}{2} \right) \right], \tag{61c}$$

$$I_j = \frac{\Gamma_j \exp(-\Gamma_j/2)}{2(1+\tau f)} \sqrt{G_j} \int_0^1 \mathrm{d}x \times \exp\left(-\frac{\tau f}{1+\tau f}\left\{\frac{\Gamma_j}{2}x + \ln[y_j + \exp(-\Gamma_j)]\right\}\right). \tag{61d}$$

The dispersion equation takes the form

$$(\sqrt{R_1}T_{11} - 1)(\sqrt{R_2}T_{22} - 1) - \sqrt{R_1 R_2}T_{12}T_{21} = 0. \tag{62}$$

Equation (62) has only stable (Re $f < 0$) roots. The proof of this statement is similar to that carried out in the case of Eqs. (42). We start by assuming that Eq. (62) has an unstable solution (Re $f > 0$). In this case both integrals I_j in Eqs. (61) can be neglected in Eq. (62), and the rest yields

$$\tau f_j = -\frac{2}{\Gamma_j}[\Gamma_j + \ln(R_j)]. \tag{63}$$

Inasmuch as the conditions for existence of the stationary state under investigation are $\Gamma_j + \ln(R_j) > 0$, both solutions (63) are negative, which leads to a contradiction. Expressions for the damping rates should in general be obtained by keeping integrals I_j in Eq. (62) and are slightly more complex than Eq. (63).

We turn next to the analysis of Eqs. (59). These equations have marginally stable solutions $f = 0$ when the input pumping beams are equal: $p_{11,\mathrm{in}}^2 = p_{22,\mathrm{in}}^2$. The reason for this is that for the degenerate case $p_{11,\mathrm{in}}^2 = p_{22,\mathrm{in}}^2$ Eqs. (47) have a family of stationary solutions such that

$$\frac{r_{12}}{r_{11}} = -\frac{r_{21}}{r_{22}} = c, \tag{64}$$

where c is an arbitrary constant. The particular solution under consideration given by Eqs. (56) is a member of this family for the particular choice of $c = 0$. For any arbitrarily small difference between the intensities of the input pumps this degeneracy is broken, and solution of Eqs. (59) yields negative values of f, corresponding to the stability of the stationary state [Eqs. (56)]. To prove this statement, while avoiding cumbersome formulas, we analyze Eqs. (59) in the limit where the difference between the input pumping beam intensities is small, i.e., $\epsilon \equiv (p_{22,\mathrm{in}}^2 - p_{11,\mathrm{in}}^2)/I_T \ll 1$. We also assume that the rings have equal passive losses such that $R_1 = R_2 = R$. Introducing the new functions

$$y_1 = \frac{\delta r_{12}}{r_{11}^{(0)}} - \frac{\delta r_{21}}{r_{22}^{(0)}}, \tag{65a}$$

$$y_2 = \frac{\delta r_{12}}{r_{11}^{(0)}} + \frac{\delta r_{21}}{r_{22}^{(0)}}, \tag{65b}$$

and making use of the integral of Eqs. (59),

$$r_{11}\delta r_{12} + r_{22}\delta r_{21} + p_{11}\delta p_{12} + p_{22}\delta p_{21} = \text{const.}, \tag{66}$$

one obtains from Eqs. (59)

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = \frac{\Gamma}{2I_T}[(p_{22}^2 - p_{11}^2)y_2 - \tau f p^2 y_1], \tag{67a}$$

$$\frac{\mathrm{d}y_2}{\mathrm{d}x} = \frac{\Gamma}{2I_T}[y_{1,\mathrm{in}}(p_{22,\mathrm{in}}^2 - p_{11,\mathrm{in}}^2) + 2r_{\mathrm{in}}^2 y_{2,\mathrm{in}} - 2r^2 y_2], \tag{67b}$$

where r^2 and p^2 are the intensities of the resonator and the pumping beams, respectively, in the symmetric stationary state [Eqs. (56)], where $p_{11,\mathrm{in}}^2 = p_{22,\mathrm{in}}^2$. The boundary conditions for Eqs. (67) are $y_{1,\mathrm{out}} = y_{1,\mathrm{in}}$ and $y_{2,\mathrm{out}} = y_{2,\mathrm{in}}$.

For exactly equal intensities of the input pumps Eqs. (67) have marginally stable solutions $y_2 = 0$, $y_1 =$ constant, $\tau f = 0$. For unequal pumping beam intensities the solution of Eqs. (67) up to the second order in the expansion parameter ϵ yields

$$\tau f = -\frac{\Gamma^2}{4\ln(1/R)}\left(\frac{p_{22,\mathrm{in}}^2 - p_{11,\mathrm{in}}^2}{I_T}\right)^2. \tag{68}$$

Thus solution (56) turns out to be stable.

To summarize the results of the analysis of the feature extractor, when the input pumps are of unequal intensity and the passive losses in the two rings are different, the desired solution in which one frequency oscillates in one ring and the other frequency oscillates in the other ring is the only stable solution. When the two rings have equal passive losses the situation is identical to that of one ring discussed in Section 5, and only the stronger input signal oscillates in the rings. When the two input signals are equally strong there is a family of marginally stable solutions given by Eq. (64). The degeneracy is broken as soon as the input signals become unequal.

7. DISCUSSION

We have analyzed the stability of the flip-flop with separate and shared gain volumes. The analysis presented here considers steady states that are on resonance with the pump radiation. The analysis has been based on the assumptions of large passive losses and large small-signal gain and loss. These assumptions correspond to typical experimental conditions.[8] The assumption of large passive losses implies further that $\tau_L \gg \tau_G$, because the photorefractive time constant scales inversely with the optical intensity. In the case of the flip-flop with separate gain volumes the desired asymmetric on–off state is found to be stable, provided that the loss is large enough to satisfy inequality (25), and all other stationary states are unstable. In the flip-flop with a shared gain volume the asymmetric on–off state is always stable, and all other stationary states are unstable. The flip-flop with a shared gain volume is therefore to be preferred experimentally.

In the case of the feature extractor the theory qualitatively confirms the observed behavior. The only possible stable state has the incident signals oscillating in different rings, with zero cross talk. Stability is independent of the time constants because, in the model of Fig. 2(b) that was used for the analysis, there is only one time constant, that of the gain medium. The experimental demonstrations of the feature extractor[10,11] used the more complicated configuration of Fig. 2(a), where each ring is multimode and has reflexive coupling. In the experiments the feature extractor exhibited a small but finite cross-talk level of approximately 40:1. This level of cross talk may be due to the multimode nature of the rings or may be simply a result of finite input correlation between the signals.[20] It was also observed that when the input intensities were unequal by more than ~10% the stronger signal oscillated in both rings. The model analyzed here is not sufficiently general to permit us to predict how these experimental details depend on the circuit parameters. The analysis could in principle be extended to include the effect of the reflexive coupling and the multimode nature of the rings, although it appears that direct numerical simulations are a more appropriate method of investigating the ultimate performance limitations of this architecture.

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