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FINITE DIRECT SUMS OF CS-MODULES

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Let R be a ring and let $M = M_1 \oplus M_2$ be a right R-module which is the direct sum of submodules M_1, M_2. We are interested in conditions on M_1, M_2 which make M a CS-module. If M is a CS-module it is well-known that M_1 and M_2 are both CS-modules. In this paper, we prove that if M_1 and M_2 are relatively injective CS-modules then M is a CS-module. In consequence, we give simple proofs to show that M is (quasi-) continuous if and only if M_1, M_2 are relatively injective (quasi-) continuous modules.

Throughout we shall suppose that all rings have identities and all modules are unital right modules.

Let R be a ring and let M be an R-module. A submodule K of M is called a *complement (in M)* if K has no proper essential extension in M. The module M is called a *CS-module* if every complement is a direct summand. CS-modules have attracted considerable attention in recent years (see, for example, [1], [2],[4], [5]-[7], [9]-[11], [14]). Kamal and Müller [9] (see also [11, p. 19]) have classified which $\mathbb{Z}$-modules are CS. In particular, for any prime p, the $\mathbb{Z}$-modules $(\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^2)$ and $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}$ are both CS, but the $\mathbb{Z}$-modules $(\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$ and $(\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Q}$ are both not CS. In this note we shall consider when the direct sum of two CS-modules is CS.

In [9, Theorem 11], Kamal and Müller showed that for a commutative domain R, a torsion-free reduced R-module M is CS if and only $M = U_1 \oplus \ldots \oplus U_n$ is a finite direct sum of uniform modules U_i $(1 \leq i \leq n)$ such that $U_i \oplus U_j$ is a CS-module for all $1 \leq i \leq j \leq n$. This motivated our first theorem. In the proof of this theorem and elsewhere in this paper we make use of the following result of Chatters and Hajarnavis [2, Proposition 2.2].

Lemma 1. *Let M be any module and $K \subseteq L$ submodules of M such that K is a complement in L and L is a complement in M. Then K is a* *complement in M.*

Corollary 2. *Any direct summand of a CS-module is a CS-module.*

Proof. Clear by Lemma 1.

We now give a characterisation of CS-modules with finite Goldie dimension. (For the definition of Goldie dimension, see [3, p.8]).

Theorem 3. *Let R be any ring. An R-module M is a CS-module with finite Goldie dimension if and only if*

(i) *M is a finite direct sum of uniform submodules, and*
(ii) *every direct summand of M of uniform dimension 2 is a CS-module.*

Proof. Suppose M is a CS-module with finite non-zero Goldie dimension. Let U be a maximal uniform submodule of M. (Note that U is a complement in M.) By hypothesis, $M = U \oplus U'$ for some submodule U' of M. By induction on Goldie dimension and Corollary 2, U' is a finite direct sum of uniform submodules. This proves (i). Also, Corollary 2 proves (ii).

Conversely, suppose M satisfies (i), (ii). Let $M = U_1 \oplus \ldots \oplus U_n$, where n is a positive integer and U_i a uniform module for each $1 \leq i \leq n$. Let V be a maximal uniform submodule of M. Suppose $V \neq M$. Then $V \cap U_i = 0$ for some $1 \leq i \leq n$ (see [3, proof of Lemma 1.9]). Without loss of generality, $i = 1$. Let $U' = U_2 \oplus \ldots \oplus U_n$. There exists a complement K in M such that $V \oplus U_1$ is essential in K. Note that $K = U_1 \oplus (K \cap U')$, by the Modular Law.

Clearly $K \cap U'$ is a complement in K, and hence also in M (Lemma 1). Thus $K \cap U'$ is a complement in U'. By induction on Goldie dimension, $K \cap U'$ is a direct summand of U'. This implies at once that K is direct summand of M. Clearly K has Goldie dimension 2 (see [3, Lemma 1.9]), so that, by hypothesis, K is a CS-module. Hence V is a direct summand of K, and hence also of M.

Now let L be any complement in M. Let W be a maximal uniform submodule of L. By Lemma 1, W is a complement in M, and by the above argument W is a direct summand of M. Thus $M = W \oplus M'$ for some submodule M'. Now $L = W \oplus (L \cap M')$ and $L \cap M'$ is a complement in M (Lemma 1). By induction on the Goldie dimension of L, $L \cap M'$ is a direct summand of M, and hence also of M'. Thus L is a direct summand of M. It follows that M is a CS-module.

Two remarks can be made at this point. If A is a free Abelian group then the $\mathbb{Z}$-module A is a direct sum of uniform $\mathbb{Z}$-modules and every direct summand of A of Goldie dimension 2 is a CS-module. However A is a CS-module if and only if A has finite rank (see [9, Theorem 5] or [11, p.19]). Thus Theorem 3 is not true (as it stands) for modules with infinite Goldie dimension. The second remark is that Theorem 3 raises the following natural question: let $M = U_1 \oplus \ldots U_n$ be a finite direct sum of uniform modules U_i $(1 \leq i \leq n)$ such that $U_i \oplus U_j$ is a CS-module for all $1 \leq i \leq j \leq n$; is M a CS-module? (Compare [9, Theorem 11].)

Our second theorem is also motivated by a result of Kamal and Müller. Let M be a module. Let $Z(M)$ denote the singular submodule of M (i.e. $Z(M)$ is the set of elements m in M such that $mE = 0$ for some essential right ideal E of R), and let $Z_2(M)$ denote the second singular submodule of M (i.e. $Z_2(M)/Z(M) = Z(M/Z_2(M)))$. Kamal and Müller [9, Theorem 1] proved that a module M is a CS-module if and only if $M = Z_2(M) \oplus M'$ for some submodule $M', Z_2(M)$ and M' are CS-modules and $Z_2(M)$ is M'-injective. We shall prove the following result.

Theorem 4. *Let R be any ring. Let M be a R-module such that $M = M_1 \oplus M_2$, where M_1 and M_2 are CS-modules. Suppose further that M_1 is nonsingular and M_2 is M_1-injective. Then M is a CS-module.*

In order to prove Theorem 4, we require a result of Kamal and Müller [10, Lemma 17], which is $(i) \Rightarrow (ii)$ of the next lemma. The proof of both implications is given for completeness.

Lemma 5. *Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2. Then then the following statements are equivalent.*

(i) *M_2 is M_1-injective.*
(ii) *For each submodule N of M with $N \cap M_2 = 0$, there exists a submodule M' of M such that $M = M' \oplus M_2$ and $N \subseteq M'$.*

Proof. $(i) \Rightarrow (ii)$. For $i = 1, 2$, let $\pi_i : M \to M_i$ denote the projection mapping. Consider the following diagram

$$\begin{array}{ccccl} 0 & \longrightarrow & N & \xrightarrow{\alpha} & M_1 \text{ exact} \\ & & \beta\downarrow & & \\ & & M_2 & & \end{array}$$

where $\alpha = \pi_1 \mid_N$ and $\beta = \pi_2 \mid_N$. By (i), there exists a homomorphism $\phi : M_1 \to M_2$ such that $\phi\alpha = \beta$. Let $M' = \{x + \varphi(x) : x \in M_1\}$. It is easy to check that $M = M' \oplus M_2$ and $N \subseteq M'$.

(ii) $\Rightarrow$ *(i)*. Let K be a submodule of M, and $\vartheta : K \to M_2$ a homomorphism. Let $L = \{y - \vartheta(y) : y \in K\}$. Then L is a submodule of M and $L \cap M_2 = 0$. By (ii), $M = L' \oplus M_2$ for some submodule L' such that $L \leq L'$. Let $\pi : M \to M_2$ denote the canonical projection (for the direct sum $M = L' \oplus M_2$). Then $\chi = \pi \mid_{M_1}: M_1 \to M_2$ and, for any y in K,

$$\chi(y) = \pi\{y - \vartheta(y) + \vartheta(y)\} = \vartheta(y).$$

It follows that χ lifts ϑ to M_1. Thus M_2 is M_1-injective.

Equipped with this lemma, we now prove Theorem 4.

Proof of Theorem 4. Because M_2 is a CS-module, [8, Theorem 1] gives $M_2 = Z_2(M_2) \oplus M'$ for some nonsingular submodule M' of M_2 such that $Z_2(M_2)$ and M' are CS-modules and $Z_2(M_2)$ is M'-injective. Note that $Z_2(M) = Z_2(M_2)$ (because $Z(M_1) = 0$) and $Z_2(M)$ is M_1-injective. Thus $M = Z_2(M) \oplus (M_1 \oplus M')$, where $Z_2(M)$ is a CS-module, $Z_2(M)$ is $(M_1 \oplus M')$-injective, M_1 and M' are CS-modules and M' is M_1-injective. By [9, Theorem 1], M is a CS-module if $M_1 \oplus M'$ is a CS-module. Thus we can suppose without loss of generality that M_2 is nonsingular, and hence M is nonsingular.

Let K be a complement in M. Because M_2 is a CS-module, there exist submodules L_1, L_2 of M_2 such that $M_2 = L_1 \oplus L_2$ and $K \cap M_2$ is essential in L_1. We claim that K is essential in $K + L_1$. For, let $0 \neq x \in K + L_1$. Then $x = y + z$ for some $y \in K, z \in L_1$. Because $K \cap M_2$ is essential in L_1, there exists an essential right ideal E of R such that $zE \subseteq K$. Then M nonsingular gives

$$0 \neq xE = (y + z)E \subseteq xR \cap K.$$

It follows that K is essential in $K + L$, and hence $L_1 \subseteq K$.

Now $M = M_1 \oplus M_2 = M_1 \oplus L_1 \oplus L_2$ and, by the Modular Law, $K = L_1 \oplus K'$ where $K' = K \cap (M_1 \oplus L_2)$. By Lemma 1, K' is a complement in $M_1 \oplus L_2$. Note further that

$$K' \cap L_2 \subseteq K \cap M_2 \cap L_2 \subseteq L_1 \cap L_2 = 0.$$

By Lemma 5, $M_1 \oplus L_2 = M'' \oplus L_2$ for some submodule M'' with $K' \subseteq M''$. Clearly $M'' \cong M_1$, so that M'' is a CS-module, and K' is a complement in M''. Thus K' is a direct summand of M'', and $K = L_1 \oplus K'$ is a direct summand of M. It follows that M is a CS-module.

Theorem 4 has also been proved by Tercan [16, Corollary 2.3.10]. Following Utumi [17] and Jeremy [8], Mohamed and Müller [11, p.18] consider the following condition for a module M :

> $(C_3) K \oplus L$ is a direct summand of M whenever K and L are direct summands of M with $K \cap L = 0$.

Then they define a module M to be *quasi-continuous* if M is a CS-module satisfying (C_3). (In [8], [10], the CS condition is denoted by (C_1).) Compare the next result with Lemma 5.

Lemma 6. *The following statements are equivalent for a module M.*

(i) *M satisfies (C_3).*
(ii) *For all direct summands P, Q of M with $P \cap Q = 0$, there exists a submodule P' of M such that $M = P \oplus P'$ and $Q \subseteq P'$.*

Proof. $(i) \Rightarrow (ii)$. Let P and Q be a direct summands of M with $P \cap Q = 0$. Then $P \oplus Q$ is a direct summand of M, by (i), and hence $M = P \oplus Q \oplus Q''$ for some submodule Q'' of M. Thus $P' = Q \oplus Q''$ has the requisite properties.

$(ii) \Rightarrow (i)$. Let K, L be direct summands of M such that $K \cap L = 0$. By (ii), $M = K \oplus K'$ for some submodule K' such that $L \subseteq K'$. But $M = L \oplus L'$ for some submodule L', and hence $K' = L \oplus (K' \cap L')$. Thus $M = K \oplus L \oplus (K' \cap L')$, and $K \oplus L$ is a direct summand of M.

Proposition 7. *A CS-module M is quasi-continuous if and only if whenever $M = M_1 \oplus M_2$ is a direct sum of submodules M_1, M_2, then M_2 is M_1-injective.*

Proof. Suppose that M is quasi-continuous. Suppose $M = M_1 \oplus M_2$. Let N be a submodule of M with $N \cap M_2 = 0$. Because M is a CS-module, there exists a direct summand N' of M such that N is essential N'. Clearly $N' \cap M_2 = 0$. By Lemma 6, $M = M' \oplus M_2$ for some submodule M' such that $N' \subseteq M'$. Note that $N \subseteq M'$. By Lemma 5, M_2 is M_1-injective.

Conversely, suppose M_2 is M_1-injective whenever $M = M_1 \oplus M_2$. By Lemmas 5 and 6, M satisfies (C_3). Thus M is quasi-continuous.

The necessity part of Proposition 7 is a well-known result of Jermey [8, Theorem 4.2] (see also [11, Proposition 2.10]). Let n be a positive integer. Modules $M_1, \dots, M_n$ are called *relatively injective* if M_i is M_j-injective for all $1 \le i \ne j \le n$. Our main result is the following theorem.

Theorem 8. *Let R be a ring and M an R-module such that $M = M_1 \oplus \dots \oplus M_n$ is a finite direct sum of relatively injective modules M_i $(1 \le i \le n)$. Then M is a CS-module if and only if M_i is a CS-module for each $1 \le i \le n$.*

Proof. Corollary 2 gives the necessity immediately. Conversely suppose that M_i is a CS-module $(1 \le i \le n)$. We prove that M is a CS-module by induction on n. It is clearly sufficient to prove the case $n = 2$. Suppose $M = M_1 \oplus M_2$. Let K be a complement in M. By Zorn's Lemma there exists a submodule L of K maximal with respect to the property $L \cap M_1 = L \cap (K \cap M_1) = 0$. Clearly L is a complement in K, and hence also in M (Lemma 1). Because M_1 is M_2-injective, there exists a submodule M' of M such that $M = M_1 \oplus M'$ and $L \subseteq M'$ (Lemma 5). Note that $M' \cong M_2$, so that without loss of generality $M' = M_2$, and hence $L \subseteq M_2$. Now L is a complement in M_2 (which is a CS-module), so that $M_2 = L \oplus L'$ for some submodule L'.

Note that $M = M_1 \oplus M_2 = M_1 \oplus L \oplus L'$ and $K = L \oplus K'$, where $K' = K \cap (M_1 \oplus L')$ is a complement in $M_1 \oplus L'$ (Lemma 1). We now claim that $K' \cap M_1$ is essential in K'. It is well-known (and easy to check) that $L \oplus (K \cap M_1)$ is essential in K. Hence $[L \oplus (K \cap M_1)] \cap K'$ is essential in $K' \subseteq K$. But clearly $K' \cap M_1 = K \cap M_1$, and hence

$$[L \oplus (K \cap M_1)] \cap K' = [L \oplus (K' \cap M_1)] \cap K' = (L \cap K') \oplus (K' \cap M_1) = K' \cap M_1.$$

Thus $K' \cap M_1$ is essential in K'. But clearly

$$(K' \cap M_1) \cap (K' \cap L') \subseteq M_1 \cap L' = 0,$$

so that $K' \cap L' = 0$. By hypothesis, L' is M_1-injective and hence, by Lemma 5, $M_1 \oplus L' = M'' \oplus L'$ for some submodule M'' with $K' \subseteq M''$. Clearly $M'' \cong M_1$ (which is a CS-module) and K' is a complement in M''. Thus K' is a direct summand of $M_1 \oplus L'$, and K is a direct summand of M. It follows that M is a CS-module.

In view of Lemmas 5 and 6 it is interesting to note that a module $M = M_1 \oplus M_2$ need not be a CS-module even if M_1 and M_2 are CS-modules,

M_2 is M_1-injective and M satisfies (C_3). For example, for any prime p, the $\mathbb{Z}$-module $M = (\mathbb{Z}/\mathbb{Z}p)\oplus\mathbb{Q}$ has $M_1 = (\mathbb{Z}/\mathbb{Z}p)\oplus 0$, $M_2 = 0\oplus\mathbb{Q}$ both uniform (thus CS), M_2 is M_1-injective and M satisfies (C_3) because its only direct summands are $0, M, M_1$ and M_2. However, M is not a CS-module because $\mathbb{Z}_p(1+\mathbb{Z}p, 1)$ is a complement which is not a direct summand, where $\mathbb{Z}_p$ is the local ring (see [15, Example 10]).

Note that [9, Theorem 1] is related to Theorem 8. Let M be a module such that $M = Z_2(M) \oplus M'$ where $Z_2(M)$ and M' are CS-modules and $Z_2(M)$ is M'-injective. Immediately, M' is $Z_2(M)$-injective, so that M is a CS-module by Theorem 8.

Müller and Rizvi [13, Theorem 12] (see also [11, Theorem 2.13 and Corollary 2.14]) have given necessary and sufficient conditions for a (not necessarily finite) direct sum of modules to be quasi-continuous. We now give an alternative proof for the finite direct sum case. First we prove:

Lemma 9. *Let module $M = M_1 \oplus M_2$ be a direct sum of relatively injective submodules M_1, M_2 such that M_2 is quasi-continuous. Let K, L be a direct summands of M such that $K \cap L = 0$. Suppose further that $K \cap M_1 = 0$. Then $K \oplus L$ is a direct summand of M.*

Proof. By Lemma 5, we can suppose without loss of generality that $K \subseteq M_2$. Then $M_2 = K \oplus K'$ for some submodule K' of M_2. Note that K is K'-injective (Proposition 7). Therefore K is $(M_1 \oplus K')$-injective. Now $M = K \oplus (M_1 \oplus K')$ and $L \cap K = 0$ so that, again using Lemma 5, $M = K \oplus K''$ for some submodule K'' with $L \subseteq K''$. Now L is a direct summand of M, hence also of K''. Thus $K \oplus L$ is a direct summand of M.

Theorem 10. *(See [11, Corollary 2.14]). Let R be a ring and M an R-module such that $M = M_1 \oplus \ldots \oplus M_n$ is a finite direct sum of submodules M_i $(1 \leq i \leq n)$. Then M is quasi-continuous if and only if $M_1, \ldots, M_n$ are relatively injective quasi-continuous modules.*

Proof. For the necessity see Proposition 7 and [11, Proposition 2.7]. Conversely, suppose the M_i $(1 \leq i \leq n)$ are relatively injective and quasi-continuous. By induction on n, it is sufficient to prove the case $n = 2$. Thus suppose $M = M_1 \oplus M_2$. By Theorem 8, M is a CS-module. Let K, L be direct summands of M with $K \cap L = 0$. Then K is a CS-module, by Lemma 1, and hence $K = K_1 \oplus K_2$ for some submodules K_1, K_2, with $K \cap M_1$ essential in K_1.

Consider K_2. Note that $K_2 \cap M_1 = K_2 \cap (K \cap M_1) = 0$. By Lemma 9, $K_2 \oplus L$ is a direct summand of M. On the other hand, $(K_1 \cap M_2) \cap (K \cap M_1) = 0$ implies that $K_1 \cap M_2 = 0$. Again using Lemma 9, $K \oplus L = K_1 \oplus (K_2 \oplus L)$ is a direct summand of M. It follows that M is quasi-continuous.

Corollary 11. *A finite direct sum $M_1 \oplus \ldots \oplus M_n$ is quasi-continuous if and only if $M_i \oplus M_j$ is quasi-continuous for all $1 \leq i < j \leq n$.*

Proof. By Proposition 7, Theorem 10 and [11, Proposition 2.7].

A module M is called *continuous* if M is a CS-module such that for every direct summand K of M and every monomorphism $\varphi : K \to M$, the submodule $\varphi(K)$ is also a direct summand of M. Note that continuous modules are quasi-continuous [11, Proposition 2.2]. We can now give an elementary proof of [12, Theorem 2] (see also [13, Theorem 13] and [11, Theorem 3.16]).

Theorem 12. *Let R be a ring and M an R-module such that $M = M_1 \oplus \ldots \oplus M_n$ is a finite direct sum of submodules M_i $(1 \leq i \leq n)$. Then M is continuous if and ony if $M_1, \ldots, M_n$ are relatively injective continuous modules.*

Proof. The necessity follows by Proposition 7 and [11, Proposition 2.7].

Conversely suppose that $M = M_1 \oplus \ldots \oplus M_n$ where the M_i $(1 \leq i \leq n)$ are relatively injective continuous modules. By induction on n, it is sufficient to prove the result for the case $n = 2$. Suppose that $n = 2$. By Theorem 8, $M = M_1 \oplus M_2$ is a CS-module. Let K be a direct summand of M and let $\varphi : K \to M$ be a monomorphism.

Case 1. $K \subseteq M_1, \varphi(K) \subseteq M_2$.
Let $N = \varphi(K) \subseteq M_2$. Because M_2 is a CS-module there exists a direct summand N' of M_2 such that N is essential in N'. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{\;i\;} & N' \text{ exact} \\ & & {\scriptstyle \varphi^{-1}}\big\downarrow & & \\ & & K & & \end{array}$$

where i is the inclusion mapping. Because K is M_2-injective, there exists a homomorphism $\vartheta : N' \to K$ such that $\vartheta\,|_N = \varphi^{-1}$. It is easy to check that, because φ^{-1} is an isomophism, $N' = N \oplus \ker\vartheta$. Hence $N = N'$, a direct summand of M.

Case 2. $K \subseteq M_1, \varphi(K) \cap M_1 = 0$.
By Lemma 5, there exists a submodule M' of M such that $M = M_1 \oplus M'$ and $\varphi(K) \subseteq M'$. Clearly $M' \cong M_2$. Hence $\varphi(K)$ is a direct summand of M, by Case 1.

Case 3. $K \subseteq M_1$.
Let $L = \{x \in K : \varphi(x) \in M_2\}$. Because K is a CS-module (Corollary 2), there exists a direct summand L' of K such that L is essential in L'. Note that $\varphi(L)$ is essential in $\varphi(L')$ and hence $\varphi(L') \cap M_1 = 0$. Now $K = L' \oplus L''$, for some submodule L'' of K. Clearly $\varphi(K) = \varphi(L') \oplus \varphi(L'')$. Because $\varphi(L') \cap M_1 = 0$, Case 2 gives that $\varphi(L')$ is a direct summand of M. On the other hand, $\varphi(L'') \cap M_2 = 0$. Now M is a CS-module, and hence $\varphi(L'')$ is essential in a direct summand P of M. Note that $P \cap M_2 = 0$. Let $\pi_1 : M \to M_1$ denote the canonical projection. Then $\pi_1 |_P$ is a monomorphism. Hence $\pi_1\varphi(L'')$ is essential in $\pi_1(P)$. But M_1 is a continuous module, so that $\pi_1\varphi(L'')$ is a direct summand of $\pi_1(P)$. Thus $\pi_1\varphi(L'') = \pi_1(P)$, and hence $\varphi(L'') = P$, a direct summand of M. Thus $\varphi(K) = \varphi(L') \oplus \varphi(L'')$, where both $\varphi(L')$ and $\varphi(L'')$ are direct summands. By Theorem 10, $\varphi(K)$ is a direct summand of M.

Case 4. General case.
Let K be any direct summand of M. Recall that K is a CS-module by Corollary 2. There exist submodules K_1, K_2 of K such that $K \cap M_1$ is essential in K_1 and $K = K_1 \oplus K_2$. Note that $K_1 \cap M_2 = 0$ and $K_2 \cap M_1 = 0$. By Lemma 5 and Case 3, both $\varphi(K_1)$ and $\varphi(K_2)$ are direct summands of M. But $\varphi(K) = \varphi(K_1) \oplus \varphi(K_2)$. Hence, by Theorem 10, $\varphi(K)$ is a direct summand of M. It follows that M is continuous.

There is obvious analogue of Corollary 11 for continuous modules.

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