On the Achievable Rates of Pairwise Multiway Relay Channels

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Abstract—In this paper, we study the effect of users' transmission ordering on the common rate and sum rate of pairwise multiway relay channels (MWRCs) with functional-decode-forward strategy. To this end, we first develop a graphical model for the data transmission in a pairwise MWRC. Using this model, we then find the optimal orderings that achieve the maximum common rate and sum rate of the system. The achieved maximum common/sum rate is also found. Moreover, we show that the performance gap between optimal orderings and a random ordering vanishes when SNR increases. Computer simulations are presented for better illustration of the results.

I. INTRODUCTION

A multiway relay channel (MWRC) [1] is an extension of a two-way relay channel [2]–[6] in which $N \geq 2$ users communicate with each other by means of a relay. There is often no direct link between users and they merely communicate with the relay. Conference calls, file sharing, and multi-player gaming [7] are potential applications of MWRCs.

Depending on the relay's strategy for forming its down-link message, several relaying schemes have been considered for MWRCs, namely *amplify-and-forward* (AF), *decode-and-forward* (DF), *compress-and-forward* (CF) and *functional-decode-forward* (FDF) [1], [8]. Among these schemes, FDF is the most recent where instead of decoding users' messages separately, the relay directly decodes a function (commonly the sum) of the users' messages.

FDF is commonly employed along with a pairwise transmission scheme [8] where similar to two-way relaying, a pair of users transmit their data simultaneously to the relay in each uplink phase. This is then followed by a downlink phase in which the relay broadcasts a function of the received information in the uplink phase to all users. Pairwise transmissions continue until all users are capable of decoding the data of others. Pairwise relaying not only does have a lower decoding complexity than full decoding, but also possesses interesting capacity-achieving properties in different setups [8]–[11].

In a pairwise MWRC, the way that users are paired for transmission is referred to as *user's ordering*. As argued in [7], for an asymmetric MWRC, this ordering directly affects the achievable data rates of the users. To this end, the authors find the optimal ordering to maximize the achievable common rate of the users for an MWRC with asymmetric Gaussian channels under the assumption that each user transmits in at most two uplink phases. For relaying strategy, they considered pairwise

FDF and DF relaying and show that the optimal ordering for each strategy is different than the other.

In this work, we go one step further than the work in [7] and address the effect of ordering for a more general pairwise MWRC scenario. More precisely, we consider a pairwise FDF scenario where there is no restriction on the number of uplink transmissions by the users. In this case, we first discuss that there exist N^{N-2} distinct orderings which makes finding the optimal ordering through brute-force search expensive for large N. Then, under a reasonable assumption on user's SNR, we analytically find the optimal orderings for the common rate and the sum rate. Using the optimal ordering, we find the maximum achievable common and sum rates. Further, we study the asymptotic behavior of the sum rate for high SNR. This reveals that a randomly chosen ordering performs well for high SNR regimes while the significance of our proposed optimal orderings is more pronounced in low SNRs.

The rest of the paper is organized as follows. In Section II, we describe the system model and introduce a novel graphical interpretation for data transmission in pairwise MWRCs. The sum rate and common rate maximization problems for FDF MWRC are described in Section III. The solution to these problems along with the asymptotic study of the sum rate is presented in Section IV. We compare the performance of our proposed orderings with those of randomly chosen orderings via simulations in Section V. Finally, Section VI concludes the paper.

II. PRELIMINIARIES

A. System Model

We consider an MWRC with N users, denoted by $U_1, U_2, \ldots, U_N$, where each user U_i wants to share its message X_i with other users. Users cannot directly communicate with each other, thus, relay $\mathcal{R}$ is used to assist them. The channel from U_i to $\mathcal{R}$ is a half-duplex reciprocal channel denoted by $C_{i\mathcal{R}}$ with gain $g_{i\mathcal{R}}$. Also, transmitted signals are contaminated by a Gaussian noise with variance σ^2.

In a pairwise scheme, the users are divided into M pairs which are not necessarily disjoint. A division of the users to subsets of pairs is called an *ordering* of the users and is denoted by $O = \{\{u_{11}, u_{12}\}, \ldots, \{u_{M1}, u_{M2}\}\}$ where $u_{i1}, u_{i2} \in \{U_1, U_2, \ldots, U_N\}$. The users exchange their data in one communication round consisting of M uplink and M downlink phases. During each uplink phase, users in one of
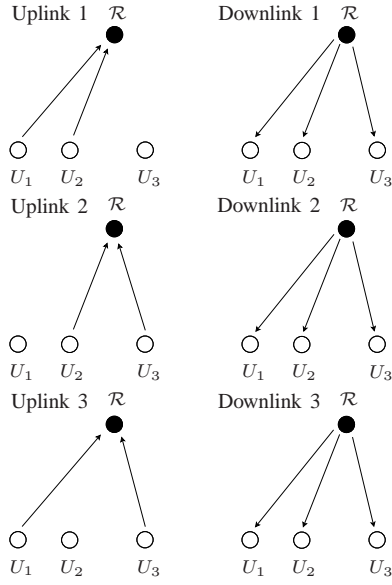
Fig. 1. A pairwise ordering with $M = N = 3$

the pairs transmit their data to the relay. After receiving the users' signal, relay directly decodes the sum of their messages [12] and broadcasts the sum to all users in a downlink phase. This means that if X_i and X_j are vectors with elements chosen from a field $\mathbb{F}$, then the relay directly decodes $X_i \oplus X_i$ where $\oplus$ means element-wise summation of X_i and X_j over $\mathbb{F}$. We consider AWGN channels such that $\oplus$ means element-wise summation over real numbers. These pairwise transmissions continue until the last pair of the ordering. Having its own data, each user is able to decode the data of others at the end of each round. The transmit power of U_i during an uplink phase is assumed to be P_i. That said, a signal to noise ratio for user U_i, namely x_i, is defined as $x_i \triangleq \frac{P_i |g_{i\mathcal{R}}|^2}{\sigma^2}$. Without loss of generality, we assume that $x_N \geq x_{N-1} \geq \cdots \geq x_1 > 0$.

Fig. 1 illustrates a pairwise MWRC when $N = 3$. After a round of communication, each user has the following set of equations:

$$
\begin{aligned}
X_1 \oplus X_2 &= C_1 \\
X_2 \oplus X3 &= C_2 \\
X_3 \oplus X_1 &= C_3
\end{aligned}
\tag{1}
$$

where C_1, C_2 and C_3 are the signals transmitted by the relay. One can see that the system of equations at each user is solvable using the knowledge of its own data. In a general N-user MWRC, if the system of equations at each user is solvable, we say that the corresponding ordering is *feasible*. This feasibility implies that M should not be less than $N - 1$ because each user needs to find $N - 1$ other users' messages.

In a pairwise MWRC with M pairs, a rate tuple $(R_1, R_2, \ldots, R_N)$ is achievable if U_i can reliably (with arbitrarily small probability of error) transmit its data to all other users with rate R_i after each round's M uplink and downlink phases. The achievable rate tuple depends on the transmit power of the users and the relay as well as the channel

gains and the noise power. Here, we assume that the data rates are limited by the uplink phase, not by the downlink phase. This commonly holds for most wireless systems where users are low-power mobile devices.

When U_i participates in a pairwise transmission, say with U_j, during an uplink phase, R_i is limited by the following achievable bound [7], [12]

$$
R_i \leq \max \left\{ 0, \frac{1}{2M} \log_2 \left(\frac{x_i}{x_i + x_j} + x_i \right) \right\}. \tag{2}
$$

and to the best of our knowledge, this is the tightest achievable bound for R_i with FDF relaying. The maximum achievable upper bound on R_i can be found by calculating upper bounds, given by (2), for R_i over all pairs that U_i is part of and then taking the minimum of these bounds. In this paper, instead of focusing on the individuals' rates, we study the system common rate and sum rate. For an achievable rate tuple $(R_1, \ldots, R_N)$, the user's common rate, C_R, and the sum rate, S_R, are defined as $C_R \triangleq \min_i R_i$ and $S_R \triangleq \sum_{i=1}^N R_i$. As seen from (2), the upper bounds on R_i's, and consequently the systems common rate and sum rate depend on the ordering of the users. Our goal in this work is to find the orderings that attains the maximum possible common rate and sum rate in the system. This is discussed in more detail later.

B. Graphical Representation

Here, we introduce the concept of *client graph* that provides a convenient representation of the users' transmission ordering. This model is later used to find the optimal ordering to maximize C_R and S_R.

A client graph $G_O = (V, E)$ for a given pairwise ordering O consists of a set of vertices $V = \{v_1, v_2, \ldots, v_N\}$ and a set of edges E. Vertex v_i is associated with U_i and $v_i v_j \in E$ iff $\{U_i, U_j\} \in O$. If $v_i v_j \in E$, we say v_j is adjacent to v_i. The set of adjacent vertices of v_i, denoted by A_i^G, is called the set of neighbors of v_i. Also the degree of node v_i is $deg(v_i) = |A_i^G|$. The adjacency matrix of $G_O(V, E)$, denoted by $\mathcal{A} = (a_{ij})$, is an $N \times N$ matrix in which $a_{ij} = 1$ iff $v_i v_j \in E$, and a_{ij} is 0 otherwise. Note that there is a one-to-one mapping between all possible client graphs and all possible orderings.

The overall energy consumed in a communication round is directly proportional to the number of pairs. As a result, we are interested in identifying feasible orderings with minimum number of pairs which, as we mentioned, is $M = N - 1$. To this end, we state the following theorem.

Theorem 1. *An ordering with $M = N - 1$ pairs is feasible iff the corresponding client graph is a tree.*

Proof: Fist, we show that if there is a cycle in the client graph G_O, the feasibility of the system will not change if we remove one of the edges from that cycle. Assume that $\mathcal{C} = \{v_{i_1} v_{i_2}, v_{i_2} v_{i_3}, \ldots, v_{i_n} v_{i_1}\}$ is a cycle in G_O. The equations

corresponding to the edges in this cycle are:

$$
\begin{aligned}
X_{i_1} \oplus X_{i_2} &= C_1 \\
X_{i_2} \oplus X_{i_3} &= C_2 \\
&\vdots \\
X_{i_n} \oplus X_{i_1} &= C_n.
\end{aligned} \tag{3}
$$

The jth equation in the system of equations (3) is not independent of the others. In other words, if we sum over all of the equations but the jth one, we wind up with:

$$
X_{i_j} \oplus X_{i_{j+1}} = \bigoplus_{i \neq j} C_i. \tag{4}
$$

This shows that removing $v_{i_j} v_{i_{j+1}}$ from the cycle $\mathcal{C}$, has no effect on the feasibility of the system of equations (3).

Then, we just need to prove the theorem for client graphs with no cycle. In order for system of equations to be feasible, each user needs to have at least $N - 1$ equations, except its own data. It means that G_O has at least $N - 1$ edges. Since G_O has no cycle, it should be a tree. ∎

In the rest of this paper, we assume $M = N - 1$ and use the terms client tree and client graph, interchangeably.

III. PROBLEM DEFINITION

In this section, we define rate maximization problems. Here, we denote the maximum achievable common rate and sum rate for a client graph G_O by $C_R(G_O)$ and $S_R(G_O)$, respectively.

By *common rate maximization* problem, we mean finding the feasible ordering that maximizes $C_R(G_O)$. More formally, if we denote the set of all feasible orderings by $\mathcal{O}$, then the common rate maximization problem translates into

$$
O_{CR} = \arg\max_{O \in \mathcal{O}} C_R(G_O) \tag{5}
$$

Similarly, a *sum rate maximization* is defined as follows

$$
O_{SR} = \arg\max_{O \in \mathcal{O}} S_R(G_O) \tag{6}
$$

One way to solve the aforementioned problems is to search over all possible client trees and find the one that maximizes the common rate and sum rate. This, according to Cayley's formula [13], necessitates searching over all N^{N-2} feasible client trees which is impractical even if the number of users is not very large. This motivates us to find efficient solutions for identifying the optimal client trees.

In order to find an ordering with maximum sum rate, we consider the case where the user's SNR is not too low which is the case for most practical settings. To this end, the upper bound on the rate of U_i when it transmits with U_j is given by

$$
R_i \leq \frac{1}{2(N-1)} \log_2 \left(\frac{x_i}{x_i + x_j} + x_i \right). \tag{7}
$$

One can easily verify that if $x_1 + \frac{x_1}{x_1 + x_N} \geq 1$, (7) and (2) are equivalent. For instance, if all SNRs of the users are more than 1, the bound in (7) is equivalent to (2). For common rate maximization, we also assume that the user's SNR is not too low and consider (7). We are not interested in cases that common rate is equal to zero.
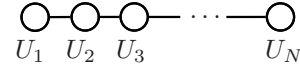

Fig. 2. Client tree that maximizes $C_R(G_O)$ for a pairwise MWRC with FDF relaying

IV. PROBLEM SOLUTION

In this section, we provide solutions to common rate and sum rate maximization problems for FDF relaying. We also show that in high SNR regimes, the performance of a randomly chosen ordering asymptotically approaches the rate performance of the optimal ordering.

A. Common Rate Maximization

Theorem 2. $C_R(G_O)$ *is maximized when the ordering is*

$$
O_{CR} = \{\{U_1, U_2\}, \{U_2, U_3\}, \{U_3, U_4\}, \dots, \{U_{N-1}, U_N\}\}.
$$

Also, the maximum achievable common rate is

$$
C_R(G_O) = \min_{i \in \{1, \dots, N\}} \left\{ \frac{1}{2(N-1)} \log_2 \left(x_i + \frac{x_i}{x_i + x_{i+1}} \right) \right\}.
$$

Proof: Here, by an optimal tree, we mean a client tree that achieves the maximum C_R with respect to (2). There are two statements regarding (2) which we use to prove the theorem:

1) The function $f(x) = x\left(1 + \frac{1}{x+\alpha}\right)$ is an increasing function of x.
2) The function $g(x) = \left(1 + \frac{1}{\alpha+x}\right)$ is a decreasing function of x.

Given a client tree, $G_O(V, E)$, with an FDF MWRC, we have

$$
C_R(G_O) = \min_{i,j} \left\{ \frac{1}{2(N-1)} \log_2 \left(x_i + \frac{x_i}{x_i + x_j} \right) \right\}. \tag{8}
$$

where $x_i \leq x_j$ and $v_i v_j \in E$. Using (8), we prove the following lemma.

Lemma 1. *There exists an optimal tree ,$G_O(V, E)$, in which* $A_1^{G_O} = \{v_2\}$.

Proof: We adapt $G_{O'}(V, E')$ from G_O such that we disconnect all of the neighbors of v_1 from v_1 and connect them to v_2. We also make v_1 and v_2 neighbors. More precisely,

$$
E' = (E - \{v_1 v_i | v_i \in A_1^{G_O}\}) \cup \{v_2 v_i | v_i \in A_1^{G_O}; i \neq 2\} \cup \{v_1 v_2\} \tag{9}
$$

Because of monotonicity of $f(x)$ and $g(x)$, to verify that $C_R(G_O) \leq C_R(G_{O'})$ we just need to show

$$
x_1 \left(1 + \frac{1}{x_1 + x_{min}}\right) \leq x_2 \left(1 + \frac{1}{x_2 + x1}\right) \tag{10}
$$

where, $x_{min} = \min\{x_i | v_i \in A_1^{G_O}\}$. After some manipulation, we find that (10) is equivalent to

$$
0 \leq (x_2 - x_1)(x_1 + x_{min})(x_2 + x_1) + x_2 x_{min} - x_1^2 \tag{11}
$$

which, according to the fact that $x_1 \leq x_{min}$, is true. ∎

We prove the theorem by induction. If $N = 2$ the theorem obviously holds. Now, assume that the statement of the theorem holds for every FDF MWRC with $N = k$. We show
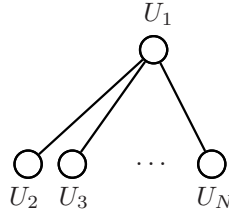
Fig. 3. Client tree that maximizes $S_R(G_O)$ for a pairwise MWRC with FDF relaying subject to the weakened upper bound given by (7)

Fig. 4. Operation of V-transform, $V(G, v_i, v_j, v_K)$

that it also holds for any FDF MWRC with $N = k + 1$. For $N = k + 1$, according to Lemma (1), there exists an optimal tree $G_O(V, E)$ in which $A_1^{G_O} = \{v_2\}$. From equation (8), we also have:

$$C_R(G_O) = \min_{i,j}\left\{ \frac{1}{2(N-1)}\log_2\left(x_i + \frac{x_i}{x_i + x_j}\right)\Big| 1 < i \leq j; v_i v_j \in E\right\}$$
$$\cup \left\{\frac{1}{2(N-1)}\log_2\left(x_1 + \frac{x_1}{x_1 + x_2}\right)\right\} \quad (12)$$

If the second term in (12) is the limiting term in all of the possible client trees with $A_1^{G_O} = \{v_2\}$, the proposed ordering is optimal. Otherwise, maximizing $C_R(G_O)$ is equivalent to maximizing $\min\left\{x_i\left(1 + \frac{1}{x_i + x_j}\right)\Big| 1 < i \leq j; v_i v_j \in E\right\}$. It is equivalent to maximizing the C_R for $G_{O'}(V', E')$, in which $V' = V - \{v_1\}$ and $E' = E - \{v_1 v_m | v_m \in A_1^{G_O}\}$. According to the induction hypothesis, it happens when $O' = \{\{v_2 v_3\}, \{v_3 v_4\}, \ldots, \{v_{N-1} v_N\}\}$ and as a reslut

$$O = \{\{v_1 v_2\}, \{v_2 v_3\}, \ldots, \{v_{N-1} v_N\}\} \quad (13)$$

∎

Fig. 2 illustrates the optimal ordering for an FDF MWRC that achieves the maximum C_R.

B. Sum Rate Maximization

Theorem 3. $O = \{\{U_2, U_1\}, \{U_3, U_1\}, \ldots, \{U_N, U_1\}\}$ is the optimal ordering maximizing the sum rate subject to (7). Moreover, the maximum sum rate for this ordering is

$$S_R(G_O) = \frac{1}{2(N-1)}\log_2\Bigg(\max\left\{1, \left(x_1 + \frac{x_1}{x_1 + x_N}\right)\right\}$$
$$\times \prod_{i=2}^{N}\max\left\{1, \frac{x_i}{x_i + x_1} + x_i\right\}\Bigg). \quad (14)$$

To prove the theorem, we first show that there is an optimal tree with $deg(v_N) = 1$ (Lemma (2)). Then we prove that in the optimal tree each node needs to have only one neighbor among nodes with a lower SNR (Lemma (3)). We then show that there exist an optimal tree with $deg(v_N) = deg(v_{N-1}) = 1$ (Lemma (4)). In the next step, we prove that in an optimal tree for two nodes of degree one, say v_i and v_j, if v_i has a higher SNR than v_j then the neighbor of v_i has a higher SNR than the neighbor of v_j (Lemma (5)). Then we prove the theorem by induction (Lemma (6))

Proof: We use the following convention for the rest of this proof:
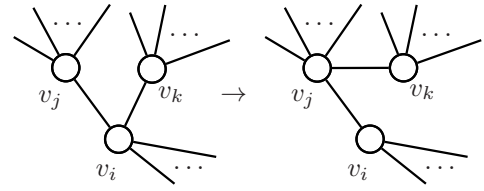
$$d_i \triangleq 2^{2(N-1)R_i}. \quad (15)$$

As a result, the bound given by (7) is equivalent to

$$d_i \leq x_i\left(1 + \frac{1}{x_i + x_j}\right). \quad (16)$$

We also define $D_s(G_O) = \max \prod_{i=1}^{N} d_i = 2^{2(N-1)S_R(G_O)}$. Assume that $G(V, E)$ is a tree such that $\{v_i, v_j, v_k\} \subseteq V$ and $\{v_i v_j, v_i v_k\} \subseteq E$. We define a V-transform on G in such a way that $V(G, v_i, v_j, v_k) = G'(V, E')$ and $E' = (E - \{v_i v_k\}) \cup \{v_j v_k\}$. Fig. 4 shows the operation of a V-transform.

Lemma 2. *There exists an optimal tree in which $deg(v_N) = 1$.*

Proof: Assume G_O is an optimal tree in which $deg(v_N) > 1$ and v_i and v_j are two neighbors of v_N and x_j is the minimum SNR value of the neighbors of V_n. Consequently, we have $x_i \geq x_j$. It is straightforward to show that by performing a V-transform on G_O and transform it to $G_{O'} = V(G_O, v_N, v_i, v_j)$, we have $\frac{D_s(G_{O'})}{D_s(G_O)} \geq 1$. It means that the sum rate of $G_{O'}$ is not less than sum rate of G_O. Note that, after applying this V-transform, we have reduced degree of v_N by one. After applying $deg(v_N) - 2$ more V-transforms, we end up with an optimal tree with $deg(v_N) = 1$. Fig. 5 illustrates an hypothetical optimal tree with $deg(v_N) = 4$. It shows how we apply 3 V-transforms to get an optimal tree with $deg(v_N) = 1$. ∎
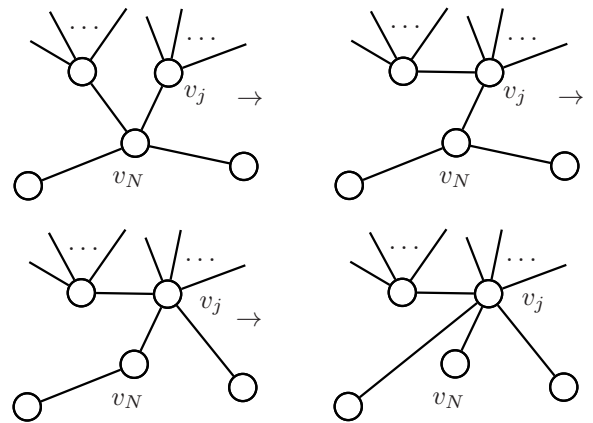

Fig. 5. Applying 3 V-transform on an optimal tree with $deg(v_N) = 4$

Lemma 3. *There exists an optimal tree, $G_O(V, E)$, such that for any $0 < i < N - 1$, $deg(v_{N-i}) \leq i + 1$. Furthermore, the number of neighbors of v_{N-i} with a lower SNR than V_{N-i} is at most one and consequently, the number of neighbors*

of v_{N-i} which have higher SNR than x_{N-i} is at least $deg(v_{N-i}) - 1$.

Proof: If the number of those neighbors of v_{N-i} that have a lower SNR value than x_{N-i} is a, after applying $a - 1$ V-transforms, we end up with an optimal tree in which $deg(v_{N-i}) \leq i + 1$. These $a - 1$ V-transforms have the form $V(G, V_{N-i}, v_i, v_k)$ and v_k has the highest SNR value among all of the neighbors of v_{N-i}.

Now, assume that $deg(v_{N-i}) \leq i + 1$ and v_{N-i} has at most one neighbor v_j such that $j < N - i$. Then, we have that the number of neighbors of v_{N-i} that have a higher SNR than x_{N-i} is $\geq |A_{N-i}^{G_O}| - 1 = deg(v_{N-i}) - 1$. ∎

Lemma 4. *There exists an optimal tree, $G_O(V, E)$, in which $deg(v_N) = deg(v_{N-1}) = 1$. Moreover, if v_j is the only neighbor of v_{N-1} and v_i is the only neighbor of v_N, then $x_i \geq x_j$*

Proof: If $deg(v_{N-1}) = 2$, according to Lemma (3) and (2), there exists an optimal tree $G_O(V, E)$ in which $deg(v_N) = 1$ and $v_N v_{N-1} \in E$. Let the other neighbor of v_{N-1} be v_j. Then, $G_{O'} = V(G_O, v_N, v_i, v_j)$ is an optimal tree in which $deg(v_{N-1}) = 1$. So, there always exists an optimal tree G_O, with $deg(v_N) = deg(v_{N-1}) = 1$. Assume that the only neighbor of v_{N-1} is v_j. If $v_j = v_N$, the graph will be disconnected. Otherwise, if the only neighbor of v_N is v_i, we want to prove that $x_i \geq x_j$. We also assume $x_N \neq x_{N-1}$; otherwise, one can rename the nodes in such a way that theorem holds. Assume that $G_{O''}(V, E'')$ is a client tree in which:

$$E'' = (E - \{v_N v_i, v_{N-1} v_j\}) \cup \{v_N v_j, v_{N-1} v_i\}. \quad (17)$$

It is easy to show that $D_s(G_{O''}) \leq D_s(G_O)$ iff $x_i \geq x_j$. ∎

Next lemma, is a generalization of Lemma (4) and we prove it in a similar way. It roughly says that in an optimal tree a node with a higher SNR has a neighbor with a higher SNR.

Lemma 5. *Assume that $G_O(V, E)$ is an optimal tree in which $deg(v_N) = deg(v_{N-1}) = \cdots = deg(v_{N-i}) = 1$ and $i < N - 1$. Also, assume that $q < p \leq i$ and $\{v_j v_{N-p}, v_k v_{N-q}\} \in E$. Then $x_j \leq x_k$.*

Proof: It is obvious that $j > N - i$ and $k > N - i$, otherwise the graph is disconnected. Now, if $x_k < x_j$, according to Lemma (4), the graph $G_{O'}(V, E')$ with $E' = (E - \{v_j v_{N-p}, v_k v_{N-q}\}) \cup \{v_j v_{N-q}, v_k v_{N-p}\}$ has a greater sum rate which contradicts the fact that G_O is optimal. ∎

Lemma 6. *Assume $G_O(V, E)$ is an optimal tree and i is the largest integer that*

$$deg(v_N) = deg(v_{N-1}) = \cdots = deg(v_{N-i}) = 1. \quad (18)$$

If $i < N - 1$, then there exists an optimal tree $G_{O'}(V, E')$ in which

$$deg(v_N) = deg(v_{N-1}) = \cdots = deg(v_{N-i+1}) = 1. \quad (19)$$
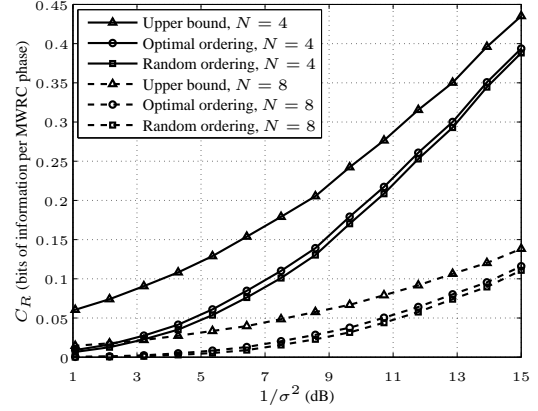

Fig. 6. Comparison between the common rate of the optimal ordering and random ordering in MWRC with FDF relaying for $N = 4$ and 8

Proof: Assume that $A_{N-i+1}^{G_O} \cap \{v_N, v_{N-1}, \ldots, v_{N-i}\} = \{v_{m_1}, v_{m_2}, \ldots, v_{m_n}\}$ where $m_1 > m_2 > \cdots > m_n$. Define

$$B = A_{N-i+1}^{G_O} - \{v_N, v_{N-1}, \ldots, v_{N-i}\}. \quad (20)$$

According to Lemma (3), we assume that $|B| \leq 1$. If $|B| = 0$, G_O is disconnected. Assume $B = \{v_j\}$. Consider $G_{O'}(V, E')$ such that

$$E' = (E - \{v_{m_1} v_{N-i+1}, v_{m_2} v_{N-i+1}, \ldots, v_{m_n} v_{N-i+1}\})$$
$$\cup \{v_{m_1} v_j, v_{m_2} v_j, \ldots, v_{m_n} v_j\}. \quad (21)$$

Then, one can conclude that $\frac{D_s(G_O)}{D_s(G_{O'})} \geq 1$. ∎

According to Lemma (6), there exists an optimal tree with respect to (7) in which

$$deg(v_N) = deg(v_{N-1}) = \cdots = deg(v_2) = 1. \quad (22)$$

As a result, O is an optimal solution with respect to (7). The muximum achievable sum rate, $S_R(G_O)$, could be found directly from (14). ∎

Fig. 3 illustrates the optimal ordering for an FDF MWRC that achieves the maximum S_R.

C. Asymptotic Behavior

Using Theorem 2, it is straightforward to show that

$$C_R(G_O) - C_R(G_{O'}) \leq \frac{1}{2(N-1)} \log_2 \left(\frac{1 + 2x_N}{2x_1} \right) \quad (23)$$

where O and O' refer to the optimal ordering and a random ordering, respectively. Now, if $x_N \sim x_1$[1], one can conclude that

$$\lim_{x_1 \to \infty} (C_R(G_O) - C_R(G_{O'})) = 0. \quad (24)$$

Similarly, for high SNR regimes, we have

$$S_R(G_O) - S_R(G_{O'}) \leq \frac{1}{2} \log_2 \left(\frac{x_N(1 + 2x_1)}{x_1(1 + 2x_N)} \right) \quad (25)$$

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[1]$f(x)$ is on the order of $g(x)$, $f(x) \sim g(x)$, if the asymptotic limit of their ratio approaches 1.
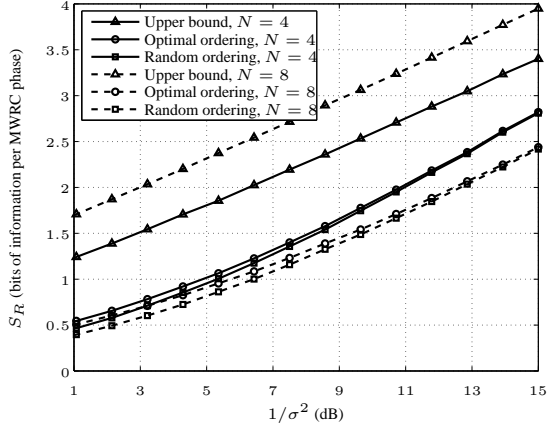
Fig. 7. Comparison between the sum rate of the optimal ordering and random ordering in MWRC with FDF relaying for $N = 4$ and 8

and consequently

$$\lim_{x_1 \to \infty} (S_R(G_O) - S_R(G_{O'})) = 0. \qquad (26)$$

In summary, equations (24) and (26) show that for FDF relaying, the performance of a randomly chosen ordering approaches the one for optimal ordering in high SNR regimes.

V. SIMULATION RESULTS

In this section, we investigate the performance of the optimal ordering in comparison with random orderings. We use Monte Carlo simulation to compare the optimal ordering and a randomly selected ordering. For each simulation round, random ordering is selected uniformly at random from all of the feasible client trees. We again assume that the data rates are limited by the uplink phase. Similar to [7], it is assumed that the channels between the users and the relay are Rayleigh fading with parameter 1. The number of users is set to $N = 4$ and 8. In order to illustrate the difference between optimal ordering and random orderings, we define the *common rate gap* [7] of random ordering and optimal ordering as $G_C = \frac{C_R(G_O) - C_R(G_{O'})}{C_R(G_O)}$ where, by abuse of notation, we denote the average of common rate over all of the simulation rounds by $C_R(\cdot)$. The subscripts O and O' denote optimal ordering and randomly chosen orderings, respectively. Similarly, we define the *sum rate gap* as $G_S = \frac{S_R(G_O) - S_R(G_{O'})}{S_R(G_O)}$.

Fig. 6 and 7 depict the comparison between the common rate and sum rate of the optimal ordering and random ordering for FDF relaying in low to high SNR regimes. The upper bounds are given by max-flow min-cut theorem [14]. Fig. 8 illustrates the aforementioned gap parameter and feature the effect of optimal ordering on both common rate and sum rate. However, these figures show that the ordering effect on FDF relaying is not significant in higher SNR regimes, as we showed earlier. The real and imaginary parts of the channel responses during each phase are modeled by independent and identically distributed zero-mean Gaussian variables with variance $1/2$. Decreasing this variance will increase

the aforementioned gap parameters in low SNR regimes. In other words, the ordering becomes more important for higher variance of channel or in lower SNR regimes. Fig. 9 illustrates the gap parameter for channel realizations with variance 1 and $\frac{1}{2}$ for $N = 4$ users.
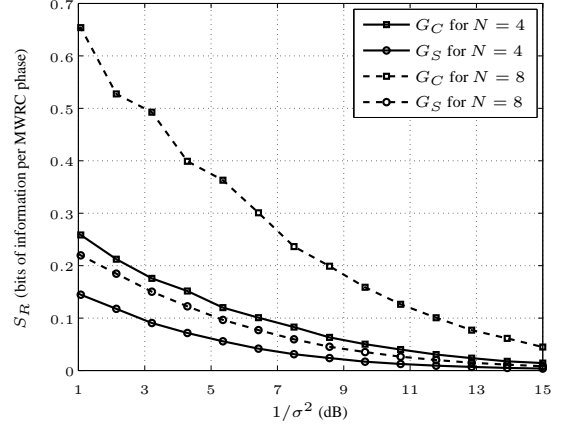

Fig. 8. Common rate and sum rate gap between optimal ordering and random ordering for $N = 4$ and 8
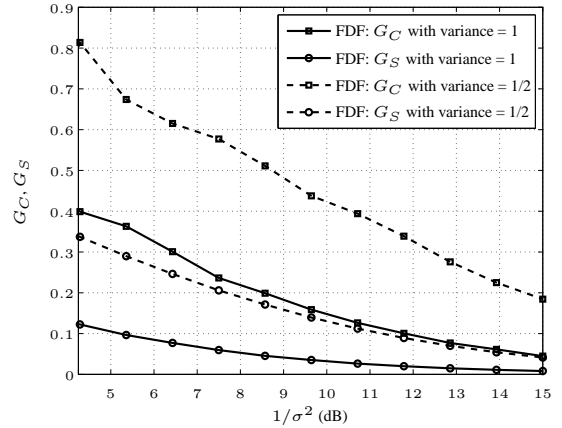

Fig. 9. Common rate and sum rate gap between optimal ordering and random ordering for 2 different channel variances with $N = 4$

VI. CONCLUSION

In this paper, we studied the effect of users' transmission ordering on the common rate and sum rate of a pairwise MWRC with FDF relaying. First, we suggested a graphical model for the data communication between the users. Then, using this model, optimal orderings were found that maximize common rate and sum rate in the system. Moreover, we showed that for high SNR regimes, the effect of ordering becomes less important. Our claims were supported and verified by computer simulations.

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