## Excitation-Induced Stability and Phase Transition: A Review


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# Excitation-Induced Stability and Phase Transition: A Review

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*Abstract:* Dynamical systems may experience undesirable behavior or instability, which can be eliminated using feedback control means. However, in the absence of feedback control, the stability of some systems may be increased by imposing parametric excitation. In other cases, the exit time of the system response from the stable to the unstable domain may be prolonged by imposing external noise, a phenomenon termed noise-enhanced stability (NES). This article presents an assessment of the mechanisms of stabilization via multiplicative noise and noise-enhanced stability. The first part deals with stabilization via deterministic parametric excitation of gravity-defying systems such as the inverted simple and spherical pendulums, aeroelastic structures, human walking, and quantum nonlinear couplers. The second part introduces the concept of noise-induced transition (NIT) in one-dimensional nonlinear systems and ship roll motion. Stabilization of originally unstable systems via multiplicative noise is treated in the third part. The fourth part addresses the influence of additive noise in delaying the exit time of system response to an unstable domain. This topic is related to the phenomenon of stochastic resonance (SR) and NES of systems with one or more metastable states and fluctuating potential. Finally, this review article also introduces some applications in other fields such as the Ising model, ecosystems, and tumor-immune system models.

*Keywords*: Stabilization, parametric excitation, additive noise, noise-induced transition, noise-induced stability, noise-enhanced stability, stochastic resonance, inverted pendulum, Langevin equation, tumor-immune systems, Ising model, ecosystems, flutter of aeroelastic structures, ship capsizing

## 1. INTRODUCTION

Stabilization of unstable systems using uncontrolled deterministic or random parametric excitation has been extensively studied in different mechanical systems, chemical reactions, nonlinear optics, and neuroscience systems. For example, the idea of stabilizing an inverted pendulum by imposing parametric excitation can be traced back to the work of Stephenson (1908). The analytical modeling of these systems ranges from the case of a simple one-dimensional nonlinear differential equation to multi-dimensional coupled oscillators. The instability state of these systems is associated with symmetry breaking due to the existence of more than one fixed point. It is the interplay of excitation and system nonlinearity that may produce unusual dynamical phenomena. For one-dimensional nonlinear systems the threshold of instability is shifted if the control parameter is noisy, with a correlation time that is much smaller than a certain characteristic time of the system.

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In nonlinear dynamical systems, parametric excitation (or noise) can shift the bifurcation threshold from one phase to another (Lücke and Schank, 1985), or induce new phase transitions in spatially-extended systems (Van den Broeck et al., 1994), or create spatial patterns (Parrondo et al., 1996; and San Miguel and Toral, 1997). If a dynamical system is excited by an additive noise, then the most probable value of the response coincides with the deterministic steady state value (fixed point) and is independent of the excitation. For parametrically excited systems, on the other hand, the most probable value of the response depends on the strength of the multiplicative noise. This feature has great influence on systems undergoing bifurcation. The threshold value of the bifurcation parameter depends on the fluctuations in multiplicative noise, but is independent of additive noise. The system response to additive noise is stable when the associated deterministic problem has a globally stable steady-state with respect to arbitrarily large fluctuations (Schenzle and Brand, 1979).

The term bifurcation refers to a qualitative change in the system's (static or dynamic) behavior under the variation of one (or more) control parameter. The idea of bifurcation may be formalized by introducing the concept of structural stability. Structural stability deals with the orbit structure (in the phase space) of a dynamical system undergoing perturbations of the system parameters. It implies the existence of a continuous coordinate transformation that transforms a nonlinear orbit into a linear orbit in the vicinity of a hyperbolic fixed point (whose Jacobian has eigenvalues with non-zero real parts). Such a transformation is called homeomorphism if it preserves the order of the points on the orbits. The orbit structure is unstable in the vicinity of a bifurcation point. Structural instability may occur in the form of static (stationary) or dynamic (Hopf) bifurcations. Static bifurcation occurs at a point for which the Jacobian of the system vector field has a simple zero eigenvalue.

The static criterion for bifurcation is not applicable to non-conservative systems, where bifurcation does not occur and dynamic criteria must be used. In the dynamic case, equilibrium is superseded by a motion represented by a limit cycle, and the oscillations are regular. This dynamic bifurcation takes the system from the static equilibrium configuration to a periodic motion regime. In the neighborhood of equilibrium, the Jacobian has a simple pair of purely imaginary eigenvalues, and the initial period of oscillation is related to the absolute value of the eigenvalue. Dynamic bifurcation is also characterized by single peak amplitude in the time domain and by a closed curve in the phase plane. As the system or excitation parameters change there can be a qualitative change in the response behavior, characterized by period doubling (two different amplitude peaks) in the time domain, and a double loop in the phase plane. The process proceeds to period tripling and higher as the system or excitation parameters change. This type of response behavior results in multiple maxima in the response probability density function (pdf).

Generally, stochastic bifurcation can be classified as bifurcation in distribution (or noise-induced transition [NIT]) or bifurcation in moments. NIT implies a change in the number of extrema in the pdf of the system response (Horsthemke and Lefever, 1984, 1989). In the vicinity of transition, a marked increase in the relaxation time occurs for cubic bi-stable systems (Sancho et al., 1985; Jackson et al., 1989). Relaxation time is an effect described as "critical slowing down," by analogy with the well-known phenomenon associated with equilibrium phase transitions. The relaxation time $T = \int_0^\infty [\langle x(t)\rangle - \langle x(\infty)\rangle]\, dt / [x(0) - \langle x(\infty)\rangle]$ was defined by Binder (1973), where $x(0)$ and $\langle x(\infty)\rangle$ are the initial and final states of the system response, respectively.

The two modes of bifurcation can be measured in terms of the sample Lyapunov exponent and the moment Lyapunov exponent, respectively. Kliemann and Arnold (1983) derived conditions for exponential sample instability of linear stochastic systems. Arnold (1984) extended the works of Molchanov (1978) and Arnold and Kliemann (1983), and developed a formula connecting the sample Lyapunov exponent and the Lyapunov exponent of response moments.

If the random excitation is additive, one would be interested in estimating the mean first passage time for the response to reach a target value. The dependence of the mean exit time on the noise intensity for metastable and unstable systems was revealed to have resonance character (Hirsch et al., 1982). Noise can modify the stability of the system in a counterintuitive way, such that the system remains in the metastable state for a longer time than in the deterministic case (Dayan et al., 1992). The escape time has a maximum at some noise intensity. Note that the phenomenon of noise-enhanced stability (NES) only results in an increase of the escape time, rather than causing absolute stabilization of an originally unstable system.

The purpose of this article is to provide an assessment of the role of parametric excitation (deterministic or random) in stabilizing, rather than destabilizing, dynamical systems. The assessment is also extended to the phenomenon of NES due to additive noise. This assessment is valuable in interpreting and understanding the influence of multiplicative and additive excitations on other dynamical systems. The assessment is divided into four main parts. The first deals with stabilization of dynamical systems via deterministic parametric excitation. The systems considered include inverted simple and spherical pendulums, aeroelastic structures, and other systems such as human walking and quantum nonlinear couplers. The second part deals with the problem of NIT, which is characterized by a qualitative change of the state of a given dynamical system as the intensity of random parametric excitation increases. The basic concept of NIT is outlined and some applications are then described. The third part addresses the stabilization of dynamical systems, including the inverted pendulum and ocean structures, via random parametric excitation. NES in metastable systems is briefly reviewed in the fourth part. The paper is closed by some conclusions and recommendations for possible future research.

## 2. STABILIZATION VIA DETERMINISTIC PARAMETRIC EXCITATION

This section is devoted to assess the main results pertaining to stabilization of some dynamical systems under sinusoidal parametric excitation. Specifically, we consider the inverted pendulum (which has been the subject of extensive studies by dynamicists, physicists, and mathematicians) as a paradigm to examine the complex dynamical phenomena encountered in other complex systems, such as ship dynamics and human walking. This is followed by other, more complex, systems such as the inverted spherical pendulum and aeroelastic structures.

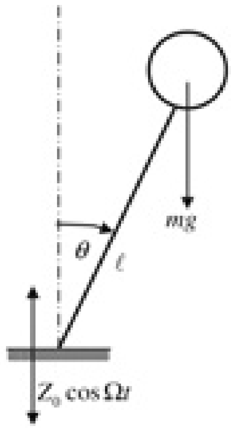



Figure 1. Inverted pendulum under parametric excitation.

### *2.1. Stabilization of the Inverted Pendulum*

#### 2.1.1. Sinusoidal Parametric Excitation

The idea of stabilizing the inverted pendulum by imposing parametric excitation can be traced back to the work of Stephenson (1908). Later, Stoker (1950) and many others (see, e.g., Kapitsa, 1951a,b; Chelomi, 1956; Bogdanoff, 1962) analyzed different versions of the inverted pendulum. Intuitively, when the pendulum support point is accelerated upwards the motion is unstable, while when it is accelerated downwards the motion can be stable. The periodic switch between these two situations can be globally stable or unstable depending on the values of some physical parameters. In particular, when the frequency of oscillation is higher than a certain threshold value the pendulum becomes stable. Note that the inverted pendulum has a negative restoring moment and its equation of motion in the absence of support oscillation (see Figure 1) is

$$\ddot{\theta} - \left(\frac{g}{\ell}\right) \sin\theta = 0 \rightarrow \quad \ddot{\theta} - \left(\frac{g}{\ell}\right) \theta = 0 \quad \text{for small } \theta \tag{1}$$

where $\ell$ is the pendulum length and $\theta$ is measured from the vertical upward position. The solution of equation (1) is a combination of $e^{(\sqrt{g/\ell})t}$ and $e^{-(\sqrt{g/\ell})t}$, which results in an unbounded motion. This means that the inverted pendulum is unstable unless a support excitation $z(t) = Z_0 \cos \Omega t$ is imposed; the equation of motion then takes the form

$$\ddot{\theta} - \omega_n^2 \left(1 + \frac{Z_0 \Omega^2}{g} \cos \Omega t\right) \theta = 0 \tag{2}$$

Setting $\tau = \Omega t$, equation (2) can be written in the standard Mathieu form

$$\theta'' + (\alpha + \beta \cos \tau)\,\theta = 0 \tag{3}$$

where a prime denotes differentiation with respect to the non-dimensional time $\tau$, $\alpha = -\omega_n^2/\Omega^2$, and $\beta = -Z_0\omega_n^2/g$. For small values of $|\beta|$, the perturbation technique adopted by Jordan and Smith (1999) may be used. This requires the solution of equation (3) to be written in the form

$$\theta(\tau) = \theta_0(\tau) + \beta\theta_1(\tau) + \beta^2\theta_2(\tau) + ... \tag{4}$$

Expressing the frequency parameter as a perturbation in $\beta$, one can then write

$$\alpha = \alpha(\beta) = \alpha_0 + \beta\alpha_1 + \beta^2\alpha_2 + ... \tag{5}$$

Substituting equations (4) and (5) into equation (3) and setting coefficients of equal powers of $\beta$ to zero gives the following set of differential equations

$$\theta_0'' + \alpha_0\theta_0 = 0 \tag{6a}$$

$$\theta_1'' + \alpha_0\theta_1 = -(\alpha_1 + \cos \tau)\theta_0 \tag{6b}$$

$$\theta_2'' + \alpha_0\theta_2 = -\alpha_2\theta_0 - (\alpha_1 + \cos \tau)\theta_1, \text{ and so on.} \tag{6c}$$

The solution of equation (6a) has a period of $2\pi$ or $4\pi$ if $\alpha_0 = n^2/4$, $n = 0, 1, 2, ....$ Considering first the case of $n = \alpha_0 = 0$, one finds that the solution $\theta_0(\tau) = \vartheta_0$ results in a periodic solution of equation (3). In this case equation (6b) takes the form

$$\theta_1'' = -(\alpha_1 + \cos \tau)\vartheta_0 \tag{7}$$

This equation has a periodic solution if $\alpha_1 = 0$, and the solution may be written in the form

$$\theta_1(\tau) = \vartheta_0 \cos \tau + \vartheta_1 \tag{8}$$

Equation (6c) becomes

$$\theta_2'' = -\alpha_2\vartheta_0 - \frac{1}{2}\vartheta_0 - \frac{1}{2}\vartheta_0 \cos 2\tau - \vartheta_1 \cos \tau \tag{9}$$

which has a periodic solution of period $2\pi$ if $\alpha_2\vartheta_0 + \frac{1}{2}\vartheta_0 = 0$, or if $\alpha_2 = -\frac{1}{2}$. Thus for small $|\beta|$ equation (5) takes the form (for solutions of period $2\pi$)

$$\alpha = -\frac{1}{2}\beta^2 + O(\beta^3) \rightarrow Z_0 = \sqrt{2}\frac{g}{\omega_n\Omega} \tag{10}$$

For $n = 1$, we have $\alpha_0 = 1/4$, and the solution of equation (6a) is

$$\theta_0(\tau) = \vartheta_0 \cos(\tau/2) + \varphi_0 \sin(\tau/2) \tag{11}$$

Thus equation (6b) takes the form

$$\begin{aligned} \theta_1'' + \frac{1}{4}\theta_1 &= -\vartheta_0 \left(\alpha_1 + \frac{1}{2}\right) \cos(\tau/2) - \varphi_0 \left(\alpha_1 - \frac{1}{2}\right) \sin(\tau/2) \\ &- \frac{1}{2}\vartheta_0 \cos(3\tau/2) - \frac{1}{2}\varphi_0 \sin(3\tau/2) \end{aligned} \tag{12}$$

A periodic solution of period $4\pi$ exists only if $\varphi_0 = 0$ and $\alpha_1 = -1/2$, or if $\vartheta_0 = 0$ and $\alpha_1 = 1/2$. To the first-order approximation, and for small $\beta$ equation (5) gives

$$\alpha = \frac{1}{4} \pm \frac{1}{2}\beta \tag{13}$$

For $\alpha = 0$ equation (13) yields $\beta \approx 1/2$.

These results can also be obtained using the harmonic balance or multiple scales methods.

Stephenson (1908) showed that the inverted pendulum can be stabilized by applying a parametric sinusoidal excitation with sufficiently small amplitude and sufficiently high frequency, i.e.,

$$\Omega > \sqrt{2g\ell}/Z_0 \tag{14}$$

which follows from condition (10). Subsequent studies, reported by Acheson (1997), revealed that the inverted state can be stabilized with lower excitation frequency and larger excitation amplitude, provided that these lie within the stability region shown in Figure 2. The first region on the left corresponds to Stephenson's criterion (14). Acheson (1997) observed two different ways in which the pendulum can avoid falling over. The first is characterized by gradually wobbling closer to the upward vertical as time goes on, while the other has the pendulum "dancing" persistently about the upward vertical with its bob at all times higher than the pivot. In region 2, the inverted pendulum bobs twice in succession on one side of the upward vertical before being flung across to the other side (see Figure 3). On the other hand, in region 3 there are three successive bobs on one side of the upward vertical before the second half of each oscillation.

Note that the negative restoring moment of the inverted pendulum can be made positive by introducing a proper periodic parametric excitation to the support point. The resulting governing equation becomes Mathieu's equation. In this case, one can construct a stability chart for which some combination of excitation amplitude and frequency always yields bounded solutions. This means that the unstable inverted pendulum can be converted into a stable one if one imposes a sinusoidal support motion with appropriate amplitude and frequency. Valeev (1971) employed an asymptotic technique and reached similar conclusion, as well as considering the case of an inverted pendulum with an elastic spring which de-

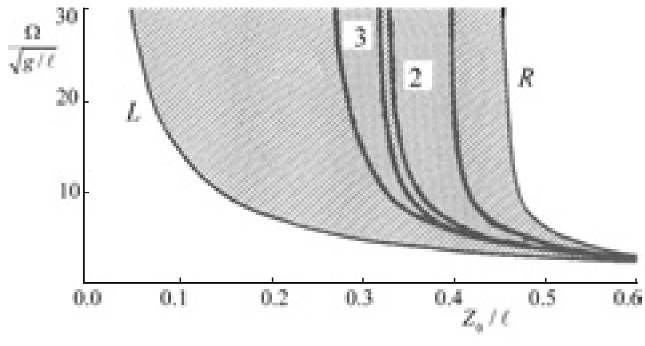



Figure 2. Stability region of the excited inverted pendulum with damping factor 0.1. Upside-down oscillation provides an alternative to the classical inverted state in regions 2 and 3 (Acheson, 1997).

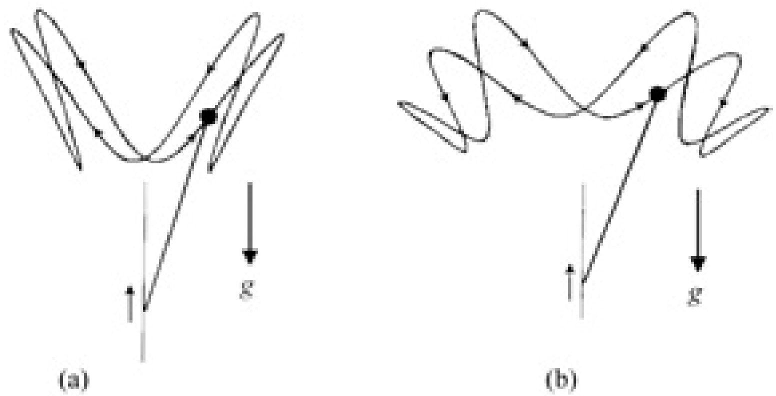



Figure 3. Upside-down oscillations of a driven inverted pendulum (a) oscillation with frequency $\Omega/4$ from region 2 of Figure 1 with $Z_0/\ell = 0.45$, and $\Omega = 5\sqrt{g/\ell}$, may be generated with, say, $\theta = \pi$ and $d\theta/dt = 1.3\sqrt{g/\ell}$ at $t = 0$. (b) An oscillation with frequency $\Omega/6$ from region 3, with $Z_0/\ell = 0.33$ and $\Omega = 10\sqrt{g/\ell}$, may be generated with, say, $\theta = \pi$ and $d\theta/dt = 2.6\sqrt{g/\ell}$ at $t = 0$ (Acheson, 1997).

forms against the pendulum's mass motion. It was shown that the inverted position can be stabilized when the support motion frequency is close to the linear natural frequency of the mass-spring system.

The stabilization of the inverted pendulum was studied experimentally under parametric sinusoidal excitation by Kalmus (1970), Michaelis (1985), and Smith and Blackburn (1992). The influence of the tilt of the excitation (relative to the gravity direction) on the stabilization of the inverted pendulum was considered by Sudor and Bishop (1996), Weibel et al. (1997), and Yabuno et al. (2004). It was found that the angle of inclination of the pendulum in the stable equilibrium state is much greater than the angle of the excitation direction to the vertical. With increasing excitation frequency, the pendulum angle was found to approach the excitation direction angle.

### 2.1.2. Periodic and Impulsed Parametric Excitations

The inverted pendulum with a periodic parametric excitation was considered as a bifurcation problem in the reversible setting by Broer et al. (1998, 1999). Levi (1988) presented a topological explanation and proof of stability of an inverted pendulum with a prescribed vertical base motion. Broer et al. (1998, 1999) showed that the planar normal form of the pendulum Hamiltonian is given by the scaled expression

$$\bar{H}(\vartheta, \theta'; \alpha, \beta) = \frac{1}{2}\theta'^2 + \frac{1}{2}\sin^2\vartheta - \alpha(1-\cos\vartheta) + O\left(|\alpha, \beta|^3\right) \tag{15}$$

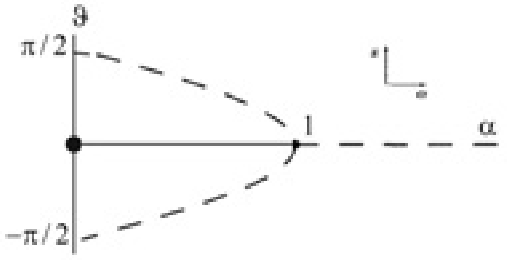
where $\theta' = \vartheta'/|Z_0/\ell|$, $\alpha = \left(\omega_n^2/\Omega^2\right)\left(\ell^2/Z_0^2\right)$, $\beta = Z_0/\ell$, a prime denotes differentiation with respect to the non-dimensional time, $\tau = \Omega^2 t$, and $\bar{H} = H/\left(m\Omega^2 Z_0^2/\ell^2\right)$ is the scaled Hamiltonian. The global dynamics of the inverted pendulum revealed the existence of a center at $\left(\vartheta, \dot{\theta}\right) = (\pi, 0)$ for all values of excitation frequency and amplitude. There is also a singularity at the origin $\left(\vartheta, \dot{\theta}\right) = (0, 0)$ whose stability type changes at a curve in the parameter plane of the form $\alpha = 1 + O(\beta)$, in the form of a pitchfork bifurcation as shown in Figure 4(a). It can be seen that the origin $\vartheta = 0$ is stable and is surrounded by two saddles in heteroclinic connection because of the spatial symmetry. The equilibrium $\vartheta = 0$ becomes unstable for $\alpha > 1$, i.e., $\Omega < \omega_n$. Figure 4(b) shows three global phase portraits corresponding (from left to right) to $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$. The only bifurcation of interest takes place at $\left(\vartheta, \dot{\theta}, \alpha\right) = (0, 0, 1)$.

The local stability diagram of the inverted pendulum in the spatially symmetric cases is shown in Figure 4(c) on the left of the plane $(\alpha, \beta)$. At $\left(\vartheta, \dot{\theta}\right) = (0, 0)$ there is a saddle point for $(\alpha, \beta)$ to the right of the curve $\alpha = \beta^2 + O(\beta^3)$. This curve represents a pitchfork bifurcation for $(\alpha, \beta)$. The shaded regions refer to the stable equilibrium of the upper position.

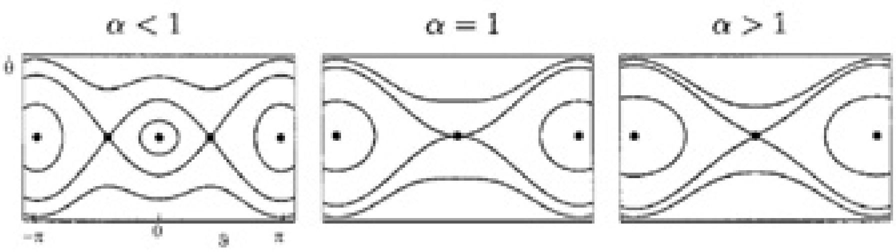
Liu and Willms (1994) showed that the undamped inverted pendulum can be stabilized by applying impulses resulting in an instantaneous change in angular velocity with no change in position. Intuitively, their result implies that whenever the pendulum starts to fall from its upright position one has to kick it back upward in order to keep it close to the unstable equilibrium position.

### 2.1.3. Chain of Inverted Pendulums

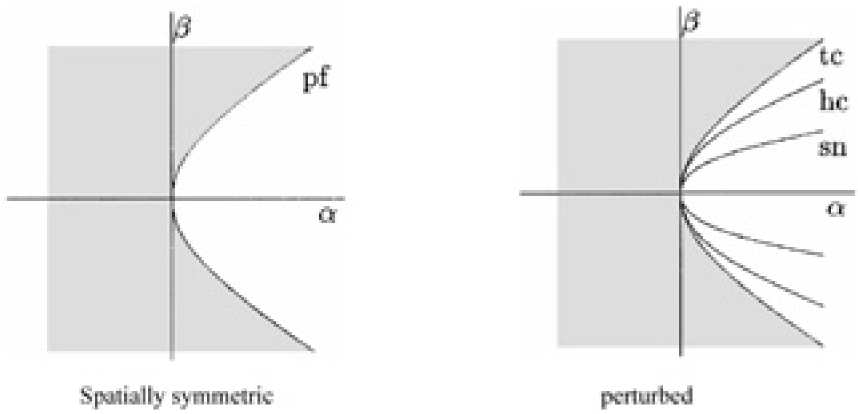
The stabilization of the equilibrium position of a chain of N inverted pendulums was considered by Otterbein (1982) and Acheson (1993, 1997) who used modal analysis and obtained $N$ uncoupled Mathieu's equations with different parameters. It was concluded that by choosing a sufficiently high frequency and sufficiently small amplitude for each mode, a finite chain can be stabilized by parametric excitation. Numerical simulation (Acheson, 1993) and experimental tests (Acheson and Mullin, 1993) for two and three pendulums showed that the stability is remarkably robust, even for quite large disturbances. Hurst (1996) adopted the limiting case of this chain, i.e., $N \to \infty$, as a possible interpretation of the so called "Indian rope trick". (The legend of an Indian magician who throws a rope to the sky. The rope does not fall back to the ground; instead it mysteriously rises until the top disappears into thin air.




(a) Bifurcation diagram in the $(\vartheta, \alpha)$ plane for $Z_0 / \ell = 0$. Dashed curve indicate unstable equilibria.




(b). Phase portraits for three different values of frequency ratio $\alpha = \omega_n^2 / \Omega^2$.




(c). Local stability for the spatial symmetric (left) and perturbed pendulum (right). $tc$ : transcritical bifurcation, $hc$ : heteroclinic bifurcation, and $sn$ : saddle-node bifurcation.

Figure 4. Bifurcation, phase, and stability diagrams of the inverted pendulum (Broer et al., 1998, 1999).

Unfortunately, this claim has never been substantiated) Unfortunately, for the case of a piece of string, the stability region becomes vanishingly small, and the explanation fails. However, a further experiment by Acheson and Mullen (1998) revealed that a piece of "bendy curtain wire," clamped at the bottom and free at the top, can be stabilized by parametric excitation. Galán et al. (2005) called this wire the "Indian rod trick."

Champneys and Fraser (2000) adopted a linearized analysis for a continuously flexible linearly elastic rod. They derived a simple lower bound on the product of the excitation frequency and amplitude necessary to stabilize the rod. This bound can be expressed solely in terms of the ratio of the column's length to the critical buckling length. Mullin et al. (2003) reported experimental observations on the stabilization of the upright wire, which would otherwise fall over under its own weight. They showed lower and upper bounds on the excitation frequency required to stabilize this gravity-defying system. The upper instability boundary was found to be a harmonic resonance rather than a subharmonic one due to a new phenomenon termed "resonance-tongue interaction." The experimental lower bound was found to be higher than the predicted one by a factor of between 2 and 4 (Champneys and Fraser, 2000). The upper bound is more subtle, due to the presence of infinitely many resonance tongues inside the stability region of parameter space. This motivated Fraser and Champneys (2002) to address these contradictions by including three-dimensional effects in addition to geometric nonlinear terms.

Stabilization was experimentally possible but there remains a lack of qualitative and quantitative fit with experimental results. Galán et al. (2005) proposed a discrete model in which small damped elastic constraints are added to the bottom joint and the joints between N identical pendulums. At the limit $N \to \infty$, and with the inclusion of realistic material damping, the system approaches the case of a continuously flexible rod. Their analysis revealed that damping has the effect of removing most of the regions of instability for a fixed amplitude of parametric excitation. The shape of instability curves and the mode shapes of the corresponding instabilities were found to be in agreement with those of an experiment on curtain wire.

### *2.2. Stabilization of the Inverted Spherical Pendulum*

The spherical pendulum differs from the simple pendulum in its ability to possess non-planar oscillations. The inverted spherical pendulum was used to model ocean articulated towers by Bar-Avi and Benaroya (1996, 1997). The stability of a spherical pendulum under sinusoidal parametric excitation was considered by Markeyev (1999). With reference to Figure 5, the Hamiltonian of the spherical pendulum can be written in the form

$$H = \frac{1}{2} m\ell^2\Omega^2 \left[ \left( \frac{\mathrm{d}\theta}{\mathrm{d}\tau} - \frac{Z_0}{\ell} \sin\theta \sin\tau \right)^2 + \left( \frac{\mathrm{d}\phi}{\mathrm{d}\tau} \sin\theta \right)^2 - 2\frac{g}{\ell\Omega^2} \cos\theta \right] \tag{16}$$

Under the assumption that $Z_0 \ll \ell$, and $\Omega \gg \sqrt{g/\ell}$, Markeyev (1999) obtained bifurcation values of the angular momentum about the $z$-axis and frequency parameter $\sqrt{2g/\ell\Omega^2}$. Figure 6 shows the bifurcation diagram on the plane $a = f_1(\mathrm{d}\phi/\mathrm{d}\tau)$, $b = f_2(\sqrt{2g/(\Omega^2\ell)}$, where $f_i(.)$, $i = 1, 2$ are functions of the indicated arguments. In region $G_1$, the pendulum

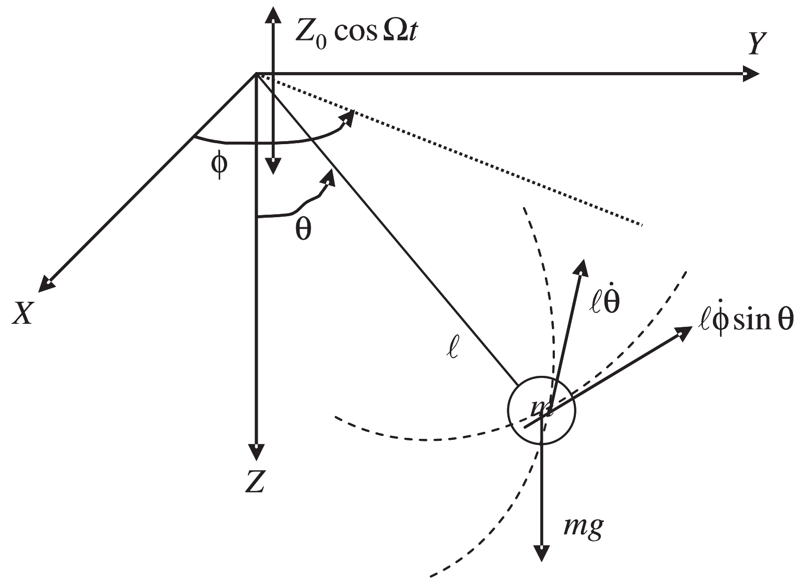



Figure 5. Schematic diagram of the spherical pendulum.

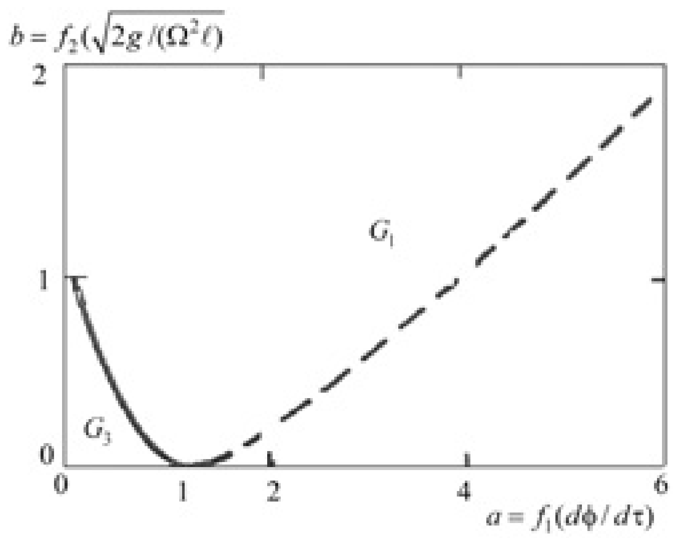



Figure 6. Bifurcation Diagram of the spherical pendulum for down and inverted positions (Markeyev, 1999).

has one equilibrium position where $\cos\theta > 0$, while in the region $G_3$ there are three equilibrium positions, for which $-1 < \cos\theta_3 < \cos\theta_2 < 0 < \cos\theta_1 < 1$. In region $G_1$ the corresponding motion is close to conical and the center of gravity of the pendulum is below the point of suspension. This motion is referred to as type-I motion. The corresponding motion of the other two solutions; $-1 < \cos\theta_3 < \cos\theta_2 < 0$, is conical, with the center of gravity above the pivot point, and the inverted position is stable. This motion is referred to as type-II motion. If the values of the parameters $a$ and $b$ lie in the region $G_3$, a single motion of type-I coexists with two motions of type-II.

### *2.3. Stabilization in Aeroelastic Structures*

The influence of parametric excitation on the flutter of aeroelastic structures has been considered by a limited number of studies. For example, Lumbantobing and Haaker (2004) studied the parametric excitation of plunge and seesaw oscillators. Both oscillators are described by a nonlinear Mathieu's equation where the nonlinearity owes it origin to the dependence on the flow angle of attack. It was found that by increasing the air flow speed above a critical value the equilibrium position is re-stabilized. However, if the parametric excitation is due to longitudinal turbulence, the frequency of coalescence occurs at a lower air flow speed (see, e.g., Poirel and Price, 2003a,b).

Chin et al. (1995) obtained the modulated equations needed to calculate the equilibrium solutions of a simply supported panel in a supersonic flow (and their stability). In the neighborhood of combination parametric resonance, they identified the excitation parameters that suppress flutter and those leading to complex motions.

The possibility of suppressing wing flutter via parametric excitation, $Y(t)$, acting along the plane of highest rigidity as shown in Figure 7 was studied by Ibrahim and Castravete (2006). The bending-torsion coupling arises mainly from the fact that the centers of mass of cross-sections undergo a small but important displacement, v, in the plane of excitation. This displacement is expressed in terms of the second-order of the fundamental bending, $u$, and torsion, $\alpha$, displacements. The displacements of the elastic axis along the $y$ and $z$ axes are $v(z,t)$ and $w(z,t)$, respectively. The airfoil is exposed to an incompressible fluid flow of speed $U_\infty$. The chord of the wing is $2b$, $\delta_1$ is the distance between the elastic axis and mid-chord, $\delta_2$ is the distance between the aerodynamic center and mid-chord, and $\delta_3$ is the distance between the elastic axis and inertia axis. The nonlinear equations of motion in the presence of incompressible fluid flow were derived using Hamilton's principle and Theodorsen's theory for modeling aerodynamic forces. For the first modes in bending and torsion, the solutions can be expressed in terms of the generalized coordinates and mode shapes as

$$u(z,t) = u_0(t) f(z); \qquad \alpha(z,t) = \alpha_0(t)\phi(z) \tag{17a,b}$$

where $f(z) = \cosh(1.875z/l) - \cos(1.875z/l) - 0.734\left[\sinh(1.875z/l) - \sin(1.875z/l)\right]$, $\phi(z) = \sin(\pi z/2l)$, and $l$ is the wing length.

By applying Galerkin's method, the following two coupled nonlinear ordinary differential equations were obtained

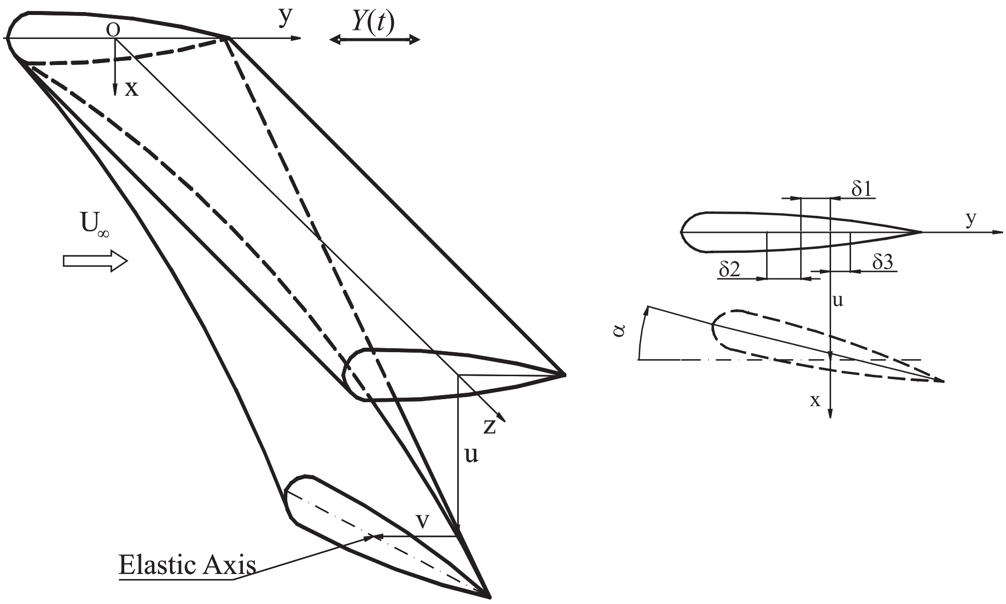



Figure 7. Schematic diagram of a wing exposed to subsonic flow under parametric excitation.

$$\begin{aligned}
&\left(1+\mu+c_1 d^2\bar{u}^2-c_2\bar{\alpha}^2\right)\bar{u}''+\left(c_6\left(x_\alpha-a\mu\right)-c_2\bar{u}\alpha\right)\alpha''+2\left(r\zeta_u+\frac{\mu}{k_\alpha}B\left(k\right)\right)\bar{u}'\\
+\;& c_6\frac{\mu}{k_\alpha}\left[1+B\left(k\right)\left(1-2a\right)\right]\alpha'+\left[r^2+c_1 d^2\bar{u}'^2\right]\bar{u}+2\left(c_6\frac{\mu B\left(k\right)}{k_\alpha^2}-c_2\bar{u}'\alpha'\right)\alpha\\
+\;& c_8 d^2 r^2\bar{u}^3+c_3\alpha\bar{Y}''(\tau)\;=\;0
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\left(c_7\left(x_\alpha-a\mu\right)+c_4\bar{u}\alpha\right)\frac{1}{r_\alpha^2}\bar{u}''+\left(1+\frac{\mu}{r_\alpha^2}\left(\frac{1}{8}+a^2\right)+\frac{c_4}{r_\alpha^2}\bar{u}^2\right)\alpha''\\
-\;& c_7\left(1+2a\right)\frac{\mu}{k_\alpha r_\alpha^2}B\left(k\right)\bar{u}'+\left[2\zeta_\alpha-\left(\frac{1}{2}-2a^2\right)\frac{\mu}{k_\alpha r_\alpha^2}B\left(k\right)+\left(\frac{1}{2}-a\right)\frac{\mu}{k_\alpha r_\alpha^2}\right]\alpha'\\
+\;& \frac{2c_4}{r_\alpha^2}\bar{u}'\alpha'\bar{u}+\left[1-\left(1+2a\right)\frac{\mu}{k_\alpha^2 r_\alpha^2}B\left(k\right)\right]\alpha+\frac{c_5}{r_\alpha^2}\bar{u}\bar{Y}''(\tau)\;=\;0
\end{aligned} \tag{19}$$

together with the boundary conditions

$$\begin{aligned}
\alpha|_{z=0} &= \frac{\partial\alpha}{\partial z}|_{z=0}=\frac{\partial\alpha}{\partial z}|_{z=l}=0,\qquad u|_{z=0}=\frac{\partial u}{\partial z}|_{z=0}=0,\\
\frac{\partial^2 u}{\partial z^2}|_{z=l} &= \frac{\partial^3 u}{\partial z^3}|_{z=l}=0
\end{aligned} \tag{20}$$

where $\bar{u} = u_0/b$, $\bar{Y} = Y/b$, $\tau = \omega_\alpha t$, $\zeta_u$ and $\zeta_\alpha$ are linear viscous damping factors in bending and torsion (respectively), $m$ is the wing mass per unit length, $a = \delta_1/b$, $k_\alpha = \dfrac{b\omega_\alpha}{U_\infty}$, $\mu = \dfrac{\pi b^2 \rho_\infty}{m}$, $x_\alpha = \dfrac{S_\alpha}{mb}$, $S_\alpha = m\delta_3$, $r_\alpha = \sqrt{I_\alpha/(mb^2)}$, $I_\alpha$ is the wing mass moment of the inertia (about the elastic axis) per unit length = $I_\alpha = I_0 + m\delta_3^2$, $I_0$ is the mass moment of the inertia (about the inertia axis) per unit length, $B(k)$ is the circulation function, which depends on the reduced frequency parameter, $k = b\omega/U_\infty$, where $\omega$ is the natural frequency of the wing coupled modes, a prime denotes differentiation with respect to the non-dimensional time $\tau$, $\rho_\infty$ is the air density, $\rho$ is the wing density, $d = \dfrac{b}{l}$, $a = \dfrac{\delta_1}{b}$, $r = \dfrac{\omega_u}{\omega_\alpha}$, $\omega_u = \sqrt{K_u/m}$, $\omega_\alpha = \sqrt{K_\alpha/I_\alpha}$, and $c_i$ are constant coefficients, $K_u = 12.3596EI_y/l^4$, $K_\alpha = \pi^2 cGJ/4l^2$, $K_{u3} = 80.8579EI_y/l^6$, $E$ is Young's modulus, $I_y$ is the area moment of inertia of the wing cross-section about $y$ axis, $J$ is the polar moment of inertia of the wing cross-section about the $z$ axis, $G$ is the modulus of rigidity, and $c$ is a correction constant due to the noncircular cross-section of the wing.

The response of the wing under different values of excitation amplitude, at the critical and post-critical flutter speeds was determined for the combination parametric resonance, $\Omega = \omega_1 + \omega_2$, in the neighborhood of one-to-one internal resonance, $\omega_1 = \omega_2$, where $\omega_i$, $i = 1, 2$, are the normal mode frequencies of the wing. The results were obtained for external excitation detuning parameter $\sigma_e = 0$, internal resonance detuning parameter $\sigma_i = 0.061$, and system parameters $\mu = 1/38$, $a = -0.36$, $x_\alpha = 0.024$, $r_\alpha = 0.62$, $r = 0.5$, $\zeta_\alpha = 0.001$, $\zeta_u = 0.001$, $d = 0.1$.

In the absence of parametric excitation, the solution of these equations revealed the co-existence of different fixed points implying different levels of limit cycle oscillations (LCO). Each fixed point is created by a certain domain of initial conditions. Under parametric excitation, the same scenario was preserved up to an excitation level $Y_0 = 0.00875$, at which the response experiences Hopf bifurcation around each fixed point in addition to a new fixed point that also experiences Hopf bifurcation. Over another range of excitation amplitude, $0.00875 < Y_0 < 0.0125$, the response experiences Hopf bifurcation for relatively high values of initial conditions below, which the response settled at the static equilibrium position.

Figure 8 presents the bifurcation diagram where the excitation amplitude is taken as the control parameter. It can be seen that for some values of the initial conditions, the parametric excitation acts as a stabilizer source for the wing flutter, and within this set of values the response achieved its static equilibrium position. Outside this domain of initial conditions, the equilibrium position loses its stability and the response may possess fixed points, Hopf bifurcation, cascades of period doubling, and eventually chaotic motion. At each bifurcation point, branches of symmetric and asymmetric periodic solutions meet in the form of supercritical symmetry-breaking bifurcations.

It is obvious that the response exhibits multiple attractors, each with its own domain of attraction. One or two of these attractors, namely the zero response or small amplitude LCO about a small mean, are superior and desirable in the post flutter region. In order to achieve this desirable performance, one may use one of the current techniques used to control chaos (see, e.g., Boccaletti et al., 2000: and Pereira-Pinto et al., 2004). For the present case, the domains of attraction of undesirable performance can be shifted away only by increasing the parametric excitation amplitude. Figure 9 shows the dependence of the percentage of

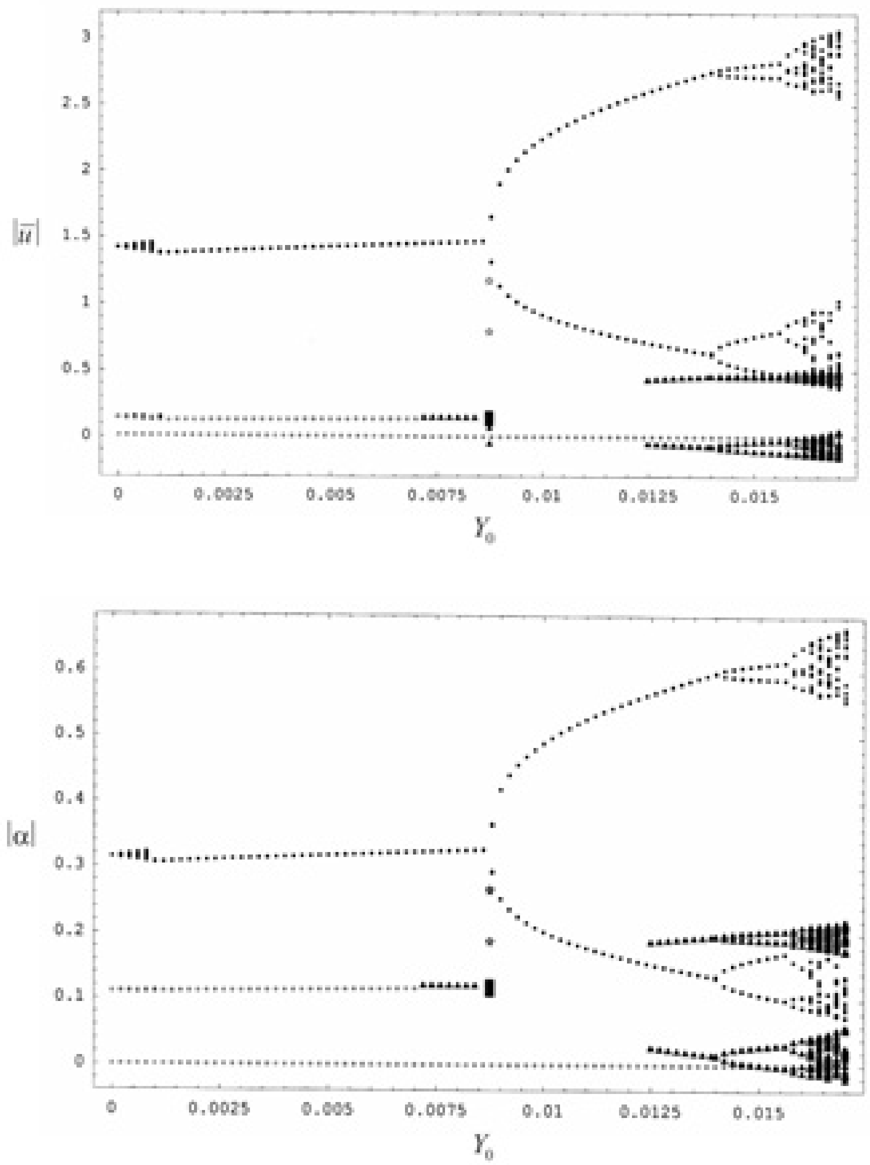



Figure 8. Bifurcation diagrams of bending and torsion responses at the critical flutter speed taking the excitation amplitude as the control parameter.

domains of attraction that lead to non-zero flutter oscillation on the excitation amplitude. It can be seen that as the excitation amplitude increases, the domains of attraction for both small and large response amplitudes decrease up to the critical value of excitation amplitude, $Y_{cr} \approx 0.01375$, above which the parametric excitation suppresses the wing flutter.

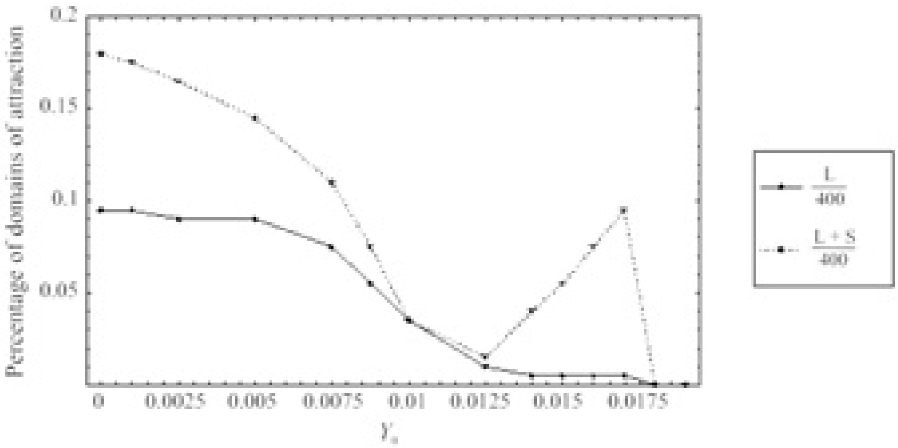
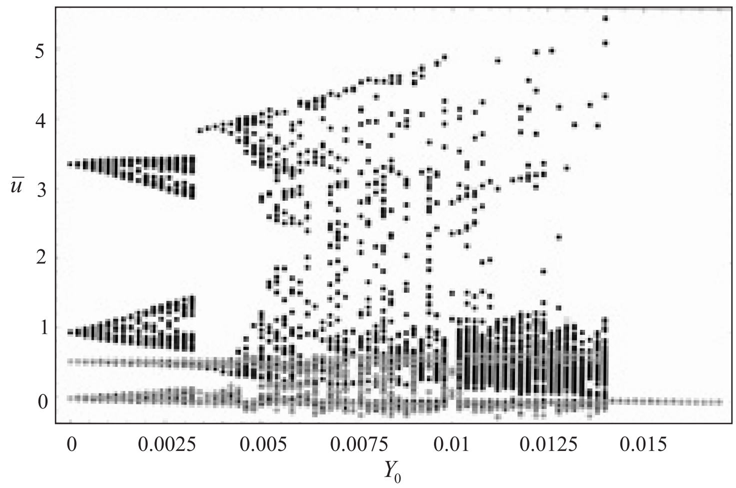


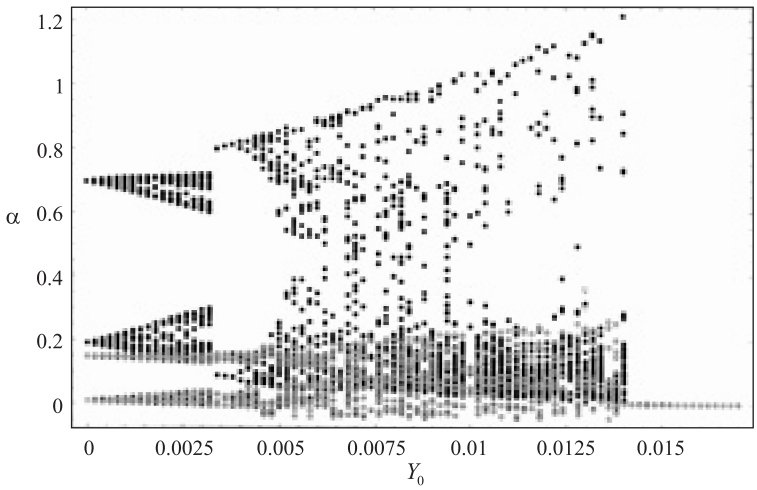

Figure 9. Percentage of domains of attraction that lead to LCO and chaotic motions.

At a flow speed that is slightly higher than the critical flutter speed, e.g., $U_\infty/b\omega_\alpha = 5.02$, the wing enters the post flutter region. In the absence of parametric excitation the wing experiences Hopf bifurcation of different values of amplitude oscillations depending on initial conditions. For all values of initial conditions, there are two possible domains of response amplitudes: small and large amplitude LCOs. Under very small excitation amplitude, the response experiences multi-frequencies. Over the excitation amplitude range $0.001 \leq Y_0 < 0.003$, the response experiences multi-frequencies with growing amplitudes. Over higher excitation amplitudes ($0.003 < Y_0 < 0.005$), there are two domains of attraction, of which one leads to small amplitude oscillations with multi-frequencies as in the previous range, while the other leads to periodic then cascade of period doubling with high amplitude oscillations. The next region of excitation amplitude, $0.005 < Y_0 < 0.01$, is characterized by chaotic motion for all possible initial conditions. However, one set of initial conditions leads to a chaotic attractor with small amplitude oscillations, while the other set leads to large amplitude oscillations for $Y_0 = 0.0075$. There is a small window around $Y_0 = 0.01$ characterized by period doubling for all possible initial conditions. A new regime of response behavior emerges at the excitation level range $0.01 < Y_0 < 0.014$, characterized by a train of spikes known as the "firing" state (Lindner et al., 2004) for certain regions of initial conditions. The remaining initial conditions lead to periodic attractors. As the excitation amplitude increases, the period between the spikes (called the "refractory" or "recovery" period) increases, and for some initial conditions the recovery period becomes infinitely large, constituting a stabilization effect.

Figures 10(a) and (b) show the bifurcation diagrams summarizing all the regimes described above. In order to better demonstrate the stabilizating effect of parametric excitation, Figure 11 delineates the dependence of the percentage of the area consisting of domains of attraction on the excitation amplitude. The numerical simulation of the original equations of motion has confirmed the multiple scales findings. Note that Theodorsen's theory is only applicable for small amplitude oscillations. This requires one to employ other aerodynamic techniques, such as the doublet-hybrid or vortex lattice methods.




(a) Dependence of bending amplitude on excitation amplitude.




(b) Dependence of torsion amplitude on excitation amplitude.

Figure 10. Bifurcation diagrams showing different regions of response behavior at post-critical flow speed $U_\infty/b\omega_\alpha = 5.02$, $\Omega = \omega_1 + \omega_2$, $\omega_1 = \omega_2$, $\sigma_e = 0$, $\sigma_i = 0.061$.

### *2.4. Stabilization of Other Systems*

The inverted pendulum has been used to model the mechanics of human walking. It predicts the arched trajectory of the body's center-of-mass (COM) and the relatively constant state of mechanical energy during single-limb support, which is assumed to reflect a continuous exchange of kinetic and potential energy as reported by Cavagna et al. (1977) and Cavagna and Margaria (1996). The inverted pendulum has also been shown to predict many facets of human walking mechanics including ratios of walking speed to frequency (Bertram and

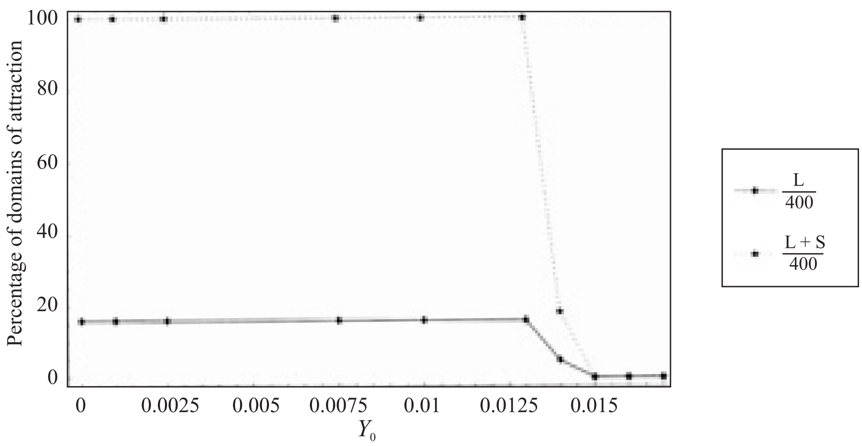



Figure 11. Stabilization effect of parametric excitation at post-critical flow speed $U_\infty/b\omega_\alpha = 5.02$ showing the percentage of domains of attraction for large amplitude response (diamond-solid curve) and large plus small amplitude responses (star-dotted curve).

Ruina, 2001), walk/run transition speeds (Kram et al., 1997), preferred walking speed/step length ratios (Kuo, 2001), the influence of preferred step width (Donelan et al., 2001), the effect of gravity (Cavagna et al., 2000; Minetti et al., 1993), stability of walking models (Garcia et al., 1998), and the relation between work-rate and stride frequency (Minetti and Saibene, 1992).

Winter et al. (1998) and Karlsson and Frykberg (2000) modeled postural sway as an inverted pendulum. The inverted pendulum model predicted that the difference between the center of pressure (COP) and COM is proportional to the horizontal acceleration of COM (Murray et al., 1976; and Geursen et al., 1976). This prediction has been validated experimentally by Winter et al. (1998). Modeling postural sway as an inverted pendulum assumes a rigid structure above the ankles. However, the body is a segmented structure capable of moving the joints superior to the ankle independently. A study by Gage et al. (2004) supported the inverted pendulum modeling of quiet standing. The condition for dynamic stability was obtained on the basis of a simple inverted pendulum model by Hof et al. (2005). It was proposed that the position of the COM plus its velocity multiplied by a factor of $\sqrt{\ell/g}$ should be within the base of support, where $\ell$ is the leg length.

The mathematical elegance of inverted pendulum analyses is confounded when the complexities of the musculoskeletal system are considered. As a result, the inverted pendulum-based models cannot identify the timing of work output by muscles (positive, negative, or net) or muscle contributions to external power. Neptune et al. (2004) found that the net muscle work output cannot be estimated from the external power trajectory, and the mechanical and metabolic energy cost of the muscles is dominated not only by the need to redirect the COM in double support, but also by the need to raise the COM afterward in single support. Brénière and Ribreau (1998) modeled the sagittal and frontal components of the human gait by means of a two inverted pendulum system formed by the head, arms, and trunk (HAT), and an equivalent two-legged system called the HCP, which links the middle (H) of hip centers to the center of pressure (CP).

The vertical position of the human posture is inherently unstable. However, the physiological control system is able to maintain it while developing operational performance under unexpected variable conditions. The control of balance is a process that involves neural latencies, feedback and random fluctuations (Moss and Milton, 2003). The presence of parametric fluctuations is crucial during the control process of an inverted pendulum allowing for control at time scales smaller than the physiological delay (Cabrera and Milton, 2002). Cabrera (2005) reproduced this behavior using a model of an inverted pendulum with time-delayed feedback in which the parametric noise forces a control parameter across a particular stability boundary. The model was built to use the opposition of two antagonistic forces plus damping; the destabilizing effect of gravity was opposed by a stabilizing delayed restoring force produced by human action.

Early attempts to characterize a biomechanical influence simplified the problem of controlling an inverted pendulum by analyzing gravitationally-driven, non-inverted pendulums with properties based on anthropometric measures of the human body (McCollum and Leen, 1989). Assuming the adult human body could behave as a two-link pendulum where the links could oscillate in phase as a single-segment system or in anti-phase as a double pendulum, McCollum and Leen (1989) predicted oscillation frequencies of 0.52 Hz for a single pendulum and 1.45 Hz for a double pendulum. This result was found to confirm the experimental measurements made by Park et al. (2004).

Other modeling includes a three-joint (ankle, knee, and hip) inverted pendulum to account for postural responses during fast forward bending at the waist on narrow and wide surfaces and in response to platform perturbations (Alexandrov et al., 2001a,b). Creath et al. (2005) demonstrated the simultaneous coexistence of in-phase and anti-phase relationships between the leg and trunk angles during quiet stance; the relationship was generally in-phase for frequencies below 1 Hz and anti-phase for frequencies above 1 Hz. Ulrich et al. (2004) used an escapement-driven damped inverted pendulum and spring model to compare the global levels of stiffness and angular impulse used by preadolescents with and without Down syndrome as they walked at their preferred speed and adapted to walking faster and slower than preferred.

It is interesting to find that the equations of motion of quantum nonlinear couplers have the structure of a classical inverted pendulum with vertically oscillating support point (Facchi et al., 2001). The amplitudes of the fields inside the coupler vary little during an optical period. The resulting equations of motion consist of two sets; one set for stable motion, and a second for unstable motion. The system is similar to other examples treated by Luis and Peřina (1996), Luis and Sánchez-Soto (1998), Thun and Peřina (1998), and Řeháček et al. (2000) in the context of the quantum Zeno effect (see, e.g., Beskow and Nilsson, 1967; Misra and Sudarshan, 1977). In these systems, the "measurement" is performed by a mode of the field on another mode. When the coupling between the two modes is large enough, the measurement becomes more effective and the dynamics more stable. This phenomenon is a manifestation of the quantum Zeno effect, which is a hindrance of quantum evolution, caused by measurements. (The quantum Zeno effect predicts a slow-down of the time development of a system under rapidly repeated ideal measurements. Experimentally, this was tested for an ensemble of atoms using short laser pulses for non-selective state measurements.) Facchi et al. (2001) found that by applying a laser field (pump) through a nonlinear coupler, a well-known unstable system could be stabilized by interspersing the nonlinear regions with regions of suitably chosen linear evolution. The generation of down-converted light can be strongly suppressed. By increasing the strength of the observation performed by the signal mode on idler mode and vice versa, the evolution is frozen and system tends to remain in its initial state. This stabilization regime can be considered a quantum Zeno effect, in that the measurements essentially affect and change the original dynamics.

## 3. NOISE-INDUCED TRANSITION

### *3.1. Basic Concept of Noise-Induced Transition*

A weak random parametric excitation can result in large deviations of some nonlinear systems away from an equilibrium state, taking the system to a new state. Physicists have been interested during the last few decades in the qualitative changes in the properties of a nonlinear system due to weak noise and the transformation of an unstable equilibrium state to a stable one and vice versa (see, e. g., Landa, 1996; Landa and McClintock, 2000). Other effects can be manifested in the occurrence of multi-stability or multimodality (Horsthemke and Lefever, 1984; Smythe et al., 1983; and Simui, 2002), and noise-induced phase transition (Landa and Zaikin, 1996, 1997, 1998). Pontryagin et al. (1933), Stratonovich (1963, 1967), and Landa and Stratonovich (1962) treated the problem of transitions from one stable state to another stable state under the influence of weak noise as a statistical problem of the probability of the first attainment of a boundary of a Brownian particle moving in a given potential field.

Noise-induced phase transitions are characterized by a qualitative change of the state of a system as the intensity of the noise increases. This change is usually manifested in the appearance of new extrema in the system response pdf or the disappearance of old ones (Horsthemke and Lefever, 1984 and Fedchenia, 1984). It also can take the form of either stabilization or destabilization of system equilibrium states (Landa, 1996; Landa and Zaikin, 1997). Kapitsa (1951a,b), Landau and Lifshitz (1969), and Landa (1996) have considered pendulum dynamics under random excitation as an example of noise-induced multi-stability.

The sharp transition of the bifurcation point in the presence of noise is interpreted as the bifurcation of a “most probable value” of the system response described by the maximum of its pdf (Horsthemke and Mansour, 1976; Horsthemke and Lefever, 1977; Arnold et al., 1978; Lefever and Horsthemke, 1979a, b; Morita et al., 1980; Brand and Schenzle, 1980; Graham, 1989). The analysis is straightforward for one-dimensional models (Kitahara et al. 1980), but becomes difficult for multi-dimensional systems (Ebeling and Schimansky-Geier, 1989). Physicists are usually interested in estimating the relaxation time in the vicinity of the transition (Leung, 1988; Jackson et al., 1989; Mannella et al., 1990; Ciuchi et al., 1993). On the other hand, engineers and mathematicians (Moshchuk et al., 1995; Roy, 1995; Arnold, 1998) are interested in estimating the probability structure of the system near transition.

Consider the state vector Itô equation

$$\frac{\mathrm{d}\mathbf{X}(t)}{\mathrm{d}t} = \mathbf{f}(\mathbf{X}, t, \Gamma) + \varepsilon \mathbf{g}(\mathbf{X}, \Gamma)\boldsymbol{\xi}(t) \tag{21}$$

where $\mathbf{X}(t) = \{x_1, x_2, ..., x_n\}^T$ is an n-dimensional vector whose elements represent displacements and velocities of the system response, $\mathbf{f} = \{f_1, f_2, ..., f_n\}^T$ is the drift vector, $\mathbf{g}$ is the diffusion matrix, and the random vector $\boldsymbol{\xi}(t) = \{\xi_1(t), \xi_2(t), ..., \xi_m(t)\}^T$ is defined as the formal derivative of the multi-dimensional Brownian motion process $\mathbf{B}(t)$, i.e., $\boldsymbol{\xi}(t) = \mathrm{d}\mathbf{B}(t)/\mathrm{d}t$, $\varepsilon$ is a small parameter depending on some physical constants of the system, $\Gamma$ is some control parameter, and $T$ denotes transpose. Let the system be in equilibrium, which may be given by the trivial solution $\mathbf{X} = 0$ for any $\Gamma$ and $\varepsilon \geq 0$. For $\varepsilon = 0$, let $\Gamma_c$ be the critical value above which the equilibrium position loses its stability and bifurcates into another nearby equilibrium or steady state solution. Since the motion is random, such a steady state cannot exist in a deterministic sense. However, as $t \to \infty$, the solution $\mathbf{X}(t)$ approaches zero for $\Gamma < \Gamma_c(\varepsilon)$ and reaches a steady state (statistically stationary) solution $\mathbf{X}(t) \neq 0$ for $\Gamma \geq \Gamma_c(\varepsilon)$, then we can say that bifurcation takes place at the bifurcation point $\Gamma_c(\varepsilon)$. If the solution $\mathbf{X}(t)$ of equations (21) has a pdf $p(\mathbf{X}, t, \Gamma)$, then it satisfies the Fokker-Planck Kolmogorov (FPK) equation:

$$
\begin{aligned}
\frac{\partial p(\mathbf{X}, t, \Gamma)}{\partial t} &= L^{\varepsilon} p(\mathbf{X}, t, \Gamma) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[a_i(\mathbf{X}, t, \Gamma) p(\mathbf{X}, t, \Gamma)\right] \\
&+ \frac{\varepsilon^2}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[b_{ij}(\mathbf{X}, t, \Gamma) p(\mathbf{X}, t, \Gamma)\right] \qquad (22)
\end{aligned}
$$

subject to the initial condition $p(\mathbf{X}_0, \Gamma, 0) = \delta(\mathbf{X} - \mathbf{X}_0)$, where $a_i(\mathbf{X}, t, \Gamma)$ and $b_{ij}(\mathbf{X}, t, \Gamma)$ are the drift and diffusion coefficients of system (21), respectively. If $\mathbf{X}(t, \Gamma)$ has a stationary response then $p(\mathbf{X}, t, \Gamma) \to p_s(\mathbf{X}, \Gamma)$ as $t \to \infty$ which satisfies the elliptic equation

$$
L^{\varepsilon} p_s(\mathbf{X}, \Gamma) = 0 \qquad (23)
$$

which always has a degenerate (trivial) solution $p_s = 0$. If $p_s$ is a stationary pdf, it must satisfy the conditions

$$
p_s(\mathbf{X}, \Gamma) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} p_s(\mathbf{X}, \Gamma) \mathrm{d}\mathbf{X} = 1 \qquad (24)
$$

Therefore, in this case, a bifurcation for system (21) occurs at $\Gamma_c$ if and only if the problem (23) and (24) has a solution for $\Gamma \geq \Gamma_c$, which ceases to exist for $\Gamma < \Gamma_c$. It must be pointed out that the existence of a solution $p_s$ to the system (23) and (24) was used as a criterion for the bifurcation of the nonlinear stochastic equation (21). It should not be confused with bifurcation of the linear equation (23) itself, since for the deterministic elliptic equation (Keller and Antman, 1968) bifurcation is normally associated with a nonlinear boundary-value problem. Note that the trivial solution $\mathbf{X} = 0$ may be regarded as having a Dirac delta pdf $\delta(\mathbf{X})$. Thus, if it exists, it bifurcates into a solution $p_s(\mathbf{X}, \Gamma)$. Therefore, the definition given above may be called "bifurcation in (stationary) distribution." Bifurcation in distribution can be estimated for nonlinear Hamiltonian systems whose steady state pdf can be determined in an exact form; see for example Soize (1991) and Ibrahim and Yoon (1996).

If $p_s(\mathbf{X}, \Gamma)$ exhibits a change in the number of extrema as a result of a change in the control parameter, $\Gamma$, the associated bifurcation is referred to as a "noise-induced transition."

### *3.2. Noise-Induced Transition in One-Dimensional Systems*

There is a significant difference between the response of systems with additive noise and systems with multiplicative noise. In chemical reactors, Caroli et al. (1981) indicated that for additive stochastic processes in a double-well potential (bi-stable), two main physical behaviors occur. The first is the exchange of population between the two wells, which leads to a switch from local to global equilibrium. If the barrier is very high with respect to the diffusion coefficient, the Kramers' relaxation time (Kramers, 1940) becomes extremely long. From the point of view of a chemist, a very high barrier means that the corresponding chemical reaction does not, for practical purposes, take place. The second is known as the activation process or the Suzuki enhancement of fluctuation (Suzuki, 1981). When starting from the top of the barrier, the time required to reach the final equilibrium is called the decay time, and it is after this that the activation process begins. Faetti et al. (1982) showed that, under multiplicative noise, the escape from a well will change from the small relaxation value of the Kramers theory into the large relaxation rate of the Suzuki regime. They also indicated that the time required to achieve equilibrium in a well after a sudden application of multiplicative noise (the activation time) is very much shorter than the Kramers relaxation time.

Under multiplicative random excitation the transition point is shifted depending on the intensity of the excitation. Now consider simple one-dimensional nonlinear systems coupled to a rapidly fluctuating noise described by the Itô stochastic differential equation

$$\mathrm{d}x(t) = f(x, \Gamma, t)\mathrm{d}t + \sigma g(x, \Gamma, t)\mathrm{d}B(t) \tag{25}$$

where $\Gamma$ is a control parameter, $\sigma$ is the intensity of the Brownian motion $B(t)$, and $f$ and $g$ are linear and nonlinear functions (respectively) of the state coordinate $x$. The FPK equation of (25) is (Horsthemke and Lefever, 1989)

$$\frac{\partial p(x, \Gamma, t)}{\partial t} = -\frac{\partial}{\partial x}\left[f(x, \Gamma)p(x, \Gamma, t)\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\left[g^2(x, \Gamma)p(x, \Gamma, t)\right] \tag{26}$$

where $p(x, \Gamma, t)$ is the transition pdf. The stationary solution of equation (26) is

$$p_s(x, \Gamma) = \frac{p_0}{g^2(x, \Gamma)} \exp\left[\frac{2}{\sigma^2}\int_0^x \frac{f(u, \Gamma)}{g^2(u, \Gamma)}\mathrm{d}u\right] \tag{27}$$

where $p_0$ is a constant obtained from the normalization condition. If the stochastic differential equation (SDE) is written in the Stratonovich sense it takes the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x, t) + \frac{\sigma^2}{2}\frac{\partial g(x, \Gamma)}{\partial x}g(x, \Gamma) + \sigma g(x, t)\xi(t) \tag{28}$$

where the Itô correction term $\frac{\sigma^2}{2}\frac{\partial g(x,\Gamma)}{\partial x}g(x,\Gamma)$ is referred to as the noise-induced drift (Risken, 1989). It owes its origin to the fact that during a change of $\xi(t) = \mathrm{d}B(t)/\mathrm{d}t$ there is a corresponding change in $x(t)$ and the average value $\langle g(x,t)\xi(t)\rangle$ does not vanish. The corresponding stationary pdf is

$$p_s(x,\Gamma) = \frac{p_0}{g(x,\Gamma)}\exp\left[\frac{2}{\sigma^2}\int_0^x \frac{f(u,\Gamma)}{g^2(u,\Gamma)}\mathrm{d}u\right] \tag{29}$$

It may happen that an Itô SDE admits a stationary solution while a Stratonovich version does not, or vice versa, since the formulae $p_s(x,\Gamma)$ differ by a factor $1/g(x,\Gamma)$.

If the deterministic equation admits more than one steady state, then stable and unstable states alternate. If there are two or more stable stationary states, then the state space is divided into non-overlapping regions known as the *domains* (or *basins*) *of attraction* of the various stable states. Obviously the steady states are the extrema of the potential. The stable steady states correspond to the minima of the potential and the unstable steady states to the maxima.

In the presence of external random excitation the sharp delta peaks of the response *pdf* will be broadened. The response *pdf* has a maximum at the coordinate that corresponds to the minimum of the potential and has a certain spread around this value, the breadth of which depends on the strength of the external excitation. If there is more than one minimum and if there is no effective upper bound on the external fluctuations, then the response pdf will be multimodal with peaks corresponding to the various minima of the potential. In order to determine when a transition occurs, the response pdf should be monitored for qualitative changes, which can be carried out by selecting a suitable indicator. This can be demonstrated by considering the first-order differential equation of the pitchfork bifurcation (Horsthemke and Lefever, 1984, 1989)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Gamma x - x^3 = f(x,\Gamma) \tag{30}$$

In the deterministic case, the steady state is given by the roots

$$x = 0, \quad \text{and} \quad x = \pm\sqrt{\Gamma} \quad \text{if} \quad \Gamma > 0 \tag{31}$$

A stability analysis shows that $x = 0$ is a stable fixed point for $\Gamma < 0$ and loses its stability at the critical value $\Gamma_c = 0$, where two new stable branches $x = \pm\sqrt{\Gamma}$ bifurcate. Introducing an additive random white noise gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Gamma x - x^3 + \sigma\xi(t) \tag{32}$$

The corresponding pdf of system (32) is

$$p(x) = p_0 \exp\left[\frac{2}{\sigma^2}\left(\Gamma\frac{x^2}{2} - \frac{x^4}{4}\right)\right] \tag{33}$$

It is not difficult to verify that for $\Gamma < \Gamma_c$, $p(x, \Gamma)$ consists of a single peak centered on $x = 0$, i.e., the stable deterministic steady state. For $\Gamma > 0$, the stationary pdf consists of two peaks centered on $+\sqrt{\Gamma}$ and $-\sqrt{\Gamma}$, i.e., on the stable states. Clearly, the system described by equation (32) undergoes a transition, at $\Gamma_c = 0$, quite similarly to the deterministic system (30). However, this transition is not reflected by the moments of the stationary pdf. The mean value and all higher odd moments are zero for all values of the bifurcation parameter, $\Gamma$, and the even-order moments are infinitely often differentiable with respect to $\Gamma$ (Horsthemke and Lefever, 1989). Thus for the SDE (28) the extrema of the steady state pdf of the system are obtained from the roots of the equation

$$f(x, \Gamma) + \frac{\sigma^2}{2} \frac{\partial g(x, \Gamma)}{\partial x} g(x, \Gamma) = 0 \tag{34}$$

If the noise is additive, i.e., $g(x, \Gamma) =$ constant, the extrema always coincide with the deterministic steady-state.

Gang and Kai-Fen (1992) studied the probability structure of the modified nonlinear Langevin equation

$$\dot{x} = x - x^3 + |x|^\beta \xi(t) \tag{35}$$

where the exponent $\beta$ is not necessarily an integer and $\xi(t)$ is a Gaussian white noise with power spectral density $2D$.The essential feature of equation (35) is that the unstable singular point of the deterministic system is identical to the singular point of the stochastic system. The FPK equation of system (35) yields the following stationary solution

$$p_s(x) = \begin{cases} p_0 \, |x|^{-\beta} \, Exp\left[\dfrac{1}{D}\left(\dfrac{1}{2-2\beta} x^{2-2\beta} - \dfrac{1}{4-2\beta} x^{4-2\beta}\right)\right] & \text{for } \beta \neq 1 \\ p_0 \, |x|^{\frac{1}{D}-1} \, Exp\left[\dfrac{-x^2}{2D}\right] & \text{for } \beta = 1 \end{cases} \tag{36}$$

The extrema of the pdf can be obtained by solving the algebraic equation

$$x^{2-2\beta} - x^{4-2\beta} - \beta D = 0 \tag{37}$$

Depending on the values of $D$ and $\beta$, Gang and Kai-Fen (1992) obtained closed form expressions for the roots of (34). The following three cases summarize the main results:

1) For $D \ll 1$ and $\beta < 0$, the maxima of $p(x)$ are repelled from the origin associated with a deep and narrow probability dip around the origin.
2) For $D \ll 1$ and $\beta > 1$, the maxima are attracted closer to the origin. The distribution around the origin is rather flat with zero value.
3) For $D \ll 1$ and $0 < \beta < 1$, the probability peak appears right at the deterministic unstable point due to the singular multiplicative noise.

Noise-induced transition can be examined using either the adiabatic elimination method (Haken, 1983, 1987; Gardiner, 1983, 1984; Risken, 1989) or stochastic averaging techniques (Stratonovich, 1963; Khasminskii, 1966, 1968; Horsthemke and Lefever, 1980). In the adiabatic elimination method the system usually possesses widely different response times and one may not be interested in the response behavior of those coordinates with a fast time scale. The fast time scale variables can be eliminated using the method of adiabatic elimination. This method is based on the slave principle (Haken, 1983) which states that "long-living systems slave short-living systems." In the modem theory of nonlinear dynamics the concept of adiabatic elimination is equivalent to the center manifold reduction (Carr, 1981; Arnold, V., 1988). In stochastic averaging the functions with higher-order multiple phase angles represent rapid oscillations or higher harmonics in the solution for the slowly varying amplitude and phase shift. When considering only the system stationary response, the high frequency oscillations have a localized effect (in time) and do not contribute significantly to the average behavior of the system over a long period of time. In this case one can therefore eliminate the oscillatory effects and simplify the equations of motion by introducing a near-identity transformation. This method has been extended to include second-order approximation by Hijawi et al. (1997a,b).

### *3.3. Noise-Induced Transition in One-Dimensional Mapping*

The onset of chaotic motion in dynamical systems can frequently be analyzed in terms of a discrete-time map with quadratic nonlinearity (Colet and Eckmann, 1980). In the logistic model, external noise induces the transition from periodic to chaotic behavior (Mayer-Kress and Haken, 1981). The effect of external noise results in broadening of the pdf and the power spectrum, and an increase in the Lyapunov exponent. Matsumoto and Tsuda (1983) studied the noise effect on other types of maps encountered in real chemical reactions. They observed that a transition from chaos to order takes place as the noise level increases. This type of transition was referred to as "noise-induced order." For a one-dimensional cubic map disturbed by a noise, Arecchi et al. (1984) showed that the Lyapunov exponent constitutes two separate contributions corresponding to the attracting and repulsive regions.

Another way to change the bifurcation structure of a quadratic map is to introduce linear time dependence into the control parameter. Kapral and Mandel (1985) and Morris and Moss (1986) examined the bifurcation structure and its postponement in the quadratic maps

$$x_{t+1} = \Gamma x_t (1 - x_t) \tag{38}$$

$$x_{t+1} = 1 - \Gamma x_t^2 \tag{39}$$

where the parameter $\Gamma$ is swept with velocity v

$$\Gamma \to \Gamma_t = \Gamma_0 + \mathrm{v}t \tag{40}$$

Note that there are no fixed point solutions when $\mathrm{v} \neq 0$ except for the trivial solution, $x_t = 0$. Each bifurcation is postponed to a larger parameter value, $\Gamma_n^*$. Without sweeping, each bifurcation occurs at a well defined parameter value, $\Gamma_n$, which corresponds to the transition from the orbit of period $2^{n-1}$ to the one of $2^n$, but with sweeping of the parameter,

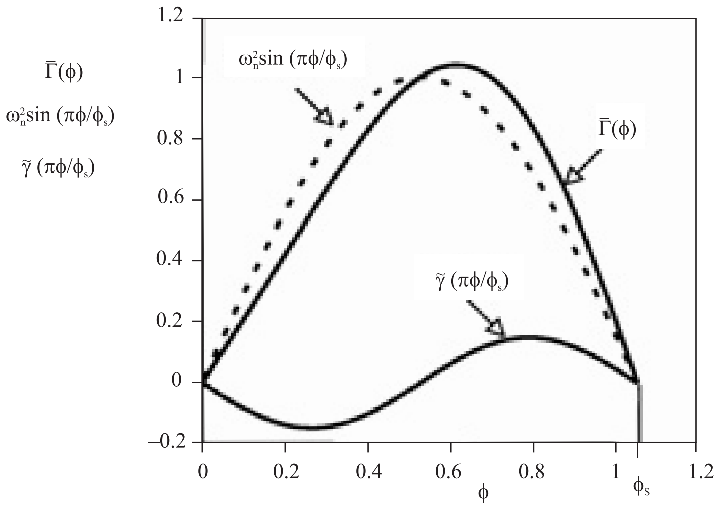



Figure 12. Dependence of the restoring moment $\bar{\Gamma}(\varphi)$ and its components on the roll angle $\phi$ showing the angle of capsizing $\phi_s = 1.05 rad$ and $\omega_n = 1$.

$\Gamma_n^* > \Gamma_n$. Kapral and Mandel (1985) and Morris and Moss (1986) studied this postponement using a linearized bifurcation theory and experimentally using an electric circuit model of the quadratic map $x_{t+1} = 1 - \Gamma x_t^2$. It was found that the postponement was dependent on the sweeping velocity, v, such that $\Delta\Gamma_n^* = \Gamma_n^* - \Gamma_n \approx \sqrt{\mathrm{v}}$. In addition, there is a hysteresis effect near the bifurcation points, with postponements occurring (in opposite directions) for both forward and reverse sweeps of $\Gamma$.

### *3.4. Noise-Induced Transition in Ship Roll Motion*

The uncoupled roll motion of a ship in irregular seas may be modeled by the nonlinear differential equation of motion

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + c_1 \frac{\mathrm{d}\phi}{\mathrm{d}t} + c_2 \frac{\mathrm{d}\phi}{\mathrm{d}t}\left|\frac{\mathrm{d}\phi}{\mathrm{d}t}\right| + M(\phi) = \xi(t) \tag{41}$$

where $\phi$ is the ship roll angle, $\xi(t)$ is the wave excitation moment, $c_1$ is the linear damping coefficient, $c_2$ is the nonlinear Morison type damping coefficient, $M(\phi)$ is the restoring moment which is nonlinear function of $\phi$ as shown in Figure 12. It indicates that the restoring moment vanishes at the capsizing angle, $\phi_s$. It is possible to represent $M(\phi)$ in the form (Moshchuk et al., 1995b)

$$M(\phi) = \omega_n^2 \sin\left(\pi \frac{\phi}{\phi_s}\right) + \tilde{\gamma}\left(\pi \frac{\phi}{\phi_s}\right) \tag{42}$$

where $\omega_n$ is the ship roll natural frequency and the function $\tilde{\gamma}$ accounts for the difference between the exact function $M(\phi)$ and $\omega_n^2 \sin(\pi\phi/\phi_s)$. Note that the first term in (42) and the conservative part of equation (41) establish the equation of motion of the simple pendulum. Introducing the non-dimensional parameters

$$\begin{aligned} \tau &= \left(\omega_n \sqrt{\frac{\pi}{\phi_s}}\right) t, \quad q = \phi \frac{\pi}{\phi_s}, \quad \varepsilon\zeta = \frac{c_1}{\omega_n \sqrt{\pi/\phi_s}}, \quad \varepsilon\eta = \frac{c_2 \pi}{\phi_s}, \\ \varepsilon\gamma(q) &= \frac{\tilde{\gamma}(q)}{\omega_n^2}, \quad \frac{\xi(t)}{\omega_n^2} = \sqrt{\varepsilon} W(\tau) + \varepsilon\mu \end{aligned} \tag{43}$$

where $0 \leq \varepsilon \ll 1$ is a small parameter, $\zeta$, $\eta$, and $\mu$ are some constants of order one, and $W(\tau)$ is a white noise of intensity $2D$. Equation (41) can be written in the form

$$q'' + \varepsilon\zeta q' + \sin q + \varepsilon\eta q' \left|q'\right| - \varepsilon\gamma(q) = \varepsilon\mu + \sqrt{\varepsilon} W(\tau) \tag{44}$$

It is convenient to write equation (44) in terms of the phase space coordinates

$$q' = u, \quad u' = -\sin q - \varepsilon\zeta u - \varepsilon\eta u\,|u| + \varepsilon\gamma(q) + \varepsilon\mu + \sqrt{\varepsilon} W(\tau) \tag{45}$$

where a prime denotes differentiation with respect to the non-dimensional time parameter $\tau$. Setting $\varepsilon = 0$ in equation (45) gives the well-known free oscillation of the simple pendulum

$$q' = u, \quad u' = -\sin q \tag{46}$$

with the Hamiltonian

$$H = \frac{u^2}{2} + 2\sin^2\frac{q}{2} \tag{47a}$$

or

$$q'^2 = 2\left(H - 2\sin^2\frac{q}{2}\right) \tag{47b}$$

Each level of energy $H$ corresponds to a periodic closed orbit in the phase space $\{q, u\}$. Level $H = 2$ corresponds to the unstable equilibrium position $q = \pm\pi$ and is characterized by two different homoclinic orbits. Periodic motion is only restricted in the domain $(q, u)|H \leq H_c$; where $H_c = 2 - \tilde{h}$ is the critical energy level above which capsizing takes place and the trajectories will be structurally unstable, and $\tilde{h} > 0$ is sufficiently small. The general solution of equations (46) for $H < 2$ can be written in the form

$$\tau = \pm \int_{q_0}^{q} \frac{\mathrm{d}q}{2\sqrt{k^2 - \sin^2(q/2)}}$$

where $k = \sqrt{H/2}$ and $q_0$ is the initial value of the angle $q$. For $k^2 < 1$, the angle $q$ can take any value within a range governed by the condition $k^2 - \sin^2(q/2) = 0$. In this case $q$ is given by the roots $q_1 = -2\sin^{-1} k$ and $q_2 = 2\sin^{-1} k$. Accordingly, the angle $q$ fluctuates between $q_1$ and $q_2$ resulting in the corresponding period of oscillation

$$\tau = \int_{q_1}^{q_2} \frac{\mathrm{d}q}{\sqrt{k^2 - \sin^2(q/2)}}$$

The general solution of equations (46) can be expressed in terms of elliptic functions. First introducing the transformation

$$\sin(q/2) = k \sin\psi, \qquad \text{and} \qquad u = 2k\cos\psi \tag{48}$$

where $\sin\psi = \pm\sqrt{2H - u^2}/\sqrt{2H}$, and differentiating the first relation in (48) with respect to $\tau$ and substituting into (47b) gives

$$\psi'^2 = 1 - k^2 \sin^2 \psi$$

The solution of this equation takes the form

$$\tau = \int_0^{\psi} \frac{\mathrm{d}\bar{\psi}}{\sqrt{1 - k^2 \sin^2 \bar{\psi}}} = F(\psi, k) \tag{49}$$

where $F(\psi, k)$ is an incomplete elliptic integral of the first kind. Solution (49) assumes that the initial value of the angle is zero. The upper limit $\psi$ is referred to as the amplitude of the function $F(\psi, k)$ and is usually written as $\psi = am\tau$. Relations (48) can be written in terms of the elliptic functions

$$u = 2k\cos(am\tau) = 2kcn\tau, \qquad q = 2\sin^{-1}(ksn\tau)$$

where $sn\tau$ and $cn\tau$ are Jacobian elliptic functions, and $k$ is the modulus. The period given by solution (49) represents one quarter of the period of one oscillation of the pendulum when $\psi = \pi/2$. In this case the period of one oscillation is given in terms of the complete elliptic integral $K(k)$ of the first kind

$$T = 4\int_0^{\pi/2} \frac{\mathrm{d}\bar{\psi}}{\sqrt{1 - k^2 \sin^2 \bar{\psi}}} = 4K(k)$$

For $k = 1$ the system is governed by two homoclinic orbits and the corresponding equilibria $\pm\pi$ are unstable. The coordinates $q$ and $u$ are related to $H$ and $\theta$ through the following relationships

$$\sin(q/2) = ksn\theta, \qquad p = 2kcn\theta$$

Note that $(H, \theta)$ and $(H, \theta + 4K(k))$ correspond to the same position of the system and $\theta = \theta \,(\mathrm{mod}\, 4K(k))$. The time derivative of $H$ is

$$H' = uu' + u\sin q = \varepsilon\left[-\zeta u^2 - \eta u^2|u| + \gamma(q)u + \mu u + D\right] + \sqrt{\varepsilon} u W(\tau) \tag{50}$$

including the Wong-Zakai correction. Averaging of the drift and diffusion expressions gives

$$\begin{aligned} < H' > &= \varepsilon\left[-\zeta < u^2 > -\eta < u^2|u| > + < \gamma(q)u > +\mu < u > +D\right] \\ &+ \sqrt{\varepsilon < u^2 >} W(\tau) \end{aligned} \tag{51}$$

where $< >$ denotes averaging over one cycle with respect to the unperturbed motion.

It is known that the phase space of the unperturbed system splits into oscillatory domain $H < 2$ and spinning domain $H > 2$ with either $u > 0$ or $u < 0$. So averaging is different in oscillatory and spinning domains. Evaluating the averages in equation (51) gives (Moshchuk et al., 1995b)

$$\begin{aligned} < u^2 > &= \frac{1}{T}\oint p^2 \mathrm{d}\tau = \frac{1}{T}\oint p\mathrm{d}q = \frac{2}{T}\int_{-2\sin^{-1}k}^{2\sin^{-1}k} \sqrt{2\left(H - 2\sin^2\frac{q}{2}\right)}\mathrm{d}q \\ &= 2(H-2) + \frac{4E\left(\sqrt{H/2}\right)}{K\left(\sqrt{H/2}\right)} \quad \text{for} \quad 0 < H < 2 \\ < u^2 > &= 2H\frac{E\left(\sqrt{2/H}\right)}{K\left(\sqrt{2/H}\right)} \quad \text{for} \quad H > 2 \\ < u > &= 0 \quad \text{for} \quad 0 < H < 2, \quad \text{or} \\ < u > &= \sqrt{H/2}\frac{\pi}{K\left(\sqrt{2/H}\right)}\,\mathrm{sgn}(u) \quad \text{for} \quad H > 2 \\ < u^2|u| > &= \frac{1}{T}\oint_{H=const} p^2|p|\,\mathrm{d}\tau = \frac{1}{T}\oint_{H=const} p|p|\,\mathrm{d}q \\ &= \frac{2}{\mathrm{K}\left(\sqrt{\mathrm{H}/2}\right)}\left[(H-1)\cos^{-1}(1-H) + \sqrt{2H - H^2}\right] \quad \text{for} \quad 0 < H < 2 \\ < u^2|u| > &= \pi\frac{\sqrt{2H}\,(H-1)}{\mathrm{K}\left(\sqrt{\mathrm{H}/2}\right)} \quad \text{for} \quad H > 2 \\ &\quad \text{and} \;\; < \gamma(q)u >= 0 \end{aligned} \tag{52}$$

The stationary solution of the FPK equation of system (51) is

$$p(H) = p_0 f(H) Exp \left[ -\frac{\zeta H}{D} - \frac{\eta}{D} \int_0^H \frac{< u^2|u| >}{< u^2 >} \mathrm{d}h + \frac{\mu}{D} \int_0^H \frac{< u >}{< u^2 >} \mathrm{d}H \right] \tag{53}$$

where $p_0$ is the normalization condition and $f(H) = (1/2)K\left(\sqrt{H/2}\right)$ for $H < 2$, or $f(H) = \sqrt{2/H}K\left(\sqrt{2/H}\right)$ for $H > 2$. The function $p(H)$ goes to infinity as $H \to 2$. It is convenient to write the response pdf in terms of the original variables $q$ and $u$ by using the transformation

$$p(q, u) = p(H, \theta) \left| \frac{\partial(H, \theta)}{\partial(q, u)} \right| \tag{54}$$

which can be determined by substituting the explicit expressions for $H$ and $\theta$ in terms of $q$ and $u$. This process results in the following expression for the stationary pdf

$$p(q, u) = \bar{p}_0 Exp \left[ -\frac{\zeta H}{D} - \frac{\eta}{D} \int_0^H \frac{< u^2|u| >}{< u^2 >} \mathrm{d}h + \chi(H) \right] \tag{55}$$

where $\chi(H) = 0$, for $0 < H < 2$, or

$$\chi(H) = \left\{ \frac{2\pi\mu}{\sqrt{2}D} \int_0^H \frac{\mathrm{d}h}{\sqrt{h}E(\sqrt{2/h}} \right\} \operatorname{sgn} u \quad \text{for} \quad H > 2 \tag{56}$$

and $H$ should be replaced by its expression in terms of $q$ and $u$.

The function $p(q, u)$ depends on the energy level $H$, and for $H > 2$ it also depends on the sign of $u$ (i.e., whether the pendulum rotates clockwise or counter-clockwise). It follows from relations (55) and (56) that for sufficiently low values of the absolute value of the moment mean $|\mu|$ there is only one maximum at $q = u = 0$. However, if $|\mu|$ exceeds a certain critical value, then there is another maximum of the pdf. Figure 13 shows three dimensional plot of $p(q, u)$ for $u > 0$, $\zeta = \eta = 0.5$, $2D = 1$ and excitation mean value $\mu = 4$. The figure reveals two peaks; one is located at the equilibrium point $(q, u) = (0.0, 0.0)$, and the other on the opposite side if $u$ was less than 0. Figure 14 shows the pdf as a function of $H$ for two values of $\mu$ (1.5 and 4). For $\mu = 1.5$ the pdf has only one peak, at $H = 0$, and exhibits singularity at $H = 2$. For $\mu = 4$, the response pdf has the same singularity but possesses two peaks; one at $H = 2$, and another at $H = 3.5$.

In order to obtain the bifurcation in probability we need to determine the condition for

$$\frac{\partial p}{\partial H} = 0 \quad \text{for} \quad H > 2 \tag{57}$$

This condition gives

$$\frac{\pi\eta}{2\zeta}\bar{k} + \frac{\pi\mu}{4\zeta}\bar{k} - \frac{\pi\eta}{\zeta}\frac{1}{\bar{k}} = E(\bar{k}) \tag{58}$$

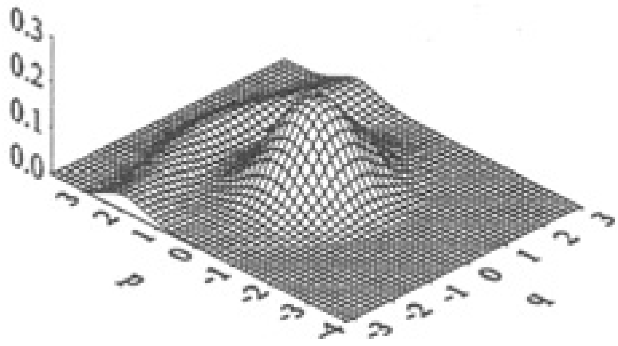


Figure 13. Probability density function in the spinning regime (Moshchuk et al., 1996).

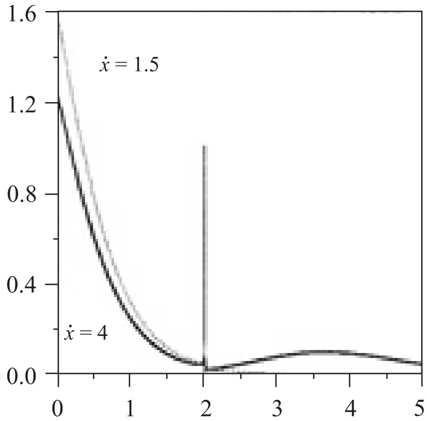



Figure 14. Probability density function showing two peaks and singularity at $H = 2$ (Moshchuk et al., 1996).

where $\bar{k} = \sqrt{2/H}$ is the modulus, $0 < \bar{k} < 1$. The behavior of the function $E(\bar{k})$ is well known. It gradually decreases from $\pi/2$ at $\bar{k} = 0$ to 1 at $\bar{k} = 1$. The left-hand side of (58) has negative value at $\bar{k} = 0$. Therefore equation (58) has a solution if and only if the left-hand side at $\bar{k} = 1$ has a value greater than 1, i.e.,

$$\frac{\pi\mu}{4\zeta} - \frac{\pi\eta}{2\zeta} > 1 \quad \text{or} \quad \mu_c = \frac{4\zeta}{\pi} + 2\eta \tag{59}$$

provided $\mu \geq \mu_c$ which is the condition for the existence of another peak in the spinning regime. If $u < 0$ then this condition becomes $-\mu > (4\zeta/\pi) + 2\eta$. Figure 15 shows the dependence of $E(\bar{k})$ and the left-hand side of (58) on the parameter $\bar{k}$ for different values of $\mu$. The points of intersection establish the points of bifurcation for each excitation mean value $\mu$.

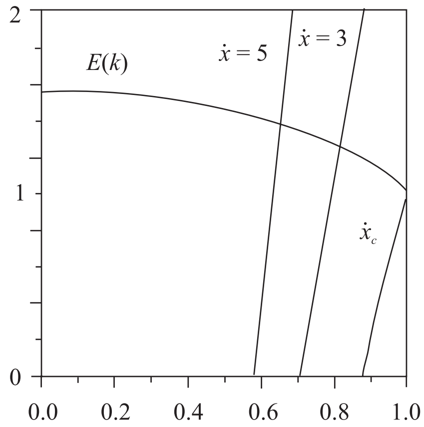



Figure 15. Regions of single and two peaks of the response pdf (Moshchuk et al., 1996).

## 4. STABILIZATION VIA MULTIPLICATIVE NOISE

### *4.1. Postponement of Bifurcation*

The influence of random fluctuations of the control parameter on the system's dynamic behavior near instability has been experimentally studied by many physicists (see, e.g., De Kepper and Horsthemke, 1978; Kabashima et al., 1979; Kabashima and Kawakub, 1979; Kai et al., 1979; Fujisaka and Grossmann, 1981; Seshadri et al., 1981; Kawakubo et al., 1981; Broggi et al., 1986; Van den Broeck and Mandel, 1987; Torrent and San Miguel, 1988a, b). For example, Kabashima et al. (1979) and Kabashima and Kawakub (1979) observed a phase transition induced by external noise, using a parametric oscillator driven by a random current, supplied in addition to the sinusoidal pumping current. Shapiro (1993) indicated that the stabilization by multiplicative noise should always be followed by an intermittency when the noise has non-zero intensity at low frequencies.

A new type of oscillatory-to-nonoscillatory transition was observed. In addition, other studies reported the following main results:

1) The transition, or instability, remains sharp after the noise is superimposed on the control parameter. The new bifurcation point is the average of the fluctuating control parameter.
2) The threshold of instability is shifted to larger values of the bifurcation parameter by the application of noise.
3) The transition point is also distinguished as the point where relaxation and fluctuations of the system response exhibit slowing down, i.e., the associated rates vanish at this point. Suzuki et al. (1981) established a general relationship between a phase transition in non-equilibrium systems and this slowing of relaxation.

For nonlinear dynamical systems with widely separated time scales, Graham and Schenzle (1982) used the adiabatic elimination method (Haken, 1983) and showed the possibility of shifting the bifurcation point using broad-band multiplicative noise. Their model is identical to the free vibration of a buckled beam represented by the Duffing oscillator with negative linear stiffness

$$\frac{1}{\gamma}\ddot{x} + \dot{x} - \Gamma x + bx^3 = 0 \tag{60}$$

where $b$ is the nonlinear stiffness coefficient, $1/\gamma = t_s$ is a short time scale, $\Gamma = \Gamma_0 + \xi(t)$, such that $1/\Gamma_0 = t_\ell$ is a long time scale, and $\xi(t)$ is a random process generated from the Ornstein-Uhlenbeck filter

$$\dot{\xi} + \Delta\omega\xi = \sqrt{2\Delta\omega D}W(t) \tag{61}$$

where $W(t)$ is a white noise process of intensity $2D$ and $\Delta\omega$ is the filter bandwidth. The oscillator described by equation (60) is characterized by a very small inertia, such that if $1/\gamma = 0$ and $\xi(t) = 0$, the instability is associated with symmetry breaking, since the states $x = \pm\sqrt{\Gamma_0/b}$ for $\Gamma_0 > 0$ are not invariant under the $x \to -x$ symmetry of

$$\dot{x} = \Gamma_0 x - bx^3 \tag{62}$$

The influence of the random process $\xi(t)$ on postponing the bifurcation point was studied for two limiting cases: $\lambda = \Delta\omega/\gamma \ll 1$ and $\lambda \gg 1$.

**Case 1:** $\lambda = \Delta\omega/\gamma \ll 1$. This case implies that the short time scale $1/\gamma = t_s$ is much smaller than the correlation time $1/\Delta\omega$ of the noise. In this case, the threshold value of the bifurcation point vanishes, i.e., $\Gamma_0 = 0$.

**Case 2:** $\lambda = \Delta\omega/\gamma \gg 1$. This case results in the bifurcation point moving to the value

$$\Gamma_0 \geq \frac{D}{\Delta\omega} \tag{63}$$

The analysis of Graham and Schenzle (1982) showed that the shift of the threshold value depends on the ratio $\lambda = \Delta\omega/\gamma$. For intermediate values of $\lambda$ neither the Itô nor Stratonovich interpretations hold and it was shown that the mean square response of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Gamma_0 x - bx^3 + \xi(t)x \tag{64}$$

for intermediate values of $\lambda$ is given by the relationship

$$\lim_{t\to\infty} E[x^2(t)] = \frac{1}{b}\begin{cases} \Gamma_0 - \dfrac{D/\gamma}{1+(\Delta\omega/\gamma)} & \text{for } \Gamma_0 > \dfrac{D/\gamma}{1+(\Delta\omega/\gamma)} \\ 0 & \text{for } \Gamma_0 \leq \dfrac{D/\gamma}{1+(\Delta\omega/\gamma)} \end{cases} \tag{65}$$

where the bifurcation occurs at

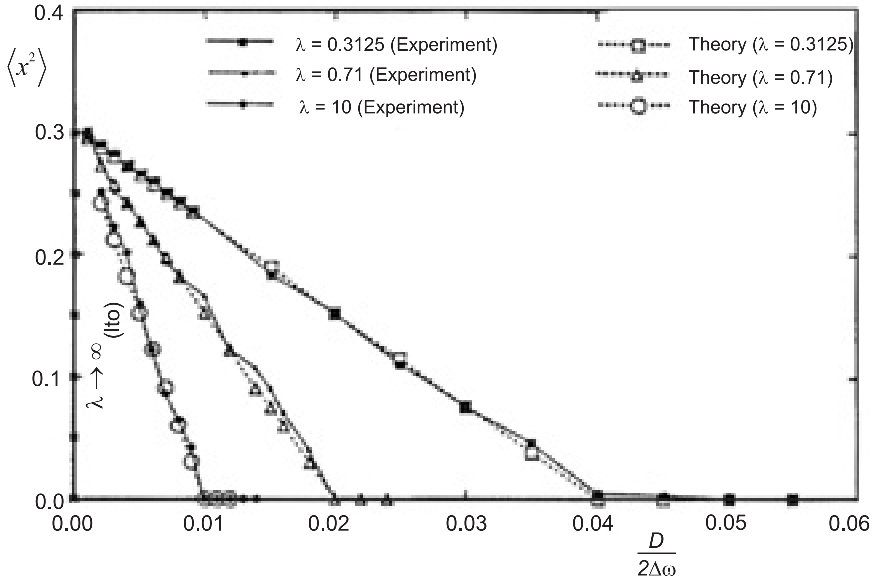



Figure 16. Dependence of the mean square response on excitation level for large values of the Stratonovich parameter $\lambda = \Delta\omega/\gamma$ as estimated numerically by Bellah and Shinozuka (1991).

$$\Gamma_0 = \frac{D/\gamma}{1 + (\Delta\omega/\gamma)} \tag{66}$$

This result agreed with the Stratonovich interpretation, producing the equivalent SDE (60) for $\gamma \to \infty$ in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(\Gamma_0 - \frac{D/\gamma}{1 + (\Delta\omega/\gamma)}\right) x - bx^3 + \xi(t)x \tag{67}$$

where $\dfrac{D/\gamma}{1 + (\Delta\omega/\gamma)}$ corresponds to the Itô correction term (see also Stratonovich, 1989).

Billah and Shinozuka (1991) considered the same system as Graham and Schenzle (1982) and presented digital simulation results to resolve the controversies reported in the literature concerning the interpretation of Stratonovich and Itô calculus and adiabatic elimination. Figures 16 and 17 show the dependence of the mean square on $D/\Delta\omega$ for large and small values of $\lambda$, respectively. It can be seen that for large values of $\lambda$ the results are close to the Itô interpretation ($\lambda \to \infty$) while the corresponding limiting case for those with small $\lambda$ represents the Stratonovich interpretation ($\lambda \to 0$).

Multiplicative noise may stabilize a dynamical system which is originally unstable. Khasminskii (1980) constructed a two-dimensional unstable system that can be stabilized by introducing two independent white noise sources. Arnold (1979) presented another class of unstable deterministic systems which can be stabilized by applying a single non-white random noise.

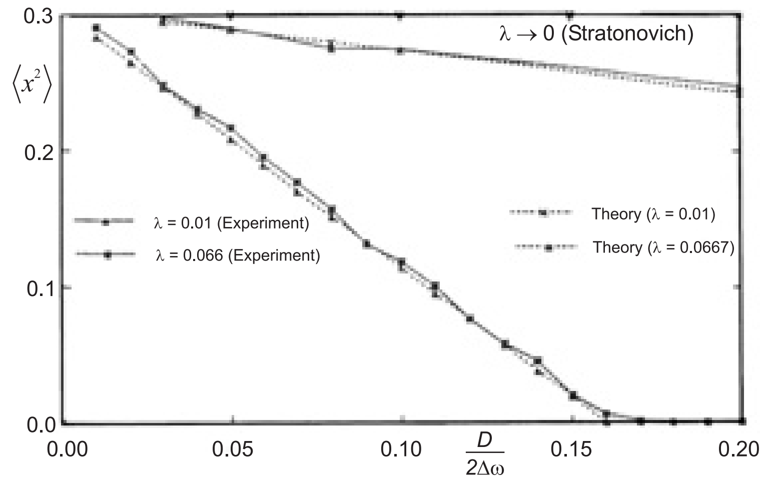



Figure 17. Dependence of the mean square response on excitation level for small values of the Stratonovich parameter $\lambda = \Delta\omega/\gamma$ as estimated numerically by Bellah and Shinozuka (1991).

Analog simulation experiments were conducted by Smythe et al. (1983a,b), and Faetti et al. (1984, 1985). The results of Smythe et al. (1983a,b) gave different trends from those predicted by Graham and Schenzle (1982) and the digital simulation by Billah and Shinozuka (1991). For example, Smythe et al. (1983a,b) observed that their results were in agreement with the Stratonovich algorithm. Faetti et al. (1984), on the other hand, obtained analog simulation results which were in agreement with those of Graham and Schenzle (1982). They also showed that the synergism of nonlinearities, energy pumping, and inertia can lead to a transition from an overdamped regime to an inertial one. Festa et al. (1984) showed that in the high noise region there are significant discrepancies between the steady state distribution of the system response described by equation (64) as predicted by Graham and Schenzle and the results of the analog simulation. Later Grigolnini (1989) and Manella et al. (1986) demonstrated that these discrepancies are due to the inherent presence of inertia in the system.

Lindner et al. (2004) presented an assessment of the dynamic behavior of some specific models driven by Gaussian white noise. The systems considered possess a single stable rest state which can be simply an equilibrium (or fixed) point or a small-amplitude (subthreshold) limit cycle. The large excursions of the system variables produced by strong enough perturbations of this state are often called spikes, and their occurrence is referred as “firing”. This terminology is borrowed from neuroscience and the associated scenarios are well described by Lindner et al. (2004). In neuroscience, the electrical discharge of a nerve cell’s membrane potential is commonly called an action potential or the firing of a spike. The generation of a spike is considered a one-way passage through a sequence of stable and unstable states. The resting state, being dynamically stable, can be left only by a sufficiently strong external excitation. The firing and the following refractory (recovery) states are unstable in the sense that the system escapes from them even in the absence of external excitations. While the escape from the resting state strongly depends on the external input, the passage through the firing and refractory states possesses only a weak dependence on the external excitation.

The occurrence of stable spiking trajectories requires a dynamical system close to a bifurcation toward a limit cycle regime. Accordingly, the system must possess at least two variables. Models exhibiting this property were first proposed for neuronal spike generation (see, e.g., Fitzhugh, 1961; Nagumo et al., 1962; Morris and Lecar, 1981). In these models a fast variable is driven above a threshold level by an external excitation. After that it keeps growing and approaching a second metastable state. Another variable, a recovery variable, acting on a slower time scale, destabilizes the excited state of the fast voltage variable, bringing it back to its rest state by means of negative feedback. When the recovery variable relaxes, the initial state is reestablished.

### *4.2. Stabilization of Inverted Pendulum*

#### 4.2.1. Parametric Random Excitation

The stabilization of the inverted pendulum via random parametric excitation has received several analytical studies and a few experimental verifications. For example, Bogdanoff and Citron (1965a,b) found experimentally that the inverted pendulum can be stabilized by a two frequency parametric excitation as the smaller frequency increases, up to a certain value above which the pendulum loses its stability. It was not possible to stabilize the inverted pendulum with Gaussian random noise, regardless of the spectral shape. However, it was possible to stabilize the pendulum for brief intervals with Gaussian random noise clipped at one sigma interval.

These observations were later confirmed analytically under high-frequency parametric excitation of the inverted pendulum (see, e.g., Levi and Weckesser, 1995; Hemp and Sethna, 1968; Sethna, 1972, 1973; Mitchell, 1972; Howe, 1974). Hemp and Sethna (1968) showed that the effect of the motion of the support at two frequencies close to each other can destabilize the pendulum. Sethna (1972) indicated that it is not possible to stabilize an inverted pendulum unless the power spectrum of the parametric fluctuations is zero in the neighborhood of the origin. Sethna (1973) extended the averaging method and obtained a criterion for the stability of a pendulum subjected to arbitrary rapid motion of the support. By including the low-frequency components of the support motion, he obtained results identical to those observed by Bogdanoff and Citron (1965a,b). Mitchell (1972) employed the method of averaging to examine the stability of an inverted pendulum subjected to deterministic and stochastic motion of the support. He found additional regions of instability when the difference between two frequencies of an almost periodic motion is small. He extended the analysis by including a second order approximation, which resulted in an additional instability region when one excitation frequency is about twice the other. His analog computer simulations revealed the possibility of stabilizing the pendulum when the damping in the support is sufficiently large and the motion of the support is a stochastic process with a high-pass power spectral density.

Howe (1974) confirmed the results of Bogdanoff and Citron (1965a,b), and derived another condition involving the power spectrum of the effective support excitation at twice the frequency of the induced stabilized oscillation of the pendulum about the upward vertical. Howe's analysis is a combination of the averaging method and an extension of the energy distribution concept used to study wave propagation in random media. He obtained an integro-differential equation that describes the energy content of each frequency component of the smoothed variable pendulum motion. The equation indicates that an interchange of energy occurs between various frequency components of the oscillation, and is caused by interactions (scattering) with support motions. Another important feature of the equation is that it accounts for resonance interactions between various frequency components of the pendulum motion and the Fourier components of the support motion. The Fourier components have twice the frequency of the pendulum.

The results of the analog computer simulation reported by Mitchell (1972) motivated Sethna and Orey (1980) to examine the possibility of stabilizing the pendulum by allowing the support motions to be sample functions of a stochastic process with a continuous spectrum. By using the averaging method they showed that the pendulum can indeed be stable in the inverted position when the support motion is stationary in the wide sense and has a continuous spectrum. This result was confirmed by Prussing (1981), who proved that the inverted pendulum can be stabilized in terms of the first- and second-order moments if the support motion is a physical white noise. However, if the excitation is a mathematical white noise, the pendulum cannot be stabilized, as proved by Nevel'son and Khasminskii (1966) and by Nakamizo and Sawaragi (1972).

The analytical treatment of the pendulum problem is well documented by Landa and McClintock (2000). They considered parametric and horizontal excitations of the pendulum whose equation of motion takes the form

$$\ddot{\theta} + 2\zeta\dot{\theta} + \left(1 + \omega^2 z(t)\right)\sin\theta = 0 \tag{68}$$

where $\omega^2 z(t)$ is the acceleration of the pendulum support in terms of the gravitational acceleration. The random process $z(t)$ is a narrow band noise generated from the second-order filter

$$\ddot{z} + 2\alpha\dot{z} + \omega^2 z = W(t) \tag{69}$$

where $W(t)$ is a white noise of intensity $D$, $1 \ll \alpha \ll \omega$, and the autocorrelation function of $z(t)$ is

$$R_z(\tau) \approx \sigma^2 e^{-\alpha\tau}\cos\omega\tau \tag{70}$$

where $\sigma^2 = D/(2\omega^2\alpha)$ is the variance of $z(t)$. Landa and McClintock (2000) showed that the pendulum in its inverted position can be stabilized if $\sigma^2$ is sufficiently large. If the power spectrum density of $z(t)$ does not contain components in zones of parametric resonance, fluctuations of the response $\theta$ caused by the random excitation will be small. This can be shown by setting $\theta = \langle\theta\rangle + \delta\theta$ in equation (68), where $\delta\theta \ll \langle\theta\rangle$, which gives the following two equations

$$\left\langle\ddot{\theta}\right\rangle + 2\zeta\left\langle\dot{\theta}\right\rangle + \sin\langle\theta\rangle + \omega^2\cos\langle\theta\rangle\langle z(t)\delta\theta\rangle = 0 \tag{71}$$

$$\delta\ddot{\theta} + 2\zeta\delta\dot{\theta} + \cos\langle\theta\rangle\,\delta\theta + \omega^2 z(t)\sin\langle\theta\rangle = 0 \tag{72}$$

The steady state solution of equations (71) and (72) corresponding to the inverted position is

$$\langle\theta\rangle = \pi, \quad \text{and} \quad \delta\theta = 0 \tag{73}$$

The stability of this solution was obtained by linearizing equations (71) and (72) with respect to small deviations from the solution (73), i.e., $\psi = \langle\theta\rangle - \pi$. The linearized equations are

$$\ddot{\psi} + 2\zeta\dot{\psi} - \psi - \omega^2 \langle z(t)\delta\theta\rangle = 0 \tag{74}$$

$$\delta\ddot{\theta} + 2\zeta\delta\dot{\theta} - \delta\theta - \omega^2 z(t)\psi = 0 \tag{75}$$

The steady state solution of equation (75) is

$$\delta\theta(t) = \frac{\omega^2}{2\sqrt{1+\zeta^2}} \int_{-\infty}^{t} \left[e^{\varpi_1(t-t')} - e^{\varpi_2(t-t')}\right] z(t')\psi(t')\mathrm{d}t' \tag{76}$$

where $\varpi_{1,2} = -\zeta \pm \sqrt{1+\zeta^2}$ are the roots of the frequency equation $\varpi^2 + 2\zeta\varpi - 1 = 0$. This solution is used to evaluate $\langle z(t)\delta\theta\rangle$ in equation (74) as

$$\langle z(t)\delta\theta\rangle = \frac{\omega^2}{2\sqrt{1+\zeta^2}} \int_{-\infty}^{t} \left[e^{\varpi_1(t-t')} - e^{\varpi_2(t-t')}\right] \left\langle z(t)z(t')\right\rangle \psi(t')\mathrm{d}t' \tag{77}$$

Introducing the time shift $\tau = t - t'$, and taking into account the fact that $\psi$ does not vary significantly during the correlation time of the random process $z(t)$, equation (77) takes the form

$$\begin{aligned} \langle z(t)\delta\theta\rangle &= \frac{\omega^2\psi(t)}{2\sqrt{1+\zeta^2}} \int_{-\infty}^{t} \left[e^{\varpi_1\tau} - e^{\varpi_2\tau}\right] \langle z(t)z(t+\tau)\rangle \,\mathrm{d}\tau \\ &= -\frac{\sigma^2\omega^2\left[\omega^2 - (\varpi_1+\alpha)(\varpi_2+\alpha)\right]}{\left[\omega^2 + (\varpi_1+\alpha)^2\right]\left[\omega^2 + (\varpi_2+\alpha)^2\right]}\psi(t) \end{aligned} \tag{78}$$

Since $\omega \gg 1, \zeta, \alpha$, equation (78) may be written in the approximate form

$$\langle z(t)\delta\theta\rangle \approx -\sigma^2\psi(t) \tag{79}$$

Substituting equation (79) into equation (74) gives the following approximate equation in the deviation, $\psi$,

$$\ddot{\psi} + 2\beta\dot{\psi} + \left(\omega^2\sigma^2 - 1\right)\psi = 0 \tag{80}$$

where $\sqrt{\omega^2\sigma^2 - 1}$ is the natural frequency of small oscillations of the pendulum about the inverted position. Equation (74) reveals that the mean deviation of the pendulum from its inverted position will decay, and the inverted position is stable provided that the frequency $\sqrt{\omega^2\sigma^2 - 1}$ is real, i.e., if

$$\omega^2\sigma^2 \geq 1 \tag{81}$$

### 4.2.2. Effect of Additive Noise

In addition to condition (81), Landa and McClintock (2000) showed the existence of a maximum and a minimum in the response pdf in the presence of additive noise $\chi(t)$ of intensity $D_1$. In this case, the pendulum equation of motion takes the form

$$\ddot{\theta} + 2\zeta\dot{\theta} + \left(1 + \omega^2 z(t)\right)\sin\theta = \chi(t) \tag{82}$$

where $\chi(t)$ is a wide-band random process with negligible spectral density at the frequency $\omega$. Equation (82) does not have a stationary pdf for its response vector coordinates $\theta$ and $\dot{\theta}$. The "slow" variable $z(t)$ and "fast" variable $\delta\theta$ are used. Since $z(t)$ is a narrow-band random process, it can be represented in the form

$$z(t) = z_1(t)\cos\omega t + z_2(t)\sin\omega t \tag{83}$$

where $z_1(t)$ and $z_2(t)$ are "slow" variables. One can also write $\delta\theta$ in the form

$$\delta\theta = A(t)\cos\omega t + B(t)\sin\omega t \tag{84}$$

where $A(t)$ and $B(t)$ are "slow" variables. Substituting $\theta(t) = \vartheta(t) + \delta\theta(t)$ and equation (83) into equation (82), and equating the slowly varying component and the coefficients of $\cos\omega t$ and $\sin\omega t$, the following three equations are obtained

$$\ddot{\vartheta} + 2\zeta\dot{\vartheta} + \sin\vartheta + \frac{1}{2}\omega^2\cos\vartheta\,\langle Az_1 + Bz_2\rangle = \chi(t) \tag{85a}$$

$$\left(\omega^2 - \cos\vartheta\right)A - 2\zeta\omega B = \omega^2 z_1\sin\vartheta \tag{85b}$$

$$\left(\omega^2 - \cos\vartheta\right)B + 2\zeta\omega A = \omega^2 z_2\sin\vartheta \tag{85c}$$

Recalling that $\omega \gg 1, \zeta$, equations (85b) and (85c) may be written in the approximate form

$$A \approx z_1\sin\vartheta, \qquad \text{and} \qquad B \approx z_2\sin\vartheta \tag{86a,b}$$

Substituting equations (86a,b) into equation (85a) gives

$$\ddot{\vartheta} + 2\zeta\dot{\vartheta} + \sin\vartheta + \frac{1}{2}\omega^2\sin 2\vartheta\,\sigma^2 = \chi(t) \tag{87}$$

where $\left\langle z_1^2 + z_2^2 \right\rangle = 2\sigma^2$. The Fokker-Planck equation of equation (87) may be written in the form

$$\begin{aligned}\frac{\partial p(\vartheta,\dot{\vartheta})}{\partial t} &= -\left[\dot{\vartheta}\frac{\partial p(\vartheta,\dot{\vartheta})}{\partial \vartheta} - \left(\sin\vartheta + \frac{\omega^2\sigma^2}{2}\sin 2\vartheta\right)\frac{\partial p(\vartheta,\dot{\vartheta})}{\partial \dot{\vartheta}}\right] \\ &+ \left[\frac{\partial\left(2\zeta\dot{\vartheta}\,p(\vartheta,\dot{\vartheta})\right)}{\partial\dot{\vartheta}} + \frac{D_1}{2}\frac{\partial^2 p(\vartheta,\dot{\vartheta})}{\partial\dot{\vartheta}^2}\right] \qquad (88)\end{aligned}$$

The stationary solution of equation (88) can be obtained by setting each bracket on the right-hand side to zero. The resulting solution is

$$p(\vartheta,\dot{\vartheta}) = p_0 e^{\dot{\vartheta}^2/\bar{D}_1} p(\vartheta) \qquad (89)$$

and

$$p(\vartheta) = p_1 e^{\left[\frac{2}{\bar{D}_1}\left(\cos\vartheta + \frac{\omega^2\sigma^2}{4}\cos 2\vartheta\right)\right]} \qquad (90)$$

where $p_0$ and $p_1$ are normalization constants, and $\bar{D}_1 = D_1/2\zeta$. Equation (89) has three extrema for $\dot{\vartheta} = 0$, $\vartheta = 0$, $\vartheta = \pi$, and ($\vartheta = \arccos(-1/\omega^2\sigma^2)$; provided $\omega^2\sigma^2 \gg 1$). For $\omega^2\sigma^2 \gg 1$, the pdf, $p(\vartheta)$, has two maxima at $\vartheta = 0$ and $\vartheta = \pi$, in addition to one minimum at $\vartheta = \arccos(-1/\omega^2\sigma^2)$. On the other had, if $\omega^2\sigma^2 < 1$, then $p(\vartheta)$ has only one maximum (at $\vartheta = 0$), and one minimum, at $\vartheta = \pi$. Figure 18 shows the pdf $p(\vartheta)/p_1$ for different values of $\omega^2\sigma^2$ and additive excitation level $\bar{D}_1 = 4$.

### *4.3. Stabilization of Nonlinear Ocean Structures*

The dynamic analysis of ocean structures involves the interaction of inertial forces, elastic forces, and hydrodynamic forces. Basically, the hydrodynamic forces induced on structural elements are nonlinear and a combination of inertial and drag contributions. Each force is time-dependent and depends on the structure's geometrical properties, fluid properties describing the flow field, and some unknown constants which must be determined experimentally. Furthermore, under large deflection these structures experience inertia and curvature nonlinearities. The hydrodynamic forces and the time variation of the system parameters are random in nature. Within the framework of the first-order stochastic averaging method (Ibrahim, 1985), one can predict stochastic stability boundaries, the first-passage problem, and the response pdf of systems with damping nonlinearity. Unfortunately, it fails to account for such factors as cubic inertia and stiffness nonlinearities because their effect is lost during the averaging procedure. The effect of such nonlinearities can only be determined by performing second-order averaging. Hijawi et al. (1997a) studied noise-induced stability (NIS) in off-shore structures subjected to random parametric excitation from vertical waves. By taking the power spectral density level of the parametric excitation as a control parame-

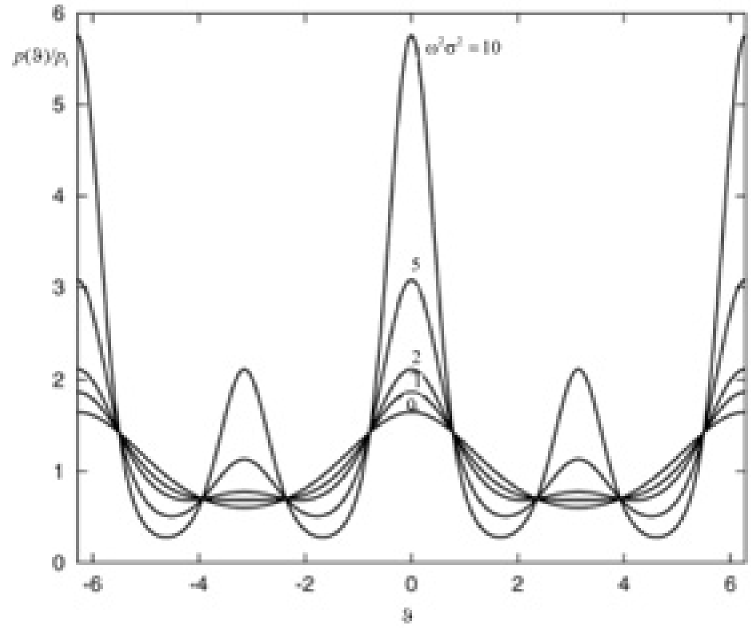


Figure 18. The response pdf for different values of parametric excitation level $\omega^2\sigma^2 = 0, 1, 2, 5$, and $10$, and for additive noise level, $D_1/2\zeta = 4$ (Landa and McClintock, 2000).

ter, the bifurcation diagram reveals a stabilizing effect of the mean square response of the structure as long as the inertia nonlinearity overcomes curvature nonlinearity.

Figure 19 shows an elastic beam of length $L$ carrying a mass $M$ at the top. The beam is subjected to two types of excitation. The first force is due to hydrodynamic loading, while the second is parametric, caused by two vertical wave loads which can result from the heaving motion of the top mass as it responds to the seaway. The equation of motion of the first mode of the system is obtained using Hamilton's principle together with Galerkin's method and takes the form

$$Y'' + Y + \varepsilon\left(\zeta Y' + c_1 Y'^2 Y + c_2 Y^3 - \lambda_1 q_0|q_0| - \lambda_2\,\mathrm{sgn}(q_0)Y'^2 + \lambda_3|q_0|Y'\right) + \sqrt{\varepsilon}Y\bar{W}(\tau) = \sqrt{\varepsilon}\lambda_4 q_0' \tag{91}$$

where $Y$ is the non-dimensional deflection of the top of the beam, a prime denotes differentiation with respect to the non-dimensional time parameter $\tau = \omega_n t$, $\omega_n$ is the first mode natural frequency, $\varepsilon = d/L$, $d$ is the characteristic width of the structure, $\varepsilon\zeta$ is the linear damping factor, and $c_1$ and $c_2$ are coefficients of nonlinear inertia and curvature, respectively. $\lambda_i$, $i = 1, 2, 3$, are the coefficients of nonlinear hydrodynamic drag, while $\lambda_4$ is the coefficient of the inertial hydrodynamic force. $\tilde{W}(\tau)$ is a dimensionless white random process representing the parametric excitation. $q_0 = u/(\omega_n d)$ is a non-dimensional fluid wave velocity at the free surface, where $u$ is the fluid velocity. $q_0$ may be represented by a

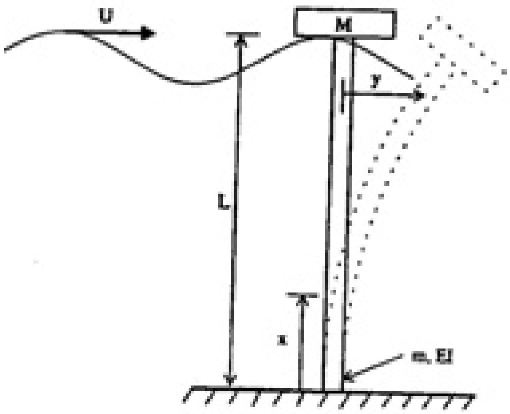



Figure 19. Schematic diagram of an elastic beam subjected to random ocean waves.

narrow-band random process and can be modeled as the output of the second-order shaping filter

$$q_0'' + 2\omega_f \zeta_f q_0' + \omega_f^2 q_0 = \xi(\tau) \tag{92}$$

where $\omega_f = \Omega_f / \omega_n$ is a non-dimensional filter frequency defined as the ratio of the filter frequency to the system natural frequency and $\zeta_f$ is the filter damping ratio. $\xi(\tau)$ is a white noise random process of intensity $\nu$. The power spectral density and mean square of the process $q_0(\tau)$ are, respectively,

$$S_{q_0}(\omega) = \frac{\nu}{2\pi} \frac{1}{\left(\omega_f^2 - \omega^2\right)^2 + \left(2\omega_f \omega \zeta_f\right)^2}, \qquad S_{q_0'}(\omega) = \omega^2 S_{q_0}(\omega) \tag{93a}$$

$$E[q_0^2] = \frac{\nu}{4\omega_f^3 \zeta_f} \tag{93b}$$

The average value $\langle |q_0| \rangle$ can be calculated by using the pdf of the filter equation (92)

$$\langle |q_0| \rangle = \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty |q_0| \exp\left(-\frac{q_0^2}{2\sigma^2}\right) \mathrm{d}q_0 \tag{94}$$

where $\sigma^2 = E[q_0^2]$ since the process $\xi(\tau)$ has a zero mean. More realistic shaping filters, such as the Pierson and Moskowitz spectrum (1964) and those reported in Moshchuk et al. (1995a,b) and Moshchuk and Ibrahim (1996) can be used to model the wave spectrum. However, the second-order filter (92) was adopted for the sake of simplification.

The solution of equation (91) may be assumed in the form

$$Y(\tau) = A(\tau)\cos\varphi(\tau), \quad \varphi(\tau) = \tau + \theta(\tau) \tag{95}$$

such that $Y'(\tau) = -A(\tau)\sin\varphi(\tau)$. Introducing equation (95) into equation (91) and following the procedure of the stochastic averaging method, the following standard amplitude and phase equations are obtained

$$\begin{Bmatrix} A' \\ \theta' \end{Bmatrix} = \varepsilon \begin{Bmatrix} f_1(A,\theta,\tau) \\ f_2(A,\theta,\tau) \end{Bmatrix} + \sqrt{\varepsilon}\begin{bmatrix} g_{11}(A,\theta,\tau) & g_{12}(A,\theta,\tau) \\ g_{21}(A,\theta,\tau) & g_{22}(A,\theta,\tau) \end{bmatrix} \tag{96}$$

where $f_1$ and $f_2$ stand for drift terms, while the functions $g_{ij}$ are diffusion terms. The nonlinear terms in $f_1$ and $f_2$ contain numerous products of sine and cosine functions with phase angle $\varphi(\tau)$. These terms may be expanded into series of sine and cosine functions at the multiple phase angles, $n\varphi$, $n = 0, 2, 4, \ldots$. We can therefore eliminate the oscillatory effects and simplify the equations of motion by introducing the near-identity transformation

$$A(\tau) = \bar{A}(\tau) + \varepsilon u(\bar{A},\bar{\theta},\tau), \qquad \theta(\tau) = \bar{\theta}(\tau) + \varepsilon\Theta(\bar{A},\bar{\theta},\tau) \tag{97}$$

where $\bar{A}(\tau)$, $\varphi(\tau) = \bar{\theta}(\tau) + \tau$, and $\bar{\theta}$ stand for non-oscillatory amplitude, phase angle, and phase shift, respectively. The process involves differentiating relations (97) and equating each result with the corresponding drift functions such that the average values are subtracted from $f_j$, $j = 1, 2$, to make sure that $u$ and $\Theta$ are oscillatory functions. Having obtained $u$ and $\Theta$, adding the contribution of the diffusion terms and following the second averaging procedure outlined in Hijawi et al. (1997a,b) yields the average equation for the amplitude

$$\bar{A}' = \frac{C_0}{\bar{A}} + C_1\bar{A} + C_3\bar{A}^3 + \sqrt{2C_0 + D_2\bar{A}^2}\,W_1(\tau) \tag{98}$$

where

$$\begin{aligned}
C_0 &= \varepsilon\left\{\frac{\pi}{2}\lambda_4^2 S_{q_0'}(1)\right\}, \quad C_3 = \varepsilon^2\left\{\frac{1}{32}(c_1 - 5c_2)\left[\zeta + \lambda_3\langle|q_0|\rangle + \pi S_{\bar{W}}(2)\right]\right\} \\
C_1 &= \varepsilon\left\{-\frac{1}{2}(\zeta + \lambda_3\langle|q_0|\rangle) + \frac{3\pi}{8}S_{\bar{W}}(2)\right\} + \varepsilon^2\left\{\frac{\pi}{32}\lambda_4^2(c_1 + 7c_2)S_{q_0'}(1)\right\} \\
D_2 &= \varepsilon\left\{\frac{\pi}{4}S_{\bar{W}}(2)\right\} - \varepsilon^2\left\{\frac{\pi}{16}\lambda_4^2(c_1 - 9c_2)S_{q_0'}(1)\right\}
\end{aligned}$$

Introducing the Hamiltonian $\bar{H} = \bar{A}^2$, and differentiating $\bar{H}$ according to the Ito formula, the averaged equation of the system energy takes the form

$$\bar{H}' = 4C_0 + (2C_1 + D_2)\bar{H} + 2C_3\bar{H}^2 + \sqrt{8C_0\bar{H} + 4D_2\bar{H}^2}\,W_1(\tau) \tag{99}$$

The stationary response pdf of equation (99) can be obtained by solving the stationary FPK equation, giving

$$p(\bar{H}) = K_0 \left(2C_0 + D_2\bar{H}\right)^{\kappa} \exp\left(\frac{C_3}{D_2}\bar{H}\right) \tag{100}$$

where $\kappa = -\left(\frac{3}{2} + 2\frac{C_0C_3}{D_2^2} - \frac{C_1}{D_2}\right)$ and $K_0$ is the normalization constant given (using the integration tables of, for example, Gradshteyn and Ryzhik, 1980) by

$$K_0 = \frac{(-C_3)^{\kappa+1} \exp\left(2C_0C_3/D_2^2\right)}{D_2^{2\kappa+1}\Gamma\left(\kappa+1, -2C_0C_3/D_2^2\right)} \tag{101}$$

where $\Gamma(a, x) = \int\limits_x^\infty e^{-t}t^{a-1}\mathrm{d}t$ is the incomplete gamma function. This function is monotonic and (for $a > 1$) rises from near zero to near unity in a range of $x$ centered on about $a-1$, and of width about $\sqrt{a}$. Note that the response pdf given by equation (100) includes the effects of inertia and curvature nonlinearities of the structure and the effects of two components of the hydrodynamic forces; $\lambda_3 |q_0| Y'$ and $\sqrt{\varepsilon}\lambda_4 q_0'$. However, it does not capture the effect of the two hydrodynamic drag components $-\lambda_1 q_0|q_0|$ and $-\lambda_2 \operatorname{sgn}(q_0)Y'^2$. These two terms have zero average; they have only a localized effect (in time) and do not contribute significantly to the average behavior of the system over a long period of time. Note that the normalization is valid provided $C_3 < 0$, or

$$(c_1 - 5c_2) < 0 \tag{102}$$

An interesting case of the response behavior of this system is the special case under parametric excitation. In the absence of hydrodynamic forces the averaged equation takes the form

$$\bar{H}' = \left(2C_1^* + D_2^*\right)\bar{H} + 2C_3^*\bar{H}^2 + \sqrt{4D_2^*\bar{H}^2}W_1(\tau) \tag{103}$$

where

$$C_1^* = \varepsilon\left\{-\frac{1}{2}\zeta + \frac{3\pi}{4}S_{\bar{W}}(2)\right\}, \quad C_3^* = \varepsilon^2\left\{\frac{1}{32}(c_1 - 5c_2)\left[\zeta + \pi S_{\bar{W}}(2)\right]\right\},$$
$$D_2^* = \varepsilon\left\{\frac{\pi}{4}S_{\bar{W}}(2)\right\}$$

The corresponding stationary pdf has the form

$$\begin{aligned} p(\bar{H}) &= K_0^*\bar{H}^{\left(\frac{C_1^*}{D_2^*}-\frac{3}{2}\right)} \exp\left(\frac{C_3^*}{D_2^*}\bar{H}\right), \\ K_0^* &= \left(-\frac{C_3^*}{D_2^*}\right)^{\left(\frac{C_1^*}{D_2^*}-\frac{1}{2}\right)} \frac{1}{\Gamma\left(\frac{C_1^*}{D_2^*}-\frac{1}{2}\right)} \end{aligned} \tag{104}$$

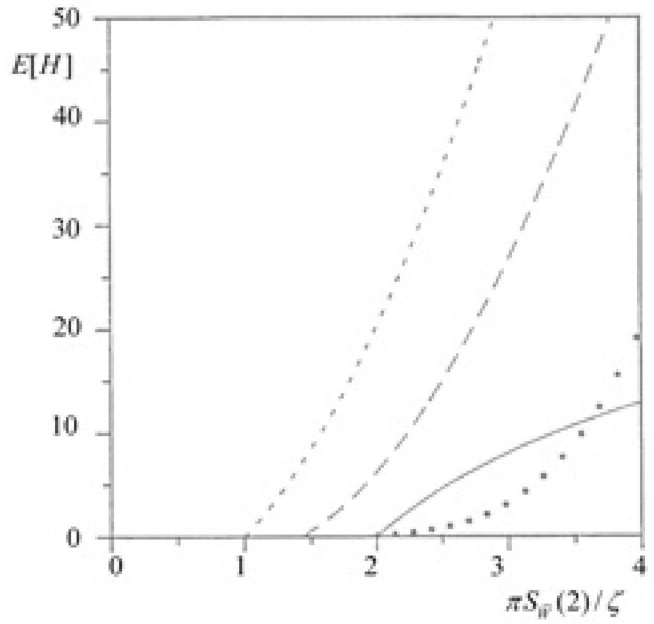



Figure 20. Dependence of the response mean energy on the parametric excitation level, for $\varepsilon = 0.005$, $\zeta = 4$, and $c_2 = 9$. - - - - Gaussian closure solution, — — —— non-Gaussian closure solution, —— second-order averaging, • • • • Monte Carlo simulation (Hijawi et al., 1997a).

The normalization constant is valid only under the two conditions

$$\text{(i)} \quad \frac{C_1^*}{D_2^*} - \frac{1}{2} > 0, \qquad \text{or} \qquad \frac{\pi S_{\bar{W}}(2)}{\zeta} > 2 \tag{105a}$$

$$\text{(ii)} \quad C_3^* < 0, \qquad \text{or} \qquad (c_1 - 5c_2) < 0 \tag{105b}$$

Finally, the expression of the mean value of the response energy has the form

$$E[\bar{H}] = -\frac{D_2^*}{C_3^*} \frac{\Gamma\left(\dfrac{C_1^*}{D_2^*} + \dfrac{1}{2}\right)}{\Gamma\left(\dfrac{C_1^*}{D_2^*} - \dfrac{1}{2}\right)} \tag{106}$$

The dependence of the response mean square $E[\bar{H}]$ on the parametric excitation level is shown by the solid curve in Figure 20. Figure 20 also shows the mean energy level as predicted by Gaussian and non-Gaussian closures and Monte Carlo simulation. It can be seen that there is a critical excitation level, give by relation (105a), above which the response mean square bifurcates to non-zero value. This level corresponds to the well-known ensemble stochastic stability condition of dynamic systems parametrically excited by a white noise.

The bifurcation point predicted by second-order averaging is in good agreement with the one estimated by the Monte Carlo simulation (shown by solid small circles).

In order to study the problem of NIT, one should consider the averaged equation of the response amplitude by setting the external excitation to zero in equation (98)

$$\bar{A}' = C_1^* \bar{A} + C_3^* \bar{A}^3 + \sqrt{D_2^* \bar{A}^2} W_1(\tau) \tag{107}$$

The corresponding stationary pdf is given by the formula

$$\begin{aligned} p(\bar{A}) &= K_1^* \bar{A}^{2\left(\frac{C_1^*}{D_2^*}-1\right)} \exp\left(\frac{C_3^*}{D_2^*}\bar{A}^2\right), \\ K_1^* &= 2\left(-\frac{C_3^*}{D_2^*}\right)^{\left(\frac{C_1^*}{D_2^*}-\frac{1}{2}\right)} \frac{1}{\Gamma\left(\frac{C_1^*}{D_2^*}-\frac{1}{2}\right)} \end{aligned} \tag{108}$$

In terms of system parameters the response pdf takes the form

$$\begin{aligned} p(\bar{A}) &= 2\left(-\frac{\varepsilon}{8}(c_1 - 5c_2)\left(1 + \frac{\zeta}{\pi S_{\bar{W}}(2)}\right)\right)^{\left(1-\frac{2\zeta}{\pi S_{\bar{W}}(2)}\right)} \bar{A}^{\left(1-\frac{4\zeta}{\pi S_{\bar{W}}(2)}\right)} \\ &\times \exp\left(\frac{\varepsilon}{8}(c_1 - 5c_2)\left(1 + \frac{\zeta}{\pi S_{\bar{W}}(2)}\right)\bar{A}^2\right) / \Gamma\left(1 - \frac{2\zeta}{\pi S_{\bar{W}}(2)}\right) \end{aligned} \tag{109}$$

Inspecting this result, one may establish the following three different response regimes, depending on the parametric excitation level $\pi S_{\bar{W}}(2)/\zeta$

1) *Zero motion* occurs for all excitation levels within the range $0 < \pi S_{\bar{W}}(2)/\zeta < 2$. This condition preserves the positive property of the response pdf as dictated by the argument of the Gamma-function in equation (109), from condition (105a). Physically the system linear damping overcomes the excitation energy input and the equilibrium remains stable.
2) *Partially developed random motion* (or *on-off intermittency*) takes place for all excitation levels within the range $2 < \pi S_{\bar{W}}(2)/\zeta < 4$. Within this range the peak of the response pdf occurs at zero response amplitude, as determined by the exponent of $\bar{A}$. The time evolution of the response passes through a sequence of zero and non-zero response periods, as revealed by Monte Carlo simulation. The periods of zero response decrease in length as the excitation level increases. This statement is based on numerical simulation results and experimental observations (Ibrahim and Heinrich, 1988).
3) *Fully developed random motion* occurs for all excitation levels exceeding 4, i.e., when $\pi S_{\bar{W}}(2)/\zeta > 4$. This regime is characterized by continuous random motion and the peak of the response pdf is shifted from zero amplitude.

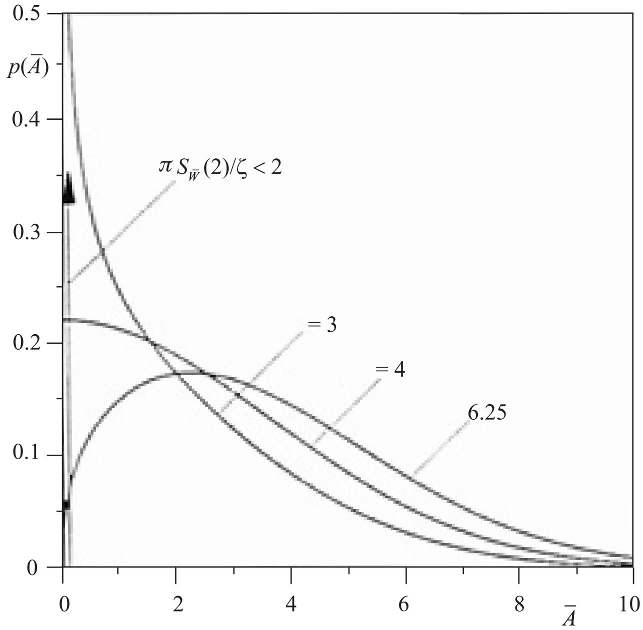



Figure 21. Probability density function of response amplitude for different values of parametric excitation level for $\varepsilon = 0.005$, $\zeta = 4$, $\lambda_4 = 10$, and $c_2 = 9$ (Hijawi et al., 1997a).

The response pdf $p(\bar{A})$ is shown in Figure 21 for different parametric excitation levels. Note that both on-off intermittency and NIT cannot be predicted by estimating the response moments because a great deal of information about the state of the system is not contained in the moments. The on-off intermittency can only be uncovered by using Monte Carlo simulation or experimental tests (Ibrahim, 1991). The problem of NIT can be examined using the qualitative change in the state of the system as outlined by Horsthemke and Lefever (1984, 1989). The extrema of the stationary pdf of the system are determined from the condition

$$C_1^* \bar{A} + C_3^* \bar{A}^3 - \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\bar{A}} \left( D_2^* \bar{A}^2 \right) = 0 \tag{110}$$

This equation yields

$$C_3^* \bar{A}^3 - \bar{A} \left( D_2^* - C_1^* \right) = 0 \tag{111}$$

which has three solutions given in terms of the system parameters in the form

$$\bar{A}_1 = 0, \qquad \bar{A}_{2,3} = \pm \sqrt{4 \frac{(4 - (\pi S_{\bar{W}}(2)/\zeta))}{\left[\varepsilon \left(c_1 - 5c_2\right) \left[1 + (\pi S_{\bar{W}}(2)/\zeta)\right]\right]}} \tag{112}$$

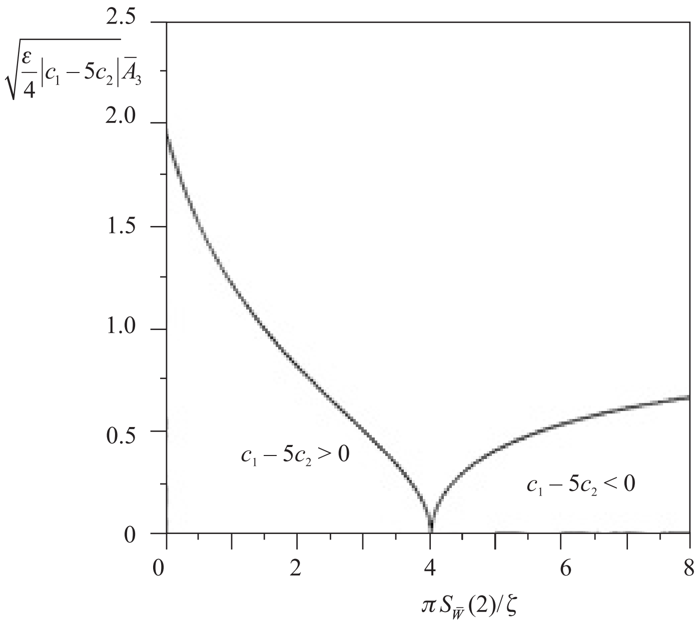
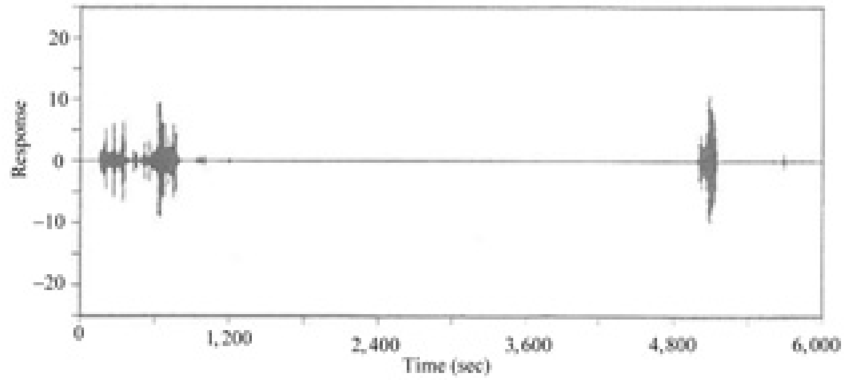



Figure 22. Bifurcation diagram for the extrema transition of the response pdf showing regions of stabilization/destabilization under parametric excitation (Hijawi et al., 1997a).

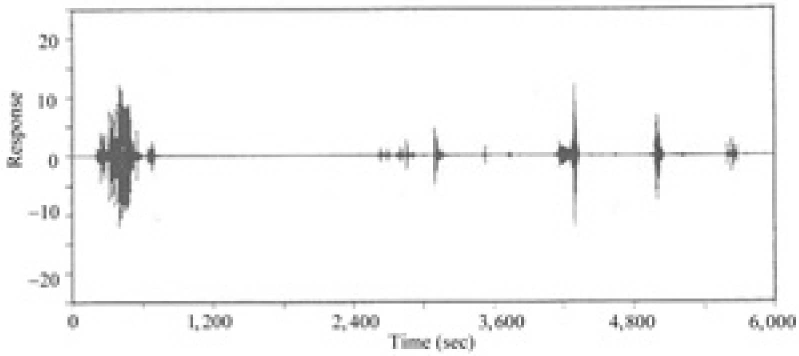
For $(c_1 - 5c_2) > 0$, there is one peak of the response pdf at $\bar{A}_1 = 0$ as long as $\pi S_{\bar{W}}(2)/\zeta < 4$. Above this level a transition of the response pdf takes place and the peak is determined by the second root of (112). A transition to the peak $\bar{A}_2$ takes place due to the multiplicative noise, and its value depends on whether the net value of the nonlinear inertia and stiffness expression $(c_1 - 5c_2)$ is greater or less than zero. A bifurcation diagram showing the dependence of the extrema on the excitation level is shown in Figure 22. It can be seen that $\pi S_{\bar{W}}(2)/\zeta = 4$ separates between the extrema of positive and negative nonlinear parameter $(c_1 - 5c_2)$. Figure 22 reveals the stabilization effect of the multiplicative noise on the originally unstable system when $(c_1 - 5c_2)$ only assumes positive values.

The effect of parametric colored noise on the stabilization of nonlinear bi-stable systems or on delaying the bifurcation point has been examined analytically (Graham and Schenzle, 1982; Zaghlache et al., 1989; Stocks, et al., 1989, 1990) and numerically (Billah and Shinozuka, 1990; Shinozuka and Billah, 1991).

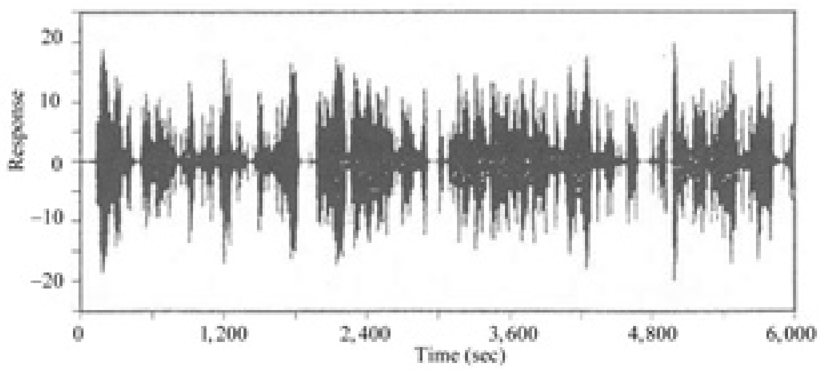
The time history records of the response as estimated by Monte Carlo simulation exhibit the well known phenomenon of on-off intermittency, which is similar to the one observed experimentally and referred to as “uncertain response motion” by Ibrahim and Heinrich (1988) and Yoon and Ibrahim (1995) under relatively small parametric excitation levels. Figures 23(a), (b), and (c) show a set of time history records of response for excitation levels $\pi S_{\bar{W}}(2)/\zeta = 1.95$, 2.0 and 3.0, respectively. This type of response corresponds to the class of partially developed motion discussed earlier. It can be seen that the periods of zero response decrease as the excitation level increases, until the response exhibits fully developed random motion where the excitation level is 6.25 (not shown). This type of on-off




(a) $\pi S_{gg}(2)/\zeta = 1.95$




(b) $\pi S_{gg}(2)/\zeta = 2.0$




(c) $\pi S_{gg}(2)/\zeta = 3.0$

Figure 23. Samples of time history records of response to different values of excitation level (Hijawi et al., 1997a).

intermittency is similar to a great extent to the type reported by Platt et al. (1993, 1994) and Heagy et al. (1994) who reported the on-off intermittency in a class of one dimensional maps that are multiplicatively coupled to either random or chaotic signals. In this case the on-off intermittency is a result of time-dependent forcing of a bifurcation parameter through a bifurcation point.

## 5. NOISE-ENHANCED STABILITY (NES)

The previous sections addressed some effects of multiplicative noise such as noise-induced phase transition, and stabilization (or destabilization) of originally unstable (or stable) systems. However, if the random excitation is additive, one would be interested in estimating the mean first passage time for the response to reach a target value. With an additive random noise, the mean exit time of a Brownian particle moving in a potential field usually decreases with noise intensity according to the Kramers formula (Kramers, 1940; Hänggi et al., 1990) or some universal scaling function of the same parameters (Colet et al., 1991; Miguel and Toral, 1997). Kuznetsov et al. (1997) and Kuznetsov (2002) presented some analyses of convective and absolute instabilities in one-dimensional Brusselator flow model and reaction-diffusion systems. The transition from one type of instability to another occurs at a critical value of the control parameter.

The dependence of the mean exit time on the noise intensity for metastable and unstable systems was revealed to have a resonance character (Hirsch et al., 1982). Noise can modify the stability of the system in a counterintuitive way such that the system remains in the metastable state for a longer time than in the deterministic case (Dayan et al., 1992; Casado and Morillo, 1994; Agudov and Malakhov, 1995, 1999; Agudov, 1998; Malakhov and Pankratov, 1996). The escape time has a maximum at some noise intensity. Note that the phenomenon of NES only results in an increase of the escape time rather than causing absolute stabilization of the originally unstable system. The escape rate of a particle over a fluctuating barrier, due to an external Ornstein-Uhlenbeck noise, in a double-well potential exhibits resonance at an optimum value of correlation time of fluctuation (Ghosh et al., 2005).

Related work involving noise-induced stability (NIS) includes several different types of applications: (i) Physical systems such as tunnel diodes (Mantegna and Spagnolo, 1996, 1998) and Josephson junctions (Malakhov and Pankratov, 1996; Al-Khawaja and Alsous, 2005) where the influence of thermal fluctuations on the superconductive state lifetime and the turn-on delay time for a single Josephson element with high damping was considered; (ii) chemical systems such as the one-dimensional return map of the Belousov-Zhabotinsky reaction in which one may be interested in the behavior of the length of the laminar region as a function of the noise intensity (Matsumoto and Tsuda, 1983); and (iii) biologically motivated models (Mielke, 2000; Dan et al., 2003; Dubkov et al., 2004) in which the overdamped motion of a Brownian particle moving in an asymmetric fluctuating potential exhibited NIS.

In an analytical study of an over-damped system with a time potential, Dayan et al. (1992) showed that the stability of an unstable system is enhanced in the presence of a noise of finite intensity. Mantegna and Spagnolo (1996) showed that this phenomenon is not crucially dependent on the exact shape of the potential of a system comprised of a series of biasing resistors with tunnel diode in parallel to a capacitor (see also Landauer, 1962, 1978; Mantegna and Spagnolo, 1994, 1995). Other studies (Dan et al., 1999; Mahato and Jayannavar, 1997, 1998; Wackerbrauer, 1998, 1999; Mielke, 2000; Mantegna and Spagnolo, 1996, 1998) revealed that the noise-induced slowing down and noise-induced stabilization are related to NES. Agudov and Spagnolo (2001) analytically estimated the escape time from a periodically driven metastable state for a piecewise linear potential. They found that for a fixed potential, the decay time of the unstable initial state can be dramatically increased by the presence of a small noise, depending on the initial condition of the system.

In addition to NES, the presence of external noise in nonlinear dynamical systems may result in some resonance-like phenomena such as SR (Gammaitoni et al., 1998; Anishchenk et al., 1999; Mantegna et al., 2001; and Wellens et al., 2004) and resonant activation (Doering and Gadoua, 1992; and Mantegna and Spagnolo, 2000).

We will focus on the counterintuitive dynamics of nonlinear systems in the presence of random noise. Specifically, this section will address the phenomenon of NES in some applications. The door to this phenomenon is the SR, the basic concept of which is discussed in the next subsection.

### *5.1. Stochastic Resonance (SR)*

The effect of very small noise excitations on the behavior of dynamical systems may cause a significant difference in the observable behavior of some systems, in which such systems act as a noise amplifier. Noise enhancement of the transmission of periodic or aperiodic signals via SR has been demonstrated in nonlinear systems. The concept of SR differs from the traditional definitions of linear and nonlinear resonances; it arises as a result of a cooperative effect in which a small periodic influence entrains external random noise. The cooperation between the internal noise and the external periodic force is manifested by a peak in the response power spectrum around the forcing frequency. The concept of SR was originally introduced independently by Benzi et al. (1981, 1982, 1983), and Nicolis (1982, 1993) (see also Nicolis and Nicolis (1981)) in dealing with the problem of periodically recurrent ice ages. It was pointed out by Benzi et al. (1981, 1982) that a weak periodic eccentricity in the earth's orbit can lead to more drastic weather changes than expected due to nonlinear coupling with environmental fluctuations. For a comprehensive assessment of SR and related issues the reader may consult Nicolis (1995), Moss (1994), Moss et al. (1993, 1994), Wiesenfeld and Moss (1995), and Gammaitoni et al. (1998). Some measures have been adopted in the literature to measure the transmission process receiving enhancement from noise. For periodic signals, these measures include the signal-to-noise ratio the amplitude of the coherent component in the noisy output (see, e.g., McNamara and Wiesenfeld, 1989; Gammaitoni et al., 1989; Jung and Hänggi, 1991; Moss et al., 1994; Chapeau-Blondeau, 1997b).

Benzi et al. (1981) considered the Langevin equation, subject to a periodic forcing $A\cos\Omega t$,

$$\mathrm{d}x = [x(\Gamma - x^2) + A\cos\Omega t]\mathrm{d}t + \sigma\,\mathrm{d}B(t) \tag{113}$$

where $B(t)$ is a Brownian-Weiner process. In the absence of the sinusoidal excitation, there are two stable solutions ($x_{1,2} = \pm\sqrt{\Gamma}$) and one unstable one ($x = 0$) at the bifurcation point.

Due to the random noise, the solution of equation (113), with $A = 0$, jumps at random times between the two stable steady states $x_{1,2} = \pm\sqrt{\Gamma}$. Benzi et al. (1981) estimated the exit time when the excitation amplitude $A$ is small compared with the bifurcation parameter $\Gamma^{3/2}$. The influence of the periodic force was studied for $t = 0$ and $t = \pi/\Omega$. In other words, they considered the two stochastic equations

$$\mathrm{d}x = [x(\Gamma - x^2) + A]\mathrm{d}t + \sigma \mathrm{d}B(t) \tag{114a}$$

$$\mathrm{d}x = [x(\Gamma - x^2) - A]\mathrm{d}t + \sigma \mathrm{d}B(t) \tag{114b}$$

Let the corresponding fixed points of equations (114a,b) be $\tilde{x}_1$, and $\tilde{x}_2$, respectively. The first moment of the first passage time of equations (114a,b) was estimated using the Kolmogorov backward equation (Pontryagin equation)

$$[x_0(\Gamma - x_0^2) \pm A]\frac{\mathrm{d}\langle T(x_0)\rangle}{\mathrm{d}x_0} + \frac{\sigma^2}{2}\frac{\mathrm{d}^2\langle T(x_0)\rangle}{\mathrm{d}x_0^2} = -1 \tag{115}$$

where $T(x_0) = \inf(t : x(t) = 0 \text{ and } x(0) = x_0 \in (-\infty, 0))$. The solutions of equation (115) corresponding to the fixed points $\tilde{x}_1$, and $\tilde{x}_2$ are, respectively,

$$\langle T(\tilde{x}_1)\rangle \cong \frac{\pi}{\Gamma\sqrt{2}} Exp\left[\frac{\Gamma^2}{2\sigma^2}\left(1 + \frac{4A}{\Gamma^{3/2}}\right)\right] \tag{116a}$$

$$\langle T(\tilde{x}_2)\rangle \cong \frac{\pi}{\Gamma\sqrt{2}} Exp\left[\frac{\Gamma^2}{2\sigma^2}\left(1 - \frac{4A}{\Gamma^{3/2}}\right)\right] \tag{116b}$$

where $\langle T(\tilde{x}_1)\rangle$ and $\langle T(\tilde{x}_2)\rangle$ are the mean exit times from the basins of attraction belonging to $\tilde{x}_1$ and $\tilde{x}_2$, respectively. The qualitative behavior of equation (113) can be described if one starts to observe the motion of the system at $t = 0$ with $x = \tilde{x}_1$. If $\langle T\rangle$ is the mean exit time from the basin of attraction, it follows that $\langle T(\tilde{x}_2)\rangle < \langle T\rangle < \langle T(\tilde{x}_1)\rangle$. If

$$\langle T(\tilde{x}_1)\rangle \geq \pi/\Omega \quad \text{and} \quad \langle T(\tilde{x}_2)\rangle \ll \pi/\Omega \tag{117a,b}$$

then $\langle T\rangle \approx \pi/\Omega$, while the variance of the exit time is of order $\langle T(\tilde{x}_2)\rangle$. Thus, with probability 1, the solution of equation (113) with initial condition $x(t = 0) = \tilde{x}_1$ will jump to the point $x = \tilde{x}_2$ at $t = \pi/\Omega$. It can also be seen that the solution will spend a period of time $\pi/\Omega$ in the new basin of attraction and at $t = 2\pi/\Omega$ the motion will jump to the point, $x(t = 2\pi/\Omega) = \tilde{x}_1$. In this case, the motion will jump between two stable steady states nearly periodically in phase with the periodic force. Benzi et al. (1981) found that in order to satisfy the inequalities, the variance of the noise has to be confined in the interval $(\sigma_1, \sigma_2)$, where

$$\sigma_1 = \Gamma\sqrt{\frac{1 - 4(A/\Gamma^{3/2})}{2\ln(2\Gamma\sqrt{2}/\Omega)}}, \qquad \sigma_2 = \Gamma\sqrt{\frac{1 + 4(A/\Gamma^{3/2})}{2\ln(2\Gamma\sqrt{2}/\Omega)}} \tag{118a,b}$$

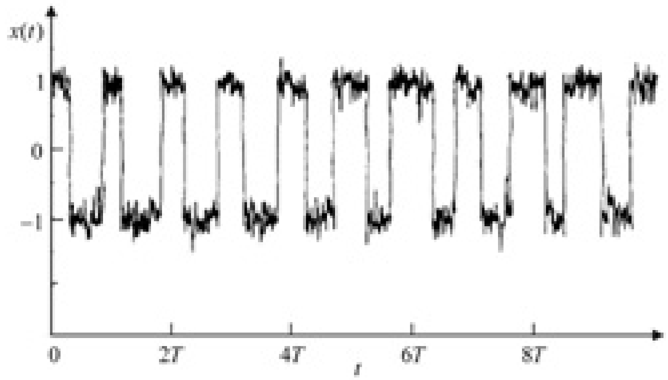



Figure 24. Time history record of the Langevin equation with periodic excitation (Benzi et al., 1981).

For a given value of $A$, small compared with $\Gamma^{3/2}$, the power structure of $x(t)$ exhibits a peak at the frequency when $\sigma$ is confined between the values $\sigma_1$ and $\sigma_2$. This is the condition of SR. Figure 24 shows a time history record of the response of system (113) for $\Gamma = 1$, $A = 0.12$, $\Omega = 2\pi/T = 0.001$, and $\sigma = 0.25$. The corresponding values of equation (111) are $\sigma_1 = 0.18$ and $\sigma_2 = 0.31$.

SR is manifested in electronic systems (Fauve and Heslot, 1983; Harmer et al., 2002), tunnel diodes (Mantegna and Spagnolo, 1995; Mantegna et al., 2001), and operational amplifiers (Harmer and Abbott, 2000). SR has also been demonstrated in a wide range of physical and biological systems (see, e.g., Gammaitoni et al., 1998; Wiesenfeld and Jaramillo, 1998; Yang et al., 1998a,b) and social systems (Babinec, 1997). SR has been extended to aperiodic signals by Collins et al. (1995), Bulsara and Zador (1996), and Chapeau-Blondeau (1997a), and has been used to improve the transmission of actual useful information by implementing a broadband aperiodic signal in place of the periodic one (Kiss, 1995; Collins et al. 1995; Heneghan et al., 1996; Chapeau-Blondeau, 1997a). Casado-Pascual et al. (2005) adopted some analytical and numerical tools to study the SR in the nonlinear regime of two simple models. The first model is an overdamped bistable system with white noise and a time periodic external force. The second model is a discrete two-state system with NIT rates modulated by the external driving excitation.

Abbott (2001) highlighted some unsolved problems in SR. In particular, the underlying principles of SR have not been unified. For example, SR was thought to be inherent (and restricted) to bistable systems until Gingl et al. (1995) observed SR in a simple threshold element; however, Bezrukov and Vodyanov (1997) have shown that not even a threshold operation is required. Others (Klimontovich, 1999; McClintock, 1999) generalized SR as a nonlinear filtering phenomenon, or a natural feature of any nonlinear system with a large, strongly noise-dependent susceptibility. Other works by Bezrukov (1998) and Bezrukov and Vodyanov (2000) demonstrated that these features are not in fact necessary for SR to occur. Among other unsolved problems is the need to develop suitable metrics to quantify SR of aperiodic signals, since the signal-to-noise ratio is not useful for aperiodic signals. Some attempts have been made by Kish et al. (2001), based on a threshold potential.

### *5.2. Selected Applications of NES*

#### 5.2.1. Systems Described by Langevin Equation

NES is usually manifested as an increase of the average escape (exit) time as the noise intensity increases. Mantegna and Spagnolo (1996) conducted an experimental investigation on a system consisting of a biasing resistor with a tunnel diode parallel to a capacitor when the system exhibited a strongly asymmetric bistable potential. The initial conditions were taken as identical to the minimum of the well characterized by the smaller barrier. The difference between the barriers of the two wells was such that the probability of the system returning to the starting well after the first escape is totally negligible for the noise intensities used in the experiments. The system was designed to remain in the metastable state in all cases. Mantegna and Spagnolo (1996) applied across the network the voltage signal

$$v(t) = V_b + V_s \sin \Omega_s t + V_n(t) \tag{119}$$

where $V_b$ is a biasing voltage, $V_s \sin \Omega_s t$ is a sinusoidal modulating signal, and $V_n(t)$ is the noise voltage. Under this condition, the equation of motion for the voltage across the tunnel diode $v_d$ was found to be a Langevin equation, in the overdamped regime, of a system characterized by a time dependent potential. By properly selecting $V_b$ and $V_s$ one can obtain: (a) an overall-stable system, i.e., a system with a time-dependent potential well which has a finite barrier at any time; (b) an overall-unstable system with a time-dependent potential well, which has no barrier for a short interval of time within each period of the modulated signal. The degree of stability of the time modulated system was quantified by introducing the variable

$$\delta \equiv \frac{V_s - V_{sm}}{V_{sm}} \tag{120}$$

where $V_{sm}$ is the value of the amplitude of the modulated signal driving the system into the marginal state of the potential at $t = T_s/4$, $T_s$ is the period of the modulated signal. The point at which the unstable and locally stable branches of the potential coalesce (Arecchi et al., 1982; Colet et al., 1989) has been taken to identify the marginal state of the system. Figure 25 shows the dependence of the average escape time on the noise root-mean-square amplitude $V_{rms}$ measured for six different values of $\delta$, of which three belong to the stable case and three to the unstable case. It can be seen that for an overall stable system ($\delta < 0$) the average escape time monotonically decreases with the noise amplitude. On the other hand, for an overall unstable system ($\delta > 0$), the average escape time increases over an interval of $V_{rms}$, and $\langle T \rangle > T_s/4$, i.e., longer than the deterministic escape time observed in the absence of noise. Over this interval the presence of noise enhances the stability of an unstable modulated system.

Fiasconaro et al. (2003a,b) considered more realistic noise sources such as colored noise with a finite correlation time. They found that the value of the noise intensity at which the maximum of the average exit time occurs is larger than that corresponding to the white noise case. Fiasconaro et al. (2005) analyzed the dynamic behavior of a Brownian particle governed by the Langevin equation

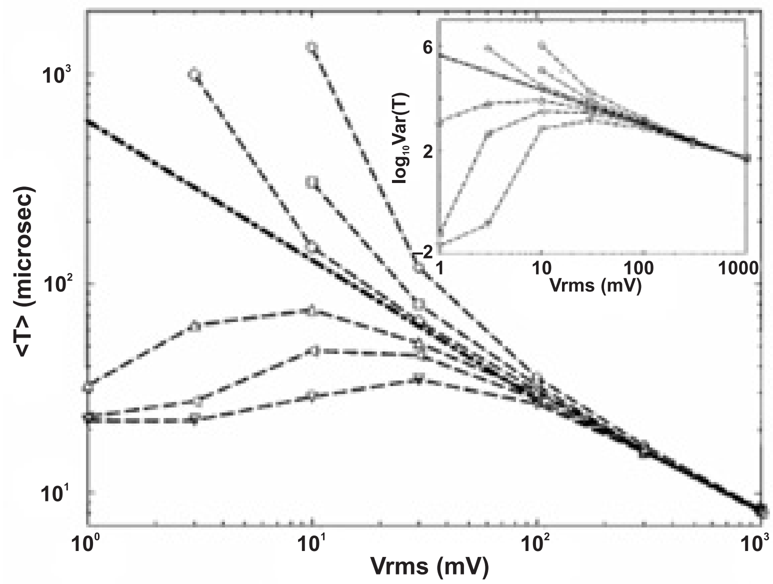



Figure 25. Dependence of the average escape time on the noise level for six different values of stability parameter $\delta$. The overall-stable cases ($\bigcirc$)$\delta = -0.0195$, ($\square$)$\delta = -0.009$, and ($\diamondsuit$)$\delta = -0.0039$, show monotonic decrease of escape time. The overall-unstable cases ($\triangle$)$\delta = +0.0039$, ($\triangleleft$)$\delta = +0.0091$, and ($\bigtriangledown$)$\delta = +0.0195$ show a non-monotonic change. The dot-dashed curve is a scaling behavior. The inset is the dependence of the variance of the escape time (in microsecond square) as a function of the noise amplitude for six different values of $\delta$. (Mantegna and Spagnolo, 1996).

$$\dot{x} = -\frac{\mathrm{d}U(x)}{\mathrm{d}x} + \sqrt{D}\xi(t) \tag{121}$$

where $\xi(t)$ is a white Gaussian random process (with a zero mean), which is delta-correlated in time, and $U(x) = 0.3x^2 - 0.2x^3$ is a cubic potential. The potential has a local stable state at $x_s = 0$, an unstable state at $x_u = 1$, and intersects the x-axis at $x_c = 1.5$ as shown in the inset of Figure 26. Let $x_F > x_c$ be the final target position. Hänggi et al. (1990) and Gardiner (1983) showed that the average time for a particle starting from an initial position $x_0$ to reach the target value $x_F$ is given by the following closed form expression

$$\langle T(x_0, x_F)\rangle = \frac{2}{D}\int_{x_0}^{x_F} e^{2\bar{U}(z)} \int_{-\infty}^{z} e^{-2\bar{U}(y)}\mathrm{d}y\mathrm{d}z = \frac{2}{D}\int_{x_0}^{x_F} e^{2\bar{U}(z)} G(z)\mathrm{d}z \tag{122}$$

where $\bar{U}(x) = U(x)/D$ is the dimensionless potential,

$$G(z) = 0.6046e^{-a}\left[I_{-1/3}(z) + I_{1/3}(z)\right] - \frac{1}{2}{}_2F_2\left(\frac{1}{2}, 1; \frac{2}{3}, \frac{4}{3}; -2z\right) + \int_0^z e^{-2\bar{U}(y)}\mathrm{d}y$$

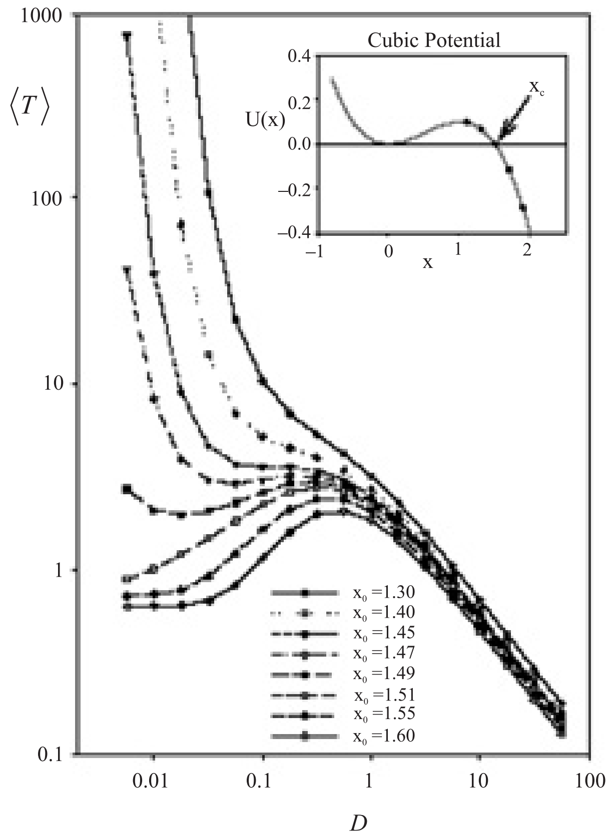



Figure 26. Dependence of the mean first passage time on the noise intensity level for target value $x_F = 2.2$ and for different values of initial conditions: —•— $x_0 = 1.3$, .....○..... $x_0 = 1.4$, – – –▼– – – $x_0 = 1.45$, –.. – .. – ▽ – .. – ..– $x_0 = 1.47$, – – – ■ – – – $x_0 = 1.49$, –. – . – □ – . – .– $x_0 = 1.51$, – – ◆ – – $x_0 = 1.55$, – – ◇ – – $x_0 = 1.60$. Inset is the cubic potential $U(x)$ with the metastable state at $x = 0$. (Fiasconaro et al., 2005).

$z = \dfrac{1}{10D}$, and ${}_pF_q\,(a_1, a_2; b_1, b_2; z)$ is the generalized hypergeometric function (Morse and Feshbach, 1953).

Figure 26 shows the dependence of mean first passage time $\langle T(x_0, x_F)\rangle$ on the noise intensity $D$, for $x_F = 2.2$ and different values of $x_0$. The figure reveals two different regimes. The most interesting regime is characterized by NES and is only valid for all initial conditions where $x_0 > x_c$. The second regime is referred to as the divergent regime, and takes place for initial conditions occupying the range $x_u \leq x_0 < x_c$, where $x_u = 1$ is the location of the relative maximum of the potential. Note that for all initial positions $x_0$ smaller than (but sufficiently close to) $x_c$ the mean exit time displays a non-monotonic behavior with a minimum and a maximum. For very low values of noise intensity, the Brownian particle is trapped in the potential well, as a consequence of the divergence of the mean exit time at the limit $D \to 0$. As the noise intensity increases the particle can escape more easily and the mean exit time decreases. When $D \approx \Delta U = 0.1$ (corresponding to the potential barrier height), the concavity of the mean exit time curves changes and the escape process is slowed down. At higher noise intensities the mean exit time exhibits a monotonic decrease.

It was indicated by Dubkov et al. (2004) that the estimation of the mean first passage time (MFPT) requires the implication of absorbing boundary in the system. Alternatively, the nonlinear relaxation time (NLRT) introduced by Agudov and Malakhov (1999) does not have that restriction. The NLRT measures the average residence time or the mean lifetime of Brownian particles within a certain domain. The general equations for the NLRT for potentials randomly switching between two arbitrary configurations with a sink were derived by Dubkov et al. (2004). The noise enhanced stability phenomenon for the NLRT in a randomly switching metastable state was studied by considering a modified system of equation (121) in which the potential includes dichotomous noise, i.e.,

$$\dot{x} = -\frac{\mathrm{d}}{\mathrm{d}x}\left[U(x) + V(x)\varsigma(t)\right] + \sqrt{D}\xi(t) \tag{123}$$

where the Markovian dichotomous noise $\varsigma(t)$ takes the values $\pm 1$ with the mean flipping rate $\nu$. By introducing the following expression for the pdf in terms of the average

$$p(x,t) = \langle \delta(x - x(t) \rangle \tag{124}$$

together with the auxiliary function $Q(x,t)$,

$$Q(x,t) = \langle \varsigma(t)\delta(x - x(t)) \rangle \tag{125}$$

Dubkov et al. (2003, 2004) obtained the following set of the time evolution equations

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x}p + \frac{\partial V}{\partial x}Q\right] + D\frac{\partial^2 p}{\partial x^2} \tag{126a}$$

$$\frac{\partial Q}{\partial t} = -2\nu Q + \frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x}Q + \frac{\partial V}{\partial x}p\right] + D\frac{\partial^2 Q}{\partial x^2} \tag{126b}$$

Dubkov et al. (2004) considered the potential profiles $U(x) \pm V(x)$ shown in Figure 27, with a wall at $x \to -\infty$ and a sink at $x \to +\infty$. The potential profile $U(x) + V(x)$ corresponds to a metastable state, while $U(x) - V(x)$ corresponds to an unstable state. Agudov and Malakhov (1999) defined the nonlinear relaxation time using the expression

$$T(x_0) = \int_0^\infty \mathrm{d}t \int_{X_1}^{X_2} p(x, t|x_0, 0)\mathrm{d}x \tag{127}$$

where $x_0 \in (X_1, X_2)$. For the special case of a piece-wise linear potential profile with $V(x) = ax$, $(x > 0, 0 < a < k)$, and

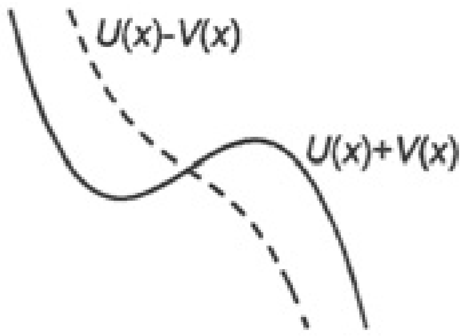



Figure 27. Switching potential with metastable state (Dubkov et al., 2004).

$$U(x) = \begin{cases} +\infty & x < 0 \\ 0 & 0 \leq x \leq X \\ k(X - x) & x > X \end{cases} \tag{128}$$

Dubkov et al. (2004) estimated the dependence of the normalized mean life-time $T_{-}(0)/T_0$ on the noise intensity $D$, where $T_{-}(0)$ represents the mean life-time corresponding to the potential profile $U(x) - V(x)$, with $x_0 = 0$, and $T_0$ is the mean life-time corresponding to very slow switching $\nu \to 0$. Figure 28 shows such dependence for three different values of the dimensionless mean flipping rate $\nu X/k = 0.1$, 0.05, and 0.01. It can be seen that the maximum value of the NLRT and the range of noise intensity values where NES occurs both increase when the mean flipping rate decreases.

### 5.2.2. Ising Model

The Ising model was originally proposed by Ernst Ising (1924) in his doctoral thesis, to explain the physical alignment of particles within a magnetically-charged material such as iron. This model has been used as a prototype for the metastable dynamics of short-range systems using Monte Carlo simulations in both two and three dimensions (Stoll and Schneider, 1972, 1977; Binder, 1973; Binder and Stoll, 1973; Stauffer et al., 1982; Rikvold et al., 1994). It was demonstrated that the metastable lifetimes for impurity-free kinetic Ising models exhibit system-size and field dependence. Hurtado et al. (2004a,b, 2006) studied the simplest non-equilibrium Ising model with dynamic impurities and the interfaces that separate non-equilibrium phases. They predicted noise-enhanced stabilization of the non-equilibrium metastable states at low temperature. This is caused by the nonlinear interplay between thermal and non-equilibrium fluctuations, which induces anomalous, non-monotonous dependence on the temperature of the surface tension.

The analytical modeling of the Ising model was based on a two-dimensional square lattice with periodic boundary conditions and a spin variable at each node. Two different heat paths were found to compete in the model. Both induce completely random spin-flips;

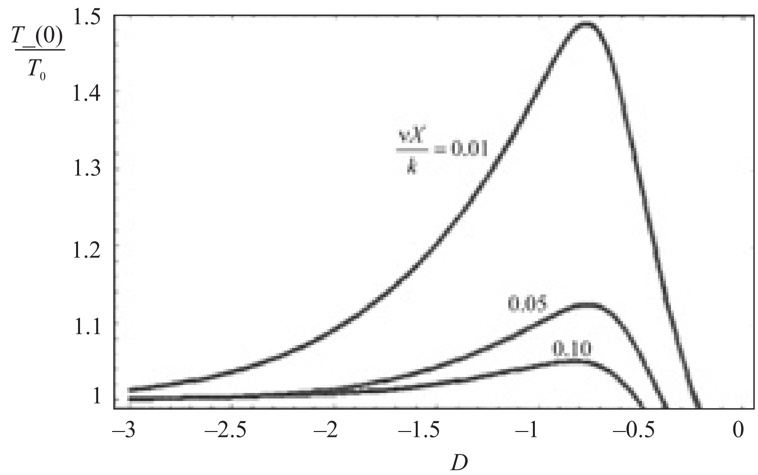



Figure 28. Dependence of the normalized mean life-time on the white noise intensity for three different values of mean flipping rate parameter $\nu X/k$, for $X = 1$, $k = 1$, $X_1 = 0$, $X_2 = 2$, and $a = 0.995$ (Dubkov et al., 2003).

one with probability $p$, and the other with probability $(1 - p)$. This competition resulted in non-equilibrium steady state sets. The strength of fluctuations affecting a spin was found to increase with the local order (the number of neighbors with the same state). It was implied that such a situation implies the presence of multiplicative noise and allows one to use a Langevin equation, which can capture the essential physics of the model (Hurtado et al., 2006) in the form

$$\frac{\partial \psi}{\partial t} = \psi - \psi^3 + h + \sqrt{D + \mu \psi^2}\xi(t) \tag{129}$$

where $\psi$ is the transition rate of spin-flip, $h$ is the external magnetic field, $\xi(t)$ is a Gaussian white noise with zero mean and autocorrelation function $\langle \xi(t)\xi(t') \rangle = 2\delta(t - t')$, $D$ is the strength of the noise, and $\mu$ is the renormalized version of $p$. The steady state probability density function, $p_{st}(\psi)$, obtained by solving the Fokker-Planck equation of system (129) is

$$p_{st}(\psi) = \frac{p_0}{2\sqrt{\mu\left(D + \mu\psi^2\right)}} Exp\left\{-\frac{1}{\mu}\left[\psi^2 - (1 + D/\mu)\ln\left(\psi^2 + D/\mu\right)\right]\right\} \tag{130}$$

where $p_0$ is the normalization constant.

The extrema of the effective potential $\left[\psi^2 - (1 + D/\mu)\ln\left(\psi^2 + D/\mu\right)\right]$ are:

$$\psi_k = \frac{2}{\sqrt{3}}\cos(\theta_k) \tag{131}$$

where $\theta_k = \frac{1}{3}\left[\cos^{-1}\left(-\frac{\sqrt{27}}{2}h\right) + 2k\pi\right]$, $k = 0, 1, 2.$

For $h < 0$, $\psi_0$, $\psi_1$, and $\psi_2$ correspond the metastable, stable, and unstable extrema, respectively. The escape time from the metastable minimum was obtained by Hurtado et al. (2006). This modeling was shown to recover the thermal NES phenomenon for the escape time.

### 5.2.3. Ecosystems

Ecosystems such as dry-land plants (D'Odorico et al., 2005) and interacting species (La Barbera and Spagnolo, 2002; Spagnolo and La Barbera, 2002; Spagnolo et al., 2002; Valenti et al., 2004) are examples demonstrating NIS. Dry-land plant systems tend to exhibit bistable dynamics with two preferential configurations of bare and vegetated soils. Climate fluctuations are usually believed to act as a source of disturbance on these systems and to reduce their stability and resilience. Random inter-annual fluctuations of precipitation may lead to the emergence of an immediate statistically stable condition between the two stable states (Walker et al., 1981; Scheffer et al., 2001) of the deterministic vegetation dynamics. The existence of these two stable states is usually associated with positive feedbacks between vegetation and its most limiting resources.

The dynamics of interacting species can be described by the generalized Lotka-Volterra equations. Valenti et al. (2004) studied the role of the noise in the dynamics of two competing species. They modeled the interaction between the species and the environment using generalized Lotka-Volterra equations in the presence of a multiplicative noise. The interaction parameter between the species was assumed to be a random process obeying a stochastic differential equation with a generalized bistable potential in the presence of a periodic driving term. This term accounted for the environmental temperature variation. It was found that noise induced periodic oscillations of the species concentrations and SR phenomenon. A non-monotonic behavior of the mean extinction time of one of the two competing species was found as a function of the additive noise intensity.

### 5.2.4. NES in Tumor-Immune Systems

Instability phenomena encountered in nonlinear systems can take place in tumor systems as well. This particular problem has been the subject of extensive research by oncologists, physicists, and mathematicians. Significant research funds are currently directed to the treatment of cancer and its prevention. The related tumor behavior is called tumor dormancy, prolonged arrest, or spontaneous remission. Kaiser (1990) indicated that spontaneous regression and very long-term arrest of cancer is a most paradoxical, captivating, and promising phenomenon of immunology. Different types of analytical models of cancer-immune system interactions have been reported by Stepanova (1980), Kuznetsov et al. (1994), Bellomo and Preziosi (2000), Nani and Freedman (2000), Szymanska (2003), De Vladar and Gonzalez (2004), Bellomo et al. (2004) and D'Onofrio (2005). The main features of these models are the existence of a tumor free equilibrium and that, depending on the values of model parameters, the tumor may tend either to $+\infty$ or to a macroscopic value. Additional features include the existence of limit cycles as indicated by Kirschner and Panetta (1998).

Galach (2003) and Sotolongo-Costa et al. (2003) proposed a model for the interaction between a population of tumor cells whose number is $X$ and a population of lymphocyte cells whose number is $Y$. The rate of increase of the malignant cells, $\mathrm{d}X/\mathrm{d}t$, was assumed to be proportional to $X$, which is justified for the initial phases of the growth. The rate was also assumed to decrease with increasing frequency of interaction with lymphocytes. This was described by the equation

$$\frac{\mathrm{d}X}{\mathrm{d}t} = aX - bXY \tag{132}$$

where $a$ and $b$ are constant coefficients, $a$ being tissue-dependent. The growth rate of lymphocytes, $\mathrm{d}Y/\mathrm{d}t$, was described by the equation

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = dXY - fY - KX + u + P(t) \tag{133}$$

where $d$, $f$, and $K$ are constant coefficients, and $u$ represents the diffusion process of lymphocytes that takes place in the tissue surrounding the tumor, assuming a constant flux of lymphocytes (Kuznetsov et al., 1994). The last term in equation (133) represents the effects produced by the treatment with cytokines in the process of activation of the immune system. Sotolongo-Costa et al. (2003) proposed periodic treatment represented by the function $P(t) = \mathrm{P}\cos^2\Omega t$, where $\Omega$ is the frequency of the periodic behavior of the cytokines inside the body. Equation (133) shows that the growth rate of lymphocytes, $\mathrm{d}Y/\mathrm{d}t$, is proportional to their rate of interaction with malignant cells and also to the flux per unit time of lymphocytes to the place of interaction. These effects are represented by the first and fourth terms on the right-hand sides of equation (133).

Note that in the absence of treatment, i.e., $P(t) = 0$, equations (132) and (133) take the autonomous non-dimensional form

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \alpha x - xy \tag{134}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = xy - \frac{1}{\alpha}y - kx + \gamma \tag{135}$$

where $\tau = t\sqrt{af}$, $k = Kb/d\sqrt{af}$, $\alpha = \sqrt{a/f}$, and $\gamma = ub/af$. Equations (134) and (135) can be combined into the second-order nonlinear differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + \left(\frac{1}{\alpha} - x - \frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)\frac{\mathrm{d}x}{\mathrm{d}\tau} - (k-\alpha)\,x^2 - (1-\gamma)\,x = 0 \tag{136}$$

This equation describes the motion of a particle in a force field whose potential is

$$U(x) = -\frac{1}{3}(k-\alpha)\,x^3 - \frac{1}{2}(1-\gamma)\,x^2 \tag{137}$$

This potential possesses the two extrema

$$x_1 = 0, \quad \text{and} \quad x_2 = \frac{\gamma - 1}{k - \alpha} \tag{138}$$

These extrema depend on the values of $\gamma$, $k$, and $\alpha$. Sotolongo-Costa et al. (2003) classified them as follows

$$\gamma > 1 \rightarrow \begin{cases} x_1 = 0 & \text{minimum} \\ \dfrac{k}{\alpha} > 1,\ x_2 > 0 & \text{maximum} \\ \dfrac{k}{\alpha} < 1,\ x_2 < 0 & \text{maximum} \end{cases}, \quad \text{and}$$

$$\gamma < 1 \rightarrow \begin{cases} x_1 = 0 & \text{maximum} \\ \dfrac{k}{\alpha} < 1,\ x_2 > 0 & \text{minimum} \\ \dfrac{k}{\alpha} > 1,\ x_2 < 0 & \text{minimum} \end{cases} \tag{139}$$

It is not difficult to study the stability of fixed points in the phase space. A saddle point occurs for values of $\gamma$ in the range $1 < \gamma < k/\alpha$ and $1 < k/\alpha < \gamma$. The corresponding time history records of unstable and stable trajectories are shown in Figures 29(a) and (b), respectively. The system can evolve towards a state of uncontrollable tumor growth as shown in Figure 29(a). This case can be interpreted as a recurrence-like behavior (Forys, 1995; Wheldom, 1988). On the other hand, Figure 29(b) corresponds to the case where the system evolves towards a controllable mass of malignant cells in a damped oscillatory trend. This is known as a dormant state (Kuznetsov et al., 1994; Kuznetsov and Makalkin, 1992; Kuznetsov and Knott, 2001; Michelson and Leith, 1994).

With periodic treatment, $P(t) = \mathrm{P}\cos^2\Omega t$, equations (132) and (133) should be considered. For the case $k/\alpha < 1$, $\gamma < k/\alpha$, and for effective doses and frequencies higher than a certain threshold value, the system was found to revert from uncontrollable growth to a treatment controlled population. Sotolongo-Costa et al. (2003) studied this situation and found that for every set of parameters $\gamma$, $k$, and $\alpha$, there are threshold values for the treatment frequency $\Omega$ and amplitude $P$ which split the parameter space into two zones corresponding to uncontrollable and controllable growth of malignant cells, as shown in Figure 30. The existence of threshold values reflects the fact that reaching controllable populations of malignant cells is only possible by maintaining a minimal dose above a certain threshold value given by the continuous line shown in Figure 30 and fitting the function

$$\frac{Pb}{af} = 0.10478 + \frac{0.00044}{\left(-0.05343 + \dfrac{\Omega}{\sqrt{af}}\right)^{2.7313}} \tag{140}$$

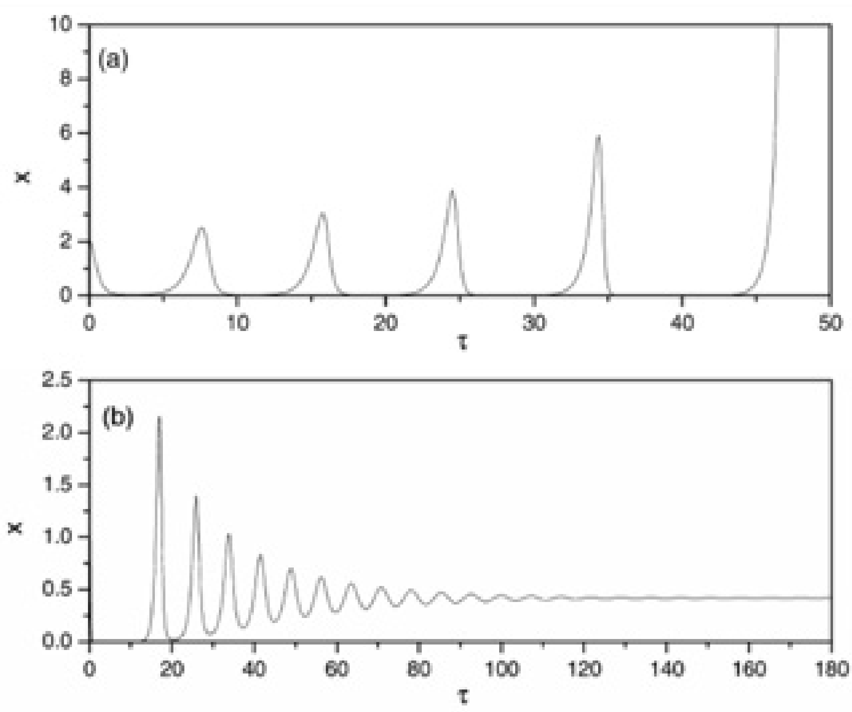



Figure 29. Time history records of malignant cells without treatment (a) $\alpha = 2$, $k = 0.2$, $\gamma = 0.05$ with initial conditions $x_0 = 2.1$ and $y_0 = 2.7$ (b) $\alpha = 2$, $k = 0.2$, $\gamma = 0.25$ with initial conditions $x_0 = 5.3$ and $y_0 = 6.7$ (Sotolongo-Costa et al., 2003).

Figure 30 contains some questionable features. For example, the growth of malignant cells can be controlled with low treatment frequencies (below the threshold values). Furthermore, the growth becomes uncontrollable at given frequencies above those localized in the region of controllable growth malignant cells.

De Vita et al. (1995), Heim and Kobele (1998), and These (2000) conducted different studies pertaining to the role of lymphocytes in tumor instability and spontaneous tumor regression and progression from information processing, respectively, in terms of reception, storage, processing, and transmission. During the working-resting cycle of the lymphocytes, the immune cells or lymphocytes have two alternating functional phases; active (A) and passive (P). The actual phase (A or P) depends on whether the lymphocyte is in the working mode (hunting or counteracting tumor cells) or in the resting mode (recuperating state after a tumor cell has been counteracted). Each mode consists of two of the informational operations, and these cyclically repeat as further new tumor cells are presented to the lymphocyte. The interaction between the lymphocyte and the tumor cell was considered by Roy et al. (2002). Let the durations of the active and passive phases be $t'$ and $t''$, respectively, and the corresponding rate constants of predatory and resting phases be $k_1 = 1/t'$ and $k_2 = 1/t''$, respectively. Let the density of tumor cells and that of cytotoxic lymphocytes in phase A be

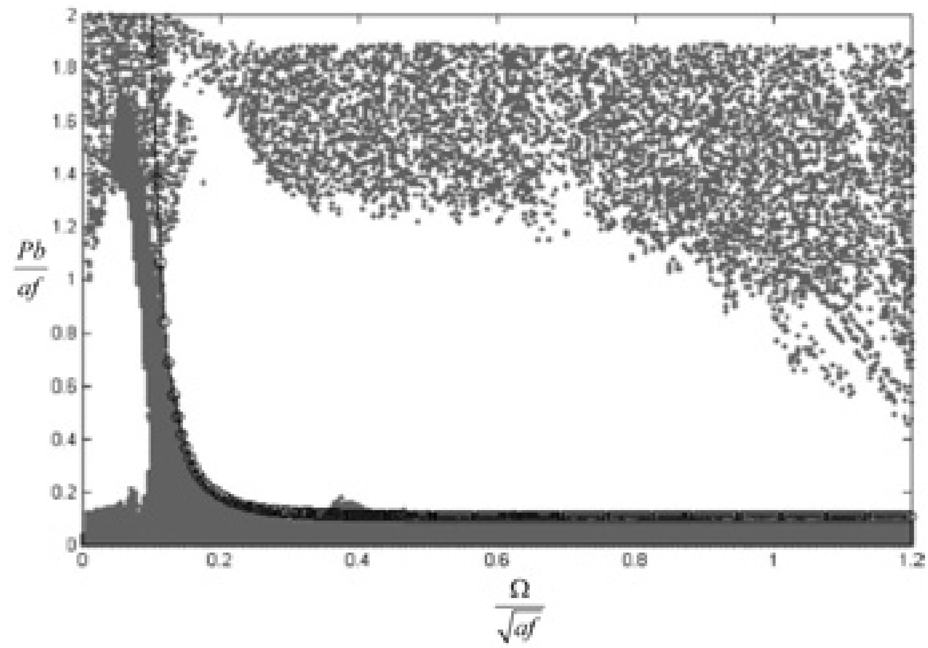



Figure 30. Growth behavior of malignant cells with periodic treatment for $\alpha = 2$, $k = 0.2$, $\gamma = 0.05$, with initial conditions $x_0 = 5.3$ and $y_0 = 6.7$. Uncontrollable growth is shown by gray points, controllable growth by white points. Solid curve represents a fitting function. (Sotolongo-Costa et al., 2003).

$M$ and $N$, respectively. The intensity $Q$ of tumor cells undergoing processing and extinction by lympocytes is given by the expression (Roy et al., 2002)

$$Q = -(MN)k_1 = -(MN)/t' \tag{141}$$

Note that the negative sign signifies extinction or subtraction of tumor cells. The rate of increase of the predatory lymphcytes (in phase A) and the rate of increase of tumor cells are given, respectively, by the two equations

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -MN/t' + Z/t'' \tag{142}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = q + \left[sM\{1 - (M/K)\}\right] - rf(M) \tag{143}$$

where $Z/t''$ denotes the generation of A-phase lymphocytes from the P-phase lymphocytes whose density is $Z$, and $q$ is the rate of conversion of normal cells to malignant ones (malignant cellular transformation). The expression in square brackets is known as the Fisher logistic growth term, describing the increase of tumor cells with replication rate $s$ and maximum carrying capacity or packing capacity $K$. The last term, $rf(M)$, is the intensity of tumor cells killed ($Q$), where $r$ is the rate of tumor cell destruction and $f(M)$ is a saturating function of tumor cell population $M$. The following dimensionless equation of the scaled tumor cell density $m$ in terms of the rescaled time $t = t(s - q)$ was obtained by Roy et al. (2002)

$$dm = \left\{v + m(1 - um) - r\left[m/(1 + m)\right]\right\} \mathrm{d}t \tag{144}$$

where $v = qt''/st'$, $u = t'/Kt''$, $m = Mt''/t'$, and $r = (M + Z)/st'$. The steady-state solution of equation (144) exhibits a remarkable feature of bistability, i.e. a cusp catastrophe with critical points $v'$, $r'$, and $m'$.

Garay (1978) and Perelson (1988) have shown that the fluctuation of the tissue environment and the stochastic nature of the immune system have a crucial effect on the immunological interactions. In this case, the parameter $r$ will be fluctuating with time and can be represented by a mean value $\bar{r}$ plus a random component, i.e.,

$$r(t) = \bar{r} + \sigma H(t) \tag{145}$$

where $\sigma H(t)$ is the perturbation with standard deviation $\sigma$. Equation (144) can be written as a stochastic differential equation in the form

$$dm(t) = \left\{v + m(1 - um) - r\left[\frac{m}{1 + m}\right]\right\} \mathrm{d}t + \left[\frac{\sigma m}{1 + m}\right] \mathrm{d}B(t) \tag{146}$$

where $B(t)$ is the Brownian (Wiener) motion process. The corresponding Fokker-Planck equation of equation (146) is

$$\begin{aligned} \frac{\partial p(m,t)}{\partial t} &= -\frac{\partial}{\partial m}\left[\left\{v + m(1 - um) - r\left[\frac{m}{1 + m}\right]\right\} p(m,t)\right] \\ &+ \frac{\sigma^2}{2}\frac{\partial^2}{\partial m^2}\left\{\left[\frac{m}{1 + m}\right]^2 p(m,t)\right\} \end{aligned} \tag{147}$$

The stationary solution of equation (147) is

$$\begin{aligned} p(m) &= e^{2/\sigma^2}\left[-\frac{v}{m} + (v + 2 - u - r)m - \frac{m^2}{2}(1 - 2u)\right. \\ &- \left.\frac{1}{3}um^2 + \left(2v + 1 - r - \sigma^2\right)\ln m + \sigma^2 \ln(1 + m)\right] \end{aligned} \tag{148}$$

This solution can provide useful information regarding the way in which the peak of the probability density function of tumor cell shifts as one increases the non-equilibrium perturbation intensity, $\sigma$, which implies a non-equilibrium phase transition.

This subsection did not exhaust the available research results and the presented results are only a sample of hundreds of papers published in this field.

## 6. CONCLUSIONS

Stabilization and delay of instability of dynamical systems using deterministic or stochastic excitation has been the focus of many studies by dynamicists, physicists, mathematicians, and oncologists. The excitation can be additive or multiplicative. This review article has addressed three major topics: Noise-induced transition, noise-induced stability and noise-enhanced stability. One should take care not to confuse NIS with NES: NIS implies complete stabilization of an unstable system, while NES is only a postponement of the system instability. Deterministic and stochastic stability conditions, in the absence of feedback control, have been developed analytically and verified experimentally. The early paradigm of stabilization without feedback control is the inverted pendulum and has been considered extensively in the literature. However, some studies have been devoted to the stabilization of aeroelastic structures, human walking, and quantum nonlinear couplers.

Significant research activity by physicists and mathematicians has dealt with the problem of NIT in one-dimensional systems, one-dimensional mappings and nonlinear oscillators such as ship roll dynamics. Stabilization via multiplicative noise includes postponement of bifurcation. For nonlinear dynamical systems with widely separated time scales the adiabatic elimination method has proven to be very effective. Applications of NIS have been limited to few dynamical systems such as the inverted pendulum, ocean structures, and ship roll capsizing.

Conditions of NES have been obtained analytically and numerically. Applications of NES covered systems described by Langevin equation such as tunnel diodes, chemical systems, the Ising model, biologically motivated models, and tumor-immune systems. The modern nonlinear theory of dynamics has been extended to the phenomena of tumor destabilization or spontaneous biological regression of malignant focus-lymphocyte interactive systems. The available models of cancer-immune system interactions have provided some guidelines to the threshold values of the effective doses and their frequencies. However, some questionable features were reported in the parameter plane. The presence of random fluctuations of the immune system has a crucial effect on the immunological interactions. In addition to NES, the presence of small noise in nonlinear dynamical systems may result in what is known as stochastic resonance (SR). SR is manifested in electronic systems, tunnel diodes, and operational amplifiers and a wide range of physical and biological systems.

Noise-enhanced/induced stability needs to be explored in structural systems subjected to earthquakes, wind forces, and non-Gaussian ocean waves. These systems have been treated in the literature within the framework of the first passage problem. However, only a limited number of studies have considered the problem of NIS. The difficulty in establishing the conditions of NES arises in estimating the mean exit time using Pontryagin's equation. Alternatively, one has to develop numerical simulations for different levels of the additive noise to establish the threshold level for stabilizing such systems. Particular attention should be directed to developing reliable phenomenological models of cancer-immune system interactions in collaboration with oncologists.

More research needs to be directed to the study of ocean and aerospace systems subjected to irregular impact loading and non-smooth contact forces. Equally important is the need to conduct experimental investigations into simple models, with the purpose of developing phenomenological modeling of boundary condition relaxation. In real applications, most of the boundary conditions are not ideal, since one cannot achieve infinite stiffness for clamped ends. Relaxation of boundary conditions will certainly affect the stability of dynamic systems under severe environmental conditions and the system dynamical properties become time-dependent.

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## REFERENCES

Abbott, D., 2001, "Overview: Unsolved problems of noise and fluctuations," *Chaos* **11(3)**, 526–538.

Acheson, D. J., 1993, "A pendulum theorem," *Proceedings of the Royal Society of London A* **443**, 239–245.

Acheson, D. J., 1997, *From Calculus to Chaos, an Introduction to Dynamics*, Oxford University Press, Oxford, UK.

Acheson, D. J. and Mullin, T., 1993, "Upside-down pendulums," *Nature* **366**, 215–216.

Acheson, D. J. and Mullin, T., 1998, "Ropy magic," *New Scientist* **21 February**, 32–33.

Agudov, N. V., 1998, "Noise delayed decay of unstable states," *Physical Review E* **57(3)**, 2618–2625.

Agudov, N. V. and Malakhov, A. N., 1995, "On the effect of fluctuations on an intermittent laminar motion," *International Journal of Bifurcation and Chaos* **5(2)**, 531–536.

Agudov, N. V. and Malakhov, A. N., 1999, "Decay of unstable equilibrium and nonequilibrium states with inverse probability current taken into account," *Physical Review E* **60(6A)**, 6333–6342.

Agudov, N. V. and Spagnolo, B., 2001, "Noise enhanced stability of periodically driven metastable systems," *Physical Review E* **64**, 035102R, 1–4.

Alexandrov, A. V., Frolov, A. A., and Massion, J., 2001a, "Biomechanical analysis of movement strategies in human forward trunk bending, I: Modeling," *Biological Cybernetics* **84(6)**, 425–434.

Alexandrov, A. V., Frolov, A. A., and Massion, J., 2001b, "Biomechanical analysis of movement strategies in human forward trunk bending, II: Experimental study," *Biological Cybernetics* **84(6)**, 435–443.

Al-Khawaja, S. and Alsous, M. B., 2005, "Noise-stabilized chaotic substructure on the I-V staircase curve of a rocked superconducting Josephson junction," *Chaos, Solitons and Fractals* **26**, 723–729.

Anishchenko,V. S., Neiman, A. B., Moss, F., and Schimansky-Geier, 1999, "Stochastic resonance: Noise-enhanced order," *Physics-Uspekhi* **42**, 7–36.

Arecchi, F. T., Badii, R., and Politi, A., 1984, "Low-frequency phenomena in dynamical systems with many attractors," *Physical Review A* 29, 1006–1009.

Arecchi, F. T., Politi, A., and Ulivi, L., 1982, "Stochastic time approach to the decay of unstable states: Failure of the asymptotic approximation," *Physics Letters A* **47** A(7), 333–335.

Arnold, L., 1979, "A new example of an unstable system being stabilized by random parameter noise," *Match* **7**, 133–140.

Arnold, L., 1984, "A formula connecting sample and moment stability of linear stochastic systems," *SIAM Journal of Applied Mathematics* **44(4)**, 793–802.

Arnold, L., 1998, *Random Dynamical Systems*, Springer-Verlag, New York, NY.

Arnold, V. I., 1988, *Geometrical Methods in the Theory of Ordinary Differential Equations,* Springer-Verlag, New York, NY.

Arnold, L. and Kliemann, W., 1983, "Qualitative theory of stochastic systems," in *Probabilistic Analysis and Related Topics*, A. T. Bharucha-Reid, ed., Academic Press, New York, NY, Vol. 3.

Arnold, L., Horsthemke, W., and Lefever, R., 1978, "White and colored external noise and transition phenomena in nonlinear systems," *Zeitschrift fur Physik B* **29**, 367–373.

Babinec, P., 1997, "Stochastic resonance on the Weidlich model of public opinion formation," *Physics Letters A* **225**, 179–191.

Bar-Avi, P. and Benaroya, H., 1996, "Response of a two degrees of freedom articulated tower to different environmental conditions," *International Journal of Nonlinear Mechanics* **31**, 717–741.

Bar-Avi, P. and Benaroya, H., 1997, "Stochastic response of a two DoF articulated tower," *International Journal of Nonlinear Mechanics* **32**, 639–655.

Bellomo, N. and Preziosi, L., 2000, "Modeling and mathematical problems related to tumor evolution and its interaction with the immune system," *Mathematical Computation and modeling* **32**, 413–452.

Bellomo, N., Bellouquid, A., and Delitala, M., 2004, "Mathematical topics the modeling complex multicellular systems and tumor immune cells competition," *Mathematical Models and Methods in Applied Science* **14**, 1683–1733.

Benzi, R., Parisi, G., Sutera, A, and Vulpiani, A., 1982, "Stochastic resonance in climatic-change," *Tellus* **34**, 10–16.

Benzi, R., Sutera, A, Parisi, G., and Vulpiani, A., 1983, "A theory of stochastic resonance in climate change," *SIAM Journal of Applied Mathematics* **43**, 565–578.

Benzi, R., Sutera, A, and Vulpiani, A., 1981, "The mechanism of stochastic resonance," *Journal of Physics A* **14**, L453–L457.

Bertram, J. E. and Ruina, A., 2001, "Multiple walking speed-frequency relations are predicted by constrained optimization," *Journal of Theoretical Biology* **209**, 445–453.

Beskow, A. and Nilsson, J., 1967, "The concept of wave function and the irreducible representations of the Poincaré group: The experimental delay law," *Arkiv für Fisik* **34**, 561–569.

Bezrukov, S. M., 1998, "Stochastic resonance as an inherent property of rate-modulated random series of events," *Physics Letters A* **248**, 29–38.

Bezrukov, S. M. and Vodyanov, I., 1997, "Stochastic resonance in non-dynamical systems without response thresholds," *Nature* **385**, 319–321.

Bezrukov, S. M. and Vodyanov, I., 2000, "In search for a possible statistical interpretation of stochastic resonance," in *Proceedings of the 2nd International Conference on Unsolved Problems of Noise and Fluctuations (UPoN'99)*, D. Abbott and L. B. Kish (eds.), AIP, New York, NY, Vol. 511.

Billah, K. Y. and Shinozuka, M., 1990, "Numerical method for colored-noise generation and its application to a bi-stable system," *Physical Review A* **42(12)**, 7492–7495.

Billah, K. Y. and Shinozuka, M., 1991, "Stabilization of a nonlinear system with multiplicative noise," *Physical Review A* **44(8)**, R4779-R4781.

Binder, K., 1973, "Time-dependent Ginzburg-Landau theory of non-equilibrium relaxation," *Physical Review B* **8**, 3423–3438.

Binder, K. and Stoll, E., 1973, "Scaling theory for metastable states and their lifetime," *Physical Review Letters* **31**, 47–51.

Boccaletti, S., Grebogi, C., Lai, Y.-C., Mancini, H., and Maza, D., 2000, "The control of chaos: Theory and applications," *Physics Reports* **329**, 103–197.

Bogdanoff, J. L., 1962, "Influence on the behavior of a linear dynamical systems of some imposed motion of small amplitude," *Journl of Acoustical Society of America* **34**, 1055–1062.

Bogdanoff, J. L. and Citron, S. J., 1965a, "On the stabilization of the inverted pendulum," in *Proceedings of the 9th Midwestern Mechanics Conference on Development in Mechanics,* John Wiley & Sons, New York, NY, pp. 3–15.

Bogdanoff, J. L. and Citron, S. J., 1965b, "Experiments with an inverted pendulum subjected to random parametric excitation," *Journal of Acoustical Society of America* **38**, 447–452.

Brand, H. and Schenzle, A., 1980, "Comment on 'Effect of white noise on electro-hydrodynamic instabilities in nematics,"' *Journal of the Physical Society of Japan* **48(4)**, 1382–1383.

Brénière, Y. and Ribreau, C., 1998, "A double-inverted pendulum approach of the human gait," *Journal of Biomechanics* **31**, 86. Also in *Proceedings of the 11th Conference of the European Society of Biomechanics*.

Broer, H. W., Hovejin, I., and van Noort, M., 1998, "A reversible bifurcation analysis of the inverted pendulum," *Physics D* **112(1–2)**, 50–63.

Broer, H. W., Hovejin, I., van Noort, M., and Vegter, G., 1999, "The inverted pendulum: A singularity theory approach," *Journal of Differential Equations* **157**, 120–149.

Broggi, G., Colombo, A., Lugiato, L. A., and Mandel, P., 1986, "Influence of white noise on delayed bifurcation," *Physical Review A* **33**, 3635–3637.

Bulsara, A. R. and Zador, A., 1996, "Threshold detection of wideband signals: A noise-induced maximum in the mutual information," *Physical Review E* **54(3)**, R2185-R2188.

Cabrera, J. L., 2005, "Controlling instability with delayed antagonistic stochastic dynamics," *Physica A* **356**, 25–30.

Cabrera, J. L. and Milton, J., 2002, "On-off intermittency in a human balancing task," *Physical Review Letters* **89**, 158702-1.

Caroli, B., Caroli, C, Roulet, B, and Saint-James, D., 1981, "On fluctuations and relaxation in systems described by a one-dimensional Fokker-Planck equation with time-dependent potential," *Physica A* **108**, 233–256.

Carr, J., 1981, *Applications of Center Manifold Theory*, Springer-Verlag, New York, NY.

Casado, J. M. and Morillo, M., 1994, "Distribution of escape times in a driven stochastic model," *Physical Review E* **49**, 1136–1139.

Casado-Pascual, J., Gómez-Ordóñez, J., and Morillo, M., 2005, "Stochastic resonance: Theory and numerics," *Chaos* **15**, 026115(1)–026115(12).

Cavagna, G. A. and Margaria, R., 1966, "Mechanics of walking," *Journal of Applied Physiology* **21**, 271–278.

Cavagna, G. A., Heglund, N. C., and Taylor, C. R., 1977, "Mechanical work in terrestrial locomotion: Two basic mechanisms for minimizing energy expenditure," *American Journal of Physiology* **233**, R243-261.

Cavagna, G. A., Willems, P. A., and Heglund, N. C., 2000, "The role of gravity in human walking: Pendular energy exchange, external work and optimal speed," *Journal of Physiology* **528**, 657–668.

Champneys, A. R. and Fraser, W. B., 2000, "The 'Indian rope trick' for a parametrically excited rod: Linearized analysis," *Proceedings of the Royal Society of London* **456**, 553–570.

Chapeau-Blondeau, F., 1997a, "Noise-enhanced capacity via stochastic resonance in an asymmetric binary channel," *Physical Review E* **55**, 2016–2019.

Chapeau-Blondeau, F., 1997b, "Input-output gains for signal in noise in stochastic resonance," *Physics Letters A* **232**, 41–48.

Chelomi, V. N., 1956, "The possibility of increasing the stability of elastic systems using vibrations," *Doklady Akademii Nauk SSSR* **110(3)**, 345–347 (in Russian).

Chin, C., Nayfeh, A. H., and Mook, D. T., 1995, "The response of a nonlinear system with a non-semisimple one-to-one resonance to a combination resonance," *International Journal of Bifurcation and Chaos* **5(1)**, 971–982.

Ciuchi, S., de Pasuale, F., and Spagnolo, B., 1993, "Nonlinear relaxation in the presence of an absorbing barrier," *Physical Review E* **47(6)**, 3915–3926.

Colet, P. and Eckmann, J. P., 1980, *Iterated Maps on the Interval as Dynamical Systems,* Birkhauser, Boston, MA.

Colet, P., De Pasquale, F., and Miguel, M. S., 1991, "Relaxation in the subcritical pitchfork bifurcation: From critical to Gaussian scaling," *Physical Review A* **43**, 5296–5307.

Colet, P., Miguel, M. S., Casademunt, J., and Sancho, J. M., 1989, "Relaxation from a marginal state in optical bistability," *Physical Review A* **39**, 149–156.

Collins, J. J., Chow, C. C., and Imhoff, T. T., 1995, "Aperiodic stochastic resonance in excitable systems," *Physical Review E* **52**, R3321–R3324.

Creath, R., Kiemel, T., Horak, F., Peterka, R., and Jeka, J., 2005, "A unified view of quiet and perturbed stance: Simultaneous co-existing excitable modes," *Neuroscience Letters* **377**, 75–80.

Dan, D., Jayannavar, A. M., and Menon, G. I., 2003, "A biological inspired ratchet model of two coupled Brownian motors," *Physica A* **318**, 40–47.

Dan, D., Mahato, M. C., and Jayannavar, A. M., 1999, "Mobility and stochastic resonance in spatially inhomogeneous systems," *Physical Review E* **60**, 6421–6428.

Dayan, I., Gitterman, M., and Weiss, G. H., 1992, "Stochastic resonance in transient dynamics," *Physical Review Letters A* **46**, 757–761.

De Kepper and Horsthemke, W., 1978, "Chimie physique," (in French), *Comptes Rendus de L'Academie des Sciences, Paris* **287C**, 251–254.

De Vita, V., Hellman, S., and Rosenberg, S., 1995, *Biological Therapy of Cancer*, Lippincott, Philadelphia, PA.

De Vladar, H. P. and Gonzalez, J. A., 2004, "Dynamic response of cancer under the influence of immunological activity and therapy," *Journal of Theoretical Biology* **227**, 335–348.

D'Odorcio, P., Laio, F., and Ridolfi, L., 2005, "Noise-induced stability in dry-land plant ecosystems," *Proceedings of the National Academy of Science* **102(31)**, 10819–10822.

Doering, Ch. and Gaduoua, C., 1992, "Resonant activation over a fluctuating barrier," *Physical Review Letters* **69**, 2318–2321.

Donelan, J. M., Kram, R., and Kuo, A. D., 2001, "Mechanical and metabolic determinants of the preferred step with human walking," *Proceedings of the Royal Society of London B: Biological Sciences* **268**, 1985–1992.

D'Onofrio, A., 2005, "A general framework for modeling tumor-immune system competition and immunotherapy: Mathematical analysis and biomedical inferences," *Physica D* **208**, 220–235.

Dubkov, A. A., Agudov, N. V., and Spagnolo, B., 2004, "Noise-enhanced stability in fluctuating states," *Physical Review E* **69**, 061103.

Dubkov, A. A., Makhov, P. N., and Spagnolo, B., 2003, "Nonequilibrium steady-state distributions in randomly switching potentials," *Physica A: Statistical Mechanics and its Applications* **325** (1–2), 26–32, 'Transition Phenomena in Multidimensional Systems – Models of Evolution," in *Noise in Nonlinear Dynamical Systems, Vol.* 1: *Theory of Continuous Fokker-Planck Systems,* Chapter 8, F. Moss and P. V. E. McClintock (Eds.), Cambridge University Press, Cambridge.

Ebeling, W. and Schimansky-Geier, L., 1989, "Transition phenomena in high-demensional systems and models of evolution," in *Noise in Nonlinear Systems*, F. Moss and P. V. M. Mclintock (eds.), Cambridge University Press, Cambridge.

Facchi, P., Nakazato, H., Pascazio, S., Peřina, J., and Řeháček, J., 2001, "Stability and instability in parametric resonance and quantum Zeno effect," *Physics Letters A* **279**, 117–122.

Faetti, S., Festa, C., Fronzoni, L., Grigolini, P., and Martano, P., 1984, "Multiplicative stochastic processes in nonlinear systems: Noise-induced transition from the over damped to the inertial regime," *Physical Review A* **30(6)**, 3252–3263.

Faetti, S., Fronzoni, L, and Grigolini, P., 1985, "Multiplicative stochastic processes in nonlinear systems: II. Canonical and non-canonical effects," *Physical Review A* **32(2)**, 1150–1160.

Faetti, S., Grigolini, P., and Marchesoni, E, 1982, "Time behavior of nonlinear stochastic processes in the presence of multiplicative noise: From Kramers' to Suzuki's decay," *Zeitschrift fur Physik B (Condensed Matter)* **47**, 353–363.

Fauve, S. and Heslot, F., 1983, "Stochastic resonance in a bistable system," *Physics Letters* **97A**, 5–8.

Fedchenia, I. I., 1984, "Boundary stochastic problems, multi-stability in the presence of fluctuations and noise-induced phase transitions," *Physica A* **125A**, 577–590.

Festa, C., Fronzoni, L., Grigolini, P., and Marchesoni, F., 1984, "The range of validity of the current procedures of adiabatic elimination: Experimental and theoretical evidence," *Physics Letters* **102A(3)**, 95–98.

Fiasconaro, A., Spagnolo, B., and Boccaletti, S., 2005, "Signatures of noise enhanced stability in metastable states," *Physical Review E* **72(6)**, 061110.

Fiasconaro, A., Valenti, D., and Spagnolo, B., 2003a, "Role of the initial conditions on the enhancement of the escape time in static and fluctuating potentials," *Physica A* **325**, 136–143.

Fiasconaro, A., Valenti, D., and Spagnolo, B., 2003b, "Enhancement of the escape time in metastable states with colored noise," *Modern Probability of Statistical Physics* **2**, 101–107.

Fitzhugh, R. A., 1961, "Impulses and physiological states in theoretical models of nerve membrane," *Biophysics Journal* **1**, 445–466.

Forys, U., 1995, "Interleukin mathematical model of an immune system," *Journal of Biological Systems* **3**, 889–902.

Fraser, W. B. and Champneys, A. R., 2002, "The 'Indian rope trick' for a parametrically excited rod: Nonlinear and subharmonic analysis," *Proceedings of the Royal Society of London* **458**, 1353–1373.

Fujisaka, E. and Grossmann, S., 1981, "External noise on the fluctuation line width," *Zeitschrift fur Physik B (Condensed Matter)* **43**, 69–75.

Gage, W. H., Winter, D. A., Frank, W. J. S., and Adkin, A. L., 2004, "Kinematic and kinetic validity of the inverted pendulum model in quiet standing," *Gait and Posture* **19**, 124–132.

Galach, M., 2003, "Dynamics of the tumor-immune system competition: The effect of time delay," *International Journal of Applied Mathematics and Computer Science* **13(3)**, 395–406.

Galán, J., Fraser, W. B., Acheson, D. J., and Champneys, A. R., 2005, "The parametrically excited upside-down rod: An elastic joint pendulum model," *Journal of Sound and Vibration* **280**, 359–377.

Gammaitoni, L., Hänggi, P., Jung, P., and Marchesoni, F., 1998, "Stochastic resonance," *Reviews of Modern Physics* **70(1)**, 223–287.

Gammaitoni, L., Marchesoni, F., Menichella-Saetta, E., and Santucci, S., 1989, "Stochastic resonance in bistable systems," *Physical Review Letters* **62**, 349–352.

Gang, H. and Kai-Fen, H., 1992, "Bi-stable systems subject to a singular stochastic force," *Physical Review A* **45(8)**, 5447–5454.

Garay, R., 1978, "Kinetic approach to immunology of cancer," *Journal of Theoretical Biology* **73**, 417–438.

Garcia, M., Chatterjee, A., Ruina, A., and Coleman, M., 1998, "The simplest walking model: Stability, complexity, and scaling," *Journal of Biomechanical Engineering* **120**, 281–288.

Gardiner, C. W., 1983, *Handbook of Stochastic Methods,* Springer-Verlag, New York, NY.

Gardiner, C. W., 1984, "Adiabatic elimination in stochastic systems, I: Formulation of methods and application to few-variable systems," *Physical Review A (General Physics)* **29**, 2914–2822.

Geursen, J. B., Altena, D., Massen, C. H., and Verduin, M., 1976, "A model of the standing man for the description of his dynamic behavior," *Agressologie* **17**, 63–69.

Ghosh, P. K., Barik, D., Bag, B. C., and Ray, D. S., 2005, "Quantum escape kinetics over a fluctuating barrier," *Journal of Chemical and Physics* **123**, 224104(1)–224104(7).

Gingl, Z., Kiss, L. B., and Moss, F., 1995, "Non-dynamical stochastic resonance: Theory and experiments with white arbitrary coloured noise," *Europhysics Letters* **29**, 191–196.

Gradshteyn, I. and Ryzhik, I., 1980, *Tables of Integrals, Series, and Products*, Academic Press, New York, NY.

Graham, R., 1989, "Macroscopic potentials, bifurcations and noise in dissipative systems," in *Noise in Nonlinear Dynamical Systems, Vol. 1: Theory of Continuous Fokker-Planck Systems*, F. Moss and P. V. E. McClintock (Eds.), Cambridge University Press, Cambridge, UK, Chapter 7.

Graham, R. and Schenzle, A., 1982, "Stabilization by multiplicative noise," *Physical Review A* **26(3)**, 1676–1685.

Grigolini, P., 1989, "The projection approach to the Fokker-Planck equation: Application to phenomenological stochastic equation with colored noise," in *Noise in Nonlinear Dynamical Systems, Vol. 1: Theory of Continuous Fokker-Planck Systems,* F. Moss and P. V. E. McClintock (eds.) Cambridge University Press, Cambridge, UK, Chapter 5, pp. 161–190.

Haken, H., 1983, *Synergetics: An Introduction*, Springer-Verlag, New York, NY.

Haken, H., 1987, *Advanced Synergetics: Instability Hierarchies of Self-Organizing Systems and Devices*, Springer-Verlag, New York, NY.

Hänngi, P., Talkner, P., and Borkovec, M., 1990, "Reaction-rate theory: Fifty years after Kramers," *Reviews of Modern Physics* **62**, 251–341.

Harmer, G. P. and Abbott, D., 2000, "Simulation of circuits demonstrating stochastic resonance," *Microelectronics Journal* **31**, 553–560.

Harmer, G. P., Davis, B. R., and Abbott, D., 2002, "A review of stochastic resonance: Circuits and measurement," *IEEE Transactions on Instrumentation and Measurement* **51(2)**, 299–309.

Heagy, J. F., Platt, N., and Hammel, S. M., 1994, "Characterization of on-off intermittency," *Physical Review E* **49(2)**, 1140–1150.

Heim, M. and Kobbele, C., 1998, *Spontan-remissionen in der Onkologie*. Schattauer, New York, NY (in German).

Hemp, G. W. and Sethna, P. R., 1968, "On dynamical systems with high-frequency parametric excitation," *International Journal of Nonlinear Mechanics* **3**, 351–365.

Heneghan, C., Chow, C. C., Collins, J. J., Imhoff, T. T., Lowen, S. B., and Teich, M. C., 1996, "Information measures quantifying aperiodic stochastic resonance," *Physical review E* **54**, R2228–R2231.

Hijawi, M., Ibrahim, R. A., and Moshchuk, N., 1997a, "Nonlinear random response of ocean structures using first- and second-order stochastic averaging," *Nonlinear Dynamics* **12**, 155–197.

Hijawi, M., Moshchuk, N., and Ibrahim, R. A., 1997b, "Unified second-order stochastic averaging approach," *Nonlinear Dynamics* **12**, 155–197.

Hirsch, J. E., Huberman, B. A., and Scalapino, D. J., 1982, "Theory of intermittency," *Physical Review A* **25**, 519–532.

Hof, A. L., Gazendam, M. G. J., and Sinke, W. E., 2005, "The condition for dynamic stability," *Journal of Biomechanics* **38**, 1–8.

Horsthemke, W. and Lefever, R., 1977, "Phase transition induced by external noise," *Physics Letters* A **64(1)**, 19–21.

Horsthemke, W. and Lefever, R., 1980, "A perturbation expansion for external wide band Markovian noise: Application to transitions induced by Ornstein-Uhlenbeck noise," *Zeitschrift für Physik B (Condensed Matter and Quanta)* **40**(3), 241–247.

Horsthemke, W. and Lefever, R., 1984, *Noise-Induced Transition*, Springer-Verlag, New York, NY.

Horsthemke, W. and Lefever, R., 1989, "Noise-induced transition," in *Noise in Nonlinear Dynamical Systems, Vol. 2: Theory of Noise Induced Processes in Special Applications*, F. Moss and P. V. E. McClintock, eds., Cambridge University Press, Cambridge, UK, Chapter 8.

Horsthemke, W. and Mansour, M., 1976, "The influence of external noise on non-equilibrium phase transitions," *Zeitschrift fur Physik B* **24**, 307–313.

Howe, M. S., 1974, "The mean square stability of an inverted pendulum subjected to random parametric excitation," *Journal of Sound and Vibration* **32**, 407–421.

Hurst, C. A., 1996, "The Indian rope trick explained," *Australian Mathematical Society Gazette* **23**, 154–159.

Hurtado, P. I., Garrido, P. L., and Marro, J., 2004a, "Analysis of the interface in a non-equilibrium two-temperature Ising model," *Physical Review B* **70**, 245409(1)–245409(10).

Hurtado, P. I., Marro, J., and Garrido, P. L., 2004b, "Reentrant behavior of the spinodal curve in a non-equilibrium ferromagnet," *Physical Review E* **70**, 021101(1)–021101(7).

Hurtado, P. I., Marro, J., and Garrido, P. L., 2006, "Non-equilibrium metastability and noise-enhanced stabilization," E-Reprint arXiv:cond-mat/0604556.

Ibrahim, R. A., 1985, *Parametric Random Vibration*, Wiley, New York, NY.

Ibrahim, R. A., 1991, "Nonlinear random vibration: Experimental results," *ASME Applied Mechanics Reviews* **44(10)**, 423–446.

Ibrahim, R. A. and Castravete, S. C., 2006, "Flutter suppression of a plate-like wing via parametric excitation," *Nonlinear Dynamics*, in press.

Ibrahim, R. A. and Heinrich, R., 1988, "Experimental investigation of liquid sloshing under parametric random excitation," *ASME Journal of Applied Mechanics* **55**, 467–473.

Ibrahim, R. A. and Yoon, Y. J., 1996, "Response statistics of nonlinear systems to parametric filtered white noise," in *American Mathematical Society, Fields Institute Communications Nonlinear Dynamics and Stochastic Mechanics,* W. H. Kliemann, W. F. Langford, and N. S. Namachchivaya, eds., pp. 105–129.

Ising, E., 1924, *Beitrag zur Theorie des Ferro- und Paramagnetismus*, Ph.D.

Dissertation (in German), Mathematisch-Naturwissenschaftliche Fakultät der Hamburgischen Universität, Hamburg.

Jackson, P. J., Lambert, C. J., Mannella, R., Martano, P., McClintock, P. V. E., and Stocks, N. G., 1989, "Relaxation near a white noise-induced transition point," *Physical Review A* **40(5)**, 2875–2878.

Jordan, D. W. and Smith, P., 1999, *Nonlinear Ordinary Differential Equations: An Introduction to Dynamical Systems*, Third Edition, Oxford University Press, Oxford, UK.

Jung, P. and Hänggi, P., 1991, "Amplification of small signals via stochastic resonance," *Physical Review A* **44**, 8032–8042.

Kabashima, S. and Kawakub, T., 1979, "Observation of noise-induced phase transition in a parametric oscillator," *Physics letters A* **70(5–6)**, 375–376.

Kabashima, S., Kogue, S., Kawakubo, T., and Obada, T., 1979, "Oscillatory-to-non-oscillatory transition due to external noise in parametric oscillator," *Journal of Applied Physics* **50(10),** 6296–6302.

Kai, S., Kai, T., Takata, M., and Hirakawa, K., 1979, "Effect of the white noise on electro-hydrodynamic instabilities in nematics," *Journal of the Physical Society of Japan* **47(4)**, 1379–1380.

Kaiser, H., 1990, "Regression of neoplasms in man," in *The Influence of the Host on Tumor Development*, E. Heberman, ed., Kluwer, Norwell, MA.

Kalmus, H. P., 1970, "The inverted pendulum," *American Journal of Physics* **38**, 874–878.

Kapitsa, P. L., 1951a, "Dynamic stability of a pendulum with a vibrating suspension axis," (in Russian) *Zhurnal Exsperimental"noi i Teoreticheskoi Fiziki (ZhETF)* **21**, 588–597.

Kapitsa, P. L., 1951b, "Pendulum with vibrated suspension," (in Russian) *Uspekhi pizicheskikh nauk* **44**, 7–20.

Kapral, R. and Mandel, P., 1985, "Bifurcation structure of the non-autonomous quadratic map," *Physical Review A* **32,** 1076–1081.

Karlsson, A. and Fryberg, G., 2000, "Correlation between force plate measures for assessment of balance," *Clinical Biomechanics* **15**, 365–369.

Kawakubo, T., Yanagita, A., and Kabashima, S., 1981, "External noise effect on the onset of Williams domain in nematic liquid crystals," *Journal of the Physical Society of Japan* **50(5)**, 1451–1456.

Keller, J. B. and Antman, S., 1968, *Bifurcation Theory and Nonlinear Eigenvalue Problems,* Lecture Notes, Courant Institute of Mathematical Science, New York University, NY.

Khasminskii, R. Z., 1966, "A limit theorem for the solutions of differential equations with random right-hand sides," *Theory of Probability and Its Applications* **11,** 390–405.

Khasminskii, R. Z., 1968, "On averaging principle for Ito stochastic differential equations," (in Russian), *Cybernetics (Kybernetika Cislo 3 Rocnik)* **3**, 260–279.

Khasminskii, R. Z., 1980, *Stochastic Stability of Differential Equations*, Sijthoff & Noordhoof, Alphen aan den Rijn, The Netherlands.

Kirschner, D. and Panetta, J. C., 1998, "Modeling immunotherapy of the tumor-immune interaction," *Journal of Mathematical Biology* **37**, 235–252.

Kish, L. B., Harmer, G. P., and Abbott, D., 2001, "Information transfer rate of neurons: Stochastic resonance of Shannon's information channel capacity," *Fluctuation Noise Letters* **1(1)**, L13–L19.

Kiss, L. B., 1995, *Proceedings of the 3rd Technical Conference on Nonlinear Dynamics (Chaos) and Full Spectrum Processing*, Mystic, Connecticut, AIP, New York, NY.

Kitahara, K., Horsthemke, W., Lefever, R., and Inaba, Y., 1980, "Phase diagrams of noise-induced transitions," *Progress of Theoretical Physics* **64(4)**, 1233–1247.

Kliemann, W. and Arnold, L., 1983, *Lyapunov Exponent of Linear Stochastic Systems,* Report No. 93, Forschungsscwerpunkt Dynamische Universitat Bremen, Germany.

Klimontovich, Y. L., 1999, "What are stochastic filtering and stochastic resonance?" *Soviet Physics Uspecky* **42**, 37–44.

Kram, R., Domingo, A., and Ferris, D. P., 1997, "Effect of reduced gravity on the preferred walk-run transition speed," *Journal of Experimental Biology* **200(4)**, 821–826.

Kramers, H. A., 1940, "Brownian motion in a field of force and the diffusion model of chemical reactions," *Physica* **VII(4)**, 284–304.

Kuo, A. D., 2001, "A simple model of bipedal walking predicts the preferred speed-step length relationship," *Journal of Biomechanical Engineering* **123**, 264–269.

Kuznetsov, S. P., 2002, "Noise-induced absolute instability," *Mathematics and Computers in Simulation* **58**, 435–442.

Kuznetsov, V. A. and Knott, G. D., 2001, "Modeling tumor regrowth and immunotherapy," *Mathematical Computation and Modeling* **33**, 1275–1287.

Kuznetsov, V. A. and Makalkin, I. A., 1992, "Bifurcation analysis of a mathematical model of the interaction of cytotoxic lymphocytes with tumor cells: Effect of immunological enhancement of the growth of a tumor and its relationship with the other anamolous phenomena of onc-immunology," *Biophysics* **37**, 961–968.

Kuznetsov, V. A., Makalkin, I. A., Taylor, M. A., and Perelson, A. S., 1994, "Nonlinear dynamics of immunogenic tumors: Parameter estimation and global bifurcation analysis," *Bulletin of Mathematical Biology* **56**, 295–321.

Kuznetsov, S. P., Mosekilde, E., Dewel, G., and Borckmans, P., 1997, "Absolute and convective instabilities in a one-dimensional Brusselator flow model," *Journal of Chemical Physics* **106**, 7609–7616.

La Barbera, A. and Spagnolo, B.,2002, "Spatio-temporal patterns in population dynamics," *Physica A* **314**, 120–124.

Landa, P. S., 1996, *Nonlinear Oscillations and Waves in Dynamical Systems*, Kluwer, Dordrecht, The Netherlands.

Landa, P. S. and McClintock, P. V. E., 2000, "Changes in the dynamical behavior of nonlinear systems induced by noise," *Physics Reports* **323**, 1–80.

Landa, P. S. and Stratonovich, R. L., 1962, "On the theory of fluctuational transitions of different systems from one stable state to another," (in Russian) *Vestnik MGU (Physics and Astronomy)***1**, 33–45.

Landa, P. S. and Zaikin, A. A., 1996, "Noise-induced phase transitions in a pendulum with a randomly vibrating suspension axis," *Physics Review E* **54**, 3535–3544.

Landa, P. S. and Zaikin, A. A., 1997, "Nonequilibrium noise-induced phase transitions in simple systems," *Journal of Experimental and Theoretical Physics* **84**, 197–208.

Landa, P. S. and Zaikin, A. A., 1998, "Noise-induced phase transitions in nonlinear oscillators," in *AIP Conference Proceedings 465, Computing Anticipatory Systems, CASYS '98*, Liege, Belgium, pp. 419–433.

Landau, L. D. and Lifshitz, E. M., 1969, *Mechanics*, Pergamon Press, Oxford, UK.

Landauer, R., 1962, "Fluctuations in bistable tunnel diode circuits," *Journal of Applied Physics* **33**, 2209–2216.

Landauer, R., 1978, "Stability in the dissipative steady state," *Physics Today* **31(11)**, 23–26.

Lefever, R. and Horsthemke, W, 1979a, "Bi-stability in fluctuations environments: Implications in tumor immunology," *Bulletin of Mathematical Biology* **41**, 469–490.

Lefever, R. and Horsthemke, W, 1979b, "Multiple transitions induced by light intensity fluctuations in illuminated chemical systems," *Proceedings of the National Academy of Science* **76(6)**, 2490–2494.

Leung, H. K., 1988, "Critical slowing down near a noise-induced transition point," *Physical Review A* 37, 1341–1344.

Levi, M., 1988, "Stability of the inverted pendulum – a topological explanation," *Society for Industrial and Applied Mathematics Review* **30(4)**, 639–644.

Levi, M. and Weckesser, W., 1995, "Stabilization of the inverted pendulum by high-frequency vibrations," *Society for Industrial and Applied Mathematics Review* **27(2)**, 219–223.

Lima, R. and Pettini, M., 1990, "Suppression of chaos by resonant parametric perturbation," *Physical Review A* **41(2)**, 726–733.

Lindner, B., Garcia-Ojalvo, J., Neiman, A., and Schimansky-Geier, L., 2004, "Effects of noise in excitable systems," *Physics Reports* **392**, 321–424.

Liu, X. and Willms, A., 1994, "Impulsive stabilizability of autonomous systems," *Journal of Mathematical Analysis and Applications* **187**, 17–39.

Lücke, M. and Schank, F., 1985, "Response to parametric modulation near instability," *Physics Review Letters* **54**, 1465–1468.

Luis, A. and Peřina, J., 1996, "Zeno effects in parametric down conversion," *Physical Review Letters* **76**, 4340–4343.

Luis, A. and Sánchez-Soto, L. L., 1998, "Anti-Zeno effects in parametric down conversion," *Physics Review A* **57**, 781–787.

Lumbantobing, H., and Haaker, T. I., 2004, "On the parametric excitation of some nonlinear aeroelastic oscillators," *Journal of Fluids and Structures* **19**, 221–237.

Mahato, M. C. and Jayannavar, A. M., 1997, "Relation between stochastic resonance and synchronization of passages in a double-well system," *Modern Physics Letters B* **11**, 815–819.

Mahato, M. C. and Jayannavar, A. M., 1998, "Some stochastic phenomena in a driven double-well system," *Physica A* **248**, 138–154.

Malakhov, A. N. and Pankratov, A. L., 1996, "Influence of thermal fluctuations on time characteristics of a single Josephson element with high damping exact solution," *Physica C* **269**, 46–54.

Mannella, R., Faetti, S, Grigolini, P., McClintock, P. V. E., and Moss, F. E., 1986, "The effect of multiplicative noise on the relaxation time of a real non-linear physical system: A comparison of experiment and theory for the random rate model (RGRM)," *Journal of Physics A* **19**, L699–L704.

Mannella, R., Lambert, C. J., Stocks, N. G., and McClintock, P. V. E., 1990, "Relaxation of nonlinear systems driven by colored noise: An exact result," *Physical Review A* **41(6)**, 3016–3020.

Mantegna, R. N. and Spagnolo, B., 1994, "Stochastic resonance in a tunnel diode," *Physical Review E* **49**, R1792–R1795.

Mantegna, R. N. and Spagnolo, B., 1995, "Stochastic resonance in a tunnel diode in the presence of white or colored noise," *Il Nuovo Cimento D* **17(Ser1/No.7)**, 873–881.

Mantegna, R. N. and Spagnolo, B., 1996, "Noise enhanced stability in an unstable system," *Physical Review Letters* **76(4)**, 563–566.

Mantegna, R. N. and Spagnolo, B., 1998, "Probability distribution of the residence times in a periodically fluctuating metastable system," *International Journal of Bifurcation and Chaos* **4**, 783–788.

Mantegna, R. N. and Spagnolo, B., 2000, "Experimental investigation of resonant activation" *Physics Review Letters* **84**, 3025–3028.

Mantegna, R. N., Spagnolo, B., and Trapanese, M., 2001, "Linear and nonlinear experimental regimes of stochastic resonance," *Physical Review E* **63**, 011101.

Markeyev, A. P., 1999, "The dynamics of a spherical pendulum with a vibrating suspension," *Journal of Applied Mathematics and Mechanics* **63**, 205–211.

Matsumoto, K. and Tsuda, I., 1983, "Noise-induced order," *Journal of Statistical Physics* **31**, 87–106.

Mayer-Kress, G. and Haken, H., 1981, "The influence of noise on the logistic model," *Journal of Statistical Physics* **26**, 149–171.

McClintock, P. V. E., 1999, "Unresolved problems in noise," *Nature* **402**, 23–25.

McCollum, G. and Leen, T. K., 1989, "Form and exploration of mechanical stability limits in erect stance," *Journal of Motor Behavior* **21(3)**, 225–244.

McNamara, B. and Wiesenfeld, K., 1989, "Theory of stochastic resonance," *Physical Review A* **39**, 4854–4869.

Michaelis, M. M., 1985, "Stroboscopic study of the inverted pendulum," *American Journal of Physics* **53**, 1079–1083.

Michelson, S. and Leith, J. T., 1994, "Dormancy, regression and recurrence: Towards a unifying theory of tumor growth control," *Journal of Theoretical Biology* **169**, 327–338.

Mielke, A., 2000, "Noise-induced stability in fluctuating bistable potentials," *Physical Review Letters* **84**, 818–821.

Miguel, M. S. and Toral, R., 1997, "Stochastic effects in physical systems," in *Instabilities and Nonequilibrium Structures,* E. Terapegui and W. Zeller (eds.), Kluwer, Dordrecht, The Netherlands, Chapter VI.

Minetti, A. E. and Saibene, F., 1992, "Mechanical work rate minimization and freely chosen stride frequency of human walking: A mathematical model," *Journal of Experimental Biology* **170**, 19–34.

Minetti, A. E., Ardigo, L. P., and Saibene, F., 1993, "Mechanical determinants of gradient walking energetics in man," *Journal of Physiology* **472**, 725–735.

Misra, B. and Sudarshan, E. C. G., 1977, “The Zeno’s Paradox in quantum theory,” *Journal of Mathematical Physics* **18**, 756–763.

Mitchell, R. R., 1972, “Stability of the inverted pendulum subjected to almost periodic and stochastic base motion: an application to the method of averaging,” *Inernational Journal of Nonlinear Mechanics* **7**, 101–123.

Molchanov, S. A., 1978, “The structure of Eigenfunctions of one-dimensional unordered structures,” *USSR Izvestija* **12**, 69–101 (in Russian).

Morita, T., Mori, H., and Mashiyama, K. T., 1980, “Contraction of state variables in non-equilibrium open systems: II,” *Progress of Theoretical Physics* **64(2)**, 500–521.

Morris, B. and Moss, E, 1986, “Postponed bifurcations of a quadratic map with a swept parameter,” *Physics Letters A* **118(3)**, 117–120.

Morris, C. and Lecar, H., 1981, “Voltage oscillations in the barnacle giant muscle fiber,” *Biophysics Journal* **35**, 193–213.

Morse, P. M. and Feshbach, H., 1953, *Methods of Theoretical Physics*, McGraw-Hill, New York, NY.

Moshchuk, N. K. and Ibrahim, R. A., 1996, “Response statistics of ocean structures to nonlinear hydrodynamic loading, part II: Non-Gaussian ocean waves,” *Journal of Sound and Vibration* **191(1)**, 107–128.

Moshchuk, N. K., Ibrahim, R. A., and Khasminskii, R. Z., 1995a, “Response statistics of ocean structures to nonlinear hydrodynamic loading, part I: Gaussian ocean waves,” *Journal of Sound and Vibration* **184(4)**, 681–701.

Moshchuk, N K., Ibrahim, R. A., Khasminskii, R. Z., and Chow, P. L., 1995b, “Ship capsizing in random sea waves and the mathematical pendulum,” in *IUTAM Symposium on Advances in Nonlinear Stochastic Mechanics*, Trondheim, Norway, July 3–7, A. Naess and S. Krenk (Eds.), Kluwer Academic Publishers, Dordrecht, The Netherlands, pp. 299–309.

Moss, F., 1994, “Stochastic resonance: From ice ages to the monkey’s ear,” in *Contemporary Problems in Statistical Physics*, G. H. Weiss, ed., SIAM, Philadelphia, PA, pp. 205–253.

Moss, F. and Milton, J. G., 2003, “Medical technology: Balancing the unbalanced,” *Nature* **425**, 911.

Moss, F., Bulsara, A., and Shlesinger, M. F., eds., 1993, “The proceedings of the NATO advanced research workshop on stochastic resonance in physics and biology,” *Journal of Statistical Physics* **70(1–2)**, 1–512.

Moss, F., Pierson, D., and O’Gorman, D., 1994, “Stochastic resonance: Tutorial and update,” *International Journal of Bifurcation Chaos in Applied Science and Engineering* **4(6)**, 1383–1397.

Mullin, T., Champneys, A., Fraser, W. B., Galan, J., and Acheson, D., 2003, “The ‘Indian rope trick’ via parametric excitation: A comparison between theory and experiment,” *Proceedings of the Royal Society of London* **459**, 539–546.

Murray, M. P., Seireg, A., and Scholz, R. C., 1976, “Center of gravity, center of pressure, and supportive forces during human activities,” *Journal of Applied Physiology* **23**, 831–838.

Nagumo, J., Arimoto, S., and Yoshitzawa, S., 1962, “An active pulse transmission line simulating nerve axon,” *Proceedings of the Institute of the Radio Engineering* **50**, 2061–2070.

Nakamizo, T. and Sawaragi, Y., 1972 “Analytical study on n-th order linear system with stochastic coefficients,” in *Proceedings of the IUTAM Symposium on Stability of Stochastic Dynamical Systems, Lecture Notes in Mathematics* **294**, R. E. Curtain, ed., Springer-Verlag, Berlin, Germany, pp. 173–185.

Nani, F. and Freedman, H. I., 2000, “A mathematical model of cancer treatment by immunotherapy,” Mathematical Bioscience **163**, 159–199.

Neptune, R. R., Zajac, F. E., and Kautz, S. A., 2004, “Muscle mechanical work requirements during normal walking: The energetic cost of raising the body’s center-of-mass is significant,” *Journal of Biomechanics* **37**, 817–825.

Nevel’son, M. B. and Khasminskii, R. Z., 1966, “Stability of a linear system with random perturbations of its parameters,” *Applied Mathematics and Mechanics* **30**, 487–493.

Nicolis, C., 1982, “Stochastic aspect of climatic transitions – response to a periodic forcing,” *Tellus* **34(1)**, 1–9.

Nicolis, C., 1993, “Long term climatic transitions and stochastic resonance,” *Journal of Statistical Physics* **70**, 3–14.

Nicolis, C., 1995, *Introduction to Nonlinear Science*, Cambridge University Press, Cambridge, UK.

Nicolis, C. and Nicolis, G., 1981, “Stochastic aspects of climate transitions – additive fluctuations,” *Tellus* **33**, 225–234.

Ott, E., Grebogi, C., and York, J. A., 1990, “Controlling chaos,” *Physical Review Letters* **64**, 1196–1199.

Otterbein, S. 1982, “Stabilisierung des n-pendels und der indische seiltrick,” *Archive for Rational Mechanics and Analysis* **78**, 381–393 (in German).

Park, S., Horak, F. B., and Kuo, A. D., 2004, “Postural feedback responses scale with biomechanical constraints in human standing,” *Experimental Brain Research* **154(4)**, 417–427.

Parrondo, J. M. R., Van den Broeck, C., Buceta, J., and de la Rubia, F. J., 1996, “Noise-induced spatial pattern,” *Physica A* **224**, 153–161.

Pereira-Pinto, F. H. I., Ferreira, A. M., and Savi, M. A., 2004, “Chaos control in a nonlinear pendulum using a semi-continuous method,” *Chaos, Solitons and Fractals* **22**, 653–668.

Perelson, A., 1988, *Theoretical Immunology*, Vol. 1, Addison-Wesley, Reading, MA.

Pierson, W. J. and Moskowitz, L., 1964, “A proposed spectral form for fully developed wind seas based on the similarity theory of S. A. Kitaigorodskii,” *Journal of Geophysical Research* **69**, 5181–5190.

Platt, N., Hammel, S. M., and Heagy, J. F., 1994, “Effects of additive noise on on-off intermittency,” *Physical Reviews letters* **72(22)**, 3498–3501.

Platt, N., Spigel, E. A., and Tresser, C., 1993, “On-off intermittency: A mechanism for bursting,” *Physical Reviews letters* **70(3)**, 279–282.

Poirel, D. and Price, S. J., 2003a, “Random binary (coalescence) flutter of a two-dimensional linear airfoil,” *Journal of Fluids and Structures* **18**, 23–42.

Poirel, D. and Price, S. J., 2003b, “Response probability structure of a structurally nonlinear fluttering airfoil in turbulent flow,” *Probabilistic Engineering Mechanics* **18**, 185–202.

Pontryagin, L. S., Andronov, A. A., and Vitt, A. A., 1933, “The statistical treatment of dynamical systems,” *Zh. Eksp. Jeor. Fiz.* **3**, 165–180 (in Russian); English translation, 1989, in *Noise in nonlinear dynamical systems*, F. Moss and P. V. E. McClintock, Eds., Cambridge University Press, Cambridge, UK.

Prussing, J. E., 1981, “Stabilization of an unstable linear system by parametric white noise,” *ASME Journal of Applied Mechanics* **48**, 198–199.

Řeháček, J., Peřina, J., Facchi, P., Pasaczio, S., and Mišta, L., 2000, “Quantum Zeno effect in probed down-conversion process,” *Physics Reviews A* **62**, 013804.

Rikvold, P. A., Tomita, H., Miyashita, S., and Sides, S. W., 1994, “Metastable lifetime in a kinetic Ising model: Dependence on field and system size,” *Physical Review E* **49(6)**, 5080–5090.

Risken, H., 1989, *The Fokker-Planck Equation. Methods of Solution and Applications,* Springer-Verlag, New York, NY.

Romeiras, F., Grebogi, C., Ott, E., and Dayawansa, W. P., 1992, “Controlling chaotic dynamical systems,” *Physica D* **58**, 165–184.

Roy, P. K., Kozma, R., and Majumder, D. D., 2002, “From neuro-computation to immuno-computation: A model and algorithm for fluctuation-induced instability and phase transition in biological systems,” *IEEE Transactions on Evolutionary Computation* **6(3)**, 1–14.

Roy, R. V., 1995, “Noise-induced transitions in weakly nonlinear oscillators near resonance,” *ASME Journal of Applied Mechanics* **62**, 496–504.

Sancho, J. M., Mannella, R., McClintok, P. V. E., and Moss, F., 1985, “Relaxation times in a bi-stable system with parametric white noise: Theory and experiment,” *Physical Review A* **32**, 3639–3646.

San Miguel, M. and Toral, R, 1997, “Stochastic effects in physical systems,” in *Instabilities and Non-equilibrium Structures* **VI**, 35–130, E. Tirapegui and W. Zeller, eds., Kluwer, Dordrecht, The Netherlands.

Scheffer, M., Carpenter, S., Foley, J. A., Folke, C., and Walker, B., 2001, “Catastrophic shifts in ecosystems,” *Nature* **413**, 591–596.

Schenzle, A. and Brand, H., 1979, “Multiplicative stochastic processes in statistical physics,” *Physical Review A* **20(4)**, 1979, 1628–1647.

Seshadri, V., West, B. J., and Lendenberg, K., 1981, “Stability properties of nonlinear systems with fluctuating parameters,” *Physics (Utrecht)* **107A**, 219–240.

Sethna, P. R., 1972, “Ultimate behavior of a class of stochastic differential systems dependent on a parameter,” In *Proceedings of IUTAM Symposium on Stability of Stochastic Dynamic Systems, Lecture Notes in Mathematics* **294**, R. E. Curtain, ed., Springer-Verlag, Berlin, Germany, pp. 273–282.

Sethna, P. R., 1973, “Method of averaging for systems bounded for positive time,” *Journal of Mathematical Analyses and Applications* **41**, 69–96.

Sethna, P. R. and Orey, S., 1980, “Some asymptotic results for a class of stochastic systems with parametric excitations,” *International Journal of Nonlinear Mechanics* **15**, 431–441.

Shapiro, V. E., 1993, “Systems near a critical point under multiplicative noise and the concept of effective potential,” *Physical Review E* **48(1)**, 109–120.

Shinozuka, M. and Billah, K. Y., 1991, "Stochastic stabilization by multiplicative noise," In *Proceedings of IUTAM Symposium on Nonlinear Stochastic Mechanics*, Springer-Verlag, New York, NY, pp. 465–473.

Simiu, E., 2002, *Chaotic Transitions in Deterministic and Stochastic Dynamical Systems*, Princeton University Press, Princeton, NJ.

Smith, H. J. T. and Blackburn, K. A., 1992, "Experimental study of an inverted pendulum," *American Journal of Physics* **60**, 909–911.

Smythe, J., Moss, F., McClintock, P. V. E., and Clarkson, D., 1983a, "Itô versus Stratonovich revisited," *Physics Letters A* **97(3)**, 95–98.

Smythe, J., Moss, F., and McClintock, P. V. E., 1983b, "Observation of a noise-induced phase transition with an analog simulator," *Physical Review Letters* **51(12)**, 1062–1065.

Spagnolo, B. and La Barbera, A., 2002, "Role of the noise on the transient dynamics of an ecosystem of interacting species," *Physica A* **315**, 114–124.

Spagnolo, B., Cirone, M., La Barbera, A., and de Pasquale, F., 2002, "Noise-induced effects in population dynamics," *Journal of Physics: Condensed Matter* **14**, 2247–2255.

Soize, C., 1991, "Exact stationary response of multi-dimensional nonlinear Hamiltonian dynamical systems under parametric and external stochastic excitations," *Journal of Sound and Vibration* **149(1)**, 1–24.

Sotolongo-Costa, O., Morales-Molina, L., Rodriguez-Perez, D., Antonranz, J. C., and Chacon-Reyes, M., 2003, "Behavior of tumors under nonstationary therapy," *Physica D* **178**, 242–253.

Stauffer, D., Coniglio, A., and Heermann, D. W., 1982, "Monte Carlo experiment for nucleation rate in the three-dimensional Ising model," *Physical Review Letters* **49**, 1299–1302.

Stepanova, N. V., 1980, "Course of the immune reaction during the development of a malignant tumor," *Biophysics* **24**, 917–923.

Stephenson, A., 1908. "On a new type of dynamical stability," *Memoirs and Proceedings of the Manchester Literary and Philosophical Society* **52**, 1–10.

Stocks, N. G., Mannella, R., and McClintock, P. V. E., 1989, "Influence of random fluctuations on delayed bifurcations: The case of additive white noise," *Physical Review A* **40**, 5361–5469.

Stocks, N. G., Mannella, R., and McClintock, P. V. E., 1990, "Influence of random fluctuations on delayed bifurcations, II: The case of white and colored additive and multiplicative noise," *Physical Review A* **42(6)**. 3356–3362.

Stoker, J. J., 1950, *Nonlinear Vibrations in Mechanical and Electrical Systems,* Wiley & Sons, New York, NY.

Stoll, E. and Schneider, T., 1972, "Computer simulation of critical properties and metastable states in a finite square Ising system," *Physical review A* **6**, 429–432.

Stoll, E. and Schneider, T., 1997, "Size dependence of the lifetime of metastable states in the kinetic one-spine-flip Ising model," *Physica B+C* **86-88**, Part 3, 1419–1420.

Stratonovich, R. L., 1963, 1967, *Topics in the Theory of Random Noise*, Gordon and Breach, New York, NY, Vols. 1&2.

Stratonovich, R L., 1989, "Some Markov methods in the theory of stochastic processes in nonlinear dynamical systems," in *Noise in Nonlinear Dynamical Systems, Vol. 1: Theory of Continuous Fokker-Planck Systems,* F. Moss and P. V. E. McClintock, eds., Cambridge University Press, Cambridge, UK, Chapter 2, pp. 16–70.

Sudor, D. J. and Bishop, S. R., 1996, "Inverted dynamics of tilted parametric pendulum," *European Journal of Mechanics A* **18**, 517–526.

Suzuki, M., 1981, "Passage from an initial unstable state to a final stable state," *Advances in Chemical Physics* **46**, 195–278.

Suzuki, M., Kaneko, K., and Sasagawa, F., 1981, "Phase transition and slowing down in non-equilibrium stochastic processes," *Progress of Theoretical Physics* **65(3)**, 828–849.

Szymanska, S., 2003, "Analysis of the immunotherapy models in the context of cancer dynamics," *International Journal of Applied Mathematics and Computer Science* **13(3)**, 407–418.

These, J., 2000, *The Cytokine Network and Immune Function*, Oxford University Press, Oxford, UK.

Thun, K. and Peřina, J., 1998, "Zeno effect in optical parametric process with quantum pumping," *Physics Letters A* **249**, 363–368.

Torrent, M. C. and San Miguel, M., 1988a, "Stochastic-dynamics characterization of delayed laser threshold instability with swept control parameter," *Physical Review A* **38(1)**, 245–251.

Torrent, M. C. and San Miguel, M., 1988b, "Dynamics of sweeping through an instability: Passage-time statistics for colored noise," *Physical Review A* **40(11)**, 6662–6672.

Ulrich, B. D., Haehl, V., Buzzi, U. H., Kubo, M., and Holt, K. G., 2004, "Modeling dynamic resource utilization in populations with unique constraints: Preadolescents with and without Down syndrome," *Human Movement Science* **23**, 133–156.

Valeev, K. G., 1971, "Dynamic stabilization of unstable systems," *Izv. An SSSR. Mekhanika Tverdogo Tela* **16(4)**, 9–17. (in Russian).

Valenti, D., Fiasconaro, A., and Apagnolo, B., 2004, "Stochastic resonance and noise delayed extinction in a model of two competing species," *Physica A* **331**, 477–486.

Van den Broeck, C. and Mandel, P., 1987, "Delayed bifurcations in the presence of noise," *Physics Letters A* **122(1)**, 36–38.

Van den Broeck, C., Parrondo, J. M. R., and Toral, R., 1994, "Noise-induced non-equilibrium phase transition," *Physical Review Letters* **73**, 3395–3398.

Walker, B. H., Ludwig, D., Holling, C. S., and Peterman, R. M., 1981, "Stability of semi-arid savanna grazing systems," *Journal of Ecology* **69**, 473–498.

Wackerbrauer, R., 1998, "Noise-induced stabilization of one-dimensional discontinuous maps," *Physical Review E* **58**, 3036–3044.

Wackerbrauer, R., 1999, "When noise decreases deterministic diffusion," *Physical Review E* **59**, 2872–2879.

Weibel, S., Taper, J., and Baillieul, U., 1997, "Global dynamics of a rapidly forced cart pendulum," *Nonlinear Dynamics* **13**, 131–170.

Wellense, T., Shatokhin, V., and Buchleitner, A., 2004, "Stochastic resonance," *Reports on Progress in Physics* **67**, 45–105.

Wheldom, T. E., 1988, *Mathematical Models in Cancer Research*, Adam Hilger, Philadelphia, PA.

Wiesenfeld, K. and Jaramillo, F., 1998, "Minireview of stochastic resonance," *Chaos* **8**, 539–548.

Wiesenfeld, K. and Moss, F., 1995, "Stochastic resonance and the benefit of noise: From ice age to crayfish and SQUIDs," *Nature* **373**, 33–36.

Winter, D. A., Patla, A. E., Prince, F., Ishac, M., and Gielo-Perczak, K., 1998, "Stiffness control of balance in quiet standing," *Journal of Neurophysiology* **8**, 1211–1221.

Yabuno, H., Miura, M., and Aoshima, N., 2004, "Bifurcation in an inverted pendulum with tilted high-frequency excitation: Analytical and experimental investigations on the symmetry-breaking of the bifurcation," *Journal of Sound and Vibration* **273**, 493–513.

Yang, L., Hou, Z., and Xin, H., 1998a, "Stochastic resonance in surface catalytic oxidation of carbon monoxide," *Journal of Chemical Physics* **109(5)**, 2002–2005.

Yang, L., Hou, Z., Zhou, B., and Xin, H., 1998b, "Stochastic resonance in catalytic reduction of NO with CO on Pt(100)," *Journal of Chemical Physics* **109(5)**, 6456–6459.

Yoon, Y. J. and Ibrahim, R. A., 1995, "Parametric random excitation of nonlinear coupled oscillators," *Nonlinear Dynamics* **8**, 385–413.

Zaglache, H., Mandel, P., and van den Broeck, C., 1989, "Influence of noise on delayed bifurcations," *Physical Review A* **40(1)**, 286–294.