On the Four Terms in the Middle Theorem for almost split sequences

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Recently Liu [L] has proved the following result, which generalizes remarkably the wellknown Four Terms in the Middle Theorem of Bautista and Brenner [BB].

Liu's Theorem *Let Λ be an artin algebra and let $0 \to X \to \coprod_{i=1}^{r} Y_i \to Z \to 0$ be an almost split sequence in the category of finitely generated Λ-modules such that all Y_i's are indecomposable. Suppose that X has a projective predecessor and Z has an injective successor in the Auslander–Reiten quiver Γ_Λ of Λ. Then $r \leq 4$ and $r = 4$ implies that one of the Y_i's is projective-injective and the others are neither projective nor injective.*

We will show that Liu's proof can be translated into a purely combinatorial one. Before we state our combinatorial result which implies the above-mentioned theorem, let us fix some terminology.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a *quiver*, that is, a locally finite oriented graph with set of *vertices* Γ_0 and set of *arrows* Γ_1. Suppose that Γ contains neither loops nor multiple arrows. Given a vertex x, denote by x^+ the set of vertices y such that there is an arrow $x \to y$; the set x^- consists of all vertices y such that there is an arrow $y \to x$. A pair (Γ, τ) is called a *translation quiver*, if $\tau\colon \Gamma_0' \to \Gamma_0$ is an injective map for some subset $\Gamma_0' \subseteq \Gamma_0$, satisfying $(\tau x)^+ = x^-$ for all $x \in \Gamma_0'$. It is convenient to put $\tau^0 x = x$ for all $x \in \Gamma_0$. The vertices in $\Gamma_0 \setminus \Gamma_0'$ are called *projective*; those in $\Gamma_0 \setminus \tau\Gamma_0'$ are called *injective*. Let $\delta\colon \Gamma_1 \to \mathbb{N} \times \mathbb{N}$ be a map and denote the values by $\delta(\alpha) = (\delta_{x,y}, \delta'_{x,y})$ for each arrow $\alpha\colon x \to y$. The triple (Γ, τ, δ) is called a *valued translation quiver*, if the following conditions are satisfied for all non-projective vertices x.

(δ1) $\quad \delta'_{\tau x,y} = \delta_{y,x}$ for all $y \in x^-$.

(δ2) $\quad \delta_{\tau x,y} = \delta'_{y,x}$ for all $y \in x^-$.

Finally, a map $\ell\colon \Gamma_0 \to \mathbb{N}$ is called an *additive length function* for (Γ, τ, δ), if the following conditions are satified for all vertices x.

(ℓ1) $\quad \ell(x) + \ell(\tau x) = \sum_{y \in x^-} \delta_{y,x}\ell(y)$, if x is non-projective.

(ℓ2) $\quad \ell(x) > \sum_{y \in x^-} \delta_{y,x}\ell(y)$, if x is projective.

(ℓ3) $\quad \ell(x) > \sum_{y \in x^+} \delta'_{y,x}\ell(y)$, if x is injective.

The following is the main result of this note.

Theorem *Let (Γ, τ, δ) be a valued translation quiver and ℓ be an additive length function. Let x be a vertex having a projective predecessor and an injective successor. Then $\sum_{y \in x^+} \delta'_{x,y} \leq 4$ and equality implies that x is non-injective and that x^+ contains a projective vertex.*

The following consequence is the combinatorial version of Liu's theorem.

Corollary *Let (Γ, τ, δ) be a valued translation quiver and ℓ be an additive length function. Let x be a non-injective vertex such that x has a projective predecessor and $\tau^{-1}x$ has an injective successor. Then $\sum_{y \in x^+} \delta'_{x,y} \leq 4$ and equality implies that x^+ contains a projective-injective vertex y_0 with $\delta'_{x,y_0} = 1$ and $x^+ \setminus \{y_0\}$ contains neither a projective nor an injective vertex.*

It is well-known that the Auslander–Reiten quiver of an artin algebra together with the usual length function satisfies all the assumptions of the preceeding result. Applying the corollary in this situation one obtains Liu's theorem.

The rest of this paper is devoted to proving our combinatorial result, using a series of six lemmas. We stress that all the assertions of the lemmas can be found in a representation theoretic formulation in Liu's paper [L].

Recall that given a path $y = x_n \to \cdots \to x_1 \to x_0 = x$ of length $n \geq 1$, the vertex y is called a *predecessor* of x and the vertex x is called a *successor* of y. If the path is sectional, i.e. $\tau x_i \neq x_{i+2}$ for all x_i, $0 \leq i \leq n-2$, lying in Γ'_0, then the vertex y is called a *sectional predecessor* of x and the vertex x is called a *sectional successor* of y.

Lemma 1 *Let x be a vertex such that $\tau^r x$ is non-projective for all $r \geq 0$. Then all predecessors of x are non-projective if and only if all sectional predecessors of $\tau^r x$ are non-projective for all $r \geq 0$.*

Proof: Suppose there is a path $x_n \to \cdots \to x_0 = x$ with projective x_n. Choose i minimal such that $y = \tau^s x_i$ is projective for some $s \geq 0$. We obtain a sectional path $y \to \cdots \to \tau^r x$, for some $r \geq s$, by induction on i.

Lemma 2 *Let $x_n \to \cdots \to x_1 \to x_0 = x$ be a sectional path of length $n \geq 1$. If $\sum_{y \in x^-} \delta_{y,x} \ell(y) - \ell(x_1) \geq \ell(x)$, then x_n is non-projective.*

Proof: Clearly $\sum_{y\in x^-}\delta_{y,x}\ell(y)-\ell(x_1)\geq\ell(x_0)$ implies that x_0 is non-projective and $\ell(\tau x_0)=\sum_{y\in x^-}\delta_{y,x}\ell(y)-\ell(x_0)\geq\ell(x_1)$. Hence, x_1 is non-projective. Since $\tau x_0\neq x_2$, it follows that $\sum_{y\in x_1^-}\delta_{y,x_1}\ell(y)-\ell(x_2)\geq\ell(\tau x_0)\geq\ell(x_1)$. Proceeding by induction, one shows that x_n is non-projective.

Lemma 3 *Let x be a non-projective vertex satisfying $\ell(\tau x)\geq\ell(y)$ for all $y\in x^-$. Then all sectional predecessors of x are non-projective.*

Proof: Let $x_n\to\cdots\to x_1\to x_0=x$ be a sectional path of length $n\geq 1$. By assumption $\ell(\tau x)\geq\ell(x_1)$. Therefore $\sum_{y\in x^-}\delta_{y,x}\ell(y)-\ell(x_1)=\ell(x)+\ell(\tau x)-\ell(x_1)\geq\ell(x)$. Using Lemma 2 we conclude that x_n is non-projective.

Lemma 4 *Let y and y' be not necessarily distinct elements in x^+ for some vertex x. Suppose that $\delta'_{x,y}\geq 2$, if $y=y'$. Then $\ell(x)\geq\ell(y)+\ell(y')$ implies that y, y' and x are non-projective and $\ell(\tau x)\geq\sum_{z\in x^-}\delta_{z,x}\ell(z)-\ell(\tau y)-\ell(\tau y')$.*

Proof: The assumption $\ell(x)\geq\ell(y)+\ell(y')$ implies that y and y' are non-projective and $\ell(\tau y)+\ell(\tau y')\geq 2\ell(x)-\ell(y)-\ell(y')\geq\ell(x)$. Since $\delta_{\tau y,x}=\delta'_{x,y}\geq 2$, if $y=y'$, it follows that x is non-projective and $\ell(\tau x)=\sum_{z\in x^-}\delta_{z,x}\ell(z)-\ell(x)\geq\sum_{z\in x^-}\delta_{z,x}\ell(z)-\ell(\tau y)-\ell(\tau y')$.

Lemma 5 *Let $y_1,\ldots,y_n$ be non-projective vertices in x^+ for some vertex x. Let ε_i, $1\leq i\leq n$ be integers satisfying $1\leq\varepsilon_i\leq\delta'_{x,y_i}$ for all i and $\sum_i\varepsilon_i=4$. Suppose that either x is non-injective with $\ell(x)\geq\ell(\tau^{-1}x)$, or $\ell(x)\geq\sum_i\varepsilon_i\ell(y_i)$. Then x and all its predecessors are non-projective.*

Proof: By our assumptions, we have $2\ell(x)\geq\sum_i\varepsilon_i\ell(y_i)$. Choosing integers $\varepsilon'_i,\varepsilon''_i\geq 0$ satisfying $\varepsilon_i=\varepsilon'_1+\varepsilon''_i$ for all i and $\sum_i\varepsilon'_i=2=\sum_i\varepsilon''_i$, we have $\ell(x)\geq\sum_i\varepsilon'_i\ell(y_i)$ or $\ell(x)\geq\sum_i\varepsilon''_i\ell(y_i)$. Therefore x is non-projective by Lemma 4. Let $z\in x^-$ and define $\gamma_i=\varepsilon_i-1$, if $z=\tau y_i$ and $\gamma_i=\varepsilon_i$ otherwise. Then

$$\ell(\tau x)\geq\ell(z)+\sum_i\gamma_i\ell(\tau y_i)-\ell(x)\geq\ell(z)+\sum_i\gamma_i(\ell(x)-\ell(y_i))-\ell(x)\geq\ell(z).$$

Therefore any sectional predecessor of x is non-projective by Lemma 3. In particular, τy_i is non-projective for all i and

$$\ell(\tau x)\geq\sum_i\varepsilon_i\ell(\tau y_i)-\ell(x)\geq\sum_i\varepsilon_i(\ell(x)-\ell(y_i))-\ell(x)\geq\ell(x).$$

By induction, $\tau^r x$ and all sectional predecessors of $\tau^r x$ are non-projective for all $r \geq 0$. Now the assertion follows from Lemma 1.

Lemma 6 *Let x be a vertex having an injective sectional successor. Suppose that either $r = \sum_{y \in x^+} \delta'_{x,y} > 4$, or $r = 4$ and x^+ contains no projective vertex. Then all predecessors of x are non-projective.*

Proof: Let $x = x_0 \to \cdots \to x_n$ be a sectional path of length $n \geq 1$ with x_n injective. Then $\ell(x) \geq \sum_{y \in x^+} \delta'_{x,y} \ell(y) - \ell(x_1)$ by the dual of Lemma 2. If $r > 4$, then the assertion follows from Lemma 5. Therefore assume $r = 4$ and $x^+ \subseteq \Gamma'_0$. For any choice of integers ε_y, $y \in x^+$ satisfying $0 \leq \varepsilon_y \leq \delta'_{x,y}$ for all y, $\varepsilon_y \leq \delta'_{x,y} - 1$ for $y = x_1$ and $\sum_{y \in x^+} \varepsilon_y = 2$, we have $\ell(x) \geq \sum_{y \in x^+} \varepsilon_y \ell(y)$. Therefore x is non-projective and $\ell(\tau x) \geq \sum_{z \in x^-} \delta_{z,x} \ell(z) - \sum_{y \in x^+} \varepsilon_y \ell(\tau y)$ by Lemma 4. We conclude that $\ell(\tau x) \geq \ell(z)$ for all $z \in x^-$. Thus all sectional predecessors of x are non-projective by Lemma 3. In particular, $x^- \subseteq \Gamma'_0$. We have also $\sum_{y \in x^+} \delta'_{\tau x, \tau y} = \sum_{y \in x^+} \delta_{\tau y, x} = \sum_{y \in x^+} \delta'_{x,y} = 4$, and so, since x has an injective successor, $\ell(\tau x) \geq \ell(x)$ by the dual of Lemma 5. Therefore τx and all predecessors of τx are non-projective by Lemma 5. This finishes the proof.

We are now ready to prove the main result.

Proof of the Theorem: From the fact that x has an injective successor, we obtain $r \geq 0$ such that either $z = \tau^{-r} x$ has an injective sectional successor or z is injective. This follows from the dual statement of Lemma 1. Choose r minimal and assume first that z has an injective sectional successor. Then $\sum_{y \in x^+} \delta'_{x,y} = \sum_{y \in x^+} \delta'_{z, \tau^{-r} y} \leq \sum_{y \in z^+} \delta'_{z,y} \leq 4$ by Lemma 6. Moreover, equality implies that $z^+ = \tau^{-r}(x^+)$ and z^+ contains a projective vertex. But this is only possible for $r = 0$. Therefore x^+ contains a projective vertex and hence x is non-injective. Now assume that z is injective. We have $\ell(z) \geq \sum_{y \in x^+} \delta'_{z, \tau^{-r} y} \ell(\tau^{-r} y)$ and $\sum_{y \in x^+} \delta'_{x,y} = \sum_{y \in x^+} \delta'_{z, \tau^{-r} y} < 4$ follows from Lemma 5, since $\tau^{-r}(x^+) \subseteq \Gamma'_0$.

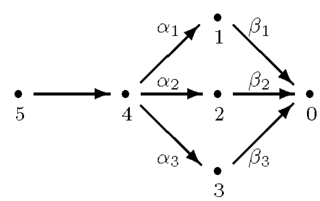
Proof of the Corollary: Combine the theorem and its dual with the following well-known observation.

Lemma 7 *Let x be a vertex such that x^+ contains a projective and an injective vertex. Then x^+ contains a projective-injective vertex y_0 with $\delta'_{x,y_0} = 1$ and $x^+ \setminus \{y_0\}$ contains neither a projective nor an injective vertex.*

Proof: Suppose $y_0 \in x^+$ is projective. Then x is non-injective and we obtain $\sum_{x^+\setminus\{y_0\}} \delta'_{x,y}\ell(y) + (\delta'_{x,y_0} - 1)\ell(y_0) = \ell(\tau^{-1}x) + \ell(x) - \ell(y_0) \leq \ell(\tau^{-1}x)$. Therefore $x^+ \setminus \{y_0\}$ contains no injective vertex and y_0 is projective-injective by assumption. Moreover, $\delta'_{x,y_0} - 1 = 0$, since y_0 is injective. The dual argument shows that $x^+ \setminus \{y_0\}$ contains no projective vertex.

The following example which was suggested by R. Betzler and R. Schmidmeier illustrates the theorem.

Example Let k be a field and denote by Λ the k-algebra given by the following quiver with relations:



, $\qquad \alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_3\beta_3.$

Denote by P_i the indecomposable projective Λ-module corresponding to the vertex i and let $S_i = P_i/\operatorname{rad} P_i$, $1 \leq i \leq 4$. There is an almost split sequence

$$0 \to \operatorname{rad} P_4 \to S_1 \coprod S_2 \coprod S_3 \coprod P_4 \to P_4/\operatorname{soc} P_4 \to 0$$

and an irreducible map $P_4 \to P_5$. The projective P_0 is a predecessor and the injective P_5 is a successor of $\operatorname{rad} P_4$ in the Auslander-Reiten quiver Γ_Λ of Λ, but $P_4/\operatorname{soc} P_4$ has no injective successor.

References

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