PRESENTATIONS OF RINGS WITH NON-TRIVIAL SELF-ORTHOGONAL MODULES

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Abstract. A result of Foxby, Reiten and Sharp says that a commutative noetherian local ring R admits a dualizing module if and only if R is Cohen–Macaulay and a homomorphic image of a local Gorenstein ring Q. We establish an analogous result by showing that such a ring R having a dualizing module admits a non-trivial finitely generated self-orthogonal module C satisfying $\mathrm{Hom}_R(C,C) \cong R$ if and only if R is the homomorphic image of a Gorenstein ring in which the defining ideal decomposes in a non-trivial way, forcing significant structural requirements on the ring R.

1. Introduction

Throughout this paper $(R, \mathfrak{m}, k)$ is a commutative noetherian local ring.

A finitely generated R-module C is *self-orthogonal* if $\mathrm{Ext}^i_R(C,C) = 0$ for all $i \geqslant 1$. Examples of self-orthogonal R-modules include the finitely generated free R-modules and the dualizing module of Grothendieck. (See Section 2 for definitions and background information.) Results of Foxby [7], Reiten [14] and Sharp [15] precisely characterize the local rings which possess dualizing modules: the ring R admits a dualizing module if and only if R is Cohen–Macaulay and there exist a Gorenstein local ring Q and an ideal $I \subset Q$ such that $R \cong Q/I$.

The point of this paper is to similarly characterize the local Cohen–Macaulay rings with a dualizing module which admit certain non-trivial self-orthogonal modules. We show that the existence of such a module imposes considerable structural implications on the ring via a Gorenstein presentation $R \cong Q/I$. The specific modules of interest are the *semidualizing* R-modules, i.e., the finitely generated self-orthogonal R-modules such that $\mathrm{Hom}_R(C,C) \cong R$. A free R-module of rank 1 is semidualizing, as is a dualizing R-module, when one exists. For this investigation, these are the *trivial* semidualizing R-modules.

Our main theorem is the following analog of the aforementioned result of Foxby, Reiten and Sharp; we prove it in Section 3.

Theorem 1.1. *Let R be a local Cohen–Macaulay ring that admits a dualizing module D. Then R admits a semidualizing module that is neither dualizing nor free if and only if there exist a Gorenstein local ring Q and ideals $I_1, I_2 \subset Q$ satisfying the following conditions:*

(1) *There are ring isomorphisms $R \cong Q/(I_1 + I_2) \cong (Q/I_1) \otimes_Q (Q/I_2)$;*

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(2) *For $j = 1,2$ the quotient ring Q/I_j is Cohen–Macaulay with a dualizing module D_j and is not Gorenstein;*
(3) *For $j = 1,2$ we have* $\mathrm{G}\text{-}\dim_{Q/I_j}(R) = 0$*;*
(4) *There is an R-module isomorphism $D_1 \otimes_Q D_2 \cong D$, and for all $i \geqslant 1$ we have* $\mathrm{Tor}_i^Q(D_1, D_2) = 0$*; and*
(5) *For all $i \geqslant 1$, we have* $\mathrm{Tor}_i^Q(Q/I_1, Q/I_2) = 0$*; in particular, there is an equality $I_1 \cap I_2 = I_1 I_2$.*

Examples of rings that do not admit non-trivial semidualizing modules are easy to come by.

Example 1.2. Let k be a field. The ring $R = k[X,Y]/(X^2, XY, Y^2)$ is local with maximal ideal $\mathfrak{m} = (X,Y)R$. It is artinian of type 2, hence Cohen–Macaulay and non-Gorenstein with a dualizing module D. From the equality $\mathfrak{m}^2 = 0$, it is straightforward to deduce that the only semidualizing R-modules, up to isomorphism, are R and D.

2. Background on Semidualizing Modules

We begin with relevant definitions. The following notions were introduced independently (with different terminology) by Foxby [7], Golod [9], Grothendieck [10, 11] Vasconcelos [16] and Wakamatsu [17].

Definition 2.1. Let C be an R-module. The *homothety homomorphism* is the map $\chi_C^R \colon R \to \mathrm{Hom}_R(C,C)$ given by $\chi_C^R(r)(c) = rc$.

The R-module C is *semidualizing* if it satisfies the following conditions:

(1) The R-module C is finitely generated;
(2) The homothety map $\chi_C^R \colon R \to \mathrm{Hom}_R(C,C)$, is an isomorphism; and
(3) For all $i \geqslant 1$, we have $\mathrm{Ext}_R^i(C,C) = 0$.

An R-module D is *dualizing* if it is semidualizing and has finite injective dimension.

Fact 2.2. The R-module R is semidualizing, so every local ring admits a semidualizing module. Examples of non-trivial semidualizing modules were given independently by Foxby [6] and Vasconcelos [16].

Fact 2.3. Let C be a semidualizing R-module. The isomorphism $R \cong \mathrm{Hom}_R(C,C)$ implies that $\mathrm{Ann}_R(C) = 0$ and $\mathrm{Ass}_R(C) = \mathrm{Ass}(R)$. It follows that $\mathrm{Supp}_R(C) = \mathrm{Spec}(R)$ and $\dim_R(C) = \dim(R)$. Furthermore, an element $x \in \mathfrak{m}$ is C-regular if and only if it is R-regular. When the element $x \in \mathfrak{m}$ is R-regular, it is straightforward to show that $\mathrm{Ext}_R^i(R/xR, C) = 0$ for all $i \neq 1$ and that the module

$$C/xC \cong \mathrm{Ext}_R^1(R/xR, C)$$

is semidualizing for R/xR. Thus, by induction on $\mathrm{depth}(R)$, we conclude that $\mathrm{depth}_R(C) = \mathrm{depth}(R)$. In particular, when R is Cohen–Macaulay, every semidualizing R-module is a maximal Cohen–Macaulay module. On the other hand, if R admits a dualizing module, then R is Cohen–Macaulay.

Fact 2.4. Let C be a semidualizing R-module. If $\mathrm{pd}_R(C) < \infty$, then $C \cong R$, as follows. Assume that $\mathrm{pd}_R(C) < \infty$. The Auslander-Buchsbaum formula yields the first equality in the following sequence while the second equality is from Fact 2.3:

$$\mathrm{pd}_R(C) = \mathrm{depth}(R) - \mathrm{depth}_R(C) = 0.$$

It follows that C is free, say $C \cong R^n$. The isomorphisms

$$R \cong \mathrm{Hom}_R(C,C) \cong \mathrm{Hom}_R(R^n,R^n) \cong R^{n^2}$$

imply that $n = 1$, so $C \cong R$.

The following definition and fact justify the term "dualizing".

Definition 2.5. Let C and B be R-modules. The natural *biduality homomorphism* $\delta_C^B\colon C \to \mathrm{Hom}_R(\mathrm{Hom}_R(C,B),B)$ is given by $\delta_C^B(c)(\phi) = \phi(c)$. When D is a dualizing R-module, we set $C^\dagger = \mathrm{Hom}_R(C,D)$.

Fact 2.6. Assume that R is Cohen–Macaulay and admits a dualizing module D. Let C be a semidualizing R-module. Fact 2.3 implies that C is a maximal Cohen–Macaulay R-module. From standard duality theory, for all $i \neq 0$ we have

$$\mathrm{Ext}_R^i(C,D) = 0 = \mathrm{Ext}_R^i(\mathrm{Hom}_R(C,D),D)$$

and the natural biduality homomorphism $\delta_C^D\colon C \to \mathrm{Hom}_R(\mathrm{Hom}_R(C,D),D)$ is an isomorphism; see, e.g., [4, (3.3.10)]. In particular, we have $\mathrm{Hom}_R(C^\dagger,D) \cong C$. Furthermore $C^\dagger$ is a semidualizing R-module by [5, (2.12)], and the evaluation map $C \otimes_R C^\dagger \to D$ given by $c \otimes \phi \mapsto \phi(c)$ is an isomorphism by [8, (3.1)].

The following construction is also known as the "idealization" of M. It was popularized by Nagata, but goes back at least to Hochschild [12]. It is the key idea for the proof of the converse of Sharp's result [15] given by Foxby [7] and Reiten [14]. It has also been very helpful in the study of G-dimensions because of the paper of Holm and Jørgensen [13]. The interested reader can find a survey of some properties of this construction in the article of Anderson and Winders [1].

Definition 2.7. Let M be an R-module. The *trivial extension* of R by M is the ring $R \ltimes M$, described as follows. As an additive abelian group, we have $R \ltimes M \cong R \oplus M$. The multiplication in $R \ltimes M$ is given by the formula

$$(r,m)(r',m') = (rr', rm' + r'm).$$

The multiplicative identity on $R \ltimes M$ is $(1,0)$. We let $\epsilon_M\colon R \to R \ltimes M$ and $\tau_M\colon R \ltimes M \to R$ denote the natural injection and surjection, respectively.

The next facts are straightforward to verify.

Fact 2.8. Let M be an R-module. The trivial extension $R \ltimes M$ is a commutative ring with identity. The maps ϵ_M and τ_M are ring homomorphisms, and $\mathrm{Ker}(\tau_M) = 0 \oplus M$. We have $(0 \oplus M)^2 = 0$, and so $\mathrm{Spec}(R \ltimes M)$ is in order-preserving bijection with $\mathrm{Spec}(R)$. It follows that $R \ltimes M$ is quasilocal and $\dim(R \ltimes M) = \dim(R)$. If M is finitely generated, then R is also noetherian and

$$\mathrm{depth}(R \ltimes M) = \mathrm{depth}_R(R \ltimes M) = \min\{\mathrm{depth}(R), \mathrm{depth}_R(M)\}.$$

In particular, if R is Cohen–Macaulay and M is a maximal Cohen–Macaulay R-module, then $R \ltimes M$ is Cohen–Macaulay as well.

Here is a discussion of the correspondence between dualizing modules and Gorenstein presentations.

Fact 2.9. Sharp [15, (3.1)] showed that if R is Cohen–Macaulay and a homomorphic image of a local Gorenstein ring Q, then R admits a dualizing module. The proof proceeds as follows. Let Q be a local Gorenstein ring equipped with a ring epimorphism $\pi\colon Q \to R$, and set $g = \operatorname{depth}(Q) - \operatorname{depth}(R) = \dim(Q) - \dim(R)$. It follows that $\operatorname{Ext}^i_Q(R,Q) = 0$ for $i \neq g$ and the module $\operatorname{Ext}^g_Q(R,Q)$ is dualizing for R. Thus, by Cohen's structure theorem, every local complete Cohen–Macaulay ring has a dualizing module.

The same idea gives the following. Let A be a local Cohen–Macaulay ring with a dualizing module D, and assume that R is Cohen–Macaulay and a module-finite A-algebra. If $h = \operatorname{depth}(A) - \operatorname{depth}(R) = \dim(A) - \dim(R)$, then $\operatorname{Ext}^i_A(R,D) = 0$ for $i \neq h$ and the module $\operatorname{Ext}^h_A(R,D)$ is dualizing for R.

Fact 2.10. Independently, Foxby [7, (4.1)] and Reiten [14, (3)] proved the converse of Sharp's result from Fact 2.9. Namely, they showed the following: If R admits a dualizing module, then it is Cohen–Macaulay and a homomorphic image of a local Gorenstein ring Q. We sketch the proof here, as the main idea forms the basis of our proof of Theorem 1.1.

Let D be a dualizing R-module. It follows that R is Cohen–Macaulay. Set $Q = R \ltimes D$, which is Cohen–Macaulay with $\dim(Q) = \dim(R)$; see Facts 2.3 and 2.8. The natural injection $\epsilon_D\colon R \to Q$ makes Q into a module-finite R-algebra. The module D is dualizing for R, so Fact 2.9 implies that the module $\operatorname{Hom}_R(Q,D)$ is dualizing for Q. There is a sequence of R-module isomorphisms

$$\operatorname{Hom}_R(Q,D) \cong \operatorname{Hom}_R(R \oplus D, D) \cong \operatorname{Hom}_R(D,D) \oplus \operatorname{Hom}_R(R,D) \cong R \oplus D \cong Q$$

and it is straightforward to show that the composition $\operatorname{Hom}_R(Q,D) \cong Q$ is actually a Q-module isomorphism. Fact 2.2 implies that Q is Gorenstein, so the natural surjection $\tau_D\colon Q \to R$ yields an presentation of R as a homomorphic image of the local Gorenstein ring Q.

The last notion we need is Golod's generalization [9] of Auslander and Bridger's G-dimension [2, 3].

Definition 2.11. Let C be a semidualizing R-module. An R-module G is *totally C-reflexive* if it satisfies the following:

(1) The R-module G is finitely generated;
(2) The biduality map $\delta^C_G\colon G \to \operatorname{Hom}_R(\operatorname{Hom}_R(G,C),C)$, is an isomorphism; and
(3) For all $i \geqslant 1$, we have $\operatorname{Ext}^i_R(G,C) = 0 = \operatorname{Ext}^i_R(\operatorname{Hom}_R(G,C),C)$.

Let M be a finitely generated R-module. Then M has *finite G_C-dimension* if it has a finite resolution by totally C-reflexive R-modules, that is, if there is an exact sequence

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$$

such that each G_i is a totally C-reflexive R-module. The *G_C-dimension* of M, when it is finite, is the length of the shortest finite resolution by totally C-reflexive R-modules:

$$\mathrm{G}_C\text{-}\dim_R(M) = \inf\left\{ n \geqslant 0 \;\middle|\; \begin{array}{c} \text{there is a finite resolution} \\ 0 \to G_n \to \cdots \to G_0 \to M \to 0 \\ \text{by totally } C\text{-reflexive } R\text{-modules} \end{array} \right\}.$$

When $C = R$, we write $\mathrm{G}\text{-}\dim_R(M)$ in place of $\mathrm{G}_R\text{-}\dim_R(M)$.

Fact 2.12. Let C be a semidualizing R-module. The AB formula [5, (3.14)] says that if M is a finitely generated R-module of finite G_C-dimension, then

$$\mathrm{G}_C\text{-}\dim_R(M) = \operatorname{depth}(R) - \operatorname{depth}_R(M).$$

Fact 2.13. Let S be a Cohen–Macaulay local ring equipped with a module-finite local ring homomorphism $\tau\colon S \to R$ such that R is Cohen–Macaulay. Let C be a semidualizing S-module. Then $\mathrm{G}_C\text{-}\dim_S(R) < \infty$ if and only if there exists an integer $g \geqslant 0$ such that $\mathrm{Ext}^i_S(R,C) = 0$ for all $i \neq g$ and $\mathrm{Ext}^g_S(R,C)$ is a semidualizing R-module; when these conditions hold, one has $g = \mathrm{G}_C\text{-}\dim_S(R)$. See [5, (6.1)].

Assume that S has a dualizing module D. If $\mathrm{G}_C\text{-}\dim_S(R) < \infty$, then $R \otimes_S C^\dagger$ is a semidualizing R-module and $\mathrm{Tor}^S_i(R, C^\dagger) = 0$ for all $i \geqslant 1$; see [5, (4.7),(5.1)]. In particular, if $\mathrm{G}\text{-}\dim_S(R) < \infty$, then $\mathrm{Tor}^S_i(R,D) = 0$ for all $i \geqslant 1$ and $R \otimes_S D$ is a semidualizing R-module.

3. Proof of Theorem 1.1

Throughout this section, we assume that R is a Cohen–Macaulay ring with dualizing module D. We divide the proof of Theorem 1.1 into several pieces. The first piece is the following lemma which covers one implication; the remaining pieces deal with the converse.

Lemma 3.1. *Assume that there exist a Gorenstein local ring Q and ideals $I_1, I_2 \subset Q$ satisfying the following conditions:*

(1) *There are ring isomorphisms $R \cong Q/(I_1 + I_2) \cong (Q/I_1) \otimes_Q (Q/I_2)$;*
(2) *For $j = 1, 2$ the quotient ring Q/I_j is Cohen–Macaulay with a dualizing module D_j and is not Gorenstein;*
(3) *For $j = 1, 2$ we have $\mathrm{G}\text{-}\dim_{Q/I_j}(R) < \infty$; and*
(4) *There is an R-module isomorphism $D_1 \otimes_Q D_2 \cong D$.*

Then R admits a semidualizing module that is neither dualizing nor free.

Proof. For $j = 1, 2$ set $R_j = Q/I_j$. Condition (3) implies that for $j = 1, 2$ we have $\mathrm{Tor}^{R_j}_i(R, D_j) = 0$ for all $i \geqslant 1$ and $C_j = R \otimes_{R_j} D_j$ is a semidualizing R-module by Fact 2.13. Since R_j is not Gorenstein, the R_j-module D_j is not cyclic. Thus, the R-module C_j is not cyclic, and hence not free.

Condition (4) provides the first isomorphism in the next sequence:

$$D \cong D_1 \otimes_Q D_2 \cong R \otimes_Q (D_1 \otimes_Q D_2) \cong (R \otimes_Q D_1) \otimes_R (R \otimes_Q D_2) = C_1 \otimes_R C_2.$$

For the second isomorphism, use the fact that D_j is annihilated by I_j for $j = 1, 2$ to conclude that $D_1 \otimes_Q D_2$ is annihilated by $I_1 + I_2$; it follows that $D_1 \otimes_Q D_2$ is naturally a module over the quotient $Q/(I_1 + I_2) \cong R$, and hence the second isomorphism. The third isomorphism is standard. This, together with the fact that each C_i is not cyclic, yields the following (in)equalities of minimal numbers of generators:

$$\mu_R(D) = \mu_R(C_1)\mu_R(C_2) > \mu_R(C_j)$$

for $j = 1, 2$. It follows that $C_j \ncong D$ for $j = 1, 2$ so each C_j is a semidualizing R-module that is not free and not dualizing, as desired. □

Assumption 3.2. For the rest of this section, assume that R admits a semidualizing module C that is neither dualizing nor free.

For the sake of readability, we include the following roadmap of the remainder of the proof.

Outline 3.3. The ring Q is constructed as an iterated trivial extension of R. As an R-module, it has the form $Q = R \oplus C \oplus C^\dagger \oplus D$ where $C^\dagger = \mathrm{Hom}_R(C, D)$. The ideals I_j are then given as $I_1 = 0 \oplus 0 \oplus C^\dagger \oplus D$ and $I_2 = 0 \oplus C \oplus 0 \oplus D$. The details for these constructions are contained in Steps 3.4 and 3.5. Conditions (1)–(3) of Theorem 1.1 are then verified in Lemmas 3.6–3.8. Theorem 1.1(5) requires more work; it is proved in Lemma 3.12, with the help of Lemmas 3.9–3.11. The proof concludes with Lemma 3.13 where we establish Theorem 1.1(4).

The following two steps contain notation and facts for use through the rest of the proof.

Step 3.4. Set $R_1 = R \ltimes C$, which is Cohen–Macaulay with $\dim(R_1) = \dim(R)$; see Facts 2.3 and 2.8. The natural injection $\epsilon_C \colon R \to R_1$ makes R_1 into a module-finite R-algebra, so Fact 2.9 implies that the module $D_1 = \mathrm{Hom}_R(R_1, D)$ is dualizing for R_1. There is a sequence of R-module isomorphisms

$$D_1 = \mathrm{Hom}_R(R_1, D) \cong \mathrm{Hom}_R(R \oplus C, D) \cong \mathrm{Hom}_R(C, D) \oplus \mathrm{Hom}_R(R, D) \cong C^\dagger \oplus D.$$

It is straightforward to show that the resulting R_1-module structure on $C^\dagger \oplus D$ is given by the following formula:

$$(r, c)(\phi, d) = (r\phi, \phi(c) + rd).$$

The kernel of the natural epimorphism $\tau_C \colon R_1 \to R$ is the ideal $\mathrm{Ker}(\tau_C) \cong 0 \oplus D$.

Fact 2.9 implies that the ring $Q = R_1 \ltimes D_1$ is local and Gorenstein. The R-module isomorphism in the next display is by definition:

$$Q = R_1 \ltimes D_1 \cong R \oplus C \oplus C^\dagger \oplus D.$$

It is straightforward to show that the resulting ring structure on Q is given by

$$(r, c, \phi, d)(r', c', \phi', d') = (rr', rc' + r'c, r\phi' + r'\phi, \phi'(c) + \phi(c') + rd' + r'd).$$

The kernel of the epimorphism $\tau_{D_1} \colon Q \to R_1$ is the ideal

$$I_1 = \mathrm{Ker}(\tau_{D_1}) \cong 0 \oplus 0 \oplus C^\dagger \oplus D.$$

As a Q-module, this is isomorphic to the R_1-dualizing module D_1. The kernel of the composition $\tau_C \circ \tau_{D_1} \colon Q \to R$ is the ideal $\mathrm{Ker}(\tau_C \tau_{D_1}) \cong 0 \oplus C \oplus C^\dagger \oplus D$.

Step 3.5. Set $R_2 = R \ltimes C^\dagger$, which is Cohen–Macaulay with $\dim(R_2) = \dim(R)$. The injection $\epsilon_{C^\dagger} \colon R \to R_2$ makes R_2 into a module-finite R-algebra, so the module $D_2 = \mathrm{Hom}_R(R_2, D)$ is dualizing for R_2. There is a sequence of R-module isomorphisms

$$D_2 = \mathrm{Hom}_R(R_2, D) \cong \mathrm{Hom}_R(R \oplus C^\dagger, D) \cong \mathrm{Hom}_R(C^\dagger, D) \oplus \mathrm{Hom}_R(R, D) \cong C \oplus D.$$

The last isomorphism is from Fact 2.6. The resulting R_2-module structure on $C \oplus D$ is given by the following formula:

$$(r, \phi)(c, d) = (r\phi, \phi(c) + rd).$$

The kernel of the natural epimorphism $\tau_{C^\dagger} \colon R_2 \to R$ is the ideal $\mathrm{Ker}(\tau_{C^\dagger}) \cong 0 \oplus D$.

The ring $Q' = R_2 \ltimes D_2$ is local and Gorenstein. There is a sequence of R-module isomorphisms

$$Q' = R_2 \ltimes D_2 \cong R \oplus C \oplus C^\dagger \oplus D$$

and the resulting ring structure on $R \oplus C \oplus C^\dagger \oplus D$ is given by

$$\begin{aligned}&(r, c, \phi, d)(r', c', \phi', d')\\ &\qquad= (rr', rc' + r'c, r\phi' + r'\phi, \phi'(c) + \phi(c') + rd' + r'd).\end{aligned}$$

That is, we have an isomorphism of rings $Q' \cong Q$. The kernel of the epimorphism $\tau_{D_2}\colon Q \to R_2$ is the ideal

$$I_2 = \operatorname{Ker}(\tau_{D_2}) \cong 0 \oplus C \oplus 0 \oplus D.$$

This is isomorphic, as a Q-module, to the dualizing module D_2. The kernel of the composition $\tau_C^\dagger \circ \tau_{D_2}\colon Q \to R$ is the ideal $\operatorname{Ker}(\tau_C^\dagger \tau_{D_2}) \cong 0 \oplus C \oplus C^\dagger \oplus D$.

We verify condition (1) from Theorem 1.1 in the next lemma.

Lemma 3.6. *With the notation of Steps 3.4 and 3.5, there are ring isomorphisms* $R \cong Q/(I_1 + I_2) \cong (Q/I_1) \otimes_Q (Q/I_2)$.

Proof. The second isomorphism is standard. For the first one, consider the following sequence of R-module isomorphisms:

$$\begin{aligned}Q/(I_1 + I_2) &\cong (R \oplus C \oplus C^\dagger \oplus D)/((0 \oplus 0 \oplus C^\dagger \oplus D) + (0 \oplus C \oplus 0 \oplus D))\\ &\cong (R \oplus C \oplus C^\dagger \oplus D)/(0 \oplus C \oplus C^\dagger \oplus D))\\ &\cong R.\end{aligned}$$

It is straightforward to check that these are ring isomorphisms. □

We verify condition (2) from Theorem 1.1 in the next lemma.

Lemma 3.7. *With the notation of Steps 3.4 and 3.5, each ring* $R_j \cong Q/I_j$ *is Cohen–Macaulay with a dualizing module and is not Gorenstein.*

Proof. It remains to show that each ring R_j is not Gorenstein, that is, that D_j is not isomorphic to R_j as an R_j-module.

For R_1, suppose by way of contradiction that there is an R_1-module isomorphism $D_1 \cong R_1$. It follows that this is an R-module isomorphism via the natural injection $\epsilon_C\colon R \to R_1$. Thus, we have R-module isomorphisms

$$C^\dagger \oplus D \cong D_1 \cong R_1 \cong R \oplus D.$$

Computing minimal numbers of generators, we have

$$\mu_R(C^\dagger) + \mu_R(D) = \mu_R(C^\dagger \oplus D) = \mu_R(R \oplus D) = \mu_R(R) + \mu_R(D) = 1 + \mu_R(D).$$

It follows that $\mu_R(C^\dagger) = 1$, that is, that $C^\dagger$ is cyclic. By Fact 2.3, we have $\operatorname{Ann}_R(C) = 0$, and hence $C^\dagger \cong R/\operatorname{Ann}_R(C^\dagger) \cong R$. It follows that

$$C \cong \operatorname{Hom}_R(C^\dagger, D) \cong \operatorname{Hom}_R(R, D) \cong D$$

contradicting the assumption that C is not dualizing for R.

Next, observe that $C^\dagger$ is not free and is not dualizing for R; this follows from the isomorphism $C \cong \operatorname{Hom}_R(C^\dagger, D)$ contained in Fact 2.6, using the assumption that C is not free and not dualizing. Hence, the proof that R_2 is not Gorenstein follows as in the previous paragraph. □

We verify condition (3) from Theorem 1.1 in the next lemma.

Lemma 3.8. *With the notation of Steps 3.4 and 3.5, we have* $\operatorname{G-dim}_{Q/I_j}(R) = 0$ *for* $j = 1, 2$.

Proof. First, note that it suffices to show that $\text{G-dim}_{R_j}(R) < \infty$. Indeed, if $\text{G-dim}_{R_j}(R) < \infty$, then the AB formula from Fact 2.12 implies that

$$\text{G-dim}_{R_j}(R) = \operatorname{depth}(R_j) - \operatorname{depth}_{R_j}(R) = \operatorname{depth}(R_j) - \operatorname{depth}(R) = 0$$

as desired.

To show that $\text{G-dim}_{R_1}(R) < \infty$, it suffices to show that $\operatorname{Ext}^i_{R_1}(R, R_1) = 0$ for all $i \geqslant 1$ and that $\operatorname{Hom}_{R_1}(R, R_1) \cong C$; see Fact 2.13. To this end, we note that there are isomorphisms of R-modules

$$\operatorname{Hom}_R(R_1, C) \cong \operatorname{Hom}_R(R \oplus C, C) \cong \operatorname{Hom}_R(C, C) \oplus \operatorname{Hom}_R(R, C) \cong R \oplus C \cong R_1$$

and it is straightforward to check that the composition $\operatorname{Hom}_R(R_1, C) \cong R_1$ is an R_1-module isomorphism. Furthermore, for $i \geqslant 1$ we have

$$\operatorname{Ext}^i_R(R_1, C) \cong \operatorname{Ext}^i_R(R \oplus C, C) \cong \operatorname{Ext}^i_R(C, C) \oplus \operatorname{Ext}^i_R(R, C) = 0.$$

Let I be an injective resolution of C as an R-module. The previous two displays imply that $\operatorname{Hom}_R(R_1, I)$ is an injective resolution of R_1 as an R_1-module. Consider the following commutative diagram of local ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{\epsilon_C} & R_1 \\ & {\scriptstyle \mathrm{id}_R}\searrow & \downarrow{\scriptstyle \tau_C} \\ & & R. \end{array}$$

It follows that

$$\operatorname{Hom}_{R_1}(R, \operatorname{Hom}_R(R_1, I)) \cong \operatorname{Hom}_R(R \otimes_{R_1} R_1, I) \cong \operatorname{Hom}_R(R, I) \cong I$$

and hence

$$\operatorname{Ext}^i_{R_1}(R, R_1) \cong \mathrm{H}^i(\operatorname{Hom}_{R_1}(R, \operatorname{Hom}_R(R_1, I))) \cong \mathrm{H}^i(I) \cong \begin{cases} 0 & \text{if } i \geqslant 1 \\ C & \text{if } i = 0 \end{cases}$$

as desired.[1]

The proof for R_2 is similar, using the fact that $C^\dagger$ is not free or dualizing. □

The next three results are for the proof of Lemma 3.12.

Lemma 3.9. *With the notation of Steps 3.4 and 3.5, one has* $\operatorname{Tor}^R_i(R_1, R_2) = 0$ *for all* $i \geqslant 1$*, and there is an* R_1*-algebra isomorphism* $R_1 \otimes_R R_2 \cong Q$.

Proof. The Tor-vanishing comes from the following sequence of R-module isomorphisms

$$\begin{aligned} \operatorname{Tor}^R_i(R_1, R_2) &\cong \operatorname{Tor}^R_i(R \oplus C, R \oplus C^\dagger) \\ &\cong \operatorname{Tor}^R_i(R, R) \oplus \operatorname{Tor}^R_i(C, R) \oplus \operatorname{Tor}^R_i(R, C^\dagger) \oplus \operatorname{Tor}^R_i(C, C^\dagger) \\ &\cong \begin{cases} R \oplus C \oplus C^\dagger \oplus D & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases} \end{aligned}$$

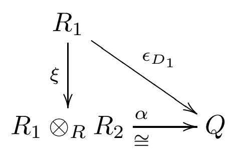
The first isomorphism is by definition; the second isomorphism is elementary; and the third isomorphism is from Fact 2.6.

[1] Note that the finiteness of $\text{G-dim}_{R_1}(R)$ can also be deduced from [13, (2.16)].

Moreover, it is straightforward to verify that (in the case $i = 0$) the isomorphism $R_1 \otimes_R R_2 \cong Q$ has the form $\alpha\colon R_1 \otimes_R R_2 \xrightarrow{\cong} Q$ and is given by

$$(r, c) \otimes (r', \phi') \mapsto (rr', r'c, r\phi', \phi'(c)).$$

It is routine to check that this is a ring homomorphism (that is, a ring isomorphism) and that the following diagram of ring homomorphisms commutes



where ξ is the natural map $x \mapsto x \otimes 1$. (To be precise, the map ξ is given by $(r, c) \mapsto (r, c) \otimes (1, 0)$, and ϵ_{D_1} is given by $(r, c) \mapsto (r, c, 0, 0)$.) It follows that $R_1 \otimes_R R_2 \cong Q$ as an R_1-algebra. □

Lemma 3.10. *Continue with the notation of Steps 3.4 and 3.5. In the tensor product* $R \otimes_{R_1} Q$ *we have* $1 \otimes (0, c, 0, d) = 0$ *for all* $c \in C$ *and all* $d \in D$.

Proof. Recall that Fact 2.6 implies that the evaluation map $C \otimes_R C^\dagger \to D$ given by $c' \otimes \phi \mapsto \phi(c')$ is an isomorphism. Hence, there exist $c' \in C$ and $\phi \in C^\dagger$ such that $d = \phi(c')$. This explains the first equality in the sequence

$$\begin{aligned} 1 \otimes (0,0,0,d) &= 1 \otimes (0,0,0,\phi(c')) = 1 \otimes [(0,c')(0,0,\phi,0)] \\ &= [1(0,c')] \otimes (0,0,\phi,0) = 0 \otimes (0,0,\phi,0) = 0. \end{aligned} \tag{3.10.1}$$

The second equality is by definition of the R_1-module structure on Q; the third equality is from the fact that we are tensoring over R_1; the fourth equality is from the fact that the R_1-module structure on R comes from the natural surjection $R_1 \to R$, with the fact that $(0, c) \in 0 \oplus C$ which is the kernel of this surjection.

On the other hand, using similar reasoning, we have

$$\begin{aligned} 1 \otimes (0,c,0,0) &= 1 \otimes [(0,c)(1,0,0,0)] = [1(0,c)] \otimes (1,0,0,0) \\ &= 0 \otimes (1,0,0,0) = 0. \end{aligned} \tag{3.10.2}$$

Combining (3.10.1) and (3.10.2) we have

$$1 \otimes (0,c,0,d) = [1 \otimes (0,0,0,d)] + [1 \otimes (0,c,0,0)] = 0$$

as claimed. □

Lemma 3.11. *With the notation of Steps 3.4 and 3.5, one has* $\operatorname{Tor}_i^{R_1}(R, Q) = 0$ *for all* $i \geqslant 1$*, and there is a* Q*-module isomorphism* $R \otimes_{R_1} Q \cong R_2$.

Proof. Let P be an R-projective resolution of R_2:

$$P = \cdots \to P_1 \to P_0 \to 0.$$

Lemma 3.9 implies that $R_1 \otimes_R P$ is a projective resolution of $R_1 \otimes_R R_2 \cong Q$ as an R_1-module. From the following sequence of isomorphisms

$$R \otimes_{R_1} (R_1 \otimes_R P) \cong (R \otimes_{R_1} R_1) \otimes_R P \cong R \otimes_R P \cong P$$

it follows that, for $i \geqslant 1$, we have

$$\operatorname{Tor}_i^{R_1}(R, Q) \cong \mathrm{H}_i(R \otimes_{R_1} (R_1 \otimes_R P)) \cong \mathrm{H}_i(P) = 0$$

where the final vanishing comes from the assumption that P is a resolution of a module and $i \geqslant 1$.

This reasoning also shows that there is an R-module isomorphism $\beta\colon R_2 \xrightarrow{\cong} R \otimes_{R_1} Q$. This isomorphism is equal to the composition

$$R_2 \xrightarrow{\cong} R \otimes_R R_2 \xrightarrow{\cong} R \otimes_{R_1} (R_1 \otimes_R R_2) \xrightarrow[R\otimes_{R_1}\alpha]{\cong} R \otimes_{R_1} Q$$

and is therefore given by

$$(r, \phi) \mapsto 1 \otimes (r, \phi) \mapsto 1 \otimes [(1, 0) \otimes (r, \phi)] \mapsto 1 \otimes (r, 0, \phi, 0).$$

We claim that β is a Q-module isomorphism. Recall that the Q-module structure on R_2 is given via the natural surjection $Q \to R_2$, and so is described as

$$(r, c, \phi, d)(r', \phi') = (r, \phi)(r', \phi') = (rr', r\phi' + r'\phi).$$

This explains the first equality in the following sequence

$$\begin{aligned}
\beta((r, c, \phi, d)(r', \phi')) &= \beta(rr', r\phi' + r'\phi) \\
&= 1 \otimes (rr', r'c, r\phi' + r'\phi, r'd + \phi'(c)) \\
&= [1 \otimes (rr', 0, r\phi' + r'\phi, 0)] + [1 \otimes (0, r'c, 0, r'd + \phi'(c))] \\
&= [1 \otimes (rr', 0, r\phi' + r'\phi, 0)].
\end{aligned}$$

The second equality is by definition; the third equality is by bilinearity; and the fourth equality is by Lemma 3.10. On the other hand, the definition of β explains the first equality in the sequence

$$\begin{aligned}
(r, c, \phi, d)\beta(r', \phi') &= (r, c, \phi, d)[1 \otimes (r', 0, \phi', 0)] \\
&= 1 \otimes [(r, c, \phi, d)(r', 0, \phi', 0)] \\
&= 1 \otimes (rr', r'c, r\phi' + r'\phi, r'd + \phi'(c)) \\
&= [1 \otimes (rr', 0, r\phi' + r'\phi, 0)] + [1 \otimes (0, r'c, 0, r'd + \phi'(c))] \\
&= 1 \otimes (rr', 0, r\phi' + r'\phi, 0).
\end{aligned}$$

The second equality is from the definition of the Q-modules structure on $R \otimes_{R_1} Q$; the third equality is from the definition of the multiplication in Q; the fourth equality is by bilinearity; and the fifth equality is by Lemma 3.10. Combining these two sequences, we conclude that β is a Q-module isomorphism, as claimed. □

We verify condition (5) from Theorem 1.1 in the next lemma.

Lemma 3.12. *With the notation of Steps 3.4 and 3.5, one has* $\operatorname{Tor}_i^Q(Q/I_1, Q/I_2) = 0$ *for all* $i \geqslant 1$*; in particular, there is an equality* $I_1 \cap I_2 = I_1 I_2$.

Proof. Let L be a projective resolution of R over R_1. Lemma 3.11 implies that the complex $L \otimes_{R_1} Q$ is a projective resolution of $R \otimes_{R_1} Q \cong R_2$ over Q. We have isomorphisms

$$(L \otimes_{R_1} Q) \otimes_Q R_1 \cong L \otimes_{R_1} (Q \otimes_Q R_1) \cong L \otimes_{R_1} R_1 \cong L$$

and it follows that, for $i \geqslant 1$ we have

$$\operatorname{Tor}_i^Q(R_2, R_1) \cong \operatorname{H}_i((L \otimes_{R_1} Q) \otimes_Q R_1) \cong \operatorname{H}_i(L) = 0$$

since L is a projective resolution.

The equality $I_1 \cap I_2 = I_1 I_2$ follows from the direct computation

$$I_1 \cap I_2 = (0 \oplus 0 \oplus C^\dagger \oplus D) \cap (0 \oplus C \oplus 0 \oplus D) = 0 \oplus 0 \oplus 0 \oplus D = I_1 I_2$$

or from the sequence $(I_1 \cap I_2)/(I_1 I_2) \cong \operatorname{Tor}_1^Q(Q/I_1, Q/I_2) = 0$. ☐

We verify condition (4) from Theorem 1.1 in the next lemma.

Lemma 3.13. *With the notation of Steps 3.4 and 3.5, there is an R-module isomorphism $D_1 \otimes_Q D_2 \cong D$, and for all $i \geqslant 1$ we have $\operatorname{Tor}_i^Q(D_1, D_2) = 0$.*

Proof. There is a short exact sequence of Q-module homomorphisms

$$0 \to D_1 \to Q \xrightarrow{\tau_{D_1}} R_1 \to 0.$$

For all $i \geqslant 1$, we have $\operatorname{Tor}_i^Q(Q, R_2) = 0 = \operatorname{Tor}_i^Q(R_1, R_2)$, so the long exact sequence in $\operatorname{Tor}_i^Q(-, R_2)$ associated to the displayed sequence implies that $\operatorname{Tor}_i^Q(D_1, R_2) = 0$ for all $i \geqslant 1$. Consider the next short exact sequence of Q-module homomorphisms

$$0 \to D_2 \to Q \xrightarrow{\tau_{D_2}} R_2 \to 0.$$

The associated long exact sequence in $\operatorname{Tor}_i^Q(D_1, -)$ implies that $\operatorname{Tor}_i^Q(D_1, D_2) = 0$ for all $i \geqslant 1$.

It is straightforward to verify the following sequence of Q-module isomorphisms

$$R \otimes_{R_1} D_1 \cong \left(\frac{R \ltimes C}{0 \oplus C}\right) \otimes_{R \ltimes C} (C^\dagger \oplus D) \cong \frac{C^\dagger \oplus D}{(0 \oplus C)(C^\dagger \oplus D)} \cong \frac{C^\dagger \oplus D}{0 \oplus D} \cong C^\dagger$$

and similarly

$$R \otimes_{R_2} D_2 \cong C.$$

These combine to explain the third isomorphism in the following sequence:

$$D_1 \otimes_Q D_2 \cong R \otimes_Q (D_1 \otimes_Q D_2) \cong (R \otimes_Q D_1) \otimes_R (R \otimes_Q D_2) \cong C^\dagger \otimes_R C \cong D.$$

For the first isomorphism, use the fact that D_j is annihilated by $D_j = I_j$ for $j = 1, 2$ to conclude that $D_1 \otimes_Q D_2$ is annihilated by $I_1 + I_2$; it follows that $D_1 \otimes_Q D_2$ is naturally a module over the quotient $Q/(I_1 + I_2) \cong R$. The second isomorphism is standard, and the fourth one is from Fact 2.6. ☐

This completes the proof of Theorem 1.1.

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