ARE THERE AN INFINITE NUMBER OF COLLATZ INTEGERS?

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ABSTRACT. A positive integer k is a *Collatz Number* if for all $n \geqslant 1$, the function

$$f(n,k) = \begin{cases} n/k, & \text{if } n \equiv 0 \pmod{k}; \\ (k+1)n + (k-x), & \text{if } n \equiv x \pmod{k}. \end{cases}$$

converges to 1. We investigate this function along with special cases when $k = 2$. We conjecture that there are infinitely many Collatz integers.

1. INTRODUCTION

In this paper, we generalize the famous $3x + 1$ problem, which is also known as *Collatz problem*, *the Syracuse problem*, *Kakutani's problem*, *Hasse's algorithm*, and *Ulam's problem*. The problem concerns the behavior of the iterates of the function

$$f(n) = \begin{cases} n/2, & \text{if } n \equiv 0 \pmod{2}; \\ 3n+1, & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

The conjecture asserts that, starting from any positive integer n, repeated iteration of $f(n)$ eventually produces the value 1. This problem is very simple to state and extremely hard to solve. For an introduction to the problem, see Lagarias [2]. For annotated bibliographies, see Lagarias [3, 4]. We generalize the problem and call a positive integer k a *Collatz Number*, if for all $n \geqslant 1$, the function

$$f(n,k) = \begin{cases} n/k, & \text{if } n \equiv 0 \pmod{k}; \\ (k+1)n + (k-x), & \text{if } n \equiv x \pmod{k}. \end{cases}$$

converges to 1. Given positive integer $k \geqslant 2$, the sequence of iterates

$$n, f(n,k), f^{(2)}(n,k), f^{(3)}(n,k), \cdots$$

is called the *trajectory* of n, namely, $T(n,k)$. The possible behaviors (as in Lagarias [2]) for such trajectories when $n \geqslant 1$ are:

(*i*) *Convergent trajectory*: $f^{(t)}(n,k) = 1$.

(*ii*) *Non-trivial cyclic trajectory*: $f^{(t)}(n,k)$ eventually becomes periodic, and $f^{(t)}(n,k) \neq 1$ for any $t \geqslant 1$.

(*iii*) *Divergent trajectory*: $\lim_{t\to\infty} f^{(t)}(n,k) = \infty$.

For $n > 1$, clearly $f^{(t)}(n,k) = 1$ cannot occur without some $f^{(t)}(n,k) < n$ occurring. Define *stopping time* $\sigma(n,k)$ as follows:

$$\sigma(n,k) = \begin{cases} \min\left\{t : f^{(t)}(n,k) < n\right\}, & \text{if such a } t \text{ occurs;} \\ \infty, & \text{otherwise.} \end{cases}$$

Similarly, difine the *total stopping time* $\sigma_\infty(n,k)$ to be the least positive t for which $f^{(t)}(n,k) = 1$, and ∞ if no such t occurs. Let $\ell(n,k)$ denote the length of the shortest trajectory when $f(n,k)$ is convergent.

We call positive integer k a *Collatz Candidate* if $f(n,k)$ has a convergent trajectory (tested computationally) for all $n \leqslant M$ for some M large enough to conjecture that k is a Collatz number. A Collatz candidate corresponding to a given M immediately loses its candidacy if for some $n > M$, $f(n,k)$ has either a divergent trajectory or a non-trivial cyclic trajectory. Define the set of Collatz candidates

$$\mathcal{C}(K,M) = \{2 \leqslant k \leqslant K : f(n,k) \text{ has a convergent trajectory for all } n \leqslant M\}\,.$$

Let the set of non-Collatz numbers be $\mathcal{NC}$ and let $\mathcal{NC}(K,M)$ be defined as

$$\{2,3,\ldots,K\} - \mathcal{C}(K,M).$$

Clearly, $\mathcal{NC}(K,M) \subseteq \mathcal{NC}$.

For example, from experimental results, we have:

$$\begin{aligned}\mathcal{NC}(100,10^6) &= \{3,4,6,9,10,11,12,15,16,17,20,23,24,\\ &\quad 25,27,29,31,48,54,57,68,72,78,85,94\}.\end{aligned}$$

For further investigation, we consider

$$\mathcal{C}(30,10^9) = \{2,5,7,8,13,14,18,19,21,22,26,28,30\}$$

and

$$\mathcal{NC}(30,10^9) = \{3,4,6,9,10,11,12,15,16,17,20,23,24,25,27,29\}\,.$$

2. On the Length of Convergent Trajectories

In this section, we get an estimate on the length of trajectories by observing that $3x+1$ can be written as the sum of $2x+1$ and x where both of these numbers are odd and that the expected number of ones (in binary) of two odd summands is less than the expected number of ones in the summands themselves.

Let $k=2$ and $n>1$, and consider n in binary. If n is even, then both n and $n/2$ have the same number of ones in the respective binary expansions. If n is odd, then $2n+1$ is also odd. Let $b_1(n)$ denote the number of ones in the binary expansion of n. Clearly, $b_1(2n) = b_1(n)$ and $b_1(2n+1) = b_1(n)+1$.

Observation 1. *Given positive integer $r \geqslant 3$, the expected number (say $e_1(r)$) of ones in the r-bit binary expansion of an odd number in $\{1,3,\ldots,2^r-1\}$ is $(r+1)/2$.*

Proof. For a given positive integer r,

$$\begin{aligned}e_1(r) &= \left(\sum_{i=0}^{2^{r-1}-1} b_1(2i+1)\right)/2^{r-1}\\ &= \left(\sum_{i=0}^{r-1}\binom{r-1}{i}(i+1)\right)/2^{r-1}\\ &= \left((r-1)2^{r-2}+2^{r-1}\right)/2^{r-1}\\ &= (r+1)/2.\end{aligned}$$

□

Observation 2. *Given positive integer $r \geqslant 3$, the expected number (say $e_2(r)$) of ones in the r-bit binary expansion of a sum of two odd summands in $\{1, 3, \ldots, 2^r - 1\}$ approaches $r/2 + 1/2^{r-1}$.*

It can be observed from computation that the expression

$$e_2(r) = \left(\sum_{i=0}^{2^{r-1}-1} \left(\sum_{j=i}^{2^{r-1}-1} b_1(i+j+1) \right) \right) / \left(\left(2^{r-1}+1\right) 2^{r-2} \right).$$

approaches $r/2 + 1/2^{r-1}$ as in the following table:

r	$e_2(r)$	$r/2 + 1/2^{r-1}$
3	1.700000000000000	1.750000000000000
4	2.111111111111111	2.125000000000000
5	2.558823529411764	2.562500000000000
6	3.030303030303030	3.031250000000000
7	3.515384615384615	3.515625000000000
8	4.007751937984496	4.007812500000000
9	4.503891050583658	4.503906250000000
10	5.001949317738791	5.001953125000000
11	5.500975609756098	5.500976562500000
12	6.000488042947779	6.000488281250000
13	6.500244081034904	6.500244140625000
14	7.000122055413158	7.000122070312500
15	7.500061031431187	7.500061035156250
16	8.000030516646831	8.000030517578125
17	8.500015258556235	8.500015258789062
18	9.000007629336324	9.000007629394531

Table 1: Convergence of $e_2(r)$

2.1. **A rough estimate for the length of trajectories with $k = 2$.** Consider an odd x on the trajectory of n. From Observations 1 and 2, we can expect the sum $(2x+1)+x$ contains $y_r = r/(r+1)$ percent as many ones as we have started with, and hence $f(n, 2)$ has a natural tendency to reduce the number of ones.

- ◇ Take the least r such that $2n + 1 < 2^r$.
- ◇ Take the smallest s such that $y_r^s \times b_1(n) < 1$.
- ◇ Then $r + s$ is the expected length of $f(n, 2)$.

n	$b_1(n)$	r	y_r	s	$s+r$	$\sigma_\infty(n,2)+1$
3	2	3	0.75000	3	6	8
4	1	4	0.80000	1	5	3
5	2	4	0.80000	4	8	6
6	2	4	0.80000	4	8	9
7	3	4	0.80000	5	9	17
8	1	5	0.83333	1	6	4
9	2	5	0.83333	4	9	20
10	2	5	0.83333	4	9	7
11	3	5	0.83333	7	12	15

12	2	5	0.83333	4	9	10
13	3	5	0.83333	7	12	10
14	3	5	0.83333	7	12	18
15	4	5	0.83333	8	13	18
16	1	6	0.85714	1	7	5
17	2	6	0.85714	5	11	13
18	2	6	0.85714	5	11	21
19	3	6	0.85714	8	14	21
20	2	6	0.85714	5	11	8
21	3	6	0.85714	8	14	8
22	3	6	0.85714	8	14	16
23	4	6	0.85714	9	15	16
24	2	6	0.85714	5	11	11
25	3	6	0.85714	8	14	24
26	3	6	0.85714	8	14	11
27	4	6	0.85714	9	15	112
28	3	6	0.85714	8	14	19
29	4	6	0.85714	9	15	19
30	4	6	0.85714	9	15	19
31	5	6	0.85714	11	17	107
32	1	7	0.87500	1	8	6

Table 2: Rough estimate vs actual lengths

3. Repeating cycles

The repeating 1-cycle for a given Collatz number k, with $k \geqslant 2$, is

$$2k, 2, 3k, 3, 4k, 4, \ldots, k^2, k, 1,$$

since

- $\diamond$ $f(1,k) = 1(k+1) + (k-1) = 2k$,
- $\diamond$ $f(ik,k) = i$ for $i = 2, 3, \ldots, k$,
- $\diamond$ $f(i,k) = i(k+1) + (k-i) = (i+1)k$ for $2 \leqslant i \leqslant k-1$,
- $\diamond$ $f(k,k) = 1$.

Note that every Collatz number k must enter a corresponding 1-cycle on the trajectory $T(n,k)$ for all $n \geqslant 1$. Suppose a number $k \in \mathcal{NC}$ has a cyclic trajectory $T(n,k)$ for some n. Then $T(n,k)$ does not contain a number x in $\left\{2k, 2, 3k, 3, 4k, 4, \ldots, k^2, k, 1\right\}$.

Lemma 3.1. *For $k \geqslant 2$ and $2 \leqslant n \leqslant k$,*

$$\sigma_\infty(n,k) = 2(k-n) + 1.$$

Proof. Let $n \equiv r \pmod{k}$. Then

$$\begin{aligned}
f^{(1)}(n,k) &= n(k+1)+(k-r) = (n+1)k+(n-r),\\
f^{(2)}(n,k) &= n+1,\\
f^{(3)}(n,k) &= (n+1)(k+1)+(k-(r+1)) = (n+2)k+(n-r),\\
f^{(4)}(n,k) &= n+2,\\
&\vdots\\
f^{(t-1)}(n,k) &= (k-1)(k+1)+(k-(k-1)) = k^2,\\
f^{(t)}(n,k) &= k.
\end{aligned}$$

It can be observed that $t = 2(k-n)$. Hence, $\sigma_\infty(n,k) = 2(k-n)+1$. □

3.1. **Repeating cycles for non-Collatz numbers.** Given a positive integer $k \geqslant 2$, if the function $f(n,k)$ enters a cyclic trajectory for some $n > k$, then k is non-Collatz. In this section, we present examples of such repeating cycles to justify that certain numbers are non-Collatz.

For each $k \in \{3,4,6,9,10,11,12,15,16,17,20,23,24,25,27,29\}$, we list the cycles on $T(n,k)$ for $n \leqslant 10^7$. For each cycle corresponding to a given k, we provide the least initial value n, for which $T(n,k)$ contains the repeating cycle.

3.1.1. $k = 3$:

$(n=5)$ 7 30 10 42 14 57 19 78 26 105 35 141 47 189 63 21. (Cycle-length=16)

3.1.2. $k = 4$:

$(n=11)$ 23 116 29 148 37 188 47 236 59 296 74 372 93 468 117 588 147 736 184 46 232 58 292 73 368 92. (Cycle-length=26)

3.1.3. $k = 6$:

$(n=7)$ 23 162 27 192 32 228 38 270 45 318 53 372 62 438 73 516 86 606 101 708 118 828 138. (Cycle-length=23)

$(n=49)$ 88 618 103 726 121 852 142 996 166 1164 194 1362 227 1590 265 1860 310 2172 362 2538 423 2964 494 3462 577 4044 674 4722 787 5514 919 6438 1073 7512 1252 8766 1461 10230 1705 11940 1990 13932 2322 387 2712 452 3168 528. (Cycle-length=48)

3.1.4. $k = 9$:

$(n=31)$ 35 351 39 396 44 441 49 495 55 558 62 621 69 693 77 774 86 864 96 963 107 1071 119 1197 133 1332 148 1485 165 1656 184 1845 205 2052 228 2286 254 2547 283 2835 315. (Cycle-length=41)

3.1.5. $k = 10$:

$(n=34)$ 42 470 47 520 52 580 58 640 64 710 71 790 79 870 87 960 96 1060 106 1170 117 1290 129 1420 142 1570 157 1730 173 1910 191 2110 211 2330 233 2570 257 2830 283 3120 312 3440 344 3790 379 4170 417 4590 459 5050 505 5560 556 6120 612 6740 674 7420 742 8170 817 8990 899 9890 989 10880 1088 11970 1197 13170 1317 14490 1449 15940 1594 17540 1754 19300 1930 193 2130 213 2350 235 2590 259 2850 285 3140 314 3460 346 3810 381 4200 420. (Cycle-length=96)

3.1.6. $k = 11$:

$(n = 588)$ 642 7711 701 8415 765 9185 835 10021 911 10934 994 11935 1085 13024 1184 14212 1292 15510 1410 16929 1539 18469 1679 20152 1832 21989 1999 23991 2181 26180 2380 28567 2597 31174 2834 34012 3092 37114 3374 40491 3681 44176 4016 48202 4382 52591 4781 57376 5216 62601 5691 68299 6209 74514 6774 81290 7390 88682 8062 96745 8795 105545 9595 115148 10468 125620 11420 137049 12459 149512 13592 163108 14828 1348 16181 1471 17655 1605 19261 1751 21021 1911 22935 2085 25025 2275 27302 2482 29788 2708 32505 2955 35464 3224 38698 3518 42218 3838 46057 4187 50248 4568 54824 4984 59818 5438 65263 5933 71203 6473 77682 7062. (Cycle-length=112)

For data regarding other non-Collatz $k \leqslant 30$, see Appendix A. Based on this data, we conjecture the following:

Conjecture 1. *Every $k \in \mathcal{NC}$ has a finite number of repeating cycles and no divergent (infinite) trajectories.*

The following conjecture is also made by Bruschi [1] with a different generalization of the Collatz porblem.

Conjecture 2. *The set of Collatz numbers is infinite.*

4. On trajectory-lengths of Collatz candidates

Consider the sequence with numbers, which as initial values, set new records on the number of steps to reach 1, on the trajectory corresponding to a given k. In this section, we use the pair (n, s) on the sequence to mean that n takes a record s steps on the trajectory $T(n, k)$ for a given k. The following pairs for the $k = 2$ case are known (A006877 [5]):

(2, 1), (3, 7), (6, 8), (7, 16), (9, 19), (18, 20), (25, 23), (27, 111), (54, 112), (73, 115), (97, 118), (129, 121), (171, 124), (231, 127), (313, 130), (327, 143), (649, 144), (703, 170), (871, 178), (1161, 181), (2223, 182), (2463, 208), (2919, 216), (3711, 237), (6171, 261), (10971, 267), (13255, 275), (17647, 278), (23529, 281), (26623, 307), (34239, 310), (35655, 323), (52527, 339), (77031, 350), (106239, 353), (142587, 374), (156159, 382), (216367, 385), (230631, 442), (410011, 448), (511935, 469), (626331, 508), (837799, 524).

We extend the $k = 2$ case with $n \leqslant 10^9$, and provide new record-setting trajectory-lengths for other Collatz candidates $k \leqslant 30$, with $n \leqslant 10^8$.

$(k = 2)$ (1117065, 527), (1501353, 530), (1723519, 556), (2298025, 559), (3064033, 562), (3542887, 583), (3732423, 596), (5649499, 612), (6649279, 664), (8400511, 685), (11200681, 688), (14934241, 691), (15733191, 704), (31466382, 705), (36791535, 744), (63728127, 949), (127456254, 950), (169941673, 953), (226588897, 956), (268549803, 964), (537099606, 965), (670617279, 986).

$(k = 5)$ (2, 7), (6, 12), (7, 31), (24, 36), (31, 129), (43, 144), (84, 155), (169, 166), (234, 181), (307, 234), (499, 266), (834, 279), (2011, 288), (4699, 316), (6528, 348), (11023, 363), (19567, 376), (21924, 467), (52561, 495), (59526, 569), (166951, 650), (194136, 667), (653209, 691), (911496, 706), (1831549, 717), (1915452, 810), (4618663, 819), (5265811, 836), (8817546, 849), (14106372, 957), (19684146, 972), (68347734, 977).

$(k = 7)$ (2, 11), (8, 18), (9, 20), (25, 35), (33, 89), (57, 112), (129, 189), (528, 198), (697, 225), (1464, 244), (3336, 292), (5961, 342), (7177, 437), (15105, 456), (23000, 510), (31736, 566), (113944, 577), (268888, 588), (274073, 594), (622632, 642), (831081, 665), (1561729, 713), (2201912, 738), (3276680, 798), (9212121, 807), (15667000, 865), (30551464, 909).

$(k = 8)$ (2, 13), (9, 100), (54, 107), (55, 216), (118, 239), (342, 258), (667, 283), (2340, 298), (3042, 438), (9451, 455), (20773, 478), (29764, 509), (69958, 530), (116371, 594), (153729, 662), (334117, 685), (862875, 706), (1001439, 739), (1083573, 774), (1392975, 807), (3666465, 826), (4181355, 828), (4974444, 857), (6340429, 890), (11289538, 1170), (24722271, 1193), (96842485, 1206).

($k = 13$) (2, 23), (14, 36), (15, 42), (29, 113), (57, 166), (98, 501), (295, 542), (476, 599), (2198, 628), (4592, 679), (20091, 710), (33446, 765), (36779, 836), (112813, 1155), (940142, 1168), (1605633, 1294), (3971969, 1410), (4475632, 1477), (11395538, 1522), (22251838, 1573), (32693710, 1634).

($k = 14$) (2, 25), (15, 52), (31, 111), (106, 154), (181, 156), (480, 341), (631, 349), (1575, 384), (2220, 390), (2610, 465), (3241, 534), (5340, 597), (28230, 624), (62085, 665), (66436, 679), (90481, 825), (231181, 862), (299115, 868), (525631, 929), (564615, 1055), (1920481, 1124), (3126120, 1173), (3134955, 1185), (5502016, 1481), (12811681, 1534), (40000831, 1656).

($k = 18$) (2, 33), (19, 74), (57, 167), (267, 196), (286, 198), (343, 210), (856, 265), (1920, 340), (2926, 350), (3535, 437), (5415, 542), (23370, 583), (23389, 595), (38912, 666), (56450, 868), (160417, 931), (240103, 1024), (1325744, 1075), (2019339, 1483), (7178752, 1544), (37078348, 1593), (51888278, 1605), (65494577, 1704).

($k = 19$) (2, 35), (20, 54), (21, 64), (40, 72), (41, 181), (401, 299), (800, 307), (2801, 374), (6521, 538), (12241, 548), (13001, 659), (55241, 686), (55840, 718), (92141, 811), (250360, 888), (261441, 892), (400441, 997), (1029240, 1076), (2027261, 1157), (2170181, 1274), (3768541, 1589), (7653021, 1654), (10097501, 1759), (27318980, 1836), (59236640, 1838).

($k = 21$) (2, 39), (22, 60), (23, 80), (111, 179), (331, 212), (397, 256), (616, 319), (704, 363), (1673, 460), (4643, 529), (4753, 547), (9284, 648), (20901, 745), (23607, 872), (97284, 937), (107295, 1068), (285780, 1125), (338339, 1155), (774995, 1214), (809688, 1460), (2081047, 1551), (5085190, 1646), (19946344, 1983), (55928708, 2130).

($k = 22$) (2, 41), (23, 76), (46, 86), (92, 157), (116, 580), (645, 645), (1312, 754), (6693, 805), (8165, 952), (16078, 1061), (21989, 1186), (164979, 1237), (329130, 1346), (670473, 1453), (1176405, 1707), (4515521, 1788), (12482952, 1883), (18581953, 2006), (43912843, 2093), (79128902, 2220).

($k = 26$) (2, 49), (27, 102), (109, 247), (243, 366), (1540, 443), (2943, 582), (6103, 717), (29781, 774), (34614, 945), (42498, 1138), (78705, 1279), (525960, 1352), (1046655, 1660), (3289141, 1958), (6325831, 2097), (57168882, 2154), (57883842, 2327).

($k = 28$) (2, 53), (29, 98), (58, 114), (116, 213), (204, 456), (494, 513), (580, 694), (5540, 841), (10353, 994), (35323, 1115), (77315, 1256), (564746, 1257), (649224, 1331), (773054, 1621), (1075553, 1687), (3143600, 1816), (7036908, 1877), (7547222, 1957), (20853639, 2006), (20948295, 2010), (24559172, 2050), (33402810, 2066), (42001889, 2241), (81081043, 2312).

($k = 30$) (2, 57), (31, 154), (124, 265), (249, 432), (279, 438), (1303, 539), (3007, 724), (6201, 863), (13486, 1024), (30753, 1617), (248589, 1698), (597216, 1853), (935736, 2044), (4019801, 2372), (16884088, 2483), (36744145, 2644).

Lemma 4.1. *If k is a Collatz candidate, then $k+1$ has a record-setting trajectory-length.*

Proof. By Lemma 3.1, the sequence $\sigma_\infty(n,k)$ with $n = 2, 3, \dots, k$ is decreasing. It remains to show that $\sigma_\infty(k+1,k) > \sigma_\infty(2,k) = 2k-3$.

With $n = k+1$ as initial value, for $i = 1, 2, \dots, \lfloor k/2 \rfloor$, we have:

$$\begin{aligned} f^{(2i-1)}(n,k) &= (k+(2i-1))\,(k+1) + (k-(2i-1)) \\ &= k^2 + (2i+1)k, \\ f^{(2i)}(n,k) &= k + (2i+1)\,. \end{aligned}$$

Now, we have the following two cases:

(a) k odd: The number of steps by far is $2\lfloor k/2 \rfloor = 2\cdot(k-1)/2 = k-1$ and the $(k-1)$-th number is $k + (2\cdot(k-1)/2+1) = 2k$. Taking $2k$ as initial value, we have the following $2(k-1)$ numbers towards 1:

$$2, 3k, 3, 4k, \dots, k^2, k, 1.$$

For $k \geqslant 3$, we have

$$\sigma_\infty(k+1,k) = (k-1) + 2(k-1) = 3k - 3 > 2k - 3.$$

(b) k even: The number of steps by far is $2\lfloor k/2 \rfloor = 2 \cdot (k)/2 = k$ and the k-th number is $k + (2 \cdot (k/2) + 1) = 2k + 1$.

Since k is a Collatz candidate, the sequence must reach jk with $3 \leqslant j \leqslant k$ on its way to 1.

(i) If $k \equiv 1 \pmod 3$, then for $i = 1, 2, \ldots, (k-1)/3$, we have:

$$\begin{aligned} f^{(2i-1)}(n,k) &= (2k + (3i-2))(k+1) + (k - (3i-2)) \\ &= 2k^2 + (3i+1)k, \\ f^{(2i)}(n,k) &= 2k + (3i+1), \end{aligned}$$

the last term of which is $2k + (3 \cdot (k-1)/3 + 1) = 3k$. Then we have the following $2(k-2)$ numbers towards 1:

$$3, 4k, \ldots, k^2, k, 1.$$

Hence,

$$\sigma_\infty(k+1,k) = k + 2 \cdot \frac{k-1}{3} + 2(k-2) = \frac{11k - 14}{3} > 2k - 3.$$

(ii) If $k \equiv 0 \pmod 3$, then after $2\lceil k/3 \rceil$ steps, we get $3k+1$; then after $2\lceil k/4 \rceil$ steps, we get $4k+1$; and so on until we get jk with $j \geqslant 5$. Note that $j \geqslant 5$ because $k \not\equiv 1 \pmod 4$ (since k is even) and $3k + (4i+1) = 4k$ does not have a solution with an integer value of i. Now, the number of steps is at least

$$k + 2(k/3 + k/4) = 2k + k/6 > 2k - 3.$$

(iii) Suppose $k \equiv 2 \pmod 3$. Then with $2k+1$ as initial value, we get the following $2(k+1)/3$ steps:

$$2k^2 + 4k, 2k + 4, 2k^2 + 7k, 2k + 7, \ldots, 3k^2 - k, 3k - 1, 3k^2 + 2k, 3k + 2.$$

Since $k \equiv 0 \pmod 2$, we have $(k \mod 4) \in \{0, 2\}$.

If $k \equiv 2 \pmod 4$, then with $3k+2$ as initial value, we get the following $2(k-2)/4$ steps: $3k^2 + 6k, 3k + 6, 3k^2 + 10k, 3k + 10, \ldots, 4k^2 - 4k, 4k - 4, 4k^2, 4k$. Hence,

$$\begin{aligned} \sigma_\infty(k+1,k) &= k + \frac{2(k+1)}{3} + \frac{2(k-2)}{4} + 2(k-3) \\ &= 4k + \frac{k-2}{6} - 6 > 2k - 3. \end{aligned}$$

If $k \equiv 0 \pmod 4$, then with $3k+2$ as initial value, we get the following $2(k/4)$ steps: $3k^2 + 6k, 3k + 6, 3k^2 + 10k, 3k + 10, \ldots, 4k^2 - 2k, 4k - 2, 4k^2 + 2k, 4k + 2$. Hence,

$$\begin{aligned} \sigma_\infty(k+1,k) &> k + \frac{2(k+1)}{3} + \frac{2k}{4} \\ &= 2k + \frac{k+4}{6} > 2k - 3. \end{aligned}$$

□

Lemma 4.2. *If k is an odd Collatz candidate, then $k+2$ has a record-setting trajectory-length.*

Proof. By Lemma 4.1, $\sigma_\infty(k+1,k) = 3k-3$. We need to show that $\sigma_\infty(k+2,k) > 3k-3$. Since k is a Collatz candidate, starting from $k+2$, the sequence must hit jk with $2 \leqslant j \leqslant k$ on the way to 1.

Note that, there are at least $2\lfloor k/(i+1)\rfloor$ numbers on the sequence between the smallest number greater than or equal to $ik+1$ and the smallest number greater than or equal to $(i+1)k$.

The lemma for $k = 5, 7, 13$ can be directly verified. Let $k \geqslant 19$.

Now, we have the following cases (using techniques similar to Lemma 4.1):

(i) If $i+1 \leqslant k$, then $\sigma_\infty(k+2,k)$ is greater than

$$
\begin{aligned}
& 2\left\{\lfloor k/2\rfloor + \lfloor k/3\rfloor + \cdots + \lfloor k/j\rfloor\right\} + 2(k-j+1) \\
\geqslant\; & 2k\left\{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{j}\right\} - (j-1) + 2(k-j+1) \\
=\; & 2k+3+2k\,(H_j - 1.0) - 3j \\
\approx\; & 2k+3+2k\left(\ln j + \frac{1}{2j} + 0.57732 - 1.0\right) - 3j \\
\geqslant\; & 2k+4+2k\,(\ln j - 0.42268) - 3j \\
=\; & 3k-3+2k\,(\ln j - 0.92268) - 3j + 7 \\
>\; & 3k-3.
\end{aligned}
$$

Note that, $2k\,(\ln j - 0.92268) + 7 > 3j$ for $k \geqslant 19$ and $3 \leqslant j \leqslant k$.

(ii) If $i+1 > k$, then $\sigma_\infty(k+2,k)$ is greater than

$$
\begin{aligned}
& 2\left\{\lfloor k/2\rfloor + \lfloor k/3\rfloor + \cdots + \lfloor k/k\rfloor\right\} + 2(k-j+1) \\
\geqslant\; & 2k\left\{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}\right\} - (k-1) + 2k - 2j + 2 \\
=\; & k+3+2k\,(H_k - 1.0) - 2j \\
\approx\; & k+3+2k\left(\ln k + \frac{1}{2j} + 0.57732 - 1.0\right) - 2j \\
\geqslant\; & k+4+2k\,(\ln k - 0.42268) - 2j \\
=\; & 3k-3+2k\,(\ln k - 0.42268) - 2k - 2j + 7 \\
>\; & 3k-3.
\end{aligned}
$$

Note that, $\ln k - 0.42268 > 2.5$ for $k \geqslant 19$ and $j \leqslant k$.

□

References

[1] M. Bruschi, A generalization of the Collatz porblem and conjecture, *Unpublished.* preprint on arXiv [arxi.org/abs/0810.5169].

[2] J. C. Lagarias, The $3x+1$ Problem and its Generalizations, *American Mathematical Monthly*, **92** (1985), 3–23.

[3] J. C. Lagarias, The $3x+1$ Problem: An Annotated Bibliography (1963–1999) (sorted by author), *Unpublished.* preprint on arXiv [arxi.org/abs/math/0309224].

[4] J. C. Lagarias, The $3x+1$ Problem: An Annotated Bibliography, II (2000–2009), *Unpublished.* preprint on arXiv [arxi.org/abs/math/0608208].

[5] Online Encyclopedia of Integer Sequences. http://oeis.org/A006877.

Appendix A. Repeating cycles for non-Collatz numbers 12, 15, 16, 17, 20, 23, 24, 25, 27, and 29

A.1. $k = 12$:

$(n = 767)$ 1348 17532 1461 18996 1583 20580 1715 22296 1858 24156 2013 26172 2181 28356 2363 30720 2560 33288 2774 36072 3006 39084 3257 42348 3529 45888 3824 49716 4143 53868 4489 58368 4864 63240 5270 68520 5710 74232 6186 80424 6702 87132 7261 94404 7867 102276 8523 110808 9234 120048 10004 130056 10838 140904 11742 152652 12721 165384 13782 179172 14931 194112 16176. (Cycle-length=63)

$(n = 793)$ 1010 13140 1095 14244 1187 15432 1286 16728 1394 18132 1511 19644 1637 21288 1774 23064 1922 24996 2083 27084 2257 29352 2446 31800 2650 34452 2871 37332 3111 40452 3371 43824 3652 47484 3957 51444 4287 55740 4645 60396 5033 65436 5453 70896 5908 76812 6401 83220 6935 90156 7513 97680 8140 105828 8819 114648 9554 124212 10351 134568 11214 145788 12149 157944 13162 171108 14259 185376 15448 200832 16736 217572 18131 235704 19642 255348 21279 276636 23053 299700 24975 324684 27057 351744 29312 381060 31755 412824 34402 447228 37269 484500 40375 524880 43740 3645 47388 3949 51348 4279 55632 4636 60276 5023 65304 5442 70752 5896 76656 6388 83052 6921 89976 7498 97476 8123 105600 8800 114408 9534 123948 10329 134280 11190 145476 12123 157608 13134 170748 14229 184980 15415 200400 16700 217104 18092 235200 19600 254808 21234 276048 23004 1917 24924 2077 27012 2251 29268 2439 31716 2643 34368 2864 37236 3103 40344 3362 43716 3643 47364 3947 51312 4276 55596 4633 60240 5020 65268 5439 70716 5893 76620 6385 83016 6918 89940 7495 97440 8120 105564 8797 114372 9531 123912 10326 134244 11187 145440 12120. (Cycle-length=189)

A.2. $k = 15$:

$(n = 49)$ 53 855 57 915 61 990 66 1065 71 1140 76 1230 82 1320 88 1410 94 1515 101 1620 108 1740 116 1860 124 1995 133 2130 142 2280 152 2445 163 2610 174 2790 186 2985 199 3195 213 3420 228 3660 244 3915 261 4185 279 4470 298 4770 318 5100 340 5445 363 5820 388 6210 414 6630 442 7080 472 7560 504 8070 538 8610 574 9195 613 9810 654 10470 698 11175 745 11925 795. (Cycle-length=81)

A.3. $k = 16$:

$(n = 35)$ 178 3040 190 3232 202 3440 215 3664 229 3904 244 4160 260 4432 277 4720 295 5024 314 5344 334 5680 355 6048 378 6432 402 6848 428 7280 455 7744 484 8240 515 8768 548 9328 583 9920 620 10544 659 11216 701 11920 745 12672 792 13472 842 14320 895 15216 951 16176 1011 17200 1075 18288 1143 19440 1215 20656 1291 21952 1372 23328 1458 24800 1550 26352 1647 28000 1750 29760 1860 31632 1977 33616 2101 35728 2233 37968 2373 40352 2522 42880 2680 45568 2848. (Cycle-length=91)

A.4. $k = 17$:

$(n = 19)$ 79 1428 84 1513 89 1615 95 1717 101 1819 107 1938 114 2057 121 2193 129 2329 137 2482 146 2635 155 2805 165 2975 175 3162 186 3349 197 3553 209 3774 222 4012 236 4250 250 4505 265 4777 281 5066 298 5372 316 5695 335 6035 355 6392 376 6783 399 7191 423 7616 448 8075 475 8551 503 9061 533 9605 565 10183 599 10795 635 11441 673 12121 713 12835 755 13600 800 14416 848 15266 898 16167 951 17119 1007 18139 1067 19210 1130 20349 1197 21556 1268 22831 1343. (Cycle-length=97)

A.5. $k = 20$:

$(n = 63)$ 71 1500 75 1580 79 1660 83 1760 88 1860 93 1960 98 2060 103 2180 109 2300 115 2420 121 2560 128 2700 135 2840 142 3000 150 3160 158 3320 166 3500 175 3680 184 3880 194 4080 204 4300 215 4520 226 4760 238 5000 250 5260 263 5540 277 5820 291 6120 306 6440 322 6780 339 7120 356 7480 374 7860 393 8260 413 8680 434 9120 456 9580 479 10060 503 10580 529 11120 556 11680 584 12280 614 12900 645 13560 678 14240 712 14960 748 15720 786 16520 826 17360 868 18240 912 19160 958 20120 1006 21140 1057 22200 1110 23320 1166 24500 1225 25740 1287 27040 1352 28400 1420. (Cycle-length=119) (n = 127) 141 2980 149 3140 157 3300 165 3480 174 3660 183 3860 193 4060 203 4280 214 4500 225 4740 237 4980 249 5240 262 5520 276 5800 290 6100 305 6420 321 6760 338 7100 355 7460 373 7840 392 8240 412 8660 433 9100 455 9560 478 10040 502 10560 528 11100 555 11660 583 12260 613 12880 644 13540 677 14220 711 14940 747 15700 785 16500 825 17340 867 18220 911 19140 957 20100 1005 21120 1056 22180 1109 23300 1165 24480 1224 25720 1286 27020 1351 28380 1419 29800 1490 31300 1565 32880 1644 34540 1727 36280 1814 38100 1905 40020 2001 42040 2102 44160 2208 46380 2319 48700 2435 51140 2557 53700 2685 56400 2820. (Cycle-length=121)

A.6. $k = 23$:

(n = 49) 82 1978 86 2070 90 2162 94 2277 99 2392 104 2507 109 2622 114 2737 119 2875 125 3013 131 3151 137 3289 143 3450 150 3611 157 3772 164 3956 172 4140 180 4324 188 4531 197 4738 206 4945 215 5175 225 5405 235 5658 246 5911 257 6187 269 6463 281 6762 294 7061 307 7383 321 7705 335 8050 350 8418 366 8786 382 9177 399 9591 417 10028 436 10465 455 10925 475 11408 496 11914 518 12443 541 12995 565 13570 590 14168 616 14789 643 15433 671 16123 701 16836 732 17572 764 18354 798 19159 833 20010 870 20884 908 21804 948 22770 990 23782 1034 24817 1079 25898 1126 27025 1175 28221 1227 29463 1281 30751 1337 32108 1396 33511 1457 34983 1521 36524 1588 38134 1658 39813 1731 41561 1807 43378 1886. (Cycle-length=143)

A.7. $k = 24$:

(n = 201) 335 8376 349 8736 364 9120 380 9504 396 9912 413 10344 431 10776 449 11232 468 11712 488 12216 509 12744 531 13296 554 13872 578 14472 603 15096 629 15744 656 16416 684 17112 713 17832 743 18576 774 19368 807 20184 841 21048 877 21936 914 22872 953 23832 993 24840 1035 25896 1079 26976 1124 28104 1171 29280 1220 30504 1271 31776 1324 33120 1380 34512 1438 35952 1498 37464 1561 39048 1627 40680 1695 42384 1766 44160 1840 46008 1917 47928 1997 49944 2081 52032 2168 54216 2259 56496 2354 58872 2453 61344 2556 63912 2663 66576 2774 69360 2890 72264 3011 75288 3137 78432 3268 81720 3405 85128 3547 88680 3695 92376 3849 96240 4010 100272 4178 104472 4353 108840 4535 113376 4724 118104 4921 123048 5127 128184 5341 133536 5564 139104 5796 144912 6038 150960 6290 157272 6553 163848 6827 170688 7112 177816 7409 185232 7718 192960 8040. (Cycle-length=155)

(n = 251) 513 12840 535 13392 558 13968 582 14568 607 15192 633 15840 660 16512 688 17208 717 17928 747 18696 779 19488 812 20304 846 21168 882 22056 919 22992 958 23952 998 24960 1040 26016 1084 27120 1130 28272 1178 29472 1228 30720 1280 32016 1334 33360 1390 34752 1448 36216 1509 37728 1572 39312 1638 40968 1707 42696 1779 44496 1854 46368 1932 48312 2013 50328 2097 52440 2185 54648 2277 56928 2372 59304 2471 61776 2574 64368 2682 67056 2794 69864 2911 72792 3033 75840 3160 79008 3292 82320 3430 85752 3573 89328 3722 93072 3878 96960 4040 101016 4209 105240 4385 109632 4568 114216 4759 118992 4958 123960 5165 129144 5381 134544 5606 140160 5840 146016 6084 152112 6338 158472 6603 165096 6879 171984 7166 179160 7465 186648 7777 194448 8102 202560 8440 211008 8792 219816 9159 228984 9541 238536 9939 248496 10354 258864 10786 269664 11236 280920 11705 292632 12193 304848 12702 317568 13232 330816 13784 344616 14359 358992 14958 373968 15582 389568 16232 405816 16909 422736 17614 440352 18348 458712 19113 477840 19910 497760 20740 518520 21605 540144 22506 562656 23444 586104 24421 610536 25439 635976 26499 662496 27604 690120 28755 718896 29954 748872 31203 780096 32504 812616 33859 846480 35270 881760 36740 918504 38271 956784 39866 996672 41528 1038216 43259 1081488 45062 1126560 46940 1173504 48896 1222416 50934 1273368 53057 1326432 55268 1381704 57571 1439280 59970 1499256 62469 1561728 65072 1626816 67784 1694616 70609 1765248 73552 1838808 76617 1915440 79810 1995264 83136 3464 86616 3609 90240 3760 94008 3917 97944 4081 102048 4252 106320 4430 110760 4615 115392 4808 120216 5009 125232 5218 130464 5436 135912 5663 141576 5899 147480 6145 153648 6402 160056 6669 166728 6947 173688 7237 180936 7539 188496 7854 196368 8182 204552 8523 213096 8879 221976 9249 231240 9635 240888 10037 250944 10456 261408 10892 272304 11346 283656 11819 295488 12312. (Cycle-length=312)

A.8. $k = 25$:

(n = 2341) 4064 105675 4227 109925 4397 114325 4573 118900 4756 123675 4947 128625 5145 133775 5351 139150 5566 144725 5789 150525 6021 156550 6262 162825 6513 169350 6774 176125 7045 183175 7327 190525 7621 198150 7926 206100 8244 214350 8574 222925 8917 231850 9274 241125 9645 250775 10031 260825 10433 271275 10851 282150 11286 293450

11738 305200 12208 317425 12697 330125 13205 343350 13734 357100 14284 371400 14856 386275 15451 401750 16070 417825 16713 434550 17382 451950 18078 470050 18802 488875 19555 508450 20338 528800 21152 549975 21999 571975 22879 594875 23795 618675 24747 643425 25737 669175 26767 695950 27838 723800 28952 752775 30111 782900 31316 814225 32569 846800 33872 880675 35227 915925 36637 952575 38103 990700 39628 1030350 41214 1071575 42863 1114450 44578 1159050 46362 1205425 48217 1253650 50146 1303800 52152 1355975 54239 1410225 56409 1466650 58666 1525325 61013 1586350 63454 1649825 65993 1715825 68633 1784475 71379 1855875 74235 1930125 77205 2007350 80294 2087650 83506 2171175 86847 2258025 90321 2348350 93934 2442300 97692 2540000 101600. (Cycle-length=165)

A.9. $k = 27$:

$(n = 196)$ 545 15282 566 15849 587 16443 609 17064 632 17712 656 18387 681 19089 707 19818 734 20574 762 21357 791 22167 821 23004 852 23868 884 24759 917 25677 951 26649 987 27648 1024 28674 1062 29754 1102 30861 1143 32022 1186 33210 1230 34452 1276 35748 1324 37098 1374 38475 1425 39906 1478 41391 1533 42930 1590 44523 1649 46197 1711 47925 1775 49707 1841 51570 1910 53487 1981 55485 2055 57564 2132 59697 2211 61911 2293 64206 2378 66609 2467 69093 2559 71658 2654 74331 2753 77085 2855 79947 2961 82917 3071 85995 3185 89181 3303 92502 3426 95931 3553 99495 3685 103194 3822 107028 3964 110997 4111 115128 4264 119394 4422 123822 4586 128412 4756 133191 4933 138132 5116 143262 5306 148581 5503 154089 5707 159813 5919 165753 6139 171909 6367 178281 6603 184896 6848 191754 7102 198882 7366 206253 7639 213894 7922 221832 8216 230067 8521 238599 8837 247455 9165 256635 9505 266166 9858 276048 10224 286281 10603 296892 10996 307908 11404 319329 11827 331182 12266 343467 12721 356211 13193 369414 13682 383103 14189 397305 14715. (Cycle-length=181)

A.10. $k = 29$:

$(n = 91)$ 111 3335 115 3451 119 3596 124 3741 129 3886 134 4031 139 4176 144 4321 149 4495 155 4669 161 4843 167 5017 173 5191 179 5394 186 5597 193 5800 200 6003 207 6235 215 6467 223 6699 231 6931 239 7192 248 7453 257 7714 266 8004 276 8294 286 8584 296 8903 307 9222 318 9541 329 9889 341 10237 353 10614 366 10991 379 11397 393 11803 407 12238 422 12673 437 13137 453 13601 469 14094 486 14587 503 15109 521 15631 539 16182 558 16762 578 17342 598 17951 619 18589 641 19256 664 19923 687 20619 711 21344 736 22098 762 22881 789 23693 817 24534 846 25404 876 26303 907 27231 939 28188 972 29174 1006 30189 1041 31233 1077 32335 1115 33466 1154 34626 1194 35844 1236 37091 1279 38396 1324 39730 1370 41122 1418 42543 1467 44022 1518 45559 1571 47154 1626 48807 1683 50518 1742 52287 1803 54114 1866 55999 1931 57942 1998 59943 2067 62031 2139 64177 2213 66410 2290 68701 2369 71079 2451 73544 2536 76096 2624 78735 2715 81461 2809 84274 2906 87203 3007 90219 3111 93351 3219. (Cycle-length=193)

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