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Brief paper

On consensus algorithms design for double integrator dynamics[☆]

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ABSTRACT

This paper considers the consensus problem of double integrator multi-agent systems where: (i) each agent is subject to input saturations, and (ii) the velocity (second state) of each agent is not available for feedback. We present new consensus algorithms that handle simultaneously the above mentioned situations. Sufficient conditions are derived such that consensus algorithms developed for first- and second-order multi-agent systems in ideal situations can be used to account for input saturations and remove the requirement of velocity measurements. To illustrate the effectiveness of the proposed approach, we propose solutions to two different second-order consensus problems in the case where the input is saturated and the velocity states are not available for feedback and simulation results are provided in each case.

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1. Introduction

The past years have witnessed an increased interest in the distributed coordination problem of dynamic multi-agent systems with its various applications in consensus, flocking/rendezvous, and formation control. The ultimate problem is how to design a consensus algorithm such that a team of agents reach an agreement on their states, or on a common objective, using local information exchange, which is generally restricted to be directed, dynamically changing and may be subject to communication delays. The consensus problem of multi-agent systems with first-order dynamics has been widely investigated and several interesting results have been proposed (Fax & Murray, 2004; Jadbabaie, Lin, & Morse, 2003; Münz, Papachristodoulou, & Allgöwer, 2011; Olfati-saber & Murray, 2004; Ren, Beard, & Atkins, 2007). Basic concepts from these results have been exploited to develop consensus algorithms for multi-agent systems with second-order dynamics leading to a myriad of papers in this field (see for instance Ren and Atkins (2007) and Yu, Chen, and Cao (2010) and references therein). This interest in double-integrator dynamics is mainly motivated by the fact that the obtained consensus algorithms can be extended to design cooperative control strategies for complex physical systems with such applications as flocking (Olfati-Saber, 2006), rigid body attitude synchronization (Abdessameud & Tayebi, 2009a; Abdessameud, Tayebi, & Polushin, 2012; Ren, 2010), formation control of unmanned vehicles (Abdessameud & Tayebi, 2009b, 2011; Lawton, Beard, & Young, 2003), and synchronization of Euler–Lagrange systems (Ren, 2009; Spong & Chopra, 2007).

In the literature related to the second-order consensus problem of linear multi-agent systems, tools from algebraic graph theory have been successfully applied to establish conditions under which consensus is reached. In directed networks, it has been shown that second-order consensus will be reached if and only if the communication graph has a spanning tree and the control gains are carefully selected (Ren & Atkins, 2007; Yu et al., 2010). Within a similar framework, several related problems to consensus have been considered such as the consensus problem with group reference velocity, Ren (2008). Also, the case of dynamically changing topologies has been discussed in Ren and Atkins (2007) and Xie and Wang (2007). The effects of communication delays that are inherently present in communication systems have been also considered in Münz, Papachristodoulou, and Allgöwer (2008), Qin, Gao, and Zhengb (2011) and Tian and Liu (2009). However, the above consensus algorithms are based on the assumption that the full state vector is available for feedback.

In practical situations, it is sometimes desirable to design consensus algorithms that do not require full state information. If we consider, for example, a group of point mass agents, an important problem is to design consensus algorithms when the velocity information (the second state) is not precisely

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measured or the agents are not equipped with velocity sensors. These velocity-free algorithms can also be used as redundant control laws in the full state information case to improve the reliability of the system to velocity-sensors failure. Another important problem arises when the input of each agent is subject to saturations, in which case the extension of traditional consensus algorithms generally fails. The complexity and the challenges of the consensus design become prominent as one attempts to handle these two problems (partial state feedback with input saturations) simultaneously, especially under general communication topologies that can be directed, dynamically changing and/or subject to communication delays. This is the main reason why there are only a few papers in the literature dealing with this problem, where only the simple case of fixed and undirected communication topology with no communication delays has been considered.

The author in Ren (2008) proposed consensus algorithms that account for input saturations, where nonlinear saturation functions have been used to define the interaction between agents. In the same reference, a consensus algorithm that removes the requirement of velocity measurements without input saturation constraints has been presented. To our best knowledge, velocity-free consensus algorithms that account for input saturations can only be found in Abdessameud and Tayebi (2010).

In the present paper, we propose an approach that extends the work of Abdessameud and Tayebi (2010) by introducing dynamic auxiliary systems that define appropriate intermediate reference trajectories for the agents in order to satisfy the constraints imposed on the input vectors without velocity feedback. As a result, the velocity-free consensus algorithm design problem with input saturations is simplified to the design of consensus algorithms in the "ideal situation" (i.e., without input saturations and in the full state feedback case). Essentially, the proposed approach provides sufficient conditions under which consensus algorithms, developed for first- and second-order multi-agent systems in ideal situations, can be used to derive solutions to the second-order consensus problem in the presence of input saturations without velocity feedback. Interestingly, the proposed consensus algorithms inherit the properties of the "ideal consensus algorithms" in terms of convergence conditions and assumptions on the graph topology for instance. It is shown that the underlying conditions of our approach are not restrictive and can be verified with an appropriate design of these ideal consensus algorithms. The application and effectiveness of the proposed consensus algorithm design approach are illustrated by two examples, where we provide solutions to the free second-order consensus problem and consensus with a group reference velocity under directed interconnection topology in the presence of the two constraints considered in this work.

2. Preliminaries and problem description

Consider a group of n-identical autonomous agents modeled by the following second-order dynamics[2]

$$\ddot{\mathbf{p}}_i = \mathbf{u}_i, \quad \text{for } i \in \mathcal{N}, \tag{1}$$

where $\mathcal{N} \triangleq \{1, \ldots, n\}$, $\mathbf{p}_i \in \mathbb{R}^m$ and $\dot{\mathbf{p}}_i$ denote respectively the position and velocity states of the ith agent, and the vector $\mathbf{u}_i \in \mathbb{R}^m$ is the control input.

The communication topology between agents is represented by a weighted graph $\mathcal{G}_n = (\mathcal{N}, \mathcal{E}, \mathcal{K})$, where $\mathcal{N}$ is the set of nodes or vertices, describing the set of vehicles in the team, $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of pairs of nodes, called edges, and $\mathcal{K} = [k_{ij}]$ is a weighted adjacency matrix. An edge $(i, j) \in \mathcal{E}$ indicates that agent i can receive information from agent j, which is designated as its neighbor. The weighted adjacency matrix of a weighted graph is defined such that $k_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$ and $k_{ij} = 0$ if and only if $(i, j) \notin \mathcal{E}$. If the communication topology is bidirectional, then $\mathcal{G}_n$ is undirected, the pairs of nodes in $\mathcal{E}$ are unordered; $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$, and $\mathcal{K}$ is symmetric. In the case of a unidirectional communication topology, $\mathcal{G}_n$ is a directed graph, $\mathcal{E}$ contains ordered pairs, and $\mathcal{K}$ is not necessarily symmetric. In the case where the communication topology is dynamically changing, due to restrictions imposed by the environment for example, the weights k_{ij} are time-varying. Also, the information exchange between agents in the team can be subject to communication delays.

We assume that all agents are subject to input saturations, such that $\|\mathbf{u}_i\|_\infty \le \mathbf{u}_{\max}$, for $i \in \mathcal{N}$, and the velocity vectors of the agents are not available for feedback. In the presence of these two constraints, the objective of our work is to present a consensus algorithm design method for the multi-agent system (1), under a certain communication topology described by $\mathcal{G}_n$, such that second-order consensus is achieved, i.e.,

$$(\mathbf{p}_i - \mathbf{p}_j) \to 0, \qquad (\dot{\mathbf{p}}_i - \dot{\mathbf{p}}_j) \to 0, \tag{2}$$

for $i, j \in \mathcal{N}$, for any initial conditions. Although this problem is generally referred to as the free-consensus problem, several related problems to second-order consensus can be discussed in a similar framework.

Before we proceed, we give some definitions and preliminary results that will be used to prove our results. We define for any vector $\mathbf{x} = (x_1, \ldots, x_m)^\top \in \mathbb{R}^m$ the saturation function

$$\boldsymbol{\chi}(\mathbf{x}) = \mathrm{col}[\sigma(x_k)] \in \mathbb{R}^m, \quad \text{for } k \in \{1, \ldots, m\}, \tag{3}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuously differentiable function satisfying the following properties:

P1. $\sigma(0) = 0$ and $x\sigma(x) > 0$ for $x \neq 0$,
P2. $|\sigma(x)| \le \sigma_b$, for $\sigma_b > 0$.
P3. The diagonal matrix $\mathbf{h}(\mathbf{x}) = \mathrm{diag}[\frac{d\sigma(x_k)}{dx_k}]$ satisfies $\|\mathbf{h}(\mathbf{x})\|_\infty \le \sigma_h$, $\sigma_h > 0$.

Note that property P3 can be verified from P1 and P2. An example of the function $\sigma(x)$ is $\tanh(x)$, with $\frac{d\sigma(x)}{dx} = 1 - \tanh^2(x)$, and $\sigma_b = \sigma_h = 1$.

Lemma 1. *Consider the second-order system: $\ddot{\boldsymbol{\zeta}} = -L^p\boldsymbol{\chi}(\boldsymbol{\zeta}) - L^d\boldsymbol{\chi}(\dot{\boldsymbol{\zeta}}) + \boldsymbol{\varepsilon}$, where $\boldsymbol{\zeta} \in \mathbb{R}^m$, the function $\boldsymbol{\chi}$ is defined in (3), and L^p and L^d are strictly positive scalars. If $\boldsymbol{\varepsilon}$ is bounded for all time and $\boldsymbol{\varepsilon} \to 0$, then $\boldsymbol{\zeta}$ and $\dot{\boldsymbol{\zeta}}$ are bounded and $\boldsymbol{\zeta} \to \dot{\boldsymbol{\zeta}} \to 0$.*

Proof. See Abdessameud and Tayebi (2010) for a similar proof with $\sigma(x) = \tanh(x)$. □

Lemma 2. *Consider the first-order system: $\dot{\boldsymbol{\delta}} = -L^p\boldsymbol{\chi}(\boldsymbol{\delta}_i) + \bar{\boldsymbol{\varepsilon}}$, where $\boldsymbol{\delta} \in \mathbb{R}^m$, the function $\boldsymbol{\chi}$ is defined in (3), and L^p is a strictly positive scalar. If $\bar{\boldsymbol{\varepsilon}}$ is bounded for all time and $\bar{\boldsymbol{\varepsilon}} \to 0$, then $\boldsymbol{\delta}$ is bounded and $\boldsymbol{\delta} \to \dot{\boldsymbol{\delta}} \to 0$.*

Proof. See Appendix A. □

3. Consensus algorithm design—method I

In this section, we present a first method for the design of second-order consensus algorithms for multi-agent system (1)

[2] For the sake of clarity of presentation, we omit throughout the paper the arguments of time dependent signals, and the limit of a signal at infinity is replaced by an arrow (e.g. $y \to c \Leftrightarrow \lim_{t\to\infty} y(t) = c$, for a constant c).

without velocity measurements and with input saturations. To this end, we associate to each agent the following auxiliary systems

$$\ddot{\boldsymbol{\zeta}}_i = \mathbf{u}_i - \ddot{\bar{\mathbf{r}}}_i + k_i^p(\mathbf{r}_i - \bar{\mathbf{r}}_i) + k_i^d(\mathbf{r}_i - \bar{\mathbf{r}}_i - \boldsymbol{\psi}_i), \quad (4)$$

$$\dot{\boldsymbol{\psi}}_i = k_i^{\psi}(\mathbf{r}_i - \bar{\mathbf{r}}_i - \boldsymbol{\psi}_i), \quad (5)$$

for $i \in \mathcal{N}$, where k_i^p, k_i^d, and k_i^{ψ} are strictly positive scalar gains, $\boldsymbol{\zeta}_i \in \mathbb{R}^m$ and $\dot{\boldsymbol{\zeta}}_i$ are, respectively, the position-like and velocity-like states of the second-order system (4) and can take arbitrary initial values, $\boldsymbol{\psi}_i(0)$ can be selected arbitrarily, $\mathbf{r}_i := (\mathbf{p}_i - \boldsymbol{\zeta}_i)$, and $\bar{\mathbf{r}}_i \in \mathbb{R}^m$ is the solution of the dynamic system

$$\ddot{\bar{\mathbf{r}}}_i = \boldsymbol{\Psi}_{i,\mathcal{G}_n}(\bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}}), \quad \text{for } i \in \mathcal{N}, \quad (6)$$

with $\bar{\mathbf{r}} = (\bar{\mathbf{r}}_1^\top, \ldots, \bar{\mathbf{r}}_n^\top)^\top \in \mathbb{R}^{nm}$, $\bar{\mathbf{r}}_i(0)$ and $\dot{\bar{\mathbf{r}}}_i(0)$ can take arbitrary values, and $\boldsymbol{\Psi}_{i,\mathcal{G}_n}(\bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}})$ is a protocol designed using the states $\bar{\mathbf{r}}$ and $\dot{\bar{\mathbf{r}}}$, as well as information on a desired global objective if assigned to the team, and satisfies the following conditions:

Design condition 1. *The multi-agent system (6) achieves consensus in the sense of (2), i.e., $(\bar{\mathbf{r}}_i - \bar{\mathbf{r}}_j) \to 0$, $(\dot{\bar{\mathbf{r}}}_i - \dot{\bar{\mathbf{r}}}_j) \to 0$, for $i, j \in \mathcal{N}$, under a certain communication topology that can be restricted to be directed, time-varying, and/or subject to communication delays, and is represented by the weighted graph $\mathcal{G}_n$.*

Design condition 2. *The protocol $\boldsymbol{\Psi}_{i,\mathcal{G}_n}(\bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}})$ can be written as: $\boldsymbol{\Psi}_{i,\mathcal{G}_n}(\bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}}) := \mathbf{f}_d + \bar{\boldsymbol{\Psi}}_{i,\mathcal{G}_n}(\bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}})$, where $\mathbf{f}_d \in \mathbb{R}^m$ satisfies $\|\mathbf{f}_d\|_\infty \le \boldsymbol{f}_{\max}$, and the solutions of (6) guarantee that $\bar{\boldsymbol{\Psi}}_{i,\mathcal{G}_n}(\bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}})$ is globally bounded and converges asymptotically to zero when the multi-agent system (6) achieves consensus.*

Note that the dynamic system (6) is considered as a multi-agent system since all agents transmit the states of their corresponding auxiliary systems (6), according to the protocol $\boldsymbol{\Psi}_{i,\mathcal{G}_n}$. In fact, (6) is an independent multi-agent system with available states and with no input saturation constraints. Therefore, Design condition 1 can always be satisfied if one is able to design a consensus algorithm for the multi-agent system (1) in ideal situations, i.e., in the full state information case and with no input saturation constraints.

Design condition 2 is mainly required in the subsequent analysis and can be satisfied for most consensus algorithms developed in ideal situations. In fact, any consensus algorithm can be decomposed into two terms. The term $\bar{\boldsymbol{\Psi}}_{i,\mathcal{G}_n}(\bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}})$ will contain relative error vectors and tracking errors, if a reference trajectory is assigned to the team. Therefore, the condition that these error terms are globally bounded and converge to zero is a natural requirement of any protocol that satisfies Design condition 1. Also, $\mathbf{f}_d$ will contain only the non-vanishing terms (if any) of the protocol $\boldsymbol{\Psi}_{i,\mathcal{G}_n}(\bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}})$, and is related to a reference trajectory if assigned to the team, such as a desired acceleration. This term can, in general, be set by the designer to be *a priori* bounded to account for input saturations.

With the above definitions, our main result in this section is stated in the following theorem and is proved in Appendix B.

Theorem 1. *Consider the multi-agent system (1) with a communication topology described by $\mathcal{G}_n$. Suppose that the protocol $\boldsymbol{\Psi}_{i,\mathcal{G}_n}(\bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}})$ in (6) satisfies Design conditions 1 and 2 with $\boldsymbol{f}_{\max} < \mathbf{u}_{\max}$. Let the control input in (1) and (4) be given by*

$$\mathbf{u}_i = \mathbf{f}_d - L_i^p \chi(\zeta_i) - L_i^d \chi(\dot{\zeta}_i), \quad (7)$$

where L_i^p, L_i^d are strictly positive scalar gains, the function $\chi(\cdot)$ is defined in (3), and ζ_i and $\dot{\zeta}_i$ are the states of the auxiliary system (4) with (5)–(6). If the control gains are selected such that

$$\sigma_b(L_i^p + L_i^d) \le \mathbf{u}_{\max} - \boldsymbol{f}_{\max}, \quad (8)$$

with σ_b is defined in P2, then the control input is guaranteed to be bounded as: $\|\mathbf{u}_i\|_\infty \le \mathbf{u}_{\max}$, for $i \in \mathcal{N}$, and the multi-agent system (1) with (7) and (4)–(6) achieves second-order consensus in the sense of (2).

The main idea in the above result is to associate to each agent in the team the two second-order systems given in (4) and (6). The dynamic system (6) is implemented to generate a first intermediary reference trajectory defined by $\bar{\mathbf{r}}_i$ to each agent in the team. The input of this system is designed without consideration of the input saturation constraints such that Design condition 1 is satisfied, i.e., all agents reach an agreement on their first reference trajectories. This requires that communicating agents transmit the states of their individual dynamic systems (6), i.e., $\bar{\mathbf{r}}_i$ and $\dot{\bar{\mathbf{r}}}_i$, rather than transmitting their position states.

Meanwhile, the dynamic system (4) is used to generate the vector $\boldsymbol{\zeta}_i$, which is a trajectory tracking error between each agent position and the time-varying vector $\mathbf{r}_i := (\mathbf{p}_i - \boldsymbol{\zeta}_i)$. This vector, $\mathbf{r}_i$, is defined here for analysis purposes and can be seen as a second intermediary reference trajectory defined for the ith agent through the dynamics (4), and is governed, in view of (1) and (4), by: $\ddot{\mathbf{r}}_i = \boldsymbol{\Gamma}_i$, with $\boldsymbol{\Gamma}_i := \left(\ddot{\bar{\mathbf{r}}}_i - k_i^p(\mathbf{r}_i - \bar{\mathbf{r}}_i) - k_i^d(\mathbf{r}_i - \bar{\mathbf{r}}_i - \boldsymbol{\psi}_i)\right)$. Note that the input $\boldsymbol{\Gamma}_i$ is designed such that the error between the two intermediate reference trajectories converges exponentially to zero. Also, the first-order filter (5) is used to achieve this last result without velocity measurements since $\dot{\mathbf{r}}_i$ is not available for feedback.

Finally, the control input $\mathbf{u}_i$ is designed as in (7) to account for input saturations and guarantee that each agent tracks the second intermediary reference signal, i.e., $(\mathbf{p}_i - \mathbf{r}_i) \to 0$ and $(\dot{\mathbf{p}}_i - \dot{\mathbf{r}}_i) \to 0$, guiding hence all agents to reach consensus. From the proof of Theorem 1 and Lemma 1, we can see that this is achieved if Design condition 2 is satisfied.

It should be noted that the control gains of the proposed consensus algorithm in Theorem 1 can be easily selected despite the presence of input saturation constraints. In fact, the control input law for each agent (7) is guaranteed to be *a priori* bounded as

$$\|\mathbf{u}_i\|_\infty \le \boldsymbol{f}_{\max} + \sigma_b(L_i^p + L_i^d), \quad (9)$$

which is independent from the number of neighbors of each agent. Consequently, with an appropriate choice of the control gains L_i^p and L_i^d, the designer can set the upper bound of the input of each agent without *a priori* knowledge on the communication topology between agents. This introduces more flexibility in the tuning of the controller gains especially in the case where $\mathbf{u}_{\max}$ is small and the number of neighbors of each agent may be large. Also, the gains k_i^p, k_i^d, and k_i^{ψ} can be tuned independently from the input constraints such that the exponential convergence to zero of $(\mathbf{r}_i - \bar{\mathbf{r}}_i)$ and $(\dot{\mathbf{r}}_i - \dot{\bar{\mathbf{r}}}_i)$ is achieved. On the other hand, the parameters of $\boldsymbol{\Psi}_{i,\mathcal{G}_n}(\bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}})$ in (6) can be selected, again without consideration of the input constraints, such that Design condition 1 is satisfied.

Also, the initial conditions of the auxiliary systems (4)–(6) can be selected arbitrarily. In some cases, it is possible to determine the consensus value of the multi-agent system (6) using only the initial states of (6), i.e., $\bar{\mathbf{r}}_i(0)$ and $\dot{\bar{\mathbf{r}}}_i(0)$. In these cases, the initial conditions of the auxiliary system (6) will specify the consensus value of the multi-agent system (1), since we have shown that $(\mathbf{p}_i - \mathbf{r}_i) \to 0$ and $(\mathbf{r}_i - \bar{\mathbf{r}}_i) \to 0$, and consequently, $(\mathbf{p}_i - \bar{\mathbf{r}}_i) \to 0$, for $i \in \mathcal{N}$. Therefore, the consensus algorithm in Theorem 1 provides a means to determine the consensus value of the multi-agent system (1) without knowing the initial states of the agents, which is advantageous in our case as the agents' states are only partly measured.

From the above discussion, we can see that Theorem 1 suggests a simple design procedure for consensus algorithms without velocity measurements in the presence of input saturation constraints. The first step is to design the input of the auxiliary multi-agent system (6), i.e., the protocol $\boldsymbol{\Psi}_{i,\mathcal{G}_n}(\bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}})$, using the available states $\bar{\mathbf{r}}$ and $\dot{\bar{\mathbf{r}}}$ and without input constraints such that Design conditions 1 and 2 are satisfied. Next, one implements the control input given in (7), with (4)–(6), by an appropriate choice of the control gains as discussed above.

As mentioned earlier, the first step above is equivalent to the design of a protocol for the multi-agent system (1) in ideal situations. If such a protocol exists, and satisfies Design conditions 1 and 2, then it is more convenient to use it as input of the multi-agent system (6), using the internally synthesized states of (6). In fact, Theorem 1 provides sufficient conditions for existing second-order consensus algorithms, developed in ideal situations, to be extended in a straightforward manner to account for input saturation constraints and handle the missing velocity vectors. Also, note that Design condition 1 can be satisfied under some assumptions on the communication topology and/or the parameters of the protocol $\boldsymbol{\Psi}_{i,\mathcal{G}_n}(\bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}})$. In this case, the derived consensus algorithms using the result of Theorem 1 will be valid under the same assumptions.

3.1. Example 1: the free consensus problem

In this subsection, we present a solution to the free consensus problem for second-order multi-agent systems. The control objective is to design a consensus algorithm that accounts for input saturations, without velocity measurements, such that consensus in the sense of (2) is achieved under a directed communication topology represented by the directed graph $\mathcal{G}_n$. Based on the result of Theorem 1, we first design the input of the auxiliary system (6) such that Design conditions 1 and 2 are satisfied. Inspired by the work of Ren and Atkins (2007) and Yu et al. (2010), we consider the input in (6) such that

$$\ddot{\bar{\mathbf{r}}}_i = -\sum_{j=1}^{n} k_{ij}\left(\alpha(\bar{\mathbf{r}}_i - \bar{\mathbf{r}}_j) + \beta(\dot{\bar{\mathbf{r}}}_i - \dot{\bar{\mathbf{r}}}_j)\right), \tag{10}$$

for $i \in \mathcal{N}$, where α, β are positive scalar gains and k_{ij} is the (i,j)th entry of the adjacency matrix of the directed graph $\mathcal{G}_n$. Following the same arguments as in Yu et al. (2010), we can show that the multi-agent system (10) achieves second-order consensus, i.e., $(\dot{\bar{\mathbf{r}}}_i - \dot{\bar{\mathbf{r}}}_j) \to 0$ and $(\bar{\mathbf{r}}_i - \bar{\mathbf{r}}_j) \to 0$ for $i, j \in \mathcal{N}$, if the directed communication graph contains a directed spanning tree and the control gains satisfy:

$$\frac{\beta^2}{\alpha} > \max_{2\le i\le n} \frac{\Im^2(\mu_i)}{\Re(\mu_i)[\Re^2(\mu_i) + \Im^2(\mu_i)]}. \tag{11}$$

In the above condition, $\Re(\mu_i)$ and $\Im(\mu_i)$ denote, respectively, the real and imaginary parts of μ_i, where μ_i, for $i = 2, \ldots, n$, are the nonzero eigenvalues of the Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n\times n}$, with $l_{ij} = -k_{ij}$, for $i \neq j$, and $l_{ii} = -\sum_{j=1, j\neq i}^{n} l_{ij}$. In addition, the solutions of (10) guarantee that the right hand side of (10) is globally bounded and $\|\dot{\bar{\mathbf{r}}}_i(t) - \sum_{j=1}^{n} q_j \dot{\bar{\mathbf{r}}}_j(0)\| \to 0$, $\|\bar{\mathbf{r}}_i(t) - \sum_{j=1}^{n} q_j \bar{\mathbf{r}}_j(0) - \sum_{j=1}^{n} q_j \dot{\bar{\mathbf{r}}}_j(0) t\| \to 0$, where $\mathbf{q} = (q_1, \ldots, q_n)^\top$ is the unique nonnegative left eigenvector of L associated with eigenvalue zero satisfying $\mathbf{q}^\top \mathbf{1}_n = 1$, with $\mathbf{1}_n \in \mathbb{R}^n$ being the vector of all ones elements (see Theorem 1 in Yu et al. (2010) for more details).

Therefore, the consensus protocol in (10) satisfies Design conditions 1 and 2 with $\mathbf{f}_d = 0$. Consequently, we conclude by Theorem 1 that the multi-agent system (1) with the input given by

$$\mathbf{u}_i = -L_i^p \boldsymbol{\chi}(\boldsymbol{\zeta}_i) - L_i^d \boldsymbol{\chi}(\dot{\boldsymbol{\zeta}}_i), \tag{12}$$

$$\ddot{\boldsymbol{\zeta}}_i = \mathbf{u}_i - \ddot{\bar{\mathbf{r}}}_i + k_i^p(\mathbf{r}_i - \bar{\mathbf{r}}_i) + k_i^d(\mathbf{r}_i - \bar{\mathbf{r}}_i - \boldsymbol{\psi}_i), \tag{13}$$

$$\dot{\boldsymbol{\psi}}_i = k_i^{\psi}(\mathbf{r}_i - \bar{\mathbf{r}}_i - \boldsymbol{\psi}_i), \tag{14}$$

where $\mathbf{r}_i = (\mathbf{p}_i - \boldsymbol{\zeta}_i)$, the control parameters are defined in Theorem 1, and $\bar{\mathbf{r}}_i$ is obtained from (10), achieves second-order consensus in the sense of (2) if the directed communication graph contains a spanning tree and (11) is satisfied. Note that these are the same conditions obtained in Yu et al. (2010) in the case of no input saturation constraints and the full state vectors are available for feedback. In addition, the final consensus value satisfies: $\|\dot{\mathbf{p}}_i(t) - \sum_{j=1}^{n} q_j \dot{\bar{\mathbf{r}}}_j(0)\| \to 0$ and $\|\mathbf{p}_i(t) - \sum_{j=1}^{n} q_j \bar{\mathbf{r}}_j(0) - \sum_{j=1}^{n} q_j \dot{\bar{\mathbf{r}}}_j(0) t\| \to 0$, for $i \in \mathcal{N}$. Also, if the control gains are selected according to (8), with $\boldsymbol{f}_{\max} = 0$, then $\|\mathbf{u}_i\|_\infty \le \mathbf{u}_{\max}$.

4. Consensus algorithm design—method II

In this section, we consider a class of second-order consensus problems where it is required to drive all agents' velocities to some known desired velocity, i.e.,

$$(\mathbf{p}_i - \mathbf{p}_j) \to 0, \qquad \dot{\mathbf{p}}_i \to \dot{\mathbf{p}}_d, \tag{15}$$

for $i, j \in \mathcal{N}$, with $\dot{\mathbf{p}}_d$ being a desired velocity available to all members of the team, which can be time-varying, constant or null, and satisfies $\|\ddot{\mathbf{p}}_d\|_\infty \le \boldsymbol{a}_{\max} < \mathbf{u}_{\max}$. It is clear that this class of problems constitutes a special case of the second-order consensus problems defined in (2).

Similar to the previous section, we associate with each agent the following auxiliary systems

$$\dot{\bar{\mathbf{r}}}_i = \dot{\mathbf{p}}_d - L_i^p \boldsymbol{\chi}(\boldsymbol{\zeta}_i), \tag{16}$$

$$\dot{\boldsymbol{\zeta}}_i = -L_i^p \boldsymbol{\chi}(\boldsymbol{\zeta}_i) + L_i^d \boldsymbol{\chi}(\boldsymbol{\delta}_i), \tag{17}$$

$$\dot{\boldsymbol{\delta}}_i = -L_i^d \boldsymbol{\chi}(\boldsymbol{\delta}_i) - \bar{\boldsymbol{\Phi}}_{i,\mathcal{G}_n}(\mathbf{r}), \tag{18}$$

where L_i^p, L_i^d are strictly positive scalar gains, the function $\boldsymbol{\chi}$ is defined in (3), $\bar{\mathbf{r}}_i(0)$, $\boldsymbol{\zeta}_i(0)$, and $\boldsymbol{\delta}_i(0)$ can be selected arbitrarily, the vector $\mathbf{r} = (\mathbf{r}_1^\top, \ldots, \mathbf{r}_n^\top)^\top \in \mathbb{R}^{nm}$, with $\mathbf{r}_i = (\bar{\mathbf{r}}_i - \boldsymbol{\zeta}_i - \boldsymbol{\delta}_i)$, and $\bar{\boldsymbol{\Phi}}_{i,\mathcal{G}_n}(\mathbf{r})$ is a protocol designed based on the states $\mathbf{r}$, and satisfies the following conditions:

Design condition 3. *The multi-agent system*

$$\dot{\mathbf{r}}_i = \dot{\mathbf{p}}_d + \bar{\boldsymbol{\Phi}}_{i,\mathcal{G}_n}(\mathbf{r}), \quad \textit{for } i \in \mathcal{N}, \tag{19}$$

achieves consensus in the sense of (15), i.e., $(\mathbf{r}_i - \mathbf{r}_j) \to 0$, $\dot{\mathbf{r}}_i \to \dot{\mathbf{p}}_d$, *for* $i, j \in \mathcal{N}$, *under a certain communication topology that can be restricted to be directed, time-varying, and/or subject to communication delays and is represented by the weighted graph* $\mathcal{G}_n$.

Design condition 4. *The solutions of (19) guarantee that* $\bar{\boldsymbol{\Phi}}_{i,\mathcal{G}_n}(\mathbf{r})$ *is globally bounded and converges to zero when the multi-agent system (19) achieves consensus.*

Note that Design condition 3 can always be satisfied if one is able to design a consensus protocol for a single-integrator multi-agent system in ideal situations. In fact, no constraints are imposed on the input of (19), and the vector $\mathbf{r}$ is available for feedback. Also, Design condition 4 implies that the error terms in the input of the multi-agent system (19) are globally bounded and converge to zero when (19) achieves first-order consensus.

Our result in this section is stated in the following theorem and is proved in Appendix C.

Theorem 2. *Consider the multi-agent system (1) with a communication topology described by* $\mathcal{G}_n$. *Suppose that Design conditions 3 and 4 are satisfied. Let the control input in (1) be given by*

$$\mathbf{u}_i = \ddot{\mathbf{p}}_d - L_i^p \mathbf{h}(\boldsymbol{\zeta}_i)\dot{\boldsymbol{\zeta}}_i - k_i^p \boldsymbol{\chi}(\mathbf{e}_i) - k_i^d \boldsymbol{\chi}(\mathbf{e}_i - \boldsymbol{\psi}_i), \tag{20}$$

$$\dot{\boldsymbol{\psi}}_i = k_i^{\psi}(\mathbf{e}_i - \boldsymbol{\psi}_i), \tag{21}$$

with (16)–(18), where $\mathbf{e}_i := (\mathbf{p}_i - \bar{\mathbf{r}}_i)$, k_i^p, k_i^d, *and* k_i^ψ *are strictly positive scalar gains, the function* $\boldsymbol{\chi}$ *is defined in (3), and the diagonal matrix* $\mathbf{h}(\cdot)$ *is defined in property P3. If the control gains are selected such that*

$$\sigma_b\left(k_i^p + k_i^d + \sigma_h L_i^p (L_i^p + L_i^d)\right) \le \mathbf{u}_{\max} - \boldsymbol{a}_{\max}, \tag{22}$$

then the control input is guaranteed to be bounded as: $\|\mathbf{u}_i\|_\infty \le \mathbf{u}_{\max}$, *and the multi-agent system (1) with (20)–(21) and (16)–(18) achieves second-order consensus in the sense of (15).*

The above result is based on the introduction of the first-order auxiliary system (16) that generates an intermediate reference trajectory, defined by $\bar{\mathbf{r}}_i$, for each agent. The input of this multi-agent system is designed such that all agents reach consensus on their intermediate reference trajectories, i.e., $(\bar{\mathbf{r}}_i - \bar{\mathbf{r}}_j) \to 0$, $\dot{\bar{\mathbf{r}}}_i \to \dot{\mathbf{p}}_d$, for $i, j \in \mathcal{N}$. The bounded control input for each agent is then designed, without velocity measurements, such that each agent tracks its corresponding reference trajectory leading hence all agents to achieve consensus. The main difficulty here is to design an *a priori* bounded intermediate reference acceleration, $\ddot{\bar{\mathbf{r}}}_i$. It is important to mention that the design of such an *a priori* bounded intermediate reference acceleration is yet a difficult task using the conventional design methods, especially in the case of general communication topologies. In our case, this is solved by the introduction of the two first-order auxiliary systems (17) and (18) to generate the auxiliary error vectors $\boldsymbol{\zeta}_i$ and $\boldsymbol{\delta}_i$. These error vectors are driven to zero once the multi-agent system (19), which is defined for analysis purposes, achieves consensus. Consequently, to apply the above results, neighboring agents should transmit their variables $\mathbf{r}_i := (\bar{\mathbf{r}}_i - \boldsymbol{\zeta}_i - \boldsymbol{\delta}_i)$.

Similar to the first design method, the initial states of the auxiliary systems (16)–(18) can take arbitrary values, and the control parameters of the protocol $\bar{\boldsymbol{\Phi}}_{i,\mathcal{G}_n}(\mathbf{r})$ can be selected independently from the input constraints such that Design condition 3 is satisfied. Also, the control input law for each agent (20) is guaranteed to be *a priori* bounded as

$$\|\mathbf{u}_i\|_\infty \le \boldsymbol{a}_{\max} + \sigma_b\left(k_i^p + k_i^d + \sigma_h L_i^p (L_i^p + L_i^d)\right), \tag{23}$$

which can be easily set with an appropriate choice of the control gains.

Furthermore, the result of Theorem 2 suggests a second procedure to the design of consensus algorithms for the class of consensus problems considered in this section. The main step in this procedure is to design a consensus algorithm for the first-order multi-agent system (19) in the full state information case and with no input constraints such that Design conditions 3 and 4 are satisfied. Note that any existent protocol satisfying Design conditions 3 and 4 can be readily used as input of the multi-agent system (19). Consequently, Theorem 2 gives sufficient conditions such that consensus algorithms developed for first-order multi-agent systems in ideal situations can be extended to solve the second-order consensus problem in the presence of the two above constraints. Note that this extension is not trivial and presents several advantages since the design of consensus protocols for first-order multi-agent systems, although challenging, is simpler than the design of second-order consensus protocols, and consensus can be achieved in general under less restrictive conditions. This can be seen in the following example.

4.1. Example 2: consensus with group reference velocity

To illustrate the application of the proposed design method in this section, we develop a solution to the consensus problem with group reference velocity, without velocity measurements and with input saturation constraints. The control objective is to design a consensus algorithm such that multi-agent system (1) achieves consensus and each agent tracks a common desired velocity, given by $\dot{\mathbf{p}}_d(t)$, which satisfies: $\|\ddot{\mathbf{p}}_d(t)\|_\infty \le \boldsymbol{a}_{\max} < \mathbf{u}_{\max}$, and the communication graph $\mathcal{G}_n$ is assumed to be directed. To this end, we first design the first-order multi-agent system (19) as

$$\dot{\mathbf{r}}_i = \dot{\mathbf{p}}_d - \sum_{j=1}^{n} k_{ij}(\mathbf{r}_i - \mathbf{r}_j), \quad \text{for } i \in \mathcal{N}, \tag{24}$$

with k_{ij} being the (i, j)th entry of the adjacency matrix of the directed graph $\mathcal{G}_n$. Let $\tilde{\mathbf{r}}_i := (\mathbf{r}_i - \int_0^t \dot{\mathbf{p}}_d(s)ds)$, and hence $\dot{\tilde{\mathbf{r}}}_i = (\dot{\mathbf{r}}_i - \dot{\mathbf{p}}_d) = -\sum_{j=1}^{n} k_{ij}(\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j)$, for $i \in \mathcal{N}$.

Then, following similar steps as in Ren, Beard, and McLain (2005), we can show that $\dot{\tilde{\mathbf{r}}}_i$ and $(\mathbf{r}_i - \mathbf{r}_j)$ are globally bounded and $(\mathbf{r}_i - \mathbf{r}_j) \to 0$, $\dot{\tilde{\mathbf{r}}}_i \to 0$, for $i \in \mathcal{N}$, if the directed communication graph contains a spanning tree. This implies that Design conditions 3 and 4 are satisfied with $\bar{\boldsymbol{\Phi}}_{i,\mathcal{G}_n}(\mathbf{r}) = -\sum_{j=1}^{n} k_{ij}(\mathbf{r}_i - \mathbf{r}_j)$.

Therefore, we can conclude from Theorem 2 that the multi-agent system (1) with the control input

$$\begin{aligned}
\mathbf{u}_i &= \ddot{\mathbf{p}}_d - L_i^p \mathbf{h}(\boldsymbol{\zeta}_i)\dot{\boldsymbol{\zeta}}_i - k_i^p \boldsymbol{\chi}(\mathbf{e}_i) - k_i^d \boldsymbol{\chi}(\mathbf{e}_i - \boldsymbol{\psi}_i),\\
\dot{\boldsymbol{\psi}}_i &= k_i^\psi(\mathbf{e}_i - \boldsymbol{\psi}_i),\\
\dot{\bar{\mathbf{r}}}_i &= \dot{\mathbf{p}}_d - L_i^p \boldsymbol{\chi}(\boldsymbol{\zeta}_i),\\
\dot{\boldsymbol{\zeta}}_i &= -L_i^p \boldsymbol{\chi}(\boldsymbol{\zeta}_i) + L_i^d \boldsymbol{\chi}(\boldsymbol{\delta}_i),\\
\dot{\boldsymbol{\delta}}_i &= -L_i^d \boldsymbol{\chi}(\boldsymbol{\delta}_i) + \sum_{j=1}^{n} k_{ij}(\mathbf{r}_i - \mathbf{r}_j),
\end{aligned} \tag{25}$$

with $\mathbf{e}_i = (\mathbf{p}_i - \bar{\mathbf{r}}_i)$, $\mathbf{r}_i = (\bar{\mathbf{r}}_i - \boldsymbol{\zeta}_i - \boldsymbol{\delta}_i)$, and the control gains being defined in Theorem 2, achieves consensus, i.e., $(\mathbf{p}_i - \mathbf{p}_j) \to 0$, $\dot{\mathbf{p}}_i \to \dot{\mathbf{p}}_d$, for $i, j \in \mathcal{N}$, under the only condition that the directed communication graph contains a spanning tree. Also, the control input of each agent is guaranteed to be *a priori* bounded as in (23).

Remark 1. A solution to the consensus problem with group reference velocity for second-order multi-agents has been proposed in Ren (2008, Theorem 5.1) in the full state information case and with no input constraints. It has been shown that consensus is achieved under the condition that the directed communication graph contains a spanning tree and the control gains satisfy some topology-dependent conditions. Note that the consensus protocol proposed in Ren (2008) can be extended using the result of Theorem 1 to account for input saturations without velocity feedback, which will lead to the same conditions obtained in Ren (2008). However, the proposed consensus algorithm in Example 2 guarantees our control objective and improves the obtained results in ideal situations by removing the topology-dependent conditions. This, with the fact that the control upper bound can be set independently from the interconnection graph, allow the effective implementation of the proposed consensus algorithm without any centralized knowledge of the directed communication topology.

5. Simulation results

We provide in this section simulation results to demonstrate the effectiveness of the proposed consensus algorithms in the two examples given above. For this purpose, we consider a group of four agents modeled as in (1), with $m = 1$, and with initial conditions: $\mathbf{P}(0) = (1, 1.5, 2, 3)^\top$ and $\dot{\mathbf{P}}(0) = (0.1, 0.02, -0.08, 0.05)^\top$, where $\mathbf{P}(t)$ and $\dot{\mathbf{P}}(t)$ are the vectors containing, respectively, $\mathbf{p}_i(t)$ and $\dot{\mathbf{p}}_i(t)$ for $i \in \mathcal{N} := \{1, 2, 3, 4\}$. We assume that all agents are constrained such that $\mathbf{u}_{\max} = 2$, and consider the function $\boldsymbol{\chi}$ defined in (3) with $\sigma = \tanh$. Also, the communication topology between agents is represented by the directed graph $\mathcal{G}_4$, given in Fig. 1, which contains a spanning tree.

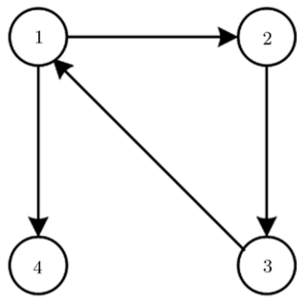

Fig. 1. Interaction graph $\mathcal{G}_4$.

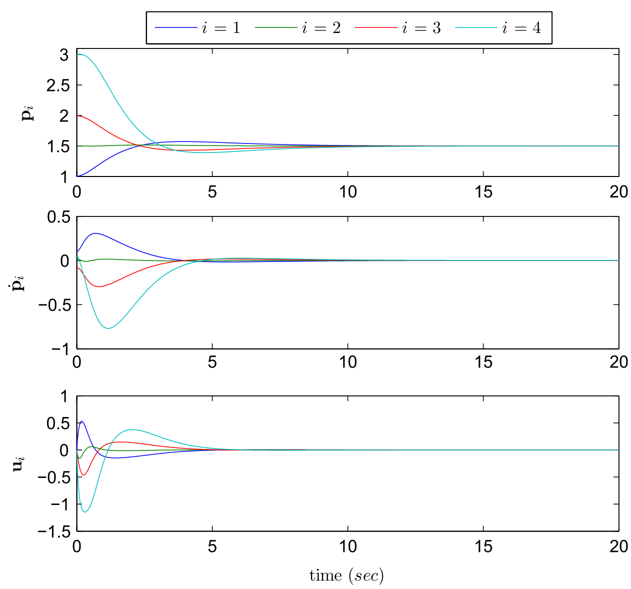

Fig. 2. Simulation results for Example 1.

First, we consider the example in Section 3.1, and implement the consensus algorithm given by (12)–(14) and (10), with the initial conditions: $\boldsymbol{\zeta}_i(0) = \dot{\boldsymbol{\zeta}}_i(0) = \bar{\mathbf{r}}_i(0) = \boldsymbol{\psi}_i(0) = 0$ and $\bar{\mathbf{r}}_i(0) = \mathbf{p}_i(0)$, for $i \in \mathcal{N}$. The control gains are selected as: $k_{ij} = 5$, for $(i, j) \in \mathcal{E}$, $(\alpha, \beta) = (1, 1)$, and $(k_i^p, k_i^d, k_i^\psi, L_i^p, L_i^d) = (1, 15, 5, 0.5, 1.5)$, for $i \in \mathcal{N}$. It is clear that this choice of the gains satisfies condition (8). Note that the eigenvalues of the Laplacian matrix of the graph $\mathcal{G}_4$ in view of the weights k_{ij} are: $0, 5, 7.5 \pm 4.3301\varsigma$, with $\varsigma^2 = -1$, and therefore, condition (11) is satisfied. It should be noted that tuning the gains of this consensus algorithm is not difficult following the discussion in Section 3.

Fig. 2 shows the obtained results in this case, where we can see that consensus is achieved without velocity measurements, and the control input for each agent satisfies $|\mathbf{u}_i| \leq \mathbf{u}_{\max}$. Note also that the final position of all agents is constant and equal to 1.5. This corresponds to the expected consensus value in this case in view of the initial conditions of the auxiliary systems (6), and knowing that the left eigenvector of the Laplacian matrix of the directed graph $\mathcal{G}_4$ associated to eigenvalue zero is obtained as: $\mathbf{q} = \frac{1}{3}(1, 1, 1, 0)^\top$.

Next, we implement the consensus algorithm presented in Section 4.1 as a solution to the consensus problem with group reference velocity, with $\dot{\mathbf{p}}_d(t) = 0.5\sin(2t/\pi)$. The initial states of the auxiliary systems are selected such that $\mathbf{r}_i(0) = \mathbf{p}_i(0)$ and $\dot{\mathbf{r}}_i(0) = 0$, with $\mathbf{r}_i = (\bar{\mathbf{r}}_i - \boldsymbol{\zeta}_i - \boldsymbol{\delta}_i)$. The control gains are selected as: $k_{ij} = 5$, for $(i, j) \in \mathcal{E}$, and $(k_i^p, k_i^d, k_i^\psi, L_i^p, L_i^d) = (0.2, 0.7, 2, 0.5, 1)$, for $i \in \mathcal{N}$, which satisfy condition (22) to guarantee the required upper bound of the control input. The obtained results are illustrated in Fig. 3, where it is clear that $|\mathbf{u}_i| \leq \mathbf{u}_{\max}$, and consensus is achieved without velocity measurements under the directed communication graph $\mathcal{G}_4$.

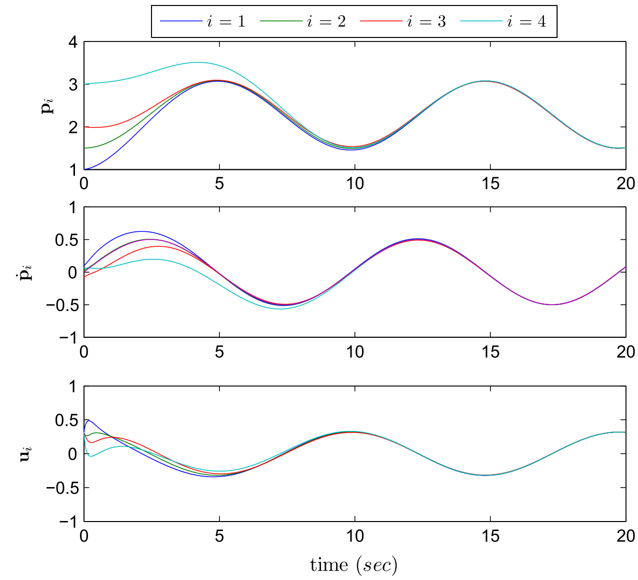

Fig. 3. Simulation results for Example 2.

6. Conclusion

New consensus algorithms have been developed for second-order linear multi-agent systems in the case where the inputs are saturated and the velocity states are not available for feedback. Our approach is based on the introduction of dynamic auxiliary systems to alter the trajectories of agents before reaching consensus. We presented in Theorem 1 a first method that only requires the design of a consensus protocol for a second-order multi-agent system in ideal situations. While this result can be applied for the general class of second-order consensus problems, we presented in Theorem 2 a second design method that can also be applied for the class of consensus problems that set the final velocities of agents. The latter result simplifies further the consensus algorithm design since it requires only the design of a consensus protocol for a multi-agent system with single-integrator dynamics in ideal situations. Each of the above design methods relies on some conditions that can be satisfied with an appropriate design of the introduced auxiliary systems.

Using the proposed methods, we developed solutions to two different second-order consensus problems and simulation results have been provided to validate our theoretical results. From these examples, it can be seen that our approach can provide solutions to several second-order consensus problems in the case of saturated inputs and without velocity measurements. As a future work, it will be interesting to extend the presented results to higher order linear multi-agent systems.

Acknowledgments

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Appendix A. Proof of Lemma 2

Consider the Lyapunov function candidate: $W = \frac{1}{2}\boldsymbol{\delta}^\top\boldsymbol{\delta}$, with its time derivative obtained as

$$\dot{W} = \boldsymbol{\delta}^\top\dot{\boldsymbol{\delta}} = -\boldsymbol{\delta}^\top\left(L^p\boldsymbol{\chi}(\boldsymbol{\delta}) - \bar{\boldsymbol{\varepsilon}}\right)$$

$$\leq -\sum_{k=1}^{m}|\delta_k|\left(L^p\sigma(|\delta_k|) - |\bar{\varepsilon}_k|\right), \quad \text{(A.1)}$$

where $\boldsymbol{\delta} = \mathrm{col}[\delta_k]$ and $\bar{\boldsymbol{\varepsilon}} = \mathrm{col}[\bar{\varepsilon}_k]$, for $k \in \{1, \ldots, m\}$, and we have used the property; $x\sigma(x) = |x|\sigma(|x|)$, for any $x \in \mathbb{R}$, to obtain the last inequality. Note that $\dot{W} \leq \|\boldsymbol{\delta}\|\|\bar{\boldsymbol{\varepsilon}}\|$, and using the fact that $\|\boldsymbol{\delta}\|^2 \leq 2W$, we can write $\dot{W} \leq \boldsymbol{\epsilon}\sqrt{W}$, with $\sqrt{2}\|\bar{\boldsymbol{\varepsilon}}\| \leq \boldsymbol{\epsilon}$, which can be rewritten as $\frac{dW}{\sqrt{W}} \leq \boldsymbol{\epsilon}dt$. Integrating this last inequality over the interval $[t_0, t]$ yields: $2\left(\sqrt{W(t)} - \sqrt{W(t_0)}\right) \leq \boldsymbol{\epsilon}(t-t_0)$, which shows that $\boldsymbol{\delta}$ cannot go to infinity in finite time.

Now, since the function $\sigma(\cdot)$ is bounded, it is easy to verify that the right hand side of inequality (A.1) is positive when $|\bar{\varepsilon}_k| > \sigma_b L^p$. However, since $\bar{\boldsymbol{\varepsilon}}$ is bounded and converges asymptotically to zero, it is clear that there exists a finite time t_1 such that $|\bar{\varepsilon}_k(t)| \leq \sigma_b L^p$ for all $t \geq t_1$. Note that $\boldsymbol{\delta}$ remains bounded on the interval $[0, t_1]$ as there is no finite-escape time. Consequently, for all $t \geq t_1$, one can conclude that the right hand side of (A.1) is negative outside the set $\mathcal{S} = \left\{\boldsymbol{\delta} \mid \sigma(|\delta_k|) \leq \frac{|\bar{\varepsilon}_k|}{L^p}, \text{ for } k = 1, \ldots, m\right\}$. Also, we can conclude that $\boldsymbol{\delta}$ is bounded outside the set $\mathcal{S}$. Since $\sigma(|\cdot|)$ is a class $\mathcal{K}$ function, $\boldsymbol{\delta}$ is ultimately bound to reach the set $\mathcal{S}$ and will be driven to zero as $\bar{\boldsymbol{\varepsilon}} \to 0$. Consequently, we conclude that $\dot{\boldsymbol{\delta}} \to \boldsymbol{\delta} \to 0$.

Appendix B. Proof of Theorem 1

First, using the properties of the function $\boldsymbol{\chi}$ in (3) we can verify that the input law in (7) can be upper bounded independently from the states as: $\|\mathbf{u}_i\|_\infty \leq \boldsymbol{f}_{\max} + \sigma_b(L_i^p + L_i^d)$. Therefore, if $\boldsymbol{f}_{\max} < \mathbf{u}_{\max}$ and the control gains are selected according to (8), the upper bound of the control input given in the theorem is obtained.

Define the vector $\bar{\mathbf{e}}_i := (\mathbf{r}_i - \bar{\mathbf{r}}_i)$ and let $\boldsymbol{\xi}_i = (\bar{\mathbf{e}}_i^\top, \dot{\bar{\mathbf{e}}}_i^\top, \bar{\mathbf{e}}_i^\top - \boldsymbol{\psi}_i^\top)^\top$, which, in view of (1) and (4)–(5), is governed by:

$$\dot{\boldsymbol{\xi}}_i = (\mathbf{A}_i \otimes \mathbf{I}_m)\,\boldsymbol{\xi}_i, \quad \text{for } i \in \mathcal{N}, \tag{B.1}$$

where $\otimes$ is the Kronecker product, $\mathbf{I}_m \in \mathbb{R}^{m\times m}$ is the identity matrix, and

$$\mathbf{A}_i = \begin{pmatrix} 0 & 1 & 0 \\ -k_i^p & 0 & -k_i^d \\ 0 & 1 & -k_i^\psi \end{pmatrix}.$$

After simple computations, we can verify that the matrix $\mathbf{A}_i$ is Hurwitz for any strictly positive gains k_i^p, k_i^d and k_i^ψ. Therefore, we conclude that $\boldsymbol{\xi}_i \to 0$ as $t \to \infty$ exponentially. Also, Design condition 1 guarantees that $(\bar{\mathbf{r}}_i - \bar{\mathbf{r}}_j) \to 0$, $(\dot{\bar{\mathbf{r}}}_i - \dot{\bar{\mathbf{r}}}_j) \to 0$, for $i, j \in \mathcal{N}$. In addition, the dynamics of the auxiliary system (4) with (7) can be written as: $\ddot{\boldsymbol{\zeta}}_i = -L_i^p\boldsymbol{\chi}(\boldsymbol{\zeta}_i) - L_i^d\boldsymbol{\chi}(\dot{\boldsymbol{\zeta}}_i) + \boldsymbol{\varepsilon}_i$, with $\boldsymbol{\varepsilon}_i = \left(-\bar{\boldsymbol{\Psi}}_{i,\mathcal{G}_n}(\bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}}) + k_i^p\bar{\mathbf{e}}_i + k_i^d(\bar{\mathbf{e}}_i - \boldsymbol{\psi}_i)\right)$. The exponential convergence of $\boldsymbol{\xi}_i$ and Design condition 2 guarantee that $\boldsymbol{\varepsilon}_i$ is globally bounded and $\boldsymbol{\varepsilon}_i \to 0$, for $i \in \mathcal{N}$. Invoking Lemma 1 leads us to conclude that $\boldsymbol{\zeta}_i$, $\dot{\boldsymbol{\zeta}}_i$ are globally bounded and $\boldsymbol{\zeta}_i \to 0$, $\dot{\boldsymbol{\zeta}}_i \to 0$ for $i \in \mathcal{N}$. Finally, from the definition of $\boldsymbol{\xi}_i$ with $\mathbf{r}_i = (\mathbf{p}_i - \boldsymbol{\zeta}_i)$, we conclude that the multi-agent system (1) achieves second-order consensus in the sense of (2).

Appendix C. Proof of Theorem 2

First, note that the control input (20) can be bounded independently from the states as: $\|\mathbf{u}_i\|_\infty \leq \mathbf{f}_{\max} + \sigma_b(k_i^p + k_i^d + \sigma_h L_i^p(L_i^p + L_i^d))$, where we have used (17) with properties P2 and P3. Therefore, if the control gains satisfy (22), then $\|\mathbf{u}_i\|_\infty \leq \mathbf{u}_{\max}$. Now, we can see from (16) and property P3 that $\ddot{\bar{\mathbf{r}}}_i = (\ddot{\mathbf{p}}_d - L_i^p\mathbf{h}(\boldsymbol{\zeta}_i)\dot{\boldsymbol{\zeta}}_i)$. Hence, the dynamics of the error vector $\mathbf{e}_i := (\mathbf{p}_i - \bar{\mathbf{r}}_i)$ can be obtained from (1) and (20) as:

$$\ddot{\mathbf{e}}_i = -k_i^p\boldsymbol{\chi}(\mathbf{e}_i) - k_i^d\boldsymbol{\chi}(\mathbf{e}_i - \boldsymbol{\psi}_i), \tag{C.1}$$

with $\dot{\boldsymbol{\psi}}_i$ given in (21). Consider the following Lyapunov function candidate

$$V = \sum_{i=1}^{n}\left(\frac{1}{2}\dot{\mathbf{e}}_i^T\dot{\mathbf{e}}_i + k_i^p\sum_{k=1}^{m}\int_0^{e_{i,k}}\sigma(s)ds\right) + \sum_{i=1}^{n}k_i^d\sum_{k=1}^{m}\int_0^{(e_{i,k}-\psi_{i,k})}\sigma(s)ds, \tag{C.2}$$

with $\mathbf{e}_i = \mathrm{col}[e_{i,k}]$ and $\boldsymbol{\psi}_i = \mathrm{col}[\psi_{i,k}]$, for $k \in \{1, \ldots, m\}$, and σ is the scalar function defined in (3). Note that V in (C.2) can be verified to be radially unbounded from the definition of σ. The time-derivative of V evaluated along the dynamics (C.1) is obtained as:

$$\dot{V} = -\sum_{i=1}^{n}k_i^dk_i^\psi(\mathbf{e}_i - \boldsymbol{\psi}_i)^T\boldsymbol{\chi}(\mathbf{e}_i - \boldsymbol{\psi}_i), \tag{C.3}$$

which is negative semi-definite, and we conclude that $\dot{\mathbf{e}}_i$, $\mathbf{e}_i$, $\dot{\boldsymbol{\psi}}_i$ and $\boldsymbol{\psi}_i$ are globally bounded. This, with property P3, leads us to conclude that $\ddot{V}$ is bounded. Invoking Barbălat Lemma, we conclude that $\dot{\boldsymbol{\psi}}_i = k_i^\psi(\mathbf{e}_i - \boldsymbol{\psi}_i) \to 0$. Furthermore, since $(\ddot{\mathbf{e}}_i - \ddot{\boldsymbol{\psi}}_i)$ is bounded, we conclude by Barbălat Lemma that $(\dot{\mathbf{e}}_i - \dot{\boldsymbol{\psi}}_i) \to 0$, and hence $\dot{\mathbf{e}}_i \to 0$. In addition, using property P3 and the above results, we can verify that $\dddot{\mathbf{e}}_i$ is bounded. Invoking Barbălat Lemma again, we conclude that $\ddot{\mathbf{e}}_i \to 0$, and hence $\mathbf{e}_i \to 0$, for $i \in \mathcal{N}$.

On the other hand, we can see that Design condition 3 guarantees that the first-order multi-agent system (19) achieves consensus, i.e., $(\mathbf{r}_i - \mathbf{r}_j) \to 0$, $\dot{\mathbf{r}}_i \to \dot{\mathbf{p}}_d$, for all $i, j \in \mathcal{N}$. In addition, the dynamics of the vector $\boldsymbol{\delta}_i$ in (18) can be written as: $\dot{\boldsymbol{\delta}}_i = -L_i^d\boldsymbol{\chi}(\boldsymbol{\delta}_i) + \bar{\boldsymbol{\varepsilon}}_i$, with $\bar{\boldsymbol{\varepsilon}}_i := -\bar{\boldsymbol{\Phi}}_{i,\mathcal{G}_n}(\mathbf{r})$. Note that Design condition 4 ensures that $\bar{\boldsymbol{\varepsilon}}_i$ is globally bounded and $\bar{\boldsymbol{\varepsilon}}_i \to 0$, for $i \in \mathcal{N}$. Therefore, the result of Lemma 2 leads us to conclude that $\boldsymbol{\delta}_i \to 0$, $\dot{\boldsymbol{\delta}}_i \to 0$, for $i \in \mathcal{N}$. As a result, the dynamics of the vector $\boldsymbol{\zeta}_i$ in (17) can be written as: $\dot{\boldsymbol{\zeta}}_i = -L_i^p\boldsymbol{\chi}(\boldsymbol{\zeta}_i) + \tilde{\boldsymbol{\varepsilon}}_i$, with $\tilde{\boldsymbol{\varepsilon}}_i := L_i^d\boldsymbol{\chi}(\boldsymbol{\delta}_i) \to 0$. Using Lemma 2 again, we conclude that $\dot{\boldsymbol{\zeta}}_i \to 0$ and $\boldsymbol{\zeta}_i \to 0$ for $i \in \mathcal{N}$.

Finally, exploiting the fact that $\mathbf{e}_i \to 0$, $\dot{\mathbf{e}}_i \to 0$, $\dot{\mathbf{r}}_i \to \dot{\mathbf{p}}_d$, and $(\mathbf{r}_i - \mathbf{r}_j) \to 0$, for all $i, j \in \mathcal{N}$, with $\mathbf{e}_i := (\mathbf{p}_i - \bar{\mathbf{r}}_i)$ and $\mathbf{r}_i = (\bar{\mathbf{r}}_i - \boldsymbol{\zeta}_i - \boldsymbol{\delta}_i)$, we conclude that $(\mathbf{p}_i - \mathbf{p}_j) \to 0$, $\dot{\mathbf{p}}_i \to \dot{\mathbf{p}}_d$, for all $i, j \in \mathcal{N}$.

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