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ON SYSTEMS OF ELLIPTIC EQUATIONS INVOLVING SUBCRITICAL OR CRITICAL SOBOLEV EXPONENTS*

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1 Introduction

Let $\Omega \subset I\!R^N$, $N \geq 3$ be bounded a domain with smooth boundary $\partial\Omega$. Depending on the parameters $a, b, c \in I\!R$; $\alpha, \beta > 1$, we shall investigate the existence and non-existence of solutions for the following system of elliptic equations

$$-\Delta u = au + bv + \frac{2\alpha}{\alpha+\beta} u|u|^{\alpha-2}|v|^{\beta}, \quad \Omega \tag{1}$$

$$-\Delta v = bu + cv + \frac{2\beta}{\alpha+\beta} |u|^{\alpha} v|v|^{\beta-2}, \quad \Omega \tag{2}$$

$$u = v = 0, \ \partial\Omega \tag{3}$$

$$u, v > 0, \ \Omega. \tag{4}$$

For this purpose, the sum $\alpha + \beta$ will be compared with the critical Sobolev exponent $2^* \equiv \frac{2N}{N-2}$ of the embedding

$$H_o^1(\Omega) \hookrightarrow L^p(\Omega). \tag{5}$$

The positive first eigenvalue of the eigenvalue problem $(-\Delta, H_o^1(\Omega))$ will be denoted by λ_1 and the respective associated eigenfunction $\phi_1$will be taken positive on Ω.

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Denoting $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in M_{2\times 2}(I\!R)$, $U = (u,v)$ and $-\overrightarrow{\Delta} U = \begin{pmatrix} -\Delta u \\ -\Delta v \end{pmatrix}$, the above system may be rewritten in the form

$$P^{+}_{(\alpha,\beta),A} : \begin{cases} -\Delta U = \nabla\left(\frac{1}{2}(AU,U)_{I\!R^2} + F(U)\right), & \Omega \\ U = 0, \partial\Omega \\ U > 0, \Omega \end{cases}$$

where $(.,.)_{I\!R^2}$ denotes the usual inner product in $I\!R^2$, $U > 0$ states (4) and $F(U) = \frac{2}{\alpha+\beta}|u|^{\alpha}|v|^{\beta}$. The sign + stands for assertion (4).

As the system is in the gradient form we will use variational methods to solve it. We shall work with the space $(H^1_o)^2 :\equiv H^1_o(\Omega) \times H^1_o(\Omega)$ endowed with the norm $||(u,v)||^2_{(H^1_o)^2} \equiv ||u||^2_{H^1_o(\Omega)} + ||u||^2_{H^1_o(\Omega)}$.

The real eigenvalues of the matrix A will be denoted by μ_1 and μ_2 and we shall assume that $\mu_1 \leq \mu_{2.}$.

Our main result is the following theorem

Theorem 1 *Let Ω be a bounded domain and suppose that the following assumptions hold*

$$\alpha + \beta = 2^* \tag{6}$$

$$b \geq 0. \tag{7}$$

Then

(i) if $N \geq 4$ and

$$0 < \mu_1 \leq \mu_2 < \lambda_1, \tag{8}$$

system $P^{+}_{(\alpha,\beta),A}$ has a solution;

(ii) if $N = 3$ and Ω is a ball,

$$\text{system } P^{+}_{(\alpha,\beta),A} \text{ has a solution if } \frac{\lambda_1}{4} < \mu_1 \leq \mu_2 < \lambda_1 \tag{9}$$

$$\text{and } P^{+}_{(\alpha,\beta),A} \text{ has no solution if } 0 < \mu_1 \leq \mu_2 < \frac{\lambda_1}{4}. \tag{10}$$

Remark 1 *The minimum and the maximum of the quadratic form $(AZ,Z)_{I\!R^N}$, $Z \in I\!R^N$ restricted to the unity sphere are μ_1 and μ_2 respectively, and we have that*

$$\mu_1|Z|^2 \leq (AZ,Z)_{I\!R^2} \leq \mu_2|Z|^2, \; Z \in I\!R^2. \tag{11}$$

Remark 2 *Considering $u = v$, $b = 0$, $a = c$ and $\alpha = \beta$ in the equations (1) and (2), we obtain the standard scalar case of this type of problems.*

This is the case treated in the well known paper [1]. The main difficulties faced in dealing with this problem with $2\alpha = 2^*$ or $\Omega = I\!R^N$, is the lack of compactness of the embedding (5). Many others results in the scalar case for this kind of problems, and also for the $p-$Laplacean operator $\Delta_p u = div(\nabla u|u|^{p-2})$, have appeared after [1]. For instance, [5], [6], [7],[8]. Theorem 1 is an extension of some results of the scalar case ([1]) to a system of elliptic equations.

Remark 3 *A weak solution for $P_{(\alpha,\beta),A}$ is a vector $U = (u,v) \in (H_o^1)^2$ such that*

$$\int_\Omega \nabla u \nabla \varphi dx + \int_\Omega \nabla v \nabla \zeta dx - \int_\Omega au\varphi dx - \int_\Omega bv\varphi dx - \int_\Omega bu\zeta dx - \int_\Omega cv\zeta dx -$$

$$-\frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}|v|^\beta u\varphi dx - \frac{2\beta}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^{\beta-2} v\zeta dx \ = 0, \quad \forall\ (\varphi,\zeta) \in (H_o^1)^2.$$

Remark 4 *By a standard bootstrap argument it is proved that a weak solution for $P_{(\alpha,\beta),A}$ is in the space $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$, and it is really a classical solution. This fact will be assumed throughout the work. We shall find weak solutions of the system.*

The proof Theorem 1 will be postponed until the last section. However, in the coming sections we will make a detailed investigation on problem $P_{(\alpha,\beta),A}$ under different sets of hypotheses. In Section 1 we prove some non-existence results for $P_{(\alpha,\beta),A}$, in Section 3 we study the subcritical case. Section 4 will be devoted to some relations which allow us to take advantage of some estimates already made in the scalar case, and in Section 5, we prove Theorem 1.

2 Non-existence results

Let us prove the following theorem

Theorem 2 *If the assertions below hold:*

$$\Omega \text{ is star shapped with respect to } 0 \tag{12}$$

$$\mu_2 \leq 0 \text{ and} \tag{13}$$

$$\alpha + \beta \geq 2^* \equiv \frac{2N}{N-2}, \tag{14}$$

then system $P_{(\alpha,\beta),A}$ has no solutions, except the trivial one.

The proof of the above Theorem is made using the following type of Pohozaev Identity adapted for systems

Lemma 1 *If $F \in C^1(I\!R^2, I\!R)$ with $F(s,z) = 0$ if $s = 0$ and $z = 0$, and $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ are solutions of the system*

$$\begin{cases} -\Delta u = F_s(u,v), \Omega \\ -\Delta v = F_z(u,v), \Omega \\ u = v = 0, \partial\Omega \end{cases}$$

then the identity holds

$$\int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx - 2^* \int_\Omega F(u,v) dx +$$

$$\frac{1}{N-2} \int\limits_{\partial\Omega} (|\frac{\partial u}{\partial \nu}|^2 + |\frac{\partial v}{\partial \nu}|^2)(x,\nu)_{I\!R^N} d\sigma = 0, \tag{15}$$

where $\frac{\partial u}{\partial \nu}(x)$ is the outward normal derivative exterior to $\partial\Omega$ at the point $x \in \partial\Omega$.

The Pohozaev Identity in the scalar case first appeared in [2] and has had various sorts of extensions since that time. The above formulation for systems may be immediately proved by an adaptation of the proof of the scalar case which may be seen in [3].

Proof of Theorem 2: Multiplying (1) and (2) by u and v respectively, integrating by parts and adding the resulting expressions we find

$$\int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx = \int_\Omega (AU, U)_{I\!R^2} dx + 2 \int_\Omega |u|^\alpha |v|^\beta dx. \tag{16}$$

On the other hand,

$$F(u,v) = \frac{1}{2}(AU,U)_{I\!R^2} + \frac{2}{\alpha+\beta}|u|^\alpha |v|^\beta. \tag{17}$$

Substituting (16) and (17) in (15) we achieve

$$\left(1 - \frac{2^*}{2}\right) \int_\Omega (AU,U)_{I\!R^2} dx + 2\left[1 - \frac{2^*}{\alpha+\beta}\right] \int_\Omega |u|^\alpha |v|^\beta dx +$$

$$\frac{1}{N-2} \int\limits_{\partial\Omega} (|\frac{\partial u}{\partial \nu}|^2 + |\frac{\partial v}{\partial \nu}|^2)(x,\nu)_{I\!R^N} d\sigma = 0. \tag{18}$$

First Case: $\alpha + \beta = 2^*$.

From (18) it follows that

$$\left(1 - \frac{2^*}{2}\right) \int_\Omega (AU,U)_{I\!R^2} dx + \frac{1}{N-2} \int_\Omega (|\frac{\partial u}{\partial \nu}|^2 + |\frac{\partial v}{\partial \nu}|^2)(x,\nu)_{I\!R^N} d\sigma = 0$$

and then $\int_\Omega (AU,U)_{I\!R^2} dx \geq 0$.

If $\mu_2 < 0$, (17) implies that $U = 0$.

On the other hand if $\mu_2 = 0$, then a, $c \leq 0$ and

$$b^2 = ac. \tag{19}$$

So, the interesting case is when a, $c < 0$. We already know that

$$\int_\Omega (AU, U)_{I\!R^2} dx = a||u||^2 + 2b(u, v) + c||v||^2 = 0, \tag{20}$$

where $||.||$ and $(.,.)$ denote, respectively, the norm and the inner product of $L^2(\Omega)$. This last equation implies that

$$a||u||^2 + 2|b|\ ||u||\ ||v|| + c||v||^2 \geq 0. \tag{21}$$

If $u, v \equiv 0$ the proof is finished. Suppose that the opposite occurs. With no loss of generality we may assume that $v \not\equiv 0$. Assertion (19) imply that strict inequality in (21) can not occur, this fact together with (20) yield that there exists $\delta \in I\!R$ such that

$$u = \delta v \tag{22}$$

since equality holds in the Cauchy-Schwartz inequality. Returning to (20) and using (22) we see that δ is the unique root of the equation

$$a\delta^2 + 2b\delta + c = 0$$

and then $\delta = -\frac{b}{a}$. Using the relation $u = \delta v$ in (2) we obtain that

$$\begin{aligned} -\Delta v &= \frac{2\beta}{2^*}|\delta|^\alpha v|v|^{2^*-2},\ \Omega \\ v &= 0,\ \partial\Omega. \end{aligned}$$

The argument in the scalar case used in [2] implies that $v \equiv 0$, which contradicts the above assertion. Hence $u \equiv v \equiv 0$.

Second Case: $\alpha + \beta > 2^*$.

As by (13) (more precisely, by (11)) $(AU, U)_{I\!R^2} \leq 0$, it follows from (18) that

$$2\left[1 - \frac{2^*}{\alpha + \beta}\right] \int_\Omega |u|^\alpha |v|^\beta dx \leq 0,$$

and hence that $\int_\Omega |u|^\alpha |v|^\beta dx = 0$. Using this fact in (16) together with (13) we obtain that

$$||(u, v)||^2_{(H^1_o)^2} = \int_\Omega (AU, U)_{I\!R^2} dx \leq 0$$

and therefore that $U = 0$. $\diamondsuit$

Theorem 3 *Suppose that*

$$b \geq 0 \text{ and } \mu_2 \geq \lambda_1 \tag{23}$$

or

$$b \leq 0 \text{ and } \mu_1 \leq \lambda_1. \tag{24}$$

Then problem $P^+_{(\alpha,\beta),A}$ has no solution.

Proof: Suppose that (u, v) is a solution for $P^+_{(\alpha,\beta),A}$. We may always assume that the eigenvector $X = (\mathrm{x}, y)$ associated to μ_2 is non-negative with $\mathrm{x} > 0$ or $y > 0$.

Multiplying equations (1) and (2) by $\mathrm{x}\phi_1$ and $y\phi_1$,respectively, we achieve that

$$(-\overrightarrow{\Delta}U, \phi_1 X)_{I\!R^2} = (AU, \phi_1 X)_{I\!R^2} +$$

$$\frac{2\alpha \mathrm{x}}{\alpha+\beta}\phi_1 u^{\alpha-1}v^{\beta} + \frac{2\beta y}{\alpha+\beta}\phi_1\ u^{\alpha}v^{\beta-1}. \tag{25}$$

Integrating by parts the left-hand side of the above expression and using the symmetry of the matrix A in the right-hand side of it, we obtain

$$(\lambda_1 - \mu_2)\int_\Omega (u\phi_1 \mathrm{x} + v\phi_1 y)dx =$$

$$\int_\Omega \left(\frac{2\alpha \mathrm{x}}{\alpha+\beta}\phi_1 u^{\alpha-1}v^{\beta} + \frac{2\beta y}{\alpha+\beta}\phi_1\ u^{\alpha}v^{\beta-1}\right)dx.$$

Therefore we have that $\lambda_1 > \mu_2$. The other case uses similar argument. $\diamondsuit$

3 Existence in the subcritical case

The Lemma below is essential in what follows .

Lemma 2 *If $\alpha + \beta \le 2^*$, then there exists a positive constant c such that*

$$\left(\int_\Omega |u|^\alpha |v|^\beta dx\right)^{\frac{1}{\alpha+\beta}} \le c||(u,v)||_{(H^1_o)^2}. \tag{26}$$

Proof: The proof follows from the definition

$$S_{\alpha+\beta}(\Omega) = \inf_{u\in H^1_o(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{\alpha+\beta}dx\right)^{\frac{2}{\alpha+\beta}}}$$

and the inequality $|u|^\alpha |v|^\beta \le |u|^{\alpha+\beta} + |v|^{\alpha+\beta}$. $\diamondsuit$

In this section let us investigate the existence of solutions for $P^+_{(\alpha,\beta),A}$ in the subcritical case, namely $\alpha + \beta < 2^*$.

Theorem 4 *Let Ω be a bounded domain. Suppose that*

$$b \ge 0 \tag{27}$$

$$\mu_2 < \lambda_1 \tag{28}$$

$$\alpha + \beta < 2^* \tag{29}$$

then system $P^+_{(\alpha,\beta),A}$ has a solution.

Proof: The idea is minimizing the functional

$$\begin{array}{rcl} I & : & (H_o^1)^2 \to I\!R \\ (u,v) & \longmapsto & I(u,v) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx - \frac{1}{2}\int_\Omega (AU,U)_{I\!R^2} dx \end{array}$$

with the constraint

$$M = \{(u,v) \in (H_o^1)^2 : \int_\Omega u_+^\alpha v_+^\beta dx = 1\},$$

and then use the Lagrange Multiplier Theorem and the homogeneity of the problem $P^+_{(\alpha,\beta),A}$ to find a solution for it.

Let

$$\inf_M I \equiv I_o$$

and $(u_n, v_n) \in M$ be a minimizing sequence for I_o. Hence

$$I(u_n, v_n) = I_o + o_n(1) \leq C, \tag{30}$$

for some $C > 0$ and $o_n(1) \to 0$ as $n \to \infty$.

On the other hand, by Poincaré inequality and (11) we achieve

$$I(u,v) \geq \frac{1}{2}\min\{1, \left(1 - \frac{\mu_2}{\lambda_1}\right)\}||(u,v)||^2_{(H_o^1)^2}. \tag{31}$$

Therefore, (30) and (31) imply that $||(u_n,v_n)||^2_{(H_o^1)^2} \leq C'$ and hence there exist subsequences u_n and v_n such that $u_n \rightharpoonup u_o$ and $v_n \rightharpoonup v_o$ in $H_o^1(\Omega)$ and the convergences still hold a.e. on Ω and in $L^{\alpha+\beta}$ (Ω) for $\alpha + \beta < 2^*$. Using these facts, we see that $(u_o, v_o) \in M$ and also that we may pass to the limit in (30) to get that $I(u_o, v_o) \leq I_o$, and therefore conclude that $I(u_o, v_o) = I_o$.

This way, if $G(u,v) = \int_\Omega u_+^\alpha v_+^\beta dx - 1$, there is a Lagrange multiplier η such that

$$I'(u_o,v_o)(\varphi,\zeta) - \eta G'(u_o,v_o)(\varphi,\zeta) = 0, \ \ \forall(\varphi,\zeta) \in (H_o^1)^2. \tag{32}$$

(Here, I' and G' are the Frechet derivatives of I and G, respectively).

Taking $(\varphi,\zeta) = (u_o^-, v_o^-)$ in (32) we obtain

$$I(u_o^-, v_o^-) - \int_\Omega (bv_o^+ u_o^- + bv_o^- u_o^+)dx = 0$$

and by (27) conclude that $I(u_o^-, v_o^-) \leq 0$ and then that $(u_o^-, v_o^-) = 0$. So, we conclude that $(u_o, v_o) \geq 0$.

Using (32) we see that $I(u_o, v_o) = \eta(\alpha+\beta)\int_\Omega u_o^\alpha v_o^\beta dx = \eta(\alpha+\beta)$,and since $I(u_o, v_o) > 0$ we get that $\eta > 0$. Equation (32) means that (u_o, v_o) is a weak solution of the problem $P_{(\frac{\alpha}{\eta},\frac{\beta}{\eta}),A}$. So, using the homogeneity of the system we find that $\left[\frac{\eta(\alpha+\beta)}{2}\right]^{\frac{1}{\alpha+\beta-2}}(u_o, v_o)$ is a weak solution for $P_{(\alpha,\beta),A}$. Assertions (27) and (28) allow us to use the Strong Maximum Principle in the scalar case for both equations, to conclude the positivity of this solution. $\diamondsuit$

4 An important relation

In order to prove Theorem 1, we shall make use of the following infima

$$S_{\alpha+\beta}(\Omega) = \inf_{u\in H_o^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{\alpha+\beta} dx\right)^{\frac{2}{\alpha+\beta}}} \tag{33}$$

and

$$\widetilde{S}_{(\alpha,\beta)}(\Omega) = \inf_{(u,v)\in (H_o^1)^2\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx}{\left(\int_\Omega |u|^\alpha |v|^\beta dx\right)^{\frac{2}{\alpha+\beta}}}. \tag{34}$$

The infimum in (34)is well-defined by Lemma 2.

Theorem 5 *Let Ω be a domain (not necessarily bounded) and $\alpha+\beta \leq 2^*$. Then we have*

$$\widetilde{S}_{(\alpha,\beta)}(\Omega) = \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}\right] S_{\alpha+\beta}(\Omega). \tag{35}$$

Moreover, if ω_o realizes $S_{\alpha+\beta}(\Omega)$ then $u_o = B\omega_o$ and $v_o = C\omega_o$ realizes $\widetilde{S}_{(\alpha,\beta)}(\Omega)$ for any real constants B and C such that $\frac{B}{C} = \sqrt{\frac{\alpha}{\beta}}$.

Proof: Consider ω_n a minimizing sequence for $S_{\alpha+\beta}(\Omega)$. Let $s,t>0$ to be chosen later. Taking $u_n = s\omega_n$ and $v_n = t\omega_n$ in the quotient (34) we have that

$$\frac{s^2+t^2}{(s^\alpha t^\beta)^{\frac{2}{\alpha+\beta}}} \frac{\int_\Omega |\nabla \omega_n|^2 dx}{\left(\int_\Omega |\omega_n|^{\alpha+\beta} dx\right)^{\frac{2}{\alpha+\beta}}} \geq \widetilde{S}_{(\alpha,\beta)}(\Omega). \tag{36}$$

Observe that

$$\frac{s^2+t^2}{(s^\alpha t^\beta)^{\frac{2}{\alpha+\beta}}} = \left(\frac{s}{t}\right)^{\frac{2\beta}{\alpha+\beta}} + \left(\frac{s}{t}\right)^{\frac{-2\alpha}{\alpha+\beta}} \tag{37}$$

and define the function

$$g(x) = x^{\frac{2\beta}{\alpha+\beta}} + x^{\frac{-2\alpha}{\alpha+\beta}}, \qquad x>0.$$

The minimum of the function g is assumed at the point $x = \sqrt{\frac{\alpha}{\beta}}$ with minimum value

$$g\left(\sqrt{\frac{\alpha}{\beta}}\right) = \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}. \tag{38}$$

Choosing s and t in (36) such that $\frac{s}{t} = \sqrt{\frac{\alpha}{\beta}}$ we have that

$$\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}} \frac{\int_\Omega |\nabla \omega_n|^2 dx}{\left(\int_\Omega |\omega_n|^{\alpha+\beta} dx\right)^{\frac{2}{\alpha+\beta}}} \geq \widetilde{S}_{(\alpha,\beta)}(\Omega)$$

and hence that

$$\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}+\left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}S_{\alpha+\beta}(\Omega)\geq\widetilde{S}_{(\alpha,\beta)}(\Omega). \tag{39}$$

To complete the proof, let (u_n,v_n) be a minimizing sequence for $\widetilde{S}_{(\alpha,\beta)}(\Omega)$. Define $z_n=s_nv_n$ for some $s_n>0$ such that

$$\int_\Omega |u_n|^{\alpha+\beta}dx=\int_\Omega |z_n|^{\alpha+\beta}dx. \tag{40}$$

By Young's Inequality

$$\int_\Omega |u_n|^\alpha|z_n|^\beta dx\leq\frac{\alpha}{\alpha+\beta}\int_\Omega |u_n|^{\alpha+\beta}dx+\frac{\beta}{\alpha+\beta}\int_\Omega |z_n|^{\alpha+\beta}dx$$

and by (40) we have that

$$\left(\int_\Omega |u_n|^\alpha|z_n|^\beta dx\right)^{\frac{2}{\alpha+\beta}}\leq\left(\int_\Omega |u_n|^{\alpha+\beta}dx\right)^{\frac{2}{\alpha+\beta}}=\left(\int_\Omega |z_n|^{\alpha+\beta}dx\right)^{\frac{2}{\alpha+\beta}}. \tag{41}$$

In this way, using (41) we have that

$$\frac{\int_\Omega(|\nabla u_n|^2+|\nabla v_n|^2)dx}{\left(\int_\Omega |u_n|^\alpha|v_n|^\beta dx\right)^{\frac{2}{\alpha+\beta}}}=\frac{s_n^{\frac{2\beta}{\alpha+\beta}}\int_\Omega(|\nabla u_n|^2+|\nabla v_n|^2)dx}{\left(\int_\Omega |u_n|^\alpha|z_n|^\beta dx\right)^{\frac{2}{\alpha+\beta}}}\geq$$

$$s_n^{\frac{2\beta}{\alpha+\beta}}\frac{\int_\Omega|\nabla u_n|^2dx}{\left(\int_\Omega |u_n|^{\alpha+\beta}dx\right)^{\frac{2}{\alpha+\beta}}}+s_n^{\frac{2\beta}{\alpha+\beta}}s_n^{-2}\frac{\int_\Omega|\nabla z_n|^2dx}{\left(\int_\Omega |z_n|^{\alpha+\beta}dx\right)^{\frac{2}{\alpha+\beta}}}\geq$$

$$g(s_n)S_{\alpha+\beta}(\Omega),$$

and hence that

$$\frac{\int_\Omega(|\nabla u_n|^2+|\nabla v_n|^2)dx}{\left(\int_\Omega |u_n|^\alpha|v_n|^\beta dx\right)^{\frac{2}{\alpha+\beta}}}\geq g\left(\sqrt{\frac{\alpha}{\beta}}\right)S_{\alpha+\beta}(\Omega).$$

Passing to the limit in the last inequality we obtain

$$\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}+\left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}S_{\alpha+\beta}(\Omega)\leq\widetilde{S}_{\alpha+\beta}(\Omega).$$

$\diamondsuit$

Remark 5 *(i)* $S_{2^*}(I\!R^N)$ *is achieved by the family of functions*

$$v_\varepsilon(x)=\frac{\left[N(N-2)\varepsilon^2\right]^{\frac{N-2}{4}}}{\left[\varepsilon^2+|x|^2\right]^{\frac{N-2}{2}}}. \tag{42}$$

See [3] for details.

(ii) The above Theorem proves that the constant $\widetilde{S}_{(\alpha,\beta)}(I\!\!R^N)$ is achieved when $\alpha+\beta=2^*$, since $S_{2^*}(I\!\!R^N)$ is achieved in this case.

(iii) Talenti in [4] computed that

$$S_{2^*}(I\!\!R^N) = 2\pi(N-2)\left[\frac{\Gamma(N/2)}{\Gamma(N)}\right]^{2/N}.$$

(iii) If Ω is bounded then $S_{2^*}(\Omega)=S_{2^*}(I\!\!R^N)$. See [3] for details.

5 Proof of Theorem 1

We shall make use of some ideas from [1]. For this purpose we need some definitions. Let

$$Q_\lambda(u) = \frac{\int_\Omega(|\nabla u|^2-\lambda|u|^2)dx}{\left(\int_\Omega |u|^{2^*}dx\right)^{\frac{2}{2^*}}},\ \ u\in H_o^1(\Omega)\backslash\{0\}, \tag{43}$$

$$\widetilde{Q}_{(\alpha,\beta),A}(u,v) = \frac{\int_\Omega(|\nabla u|^2+|\nabla v|^2)dx - \int_\Omega(AU,U)_{I\!\!R^2}dx}{\left(\int_\Omega |u|^\alpha|v|^\beta dx\right)^{\frac{2}{2^*}}}, \tag{44}$$

$$\text{for } \alpha+\beta=2^*\ ,\forall\ (u,v)\in(H_o^1)^2\backslash\{0\}$$

and

$$S_\lambda^* = \inf_{u\in H_o^1(\Omega)\backslash\{0\}} Q_\lambda(u) \tag{45}$$

$$\widetilde{S}_A^* = \inf_{(u,v)\in(H_o^1)^2\backslash\{0\}} \widetilde{Q}_{(\alpha,\beta),A}(u,v). \tag{46}$$

We also make use of the notations

$$S^* = \inf_{u\in H_o^1(I\!\!R^N)\backslash\{0\}} \frac{\int_{I\!\!R^N}|\nabla u|^2dx}{\left(\int_{I\!\!R^N}|u|^{2^*}dx\right)^{\frac{2}{2^*}}} \tag{47}$$

and

$$\widetilde{S}^* = \inf_{(u,v)\in(H_o^1)^2\backslash\{0\}} \frac{\int_{I\!\!R^N}(|\nabla u|^2+|\nabla v|^2)dx}{\left(\int_{I\!\!R^N}|u|^\alpha|v|^\beta dx\right)^{\frac{2}{2^*}}},\ \text{for } \alpha+\beta=2^*. \tag{48}$$

Remark 6 *If $\mu_2<\lambda_1$, then using Poincaré inequality,(11) and (26) we see that $\widetilde{Q}_{(\alpha,\beta),A}(u,v)\geq K>0$ and therefore that $\widetilde{S}_A^*>0$.*

The proof of Theorem 1 will be made with a series of lemmas and assertions. Our first result aiming the proof of Theorem 1 is the next Lemma

Lemma 3 *Let Ω be a bounded domain and suppose that*

$$\mu_1 > 0. \tag{49}$$

Then

$$\widetilde{S}^*_A < \widetilde{S}^*. \tag{50}$$

Proof: **The case $\mathbf{N \geq 4}$:**

We may suppose that $0 \in \Omega$.

In [1] (Lemma 1.1) they proved that if $\lambda > 0$ then $S^*_\lambda < S^*$. Their idea was to use of the function

$$u_\varepsilon(x) = \frac{\varphi(x)}{[\varepsilon^2 + |x|^2]^{\frac{N-2}{2}}}, \;\; \varepsilon > 0, \tag{51}$$

where φ is a cut-off positive function such that $\varphi(x) \equiv 1$ for x in a neighborhood of 0, to obtain the estimate

$$Q_\lambda(u_\varepsilon) < S^*, \tag{52}$$

for a sufficiently small $\varepsilon > 0$.

In our case, let $B, C > 0$ be such that

$$\left(\frac{B}{C}\right)^2 = \frac{\alpha}{\beta}. \tag{53}$$

Then by (11), (37) and (43) we have that

$$\begin{aligned} \widetilde{Q}_{(\alpha,\beta),A}(Bu_\varepsilon, Cu_\varepsilon) \;\; &\leq \;\; \frac{(B^2 + C^2)\int_\Omega(|\nabla u_\varepsilon|^2 - \mu_1|u_\varepsilon^2|)dx}{(B^\alpha C^\beta)^{\frac{2}{2^*}}\left(\int_\Omega |u_\varepsilon|^{2^*}dx\right)^{\frac{2}{2^*}}} = \\ & \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}} Q_{\mu_1}(u_\varepsilon). \end{aligned}$$

Therefore, by (52) and (35), for small ε we achieve that

$$\widetilde{Q}_{(\alpha,\beta),A}(Bu_\varepsilon, Cu_\varepsilon) < \widetilde{S}^*$$

which results in (50).

The case $N = 3$ and Ω=ball:

In this case, Brezis-Nirenberg uses the fact that u is radially symmetric (see [9]), and after some estimations they get a relation analogous to (52) for all $\lambda > \frac{\lambda_1}{4}$ and the proof follows as in the first case. In the case of systems, the radially symmetry of the solutions is guaranteed by result due to Troy [12]. $\diamondsuit$

Let us define the functional

$$
\begin{aligned}
J \quad &: \quad (H_o^1)^2 \to I\!R \\
(u,v) \quad &\longmapsto \quad J(u,v) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx - \frac{1}{2}\int_\Omega (AU,U)_{I\!R^2}dx - \\
&\qquad -\frac{2}{2^*}\int_\Omega |u|^\alpha |v|^\beta dx.
\end{aligned}
$$

The functional J is well defined and of class C^1.Our aim is to exhibit a critical point for J. We emphasize that critical points for J are precisely the weak solutions for $P_{(\alpha,\beta),A}$ (see Remark 3).

It is easy to see that J has the Mountain Pass type geometry, *i.e.*

$$\text{There exist } \rho, r > 0 \text{ such that } J(u,v) \geq \rho > 0 \text{ for } ||(u,v)||_{(H_o^1)^2} = r \tag{54}$$

and

$$\text{there exists } (u,v) \not= 0 \text{ such that } \lim_{t\to\infty} J(tu,tv) = -\infty \tag{55}$$

(to prove (54) we used the inequalities (11) and (8)).

Let (u_1,v_1) be such that $J(u_1,v_1) < 0$ and define

$$\Gamma = \{g \in C([0,1],(H_o^1)^2);\ \ g(0)=0,\ g(1)=(u_1,v_1)\} \text{ and}$$

$$\mathsf{c} = \inf_{g\in\Gamma}\max_{0\leq t\leq 1} J(g(t)). \tag{56}$$

From (54) we see that $\mathsf{c} > 0$. Using an application of the Ekeland Variational Principle, the above assertions imply the existence of a sequence $(u_n,v_n) \subset (H_o^1)^2$ (see [10]) such that

$$J(u_n,v_n) \to \mathsf{c} \tag{57}$$

$$J'(u_n,v_n) \to 0. \tag{58}$$

It is a standard procedure to use (57) and (58) to show that this sequence is bounded. Then by the Sobolev embedding Theorem, there exists a subsequence, again denoted by (u_n,v_n),such that

$$u_n \rightharpoonup u_o \text{ and } v_n \rightharpoonup v_o \text{ in } H_o^1(\Omega), \text{and they converge}$$

$$\text{pointwisely in } L^p(\Omega) \text{ for } 2 \leq p < 2^*. \tag{59}$$

From now on, our aim will be two-fold. First show that (u_o,v_o) is a solution for $P_{(\alpha,\beta),A}$, and then show that $u_o, v_o \neq 0$.

Proof that (u_o,v_o) is a solution for $P_{(\alpha,\beta),A}$:

From (58) we have that

$$J'(u_n,v_n)(\varphi,\zeta) = o_n(1), \forall\ (\varphi,\zeta) \in (H_o^1)^2; o_n(1) \to 0,\ \text{as } n\to\infty. \tag{60}$$

Due to an argument of Brezis-Lieb (Lemma 4.8 - pag. 10 - in [11]), since $w_n = |u_n|^{\alpha-2}u_n|v_n|^{\beta} \in L^{\frac{2^*}{2^*-2}}(\Omega)$ is an uniformly bounded sequence in this space and converges pointwisely to $|u_o|^{\alpha-2}u_o|v_o|^{\beta}$, we have that

$$w_n \rightharpoonup |u_o|^{\alpha-2}u_o|v_o|^{\beta}. \tag{61}$$

The same arguments applies to the sequence $|u_n|^{\alpha}v_n|v_n|^{\beta-2}$.

Now using (59), (61) and passing to the limit in (60) we have that (u_o, v_o) is a weak solution of $P_{(\alpha,\beta),A}$.

Proof that $u_o, v_o \neq 0$:

To proceed further we shall choose a special c in (56).

Lemma 4 *There exists a (u_1, v_1) such that c defined in (56) satisfies*

$$0 < \mathsf{c} < \frac{2}{N}\left(\frac{\widetilde{S}^*}{2}\right)^{N/2}. \tag{62}$$

Proof: Let B, C satisfying (53) and u_ε as in Lemma 3 with $\int_\Omega |u_\varepsilon|^{2^*}dx = 1$. Then

$$J(tBu_\varepsilon, tCu_\varepsilon) \leq \frac{1}{2}(B^2+C^2)Q_{\mu_1}(u_\varepsilon)t^2 + \frac{2}{2^*}B^{\alpha}C^{\beta}t^{2^*}.$$

Let $r(t)$ be the function in the right-hand side of the last inequality . A forward computation assures that $t_o = \left[\frac{B^2+C^2}{(B^{\alpha}C^{\beta})^{\frac{2}{2^*}}}Q_{\mu_1}(u_\varepsilon)\right]^{\frac{1}{2^*-2}}$ is the maximum point for r and that

$$r(t_o) = \frac{2}{N}\left[\frac{1}{2}\frac{B^2+C^2}{(B^{\alpha}C^{\beta})^{\frac{2}{2^*}}}Q_{\mu_1}(u_\varepsilon)\right]^{\frac{N}{2}}$$

is its maximum value. Hence,by (37), (35) e (52) we have that

$$\sup_{t\geq 0} J(tBu_\varepsilon, tCu_\varepsilon) < \frac{2}{N}\left(\frac{\widetilde{S}^*}{2}\right)^{N/2}$$

and thus, that (62) holds. $\diamondsuit$

Let us go back to the proof that $u_o, v_o \neq 0$.

Notice $u_o = 0$ if, and only if $v_o = 0$. In fact, if $u_o = 0$, then by (1) and (2) we see that $b = 0$ and then $a, c \in \{\mu_1, \mu_2\}$. Since, in this case, v_o is a solution of the equation

$$\begin{cases} -\Delta v = cv, \ \Omega \\ v = 0, \ \partial\Omega \end{cases}$$

we conclude by (8) that $v_o = 0$. The same reasoning is applied when $v_0 = 0$.

Suppose that $(u_o, v_o) = 0$. Let

$$l = \lim_{n\to\infty} \int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2) dx \geq 0.$$

Since $J'(u_n, v_n)(u_n, v_n) \to 0$ we see that $\int_\Omega |u_n|^\alpha |v_n|^\beta dx \to \frac{l}{2}$ as $n \to \infty$. On the other hand, by the above limits and (57) we obtain, for the c in Lemma 4, that

$$\mathsf{c} = \frac{l}{2}. \tag{63}$$

From (48) and Remark 5-(iii) we see that

$$\int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2) dx \geq \widetilde{S}^* \int_\Omega |u_n|^\alpha |v_n|^\beta dx.$$

Passing to the limit in the last inequality and using (63) we get that

$$\mathsf{c} \geq \frac{2}{N} \left(\frac{\widetilde{S}^*}{2} \right)^{N/2}$$

which contradicts the choice of c in (62). Therefore $(u_o, v_o) \not= 0$.

Now, since $J'(u_n, v_n)(u_n^-, v_n^-) \to 0$ this yields that $(u_o^-, v_o^-) = 0$ $(W^-(x) = \min\{0, W(x)\})$ and then that $(u_o, v_o) \geq 0$. Finally, using the hypothesis (7) and the Strong Maximum Principle in the scalar case for both equations in $P_{(\alpha,\beta),A}$ we assure that $(u_o, v_o) > 0$. Hence the first part of Theorem 1 and (9) is proved.

The proof of (10):

In order to prove (10) we need the following result:

Lemma 5 *Let ρ be a continuous function in $I = [0, \pi)$ such that $0 < \rho(r) < 1$, for all $r \in I$. Then the solution ω of the Cauchy problem:*

$$\begin{cases} \omega'' + \rho(r)\omega = 0 \\ \omega(0) = 1, \;\; \omega'(0) = 0 \end{cases} \tag{64}$$

satisfies

$$\omega'(r) < 0 \tag{65}$$

and

$$\int_o^r \omega(s) ds > 0 \tag{66}$$

for all $0 < r < \pi$.

Proof: If $\omega(r) > 0$, for all $r \in I$, the result follows from the monotonicity of ω', since $\omega''(r) = -\rho(r)\omega(r) < 0$ and $\omega'(0) = 0$. The complement case is when there is a $r_1 \in I$ such that $\omega(r_1) = 0$. By the unicity for EDO's, we have $\omega'(r_1) < 0$. We claim that $r_1 > \frac{\pi}{2}$ and r_1 is the unique zero of ω on I. In fact, define $x(r) = \cos r$. Observe that $(x\omega' - \omega x')' = x\omega'' - \omega x'' = (1-\rho(r))\omega(r)x(r)$, for all $0 < r < \pi$. Integrating this equality on $(0, r)$ we have

$$\int_o^r (1-\rho(s))\omega(s)x(s)ds = x(r)\omega'(r) - \omega(r)x'(r)$$

for all $r > 0$. Suppose that $r_1 \leq \frac{\pi}{2}$, then the left hand side of the above equality is positive in $0 < t < r_1$ and

$$0 < \int_o^{r_1} (1-\rho(s))\omega(s)x(s)ds = x(r_1)\omega'(r_1)$$

which is impossible because $\omega'(r_1) < 0$ and $\cos r > 0$ for all $0 < r < r_1$. Suppose that we have a second zero r_2 of ω in $r_1 < r < \pi$, where $\cos r$ is negative. In this case $\omega'(r_2) > 0$ and ω is negative in $r_1 < r < r_2$, then

$$0 < \int_{r_1}^{r_2} (1-\rho(s))\omega(s)x(s)ds = x(r_2)\omega'(r_2) - x(r_1)\omega'(r_1)$$

and we have another contradiction, because $x(r_2) < 0$, $\omega'(r_2) > 0$, $x(r_1) < 0$ and $\omega'(r_1) < 0$. Now observe that $\omega(r) \geq \omega'(r_1)(r - r_1)$, for all $r \in I$ and then $\int_o^r \omega(s)ds \geq \omega'(r_1)\left[\frac{(r-r_1)^2}{2} - \frac{r_1^2}{2}\right] > 0$, for all $r \in I$ (here we use the inequality $r_1 > \frac{\pi}{2}$). (66) is done if $\omega(r) \geq y(r) = \omega'(r_1)(r - r_1)$, for all $\dot{r} \in I$. Let us show this last claim. Using the relation $(y\omega' - \omega y')' = y\omega'' = -\rho(r)y(r)\omega(r)$, for all $r \in I$, we have

$$0 > -\int_r^{r_1} \rho(s))\omega(s)y(s)ds = y'(r)\omega(r) - \omega'(r)y(r) = \omega(r)^2 \frac{d}{dr}\left(\frac{y(r)}{\omega(r)}\right),$$

for all $r < r_1$, where y and ω are positive and

$$0 > -\int_{r_1}^{r} \rho(s))\omega(s)y(s)ds = y(r)\omega'(r) - \omega(r)y'(r) = y(r)^2 \frac{d}{dr}\left(\frac{\omega(r)}{y(r)}\right),$$

for all $r > r_1$, where y and ω are negative.

The quocient $\frac{y}{\omega}$ is decreasing in $(0, r_1)$ and $\frac{\omega}{y}$ is decreasing in (r_1, π) and then

$$\frac{y(r)}{\omega(r)} > \lim_{s \to r_1^-} \frac{y(s)}{\omega(s)} = \frac{y'(r_1)}{\omega'(r_1)} = 1, \text{ and } y(r) < \omega(r) \text{ for all } 0 < r < r_1.$$

Analogously $y(r) < \omega(r)$, for all $r_1 < r < \pi$.

To complete the proof let us verify (65). When $0 < r < r_1$, ω' is non-increasing because ω is positive. Since $\omega'(0) = 0$, we have $\omega'(r) < 0$ for all $0 < r < r_1$. Let $z(r) = -\sin(r - r_1)$. Using the relation

$$(z\omega'' - \omega z'') = (1-\rho(r))\omega(r)z(r)$$

we get

$$0 < \int_{r_1}^{r} (1-\rho(s))\omega(s)z(s)ds = z(r)\omega'(r) - z'(r)\omega(r) \le z(r)\omega'(r)$$

for all $r_1 < r < \pi$, since z, z' and ω are negative in this interval. This implies $\omega'(r) < 0$ for all $r_1 < r < \pi$, and the proof is done. $\diamondsuit$

Now we are ready to proof (10).

Suppose that (u, v) is a solution of (1-4), where Ω is the unitary ball $B_1(0) \subset \mathbb{R}^N$. The functions u, v are radially symmetric (see bottom of pg.11). We write $u(x) = u(r)$, $v(x) = v(r)$, where $r = |x|$. Thus

$$\begin{cases} -u'' - \frac{2}{r}u' = au + bv + \frac{\alpha}{3}u^{\alpha-1}v^{\beta}, \text{ on } (0,1) \\ -v'' - \frac{2}{r}v' = bu + cv + \frac{\beta}{3}u^{\alpha}v^{\beta-1}, \text{ on } (0,1) \\ u'(0) = v'(0) = u(1) = v(1) = 0. \end{cases} \tag{67}$$

We claim that

$$\begin{aligned} \int_o^1 \left[\frac{(u^2+v^2)}{4} r^2\psi''' + (AU,U)r^2\psi' \right] dr &= \frac{4}{3}\int_o^1 u^{\alpha}v^{\beta}\left[r\psi - r^2\psi'\right] dr \\ &\quad + \frac{\psi(1)}{2}\left[u'(0)^2 + v'(0)^2\right] \end{aligned} \tag{68}$$

for all $\psi \in C^3[0,1]$ such that $\psi(0) = 0$. In fact, multiplying the first equation in (67) by $r^2\psi u'$ and the second by $r^2\psi v'$ we obtain

$$\begin{aligned} \int_o^1 \left[u'(r)^2 + v'(r)^2\right]\left(\frac{1}{2}r^2\psi' - r\psi\right) dr &= \frac{\psi(1)}{2}\left[u'(0)^2 + v'(0)^2\right] \\ &\quad + \int_o^1 r^2\psi \frac{d}{dr}G(u,v)dr \\ &= \frac{\psi(1)}{2}\left[u'(0)^2 + v'(0)^2\right] \\ &\quad - \int_o^1 (2r\psi + r^2\psi')G(u,v)dr \end{aligned} \tag{69}$$

where $G(u, v) = \frac{1}{2}(AU, U) + \frac{1}{3}u^{\alpha}v^{\beta}$. Next we multiply the first equation of (67) by $(\frac{1}{2}r^2\psi' - r\psi)u$ and the second by $(\frac{1}{2}r^2\psi' - r\psi)v$ we obtain

$$\begin{aligned} \int_o^1 \left[u'(r)^2 + v'(r)^2\right]\left(\frac{1}{2}r^2\psi' - r\psi\right) dr &= \int_o^1 \left[\frac{(u^2+v^2)}{4}\right] r^2\psi''' dr \\ &\quad + \int_o^1 [uG_u(u,v) \\ &\quad + vG_v(u,v)]\left(\frac{1}{2}r^2\psi' - r\psi\right) dr. \end{aligned} \tag{70}$$

Using the equality $uG_u(u,v)+vG_v(u,v) = 2G(u,v)+\frac{4}{3}u^\alpha v^\beta = (AU,U)+2u^\alpha v^\beta$ and combining (69) and (70) we obtain (68). Let $\frac{\pi^2}{4} > \lambda > \mu_2$ and ψ the solution of

$$\begin{cases} \frac{(u^2+v^2)}{4}\psi''' + (AU,U)\psi' = 0 \\ \psi(0) = 0,\ \psi'(0) = 1,\ \psi''(0) = 0 \end{cases}$$

Observe that $\omega(t) = \psi'(\frac{t}{2\sqrt{\lambda}})$ satisfies (64) for $0 < t < 2\sqrt{\lambda} < \pi$ and $\rho(t) = \frac{1}{\lambda}\frac{(AU,U)}{(U,U)}(\frac{t}{2\sqrt{\lambda}})$. We have

$$0 < \rho(t) \leq \frac{\mu_2}{\lambda} < 1.$$

Using Lemma 1, we have $\psi(1) = \int_o^1 \omega(2\sqrt{\lambda}t)dt = \left(2\sqrt{\lambda}\right)^{-1}\int_o^{2\sqrt{\lambda}}\omega(t)dt > 0$ and $(\psi - r\psi')' = -r\psi''(r) = -2r\sqrt{\lambda}\omega'(2\sqrt{\lambda}r) > 0$, that is, $(\psi - r\psi')$ is non-decreasing on $(0,1)$ and then $(\psi - r\psi') > 0$ on $(0,1)$. These remarks used in (68) yields

$$\frac{4}{3}\int_o^1 u^\alpha v^\beta \left[r\psi - r^2\psi'\right] dr \leq 0$$

and then $u \equiv v \equiv 0$.

$\diamondsuit$

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References

[1] Brezis, H. & Nirenberg, N. : *Positive Solutions of Nonlinear Elliptic Equations involving Critical Sobolev Exponent*, Comm. in Pure & Applied Narh., 36, pp. 437-477 (1983).

[2] Pohozaev, S. : *Eigenfunctions of the equation* $-\Delta u + \lambda f(u) = 0$, Soviet Math. Doke.,6 ,pp. 1408-1411 (1985).

[3] Struwe, M. : *Variational Methods: Applications to Nonlinear PDE & Hamiltonion System*, Springer-Verlag (1990).

[4] Talenti, G.: *Best constants in Sobolev inequality*, Annali di Mat. ,110, pp.353-372 (1976).

[5] Azorero,J.Garcia & Alonzo,I.Peral: *Existence and non-uniqueness for the p-Laplacean nonlinear eigenvalues*, Comm. Partial Diff. Equations 12, pp.1389-1430 (1987).

[6] Brezis, H. : *Some variational problems with lack of compactness*, Proc. Symp. in Pure Math, 45, pp. 165-201 (1986).

[7] Benci, V.. & Cerami, G. : *Existence of positive solutions of the equation* $\Delta u + a(x) = u^{\frac{N+2}{N-2}}$ *in* $\mathbb{R}^N$, J. Functional Anal., 88,pp.90-117 (1990).

[8] Egnell, H.: *Existence and nonexistence results for the m-Laplacean equations involving critical Sobolev exponents*, Arch. Rational Mech. Anal., 104, pp. 57-77 (1988).

[9] Gidas, B.; Ni, W. M. & Nirenberg, L. : *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. 68, pp.209-243 (1979).

[10] J. Mawhin - M. Willem, *Critical point theory and Hamiltonian systems.* Springer Verlag, Berlin 1989.

[11] Kavian, O.: *Introduction à la theorie des points critiques et applications aux problems elliptiques*, Université de Nancy (1993).

[12] Troy, W. : *Symmetry properties in systems of semilinear elliptic equations*, J. of Diff. Equations, 42, p.p 400-413 (1981).

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