Generalized Signal Alignment: On the Achievable DoF for Multi-User MIMO Two-Way Relay Channels

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Abstract

This paper studies the achievable degrees of freedom (DoF) for multi-user multiple-input multiple-output (MIMO) two-way relay channels, where there are K source nodes, each equipped with M antennas, one relay node, equipped with N antennas, and each source node exchanges independent messages with an *arbitrary* set of other source nodes via the relay. By allowing an arbitrary information exchange pattern, the considered channel model is a unified one. It includes several existing channel models as special cases: K-user MIMO Y channel, multi-pair MIMO two-way relay channel, generalized MIMO two-way X relay channel, and L-cluster K-user MIMO multiway relay channel. Previous studies mainly considered the case $N < 2M$ and showed that the DoF upper bound $2N$ is tight under certain antenna configurations by applying signal alignment for network coding. This work aims to investigate the achievability of the DoF upper bound KM for the case $N \geq 2M$. To this end, a new transmission framework, *generalized signal alignment* (GSA) is proposed. Its notion is to form network-coded symbols by aligning every pair of signals to be exchanged in a *projected* subspace at the relay. This is realized by jointly designing the precoding matrices at all source nodes and the projection matrix at the relay. GSA is feasible when $N \geq (K-2)M + \max\{d_{i,j}\}$, where $d_{i,j}$ is the number of data streams transmitted from source node i to source node j, if any, in the considered channel model. By applying GSA, it is shown that the DoF upper bound KM is achievable when: i) $\frac{N}{M} \geq \frac{K^2-3K+3}{K-1}$ for K-user MIMO Y channel; ii) $\frac{N}{M} \geq K-1$ for multi-pair MIMO two-way relay channel; iii) $\frac{N}{M} \geq \frac{K^2-2K+2}{K}$ for generalized MIMO two-way X relay channel; iv) $\frac{N}{M} \geq \frac{(K'-1)(LK'-2)+1}{K'-1}$ for L-cluster $K' = \frac{K}{L}$-user MIMO multiway channel.

Index Terms

Multi-input multi-output, two-way relay channel, signal alignment, degrees of freedom.

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I. INTRODUCTION

Wireless relay has been an important ingredient in both ad hoc and infrastructure-based wireless networks. It shows great promises in power reduction, coverage extension and throughput enhancement. In the simplest scenario, a relay only serves a single user. This forms the classic relay channel, which includes one source, one destination and one relay. Nowadays, a relay has become very much like a wireless gateway where multiple users share a common relay and communicate with each other. A typical representative is the two-way relay channel (TWRC), where two users exchange information with each other through a relay [1]. A fundamental question that arises is what is the maximum number of data streams the relay can forward and how to achieve it. This leads to the analysis of degrees of freedom (DoF) and also drives the development of advanced relay strategies for efficient multi-user information exchange in the literature.

The success of the two-way relay channel owes to the invention of physical layer network coding (PLNC), which can almost double the spectral efficiency compared with traditional one-way relaying [2]–[5]. In specific, when each source node is equipped with M antennas and the relay node is equipped with N antennas, the maximum achievable DoF of the MIMO (multiple-input multiple-output) two-way relay channel is $2\min\{M,N\}$ [5]. When three or more users arbitrarily exchange information with each other via a common relay, it is difficult to design PLNC due to multi-user interference and hence the analysis of DoF becomes challenging. Several multi-user MIMO two-way relay channels have been investigated in the literature, such as the MIMO Y channel [6], K-user MIMO Y channel [7], multi-pair MIMO two-way relay channel [8], MIMO two-way X relay channel [9], generalized MIMO two-way X relay channel [10], L-cluster K-user MIMO multiway relay channel [11] and etc.

Based on the idea of interference alignment [12], [13], *signal alignment* (SA) is firstly proposed in [6] to analyze the maximum achievable DoF for the MIMO Y channel, where three users exchange independent messages with each other via the relay. By jointly designing the precoders at each source node, SA is able to align the signals from two different source nodes in a same subspace of the relay node. By doing so, the two data streams to be exchanged between a pair of source codes are combined into one network-coded symbol and thus the relay can forward more data streams simultaneously. It is proved that with SA and network-coding aware interference nulling, the theoretical upper bound $3M$ of DoF is achievable when $N \geq \lceil \frac{3M}{2} \rceil$ [6]. Here, again, M and N denote the number of antennas at each source node and the relay node, respectively. The extension to K-user MIMO Y channels is considered in [7], where it is shown that the DoF upper bound is $\min\{KM, 2N\}$ and the upper bound $2N$ in the case $N < 2M$ is achievable when $N \leq \lfloor \frac{2K(K-1)M}{K(K-1)+2} \rfloor$. Here K is the total number of users. The authors in [14] considered the case $N \geq 2M$ and showed that the upper bound KM of DoF is achievable when $N \geq \lceil \frac{K^2-2K}{K-1} \rceil$. Recently, the authors in [11] analyzed the multi-pair MIMO two-way relay channel and showed that the DoF upper bound $2N$ is achievable when $N \leq \lfloor \frac{2KM}{K+2} \rfloor$ and the DoF upper bound KM is achievable when $N \geq KM$. In [9], SA is applied in the MIMO two-way X relay channel, where there are two groups of source nodes and one relay node, and each of the two source nodes in one group exchange independent messages with the two source nodes in the other group via the relay node. It is shown that the DoF upper bound is $2\min\{2M, N\}$, and the upper bound $2N$ is achievable when $N \leq \lfloor \frac{8M}{5} \rfloor$ by applying SA and interference cancellation. Despite the extensive work in this topic, the DoF achievebility of multi-user MIMO two-way relay channels still remains open.

In this paper, we are interested in the achievable DoF analysis of an arbitrary multi-user MIMO two-way relay channel for the antenna configuration $N \geq 2M$. In our considered multi-user MIMO two-way relay channel, there are K users each equipped with M antennas, one relay equipped with N antennas, and each source node can arbitrarily select one or more partners to conduct independent information exchange. By allowing arbitrary information exchange pattern, our considered multi-user MIMO two-way relay channel is a unified channel model. It includes several existing channel models as special cases, namely, K-user MIMO Y channel, multi-pair MIMO two-way relay channel, generalized MIMO two-way X relay channel and L-cluster K-user MIMO multiway relay channel.

It is worth mentioning that SA is no longer feasible under such antenna configuration. The reason is as follows. Recall that the SA condition [6] is

$$\mathbf{H}_{1,r}\mathbf{V}_1 = \mathbf{H}_{2,r}\mathbf{V}_2, \tag{1}$$

where $\mathbf{H}_{i,r}$ is an $N \times M$ channel matrix from source i to relay and $\mathbf{V}_i$ is an $M \times d_{i,3-i}$ beamforming vector of source i, for $i = 1,\ 2$, where $d_{i,3-i}$ denotes the number of data streams transmitted from source node i to source node $3-i$. The above alignment condition can be rewritten as

$$[\mathbf{H}_{1,r} \quad -\mathbf{H}_{2,r}] \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = 0. \tag{2}$$

Clearly, for the above equality to hold, one must have $N < 2M$.

To achieve the maximum DoF at $N \geq 2M$ for multi-user MIMO two-way relay channels, it is not always optimal for users to utilize all the antennas at the relay. In specific, using only a subset of antennas at the relay, known as *antenna deactivation* [14], can achieve higher DoF for some cases [6], [11]. But there is still a gap from the DoF upper bound. In this work, we propose a new transmission framework, named *generalized signal alignment* (GSA), which can achieve the theoretical DoF upper bound even when $N \geq 2M$. Compared with the existing SA, the proposed GSA has the following major difference. The signals to be exchanged do not align directly in the subspace observed by the relay. Instead, they are aligned in a projected subspace after certain processing at the relay. This is done by jointly designing the precoding matrices at the source nodes and the projection matrix at the relay node.

The main results obtained in this work are as follows:

- For multi-user MIMO two-way relay channels, the GSA is feasible when $N \geq (K-2)M + \max\{d_{i,j} \mid \forall i, j\}$, where $d_{i,j}$ denotes the number of data streams transmitted from source node i to source node j, if any.
- For the special case of the K-user MIMO Y channel, the DoF upper bound KM is achievable with GSA when $\frac{N}{M} \geq \frac{K^2-3K+3}{K-1}$, which enlarges the achievable region of the DoF upper bound in [7] and [14].
- For the special case of the multi-pair MIMO two-way relay channel, the DoF upper bound KM is achievable with GSA when $\frac{N}{M} \geq K-1$, which enlarges the achievable region of the DoF upper bound in [8].
- For the special case of the generalized MIMO two-way X relay channel, the DoF upper bound KM is achievable with GSA when $\frac{N}{M} \geq \frac{K^2-2K+2}{K}$, which enlarges the achievable region of the DoF upper bound in [9]. Part of this result is also presented in [15].
- For the special case of the L-cluster K-user MIMO multiway relay channel, the DoF upper bound LKM is achievable with GSA when $\frac{N}{M} \geq \frac{(K-1)(LK-2)+1}{K-1}$, which enlarges the achievable region of the DoF upper bound in [11] and [16].

The remainder of the paper is organized as follows. In Section II, we introduce the multi-user

MIMO two-way relay channel and derive its DoF upper bound. In Section III, we introduce the principle of GSA transmission scheme and give an illustrative example for 4-user MIMO Y channel. In Section IV, we apply the GSA transmission scheme to K-user MIMO Y channel, multi-pair MIMO two-way relay channel, generalized MIMO two-way X relay channel and L-cluster K-user MIMO multiway relay channel. We also show the new DoF achievable region for these models. In Section V, we show numerical results. Section VI presents concluding remarks.

Notations: $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and the Hermitian transpose, respectively. tr($\mathbf{X}$) and rank($\mathbf{X}$) stand for the trace and rank of $\mathbf{X}$. $\varepsilon[\cdot]$ stands for expectation. span$(\mathbf{X})$ and null$(\mathbf{X})$ stand for the column space and the null space of the matrix $\mathbf{X}$, respectively. dim$(\mathbf{X})$ denotes the dimension of the column space of $\mathbf{X}$. $\lfloor x \rfloor$ denotes the largest integer no greater than x. $\lceil x \rceil$ denotes the smallest integer no less than x. $\mathbf{I}$ is the identity matrix. $[\mathbf{X}]_{i,j}$ denotes the (i,j)-th entry of the matrix $\mathbf{X}$.

II. System Model

A. Channel Model

We consider a multi-user MIMO two-way relay channel as shown in Fig. 1. It consists of K source nodes, each equipped with M antennas, and one relay node, equipped with N antennas. Each source node i, for $1 \le i \le K$, can exchange independent messages with an arbitrary set of other source nodes, denote as $\mathcal{S}_i$, with the help of the relay. The independent message transmitted from source node i to source node j, if $j \in \mathcal{S}_i$, is denoted as $W_{i,j}$. At each time slot, the message is encoded into a $d_{i,j} \times 1$ symbol vector $\mathbf{s}_{i,j} = [s_{i,j}^1, s_{i,j}^2, \cdots, s_{i,j}^{d_{i,j}}]^T$, where $d_{i,j}$ denotes the number of independent data streams transmitted from source i to source j. We define a $K \times K$ matrix $\mathbf{D}$ named as a *data switch matrix* whose (i,j)-th entry is given by

$$[\mathbf{D}]_{i,j} = \begin{cases} d_{i,j}, & j \in \mathcal{S}_i, \forall i; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Note that all the diagonal elements of $\mathbf{D}$ are zero, and when the off-diagonal element $[\mathbf{D}]_{i,j} = 0$, it means there is no information exchange between source node i and j.

It can be shown that the following existing channels are all special cases of our considered multi-user MIMO two-way relay channel.

- The K-user MIMO Y channel: For each source node i, one has $\mathcal{S}_i = \{1, 2, \cdots, K\} \backslash \{i\}$. The off-diagonal entries of $\mathbf{D}$, $\{d_{i,j} \mid i \neq j\}$, can be any nonnegative integer.

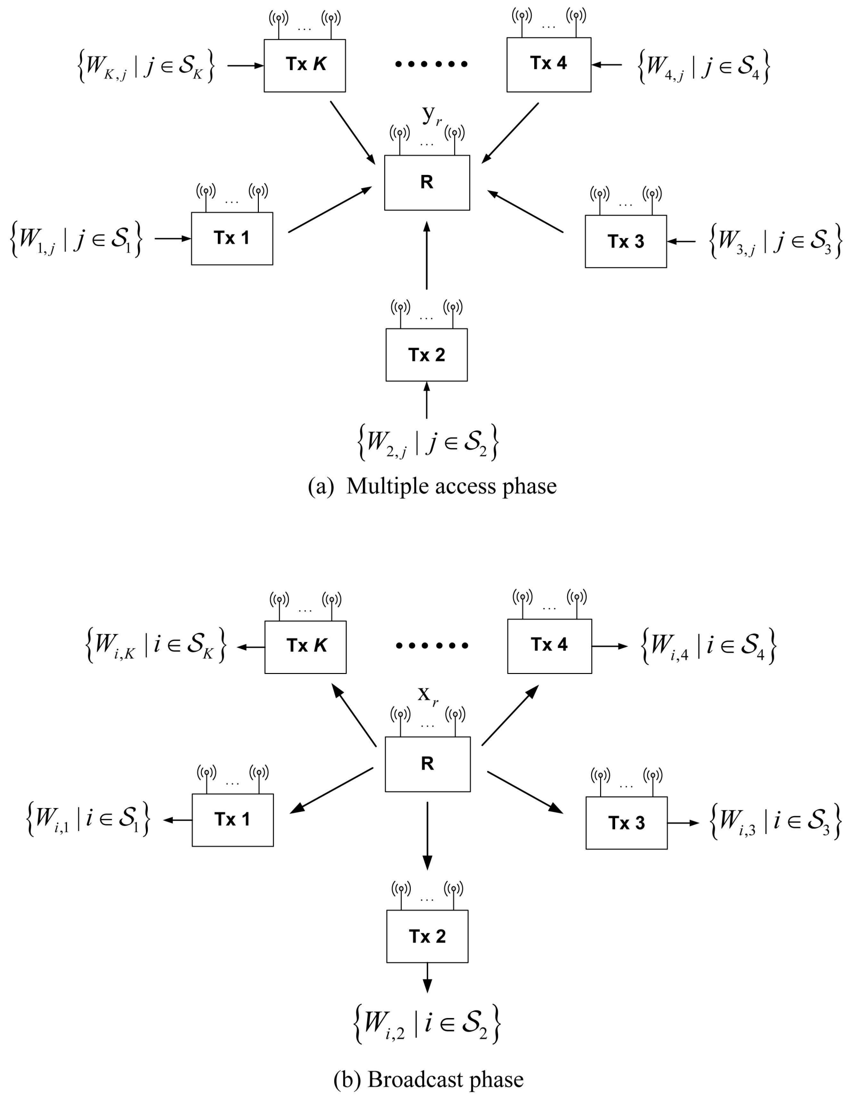

Fig. 1. Multi-user MIMO two-way relay channel

- The multi-pair MIMO two-way relay channel: Each source node i, $1 \leq i \leq \frac{K}{2}$, exchanges independent messages with its pair node $K+1-i$, and there are $\frac{K}{2}$ pairs in total. The entries of $\mathbf{D}$ which satisfy $\{[\mathbf{D}]_{i,j} \mid i+j \neq K+1\}$ must be zero. The rest can be any nonnegative integer.
- The generalized MIMO two-way X relay channel: The K source nodes are divided into two groups. Each source node i in one group exchanges independent messages with every source node in the other group. That is, $\mathcal{S}_i = \{j \mid \frac{K}{2}+1 \leq j \leq K\}$ for $1 \leq i \leq \frac{K}{2}$ and $\mathcal{S}_i = \{j \mid 1 \leq i \leq \frac{K}{2}\}$ for $\frac{K}{2}+1 \leq i \leq K$. The entries of $\mathbf{D}$ which satisfy $\{[\mathbf{D}]_{i,j} \mid 1 \leq$

$i, j \le \frac{K}{2}$ or $\frac{K}{2}+1 \le i, j \le K\}$ must be zero. The rest can be any nonnegative integer.

The difference of these channels lies at the position of “0” in the data swtich matrix $\mathbf{D}$. In this work, we unify these system models to the multi-user MIMO two-way relay channel.

The communication of the total messages takes place in two phases: the multiple access (MAC) phase and the broadcast (BC) phase. In the MAC phase, all K source nodes transmit their signals to the relay simultaneously. Let $\mathbf{x}_i$ denote the transmitted signal vector from source node i. It is given by

$$\mathbf{x}_i = \sum_{j \in \mathcal{S}_i} \mathbf{V}_{i,j}\mathbf{s}_{i,j} = \mathbf{V}_i \mathbf{s}_i, \tag{4}$$

where $\mathbf{V}_{i,j}$ is the $M \times d_{i,j}$ precoding matrix for the information symbol vector $\mathbf{s}_{i,j}$ to be sent to source node j, $\mathbf{V}_i$ is a matrix obtained by stacking $\{\mathbf{V}_{i,j} \mid j \in \mathcal{S}_i\}$ by column and $\mathbf{s}_i$ is a vector obtained by stacking $\{\mathbf{s}_{i,j} \mid j \in \mathcal{S}_i\}$ by rows. Each transmitted signal $\mathbf{x}_i$, for $i = 1, \cdots, K$, satisfies the power constraint of

$$\text{tr}(\mathbf{x}_i\mathbf{x}_i^H) \le P, \ \forall i \tag{5}$$

where P is the maximum transmission power allowed at each source node.

The received signal $\mathbf{y}_r$ at the relay is given by

$$\mathbf{y}_r = \sum_{i=1}^{K} \mathbf{H}_{i,r}\mathbf{x}_i + \mathbf{n}_r, \tag{6}$$

where $\mathbf{H}_{i,r}$ denotes the frequency-flat quasi-static $N \times M$ complex-valued channel matrix from source node i to the relay and $\mathbf{n}_r$ denotes the $N \times 1$ additive white Gaussian noise (AWGN) with variance σ_n^2.

In the BC phase, upon receiving $\mathbf{y}_r$ in (6), the relay processes it to obtain a mixed signal $\mathbf{x}_r$, and broadcasts to all the users. The transmitted signal $\mathbf{x}_r$ satisfies the power constraint of

$$\text{tr}(\mathbf{x}_r\mathbf{x}_r^H) \le P_r, \tag{7}$$

where P_r is the maximum transmission power allowed at the relay. Without loss of generality from the perspective of DoF analysis, we let $P_r = P$. The received signal at source node i can be written as

$$\mathbf{y}_i = \mathbf{G}_{r,i}\mathbf{x}_r + \mathbf{n}_i, \tag{8}$$

where $\mathbf{G}_{r,i}$ denotes the frequency-flat quasi-static $M \times N$ complex-valued channel matrix from relay to the source node i, and $\mathbf{n}_i$ denotes the AWGN at the source node i. Each user tries to obtain its desired signal from its received signal using its own transmit signal as side information.

It is assumed that the channel state information $\{\mathbf{H}_{i,r}, \mathbf{G}_{r,i}\}$ is perfectly known at all source nodes and the relay, following the convention in [6], [7], [9], [11], [14]. The entries of the channel matrices and those of the noise vectors $\mathbf{n}_r$, $\mathbf{n}_i$ are independent and identically distributed (i.i.d.) zero-mean complex Gaussian random variables with unit variance. Thus, each channel matrix is of full rank with probability 1. All the source nodes in the network are assumed to be full duplex.

B. Degrees of Freedom

Let $R_{i,j}$ denote the information rate carried in $W_{i,j}$. Since we assume the noise is i.i.d. zero-mean complex Gaussian random variables with unit variance, the average received signal-to-noise ratio (SNR) of each link is P. We define the DoF of the transmission from source node i to source node j, for $j \in \mathcal{S}_i$, as

$$d_{i,j} \triangleq \lim_{\text{SNR}\to\infty} \frac{R_{i,j}(\text{SNR})}{\log(\text{SNR})}. \tag{9}$$

The definition in (9) indicates the number of independent data streams transmitted from source node i to source node j and hence is the same as $d_{i,j}$ defined in the previous subsection. Then the total DoF of the system is

$$d_{total} = \sum_{i=1}^{K} \sum_{j\in\mathcal{S}_i} d_{i,j}. \tag{10}$$

C. DoF upper bound

By applying cut-set theorem [17], the total DoF upper bound of the multi-user MIMO two-way relay channel is given in the following theorem.

Theorem 1: The total DoF of the multi-user MIMO two-way relay channel is upper-bounded by $\min\{KM, 2N\}$.

Proof: The DoF upper bound of the source node i is

$$\begin{aligned} d_i^{upper} = & \\ \sum_{j\in\mathcal{S}_i}^{K} d_{i,j} \leq \min\{ & \underbrace{\min\{M,N\},}_{source\ node\ i\ to\ relay} \quad \underbrace{\min\{(K-1)M,N\}}_{relay\ to\ others}\} \\ & = \min\{M,N\}. \end{aligned} \tag{11}$$

Then

$$d_{total}^{upper} = \sum_{i=1}^{K} d_i^{upper} = \min\{KM, KN\}. \tag{12}$$

On the other hand, we have

$$d_{total}^{upper} \leq \underbrace{\min\{KM,N\}}_{MAC\ phase} + \underbrace{\min\{KM,N\}}_{BC\ phase} = \min\{2KM, 2N\}. \tag{13}$$

Combining (12) and (13), we obtain that

$$d_{total}^{upper} = \min\{KM, KN, 2KM, 2N\} = \min\{KM, 2N\}. \tag{14}$$

■

III. Generalized Signal Alignment

Given the antenna configuration $N \geq 2M$, the traditional SA is not feasible and, thus, a more advanced transmission strategy is desired. In this section, we propose a new transmission framework, named as *generalized signal alignment*, based on which we study the DoF achievability of the considered multi-user MIMO two-way relay channel in the next section. We first introduce its basic principles and then give an example.

A. Basic principles

We rewrite the received signal (6) at the relay during the MAC phase as

$$\mathbf{y}_r = \sum_{i=1}^{K} \mathbf{H}_{i,r}\mathbf{V}_i\mathbf{s}_i + \mathbf{n}_r. \tag{15}$$

Note that the total number of independent data streams to communicate is $\sum_{i=1}^{K}\sum_{j\in\mathcal{S}_i} d_{ij}$. When $N \geq \sum_{i=1}^{K}\sum_{j\in\mathcal{S}_i} d_{ij}$, the relay can decode all the data streams and the decode-and-forward (DF) relay is the optimal transmission strategy. When $N < \sum_{i=1}^{K} \sum_{j \in \mathcal{S}_i} d_{ij}$, it is impossible for the relay to decode all the data streams individually. However, applying physical layer network coding, we only need to obtain the network-coded symbol vector $\mathbf{s}_{\oplus}$ at the relay, where $\mathbf{s}_{\oplus}$ is a vector obtained by stacking the $\{\mathbf{s}_{i,j} + \mathbf{s}_{j,i}, \ \forall j \in \mathcal{S}_i, \ \forall i\}$ by row.

According to the signal alignment equation (1), when $N \geq 2M$, $\mathbf{s}_{\oplus}$ cannot be obtained directly by designing the precoding matrices $\mathbf{V}_{i,j}$ and $\mathbf{V}_{j,i}$. Instead, joint design of the source precoding matrices and relay projection matrix should be considered. We introduce a $J \times N$ ($J \leq N$) full-rank projection matrix $\mathbf{P}$ to project the signal $\mathbf{y}_r$ as

$$\hat{\mathbf{y}}_r = \mathbf{P}\mathbf{y}_r = \sum_{i=1}^{K} \mathbf{P}\mathbf{H}_{i,r}\mathbf{V}_i\mathbf{s}_i + \mathbf{P}\mathbf{n}_r, \tag{16}$$

so that the signals can be aligned as

$$\mathbf{P}\mathbf{H}_{i,r}\mathbf{V}_{i,j} = \mathbf{P}\mathbf{H}_{j,r}\mathbf{V}_{j,i}, \qquad \forall i,j \text{ with } [\mathbf{D}]_{i,j} \neq 0. \tag{17}$$

Note that the total DoF upper bound of the multi-user MIMO two-way relay channel is $\min\{KM, 2N\}$ from *Theorem 1*. If we can align each pair of data streams to be exchanged in a same subspace, the dimension of $\hat{\mathbf{y}}_r$ after projection must be greater than $\frac{1}{2}\min\{KM, 2N\}$. This indicates that $J \geq \min\{\frac{KM}{2}, N\}$. We refer to the above condition (17) as *generalized signal alignment equation*.

Remark 1: If $\mathbf{P}$ is not a full-rank matrix, this may reduce the dimension of signal space of the relay. To achieve a higher DoF, we assume that $\mathbf{P}$ is a full-rank matrix. In the following, we present the conditions for the existence of the precoding matrices $\{\mathbf{V}_{i,j} \mid [\mathbf{D}]_{i,j} \neq 0\}$ and the projection matrix $\mathbf{P}$ as well as their construction methods to meet the GSA equation (17).

Theorem 2 (*Existence of* $\mathbf{V}$): If there exists a projection matrix $\mathbf{P}$ such that for any source pair (i,j) with $[\mathbf{D}]_{i,j} \neq 0$ there are at least $J - 2M + d_{i,j}$ row vectors of $\mathbf{P}$ lie in the left null space of $[\mathbf{H}_{i,r} \ \ -\mathbf{H}_{j,r}]$, then there always exists a set of precoding matrices $\{\mathbf{V}_{i,j} \mid [\mathbf{D}]_{i,j} \neq 0\}$ so that the GSA equation (17) holds.

Proof: For any source pair (i,j) with $[\mathbf{D}]_{i,j} \neq 0$, the above alignment condition in (17) can be rewritten as

$$[\mathbf{P}\mathbf{H}_{i,r} \ \ -\mathbf{P}\mathbf{H}_{j,r}] \begin{bmatrix} \mathbf{V}_{i,j} \\ \mathbf{V}_{j,i} \end{bmatrix} = 0 \tag{18}$$

or

$$\mathbf{P}\left[\mathbf{H}_{i,r}\ \ -\mathbf{H}_{j,r}\right]\begin{bmatrix}\mathbf{V}_{i,j}\\ \mathbf{V}_{j,i}\end{bmatrix}=0. \tag{19}$$

Define $\mathbf{A}_{i,j}=[\mathbf{PH}_{i,r}\ \ -\mathbf{PH}_{j,r}]$. Clearly, the dimension of $\mathbf{A}_{i,j}$ is $J\times 2M$. For (18) to hold, one must have that the dimension of the null space of the matrix $\mathbf{A}_{i,j}$ should be greater than $d_{i,j}$ for signal alignment [7]. That is

$$d_{i,j}\leq 2M-\text{rank}(\mathbf{A}_{i,j})$$

or equivalently

$$\text{rank}(\mathbf{A}_{i,j})\leq 2M-d_{i,j}. \tag{20}$$

Hence, from (19), at least $J-2M+d_{i,j}$ row vectors of $\mathbf{P}$ should lie in the left null space of $[\mathbf{H}_{i,r}\ \ -\mathbf{H}_{j,r}]$.

Once we can find a projection matrix $\mathbf{P}$ to satisfy the above condition, we design the precoding matrices $\mathbf{V}_{i,j}$ and $\mathbf{V}_{j,i}$ for all source pairs (i,j) with $[\mathbf{D}]_{i,j}\neq 0$ as follows to meet the GSA equation (17):

$$\begin{bmatrix}\mathbf{V}_{i,j}\\ \mathbf{V}_{j,i}\end{bmatrix}\subseteq \textbf{Null}\ \left[\mathbf{PH}_{i,r}\ \ -\mathbf{PH}_{j,r}\right]. \tag{21}$$

■

In our proposed GSA, the signals to be exchanged do not align directly in the subspace observed by the relay. Instead, they are aligned in a projected subspace after the projection of $\mathbf{P}$ at the relay.

In this paper, we mainly analyze the achievability of the upper bound KM. It is worth mentioning that $\mathbf{PH}_{i,r}$ is a $J\times M$ matrix, where $J\geq \frac{KM}{2}$. To achieve this DoF upper bound, we need two assumptions.

Assumption 1: The total DoF of each source node is M. This indicates that $\sum\limits_{j\in\mathcal{S}_i} d_{i,j}=M$, $\forall i$.

Assumption 2: The number of data streams from source node i transmitted to source node j, $d_{i,j}$, is equal to the number of data streams from source node j transmitted to source node i. This means that the data switch matrix **D** is symmetric.

Theorem 3 (*Existence of* $\mathbf{P}$): If $N \geq (K-2)M + \max\{d_{i,j} \mid \forall i,j\}$, there exists a $\frac{KM}{2} \times N$ full-rank projection matrix $\mathbf{P}$ such that at least $\frac{KM}{2} - 2M + d_{i,j}$ row vectors of $\mathbf{P}$ will lie in the left null space of $[\mathbf{H}_{i,r} \ \ -\mathbf{H}_{j,r}]$, for any source pair (i,j) with $[\mathbf{D}]_{i,j} \neq 0$.

Proof: Let the $\frac{KM}{2} \times N$ projection matrix $\mathbf{P}$ be stacked by submatrices $\mathbf{P}_{i,j}$ by row, where $\mathbf{P}_{i,j}$ is a $d_{i,j} \times N$ matrix. Note that $\mathbf{P}_{i,j}$ exists if and only if $[\mathbf{D}]_{i,j} \neq 0$. Construct each submatrix $\mathbf{P}_{i,j}$ as

$$\mathbf{P}_{i,j}^T \subseteq \textbf{Null} \left[\mathbf{H}_{1,r} \ \cdots \ \mathbf{H}_{i-1,r} \ \mathbf{H}_{i+1,r} \ \cdots \ \mathbf{H}_{j-1,r} \ \mathbf{H}_{j,r} \ \cdots \ \mathbf{H}_{K,r}\right]^T. \tag{22}$$

From (22), we can obtain that $\mathbf{P}_{i,j}$ exists when

$$N - (K-2)M \geq d_{i,j}. \tag{23}$$

Remove the submatrices set $\{\mathbf{P}_{s,t} \mid s = i \text{ or } s = j \text{ or } t = i \text{ or } t = j\}$ from $\mathbf{P}$, the remaining submatrix of $\mathbf{P}$ is defined as $\mathbf{F}_{i,j}$. From (22), we can obtain that

$$\mathbf{F}_{i,j}^T \subseteq \textbf{Null} \ [\mathbf{H}_{i,r} \ \ -\mathbf{H}_{j,r}]^T \tag{24}$$

The number of rows of the matrix $\mathbf{F}_{i,j}$ is

$$\begin{aligned} N_{i,j} &= \frac{KM}{2} - (d_{1,i} + d_{2,i} + \cdots + d_{i-1,i} + d_{i,i+1} + \cdots + d_{i,K}) \\ &\quad - (d_{1,j} + d_{2,j} + \cdots + d_{j-1,j} + d_{j,j+1} + \cdots + d_{j,K}) + d_{i,j} \\ &= \frac{KM}{2} - M - M + d_{i,j} \\ &= \frac{KM}{2} - 2M + d_{i,j}. \end{aligned} \tag{25}$$

Therefore, when $N \geq (K-2)M + \max\{d_{i,j} \mid \forall i,j\}$, the projection matrix $\mathbf{P}$ can be constructed with the method of (22) and at least $\frac{KM}{2} - 2M + d_{i,j}$ row vectors of $\mathbf{P}$ will lie in the left null space of $[\mathbf{H}_{i,r} \ \ -\mathbf{H}_{j,r}]$, for any $[\mathbf{D}]_{i,j} \neq 0$. ■

B. An example

In this subsection, we use the 4-user MIMO Y channel, a special case of the multi-user MIMO two-way relay channels, to demonstrate the GSA. We consider the simplest case with $M = 3$

and $d_{i,j} = 1$ for any $i \neq j$. The corresponding data switch matrix $\mathbf{D}$ is

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}. \tag{26}$$

In what follows, we show how to implement generalized signal alignment when there are $N = 7$ antennas at the relay node.

We design a 6×7 projection matrix $\mathbf{P}$ at the relay using the proof of *Theorem 3*. The key idea is that the dimension of $[\mathbf{PH}_{i,r} \ \ -\mathbf{PH}_{j,r}]$ is 6 while its rank is 5 for any $i \neq j$. Let

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_{1,2} \\ \mathbf{p}_{1,3} \\ \mathbf{p}_{1,4} \\ \mathbf{p}_{2,3} \\ \mathbf{p}_{2,4} \\ \mathbf{p}_{3,4} \end{bmatrix} \tag{27}$$

where

$$\begin{aligned} \mathbf{p}_{1,2}^T &\subseteq \textbf{Null}\ \left[\mathbf{H}_{3,r}\ \mathbf{H}_{4,r}\right]^T, & \mathbf{p}_{1,3}^T &\subseteq \textbf{Null}\ \left[\mathbf{H}_{2,r}\ \mathbf{H}_{4,r}\right]^T, \\ \mathbf{p}_{1,4}^T &\subseteq \textbf{Null}\ \left[\mathbf{H}_{2,r}\ \mathbf{H}_{3,r}\right]^T, & \mathbf{p}_{2,3}^T &\subseteq \textbf{Null}\ \left[\mathbf{H}_{1,r}\ \mathbf{H}_{4,r}\right]^T, \\ \mathbf{p}_{2,4}^T &\subseteq \textbf{Null}\ \left[\mathbf{H}_{1,r}\ \mathbf{H}_{3,r}\right]^T, & \mathbf{p}_{3,4}^T &\subseteq \textbf{Null}\ \left[\mathbf{H}_{1,r}\ \mathbf{H}_{2,r}\right]^T. \end{aligned} \tag{28}$$

After that, we can design the precoding matrix for each source node using the proof of *Theorem 2*. Fig. 2 illustrates the notion of the generalized signal alignment in the MAC phase where there are 6 network coded symbols aligned at relay.

For pair (1, 2), we first define

$$\mathbf{C}_{1,2} \triangleq [\mathbf{PH}_{1,r} \ \ -\mathbf{PH}_{2,r}] = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} & c_{1,5} & c_{1,6} \\ c_{2,1} & c_{2,2} & c_{2,3} & 0 & 0 & 0 \\ c_{3,1} & c_{3,2} & c_{3,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{4,4} & c_{4,5} & c_{4,6} \\ 0 & 0 & 0 & c_{5,4} & c_{5,5} & c_{5,6} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{29}$$

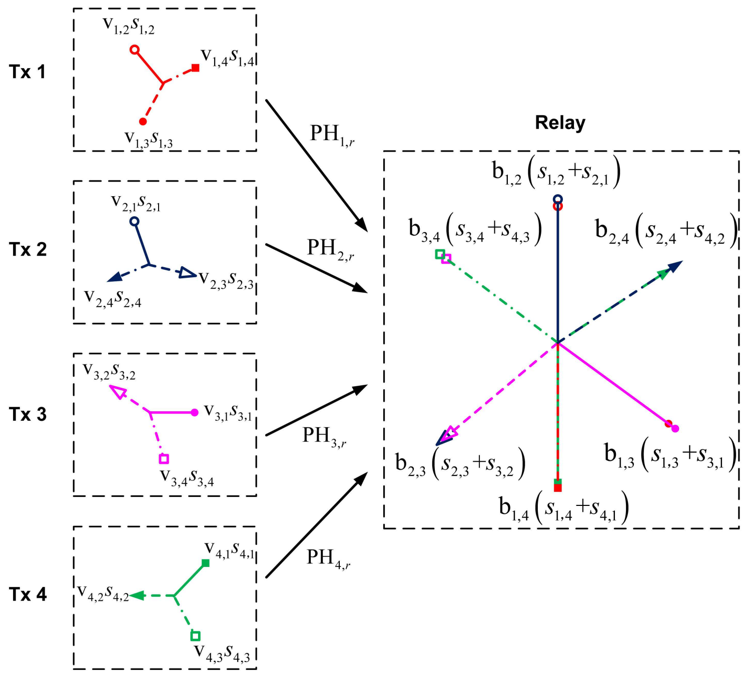

Fig. 2. Design of the precoding matrices for alignment.

Then we can choose the precoding vectors $\mathbf{v}_{1,2}$ and $\mathbf{v}_{2,1}$ as

$$\begin{bmatrix} \mathbf{v}_{1,2} \\ \mathbf{v}_{2,1} \end{bmatrix} \subseteq \textbf{Null } \mathbf{C}_{1,2}. \tag{30}$$

After simple manipulation, the equivalent channel vector seen by $s_{1,2}$ and $s_{2,1}$ becomes

$$\mathbf{b}_{1,2} = \mathbf{PH}_{1,r}\mathbf{v}_{1,2} = \mathbf{PH}_{2,r}\mathbf{v}_{2,1} = \alpha_{1,2}[1\ 0\ 0\ 0\ 0\ 0]^T, \tag{31}$$

where $\alpha_{1,2}$ is a constant.

Similarly, we can obtain all the precoding vectors $\mathbf{v}_{i,j}$ and $\mathbf{v}_{j,i}$ for each source node pair with the proof of *Theorem 2* and the direction of the aligned signals is

$$\begin{aligned}
\mathbf{b}_{1,3} &= \mathbf{PH}_{1,r}\mathbf{v}_{1,2} = \mathbf{PH}_{3,r}\mathbf{v}_{3,1} = \alpha_{1,3}[0\ 1\ 0\ 0\ 0\ 0]^T, \\
\mathbf{b}_{1,4} &= \mathbf{PH}_{1,r}\mathbf{v}_{1,2} = \mathbf{PH}_{4,r}\mathbf{v}_{4,1} = \alpha_{1,4}[0\ 0\ 1\ 0\ 0\ 0]^T, \\
\mathbf{b}_{2,3} &= \mathbf{PH}_{2,r}\mathbf{v}_{2,3} = \mathbf{PH}_{3,r}\mathbf{v}_{3,2} = \alpha_{2,3}[0\ 0\ 0\ 1\ 0\ 0]^T, \\
\mathbf{b}_{2,4} &= \mathbf{PH}_{2,r}\mathbf{v}_{2,4} = \mathbf{PH}_{4,r}\mathbf{v}_{4,2} = \alpha_{2,4}[0\ 0\ 0\ 0\ 1\ 0]^T, \\
\mathbf{b}_{3,4} &= \mathbf{PH}_{3,r}\mathbf{v}_{3,4} = \mathbf{PH}_{4,r}\mathbf{v}_{4,3} = \alpha_{3,4}[0\ 0\ 0\ 0\ 0\ 1]^T.
\end{aligned} \tag{32}$$

Therefore, the overall received signals after projection at the relay can be written as

$$\hat{\mathbf{y}}_r = \begin{bmatrix} \alpha_{1,2}(s_{1,2}+s_{2,1}) \\ \alpha_{1,3}(s_{1,3}+s_{3,1}) \\ \alpha_{1,4}(s_{1,4}+s_{4,1}) \\ \alpha_{2,3}(s_{2,3}+s_{3,2}) \\ \alpha_{2,4}(s_{2,4}+s_{4,2}) \\ \alpha_{3,4}(s_{3,4}+s_{4,3}) \end{bmatrix} + \mathbf{P}\mathbf{n}_r. \tag{33}$$

Now we can see that the signal pairs $s_{i,j}$ and $s_{j,i}$ are not only aligned in the same dimension but also in orthogonal dimensions. The following network-coded symbol vector can be easily estimated from $\hat{\mathbf{y}}_r$ at the relay:

$$\hat{\mathbf{s}}_{\oplus} = [s_{1,2}+s_{2,1}, s_{1,3}+s_{3,1}, s_{1,4}+s_{4,1}, s_{2,3}+s_{3,2}, s_{2,4}+s_{4,2}, s_{3,4}+s_{4,3}]^T . \tag{34}$$

IV. Analysis of DoF Achievability with Generalized Signal Alignment

In this section, we apply the GSA in some special cases of multi-user MIMO two-way relay channels, including K-user MIMO Y channel, multi-pair MIMO two-way relay channel, generalized MIMO two-way X relay channel, and L-cluster K-user multiway relay channel.

A. K-user MIMO Y channel

The considered K-user MIMO Y channel consists of K source nodes, each equipped with M antennas, and one relay node, equipped with N antennas. Each source node exchanges independent messages with all the other $K-1$ source nodes with the help of the relay. Previous studies [7] and [14] showed that the DoF upper bound $2N$ is achievable when $\frac{N}{M} \leq \frac{2K^2-2K}{K^2-K+2}$ and the DoF upper bound KM is achievable when $\frac{N}{M} \geq \frac{K^2-2K}{K-1}$. Our result is given in the following theorem.

Theorem 4: The DoF upper bound of KM is achievable for the K-user MIMO Y channel with generalized signal alignment when $\frac{N}{M} \geq \frac{K^2-3K+3}{K-1}$.

Proof: In the K-user MIMO Y channel, the data switch matrix $\mathbf{D}$ is

$$\mathbf{D} = \begin{bmatrix} 0 & d_{1,2} & d_{1,3} & \cdots & d_{1,K} \\ d_{2,1} & 0 & d_{2,3} & \cdots & d_{2,K} \\ d_{3,1} & d_{3,2} & 0 & \cdots & d_{3,K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{K,1} & d_{K,2} & d_{K,3} & \cdots & 0 \end{bmatrix}, \tag{35}$$

where $\{d_{i,j} \mid i \neq j\}$ can be any nonnegative integer.

From **Assumption 1** in the Section III-A, we have

$$d_i = \sum_{j=1, j\neq i}^{K} d_{i,j} = M. \tag{36}$$

It is obvious that

$$\max\{d_{i,j}\} \geq \frac{M}{K-1}. \tag{37}$$

and the equality holds when $d_{i,j} = \frac{M}{K-1}$, for $\forall i, j \neq i$.

From *Theorem 3* and (37), it can be shown that the necessary condition for the existence of the projection matrix $\mathbf{P}$ is

$$\begin{aligned} N &\geq (K-2)M + \max\{d_{i,j} \mid \forall i, j\} \\ &\geq (K-2)M + \frac{M}{K-1} \\ &= \frac{(K^2 - 3K + 3)M}{K-1}, \end{aligned} \tag{38}$$

which is equivalent to $\frac{N}{M} \geq \frac{K^2-3K+3}{K-1}$.

Next, we show how to achieve the DoF upper bound when $\frac{N}{M} = \frac{K^2-3K+3}{K-1}$. We let $d_{i,j}$ be $\frac{M}{K-1}$ for all $i \neq j$. The data switch matrix $\mathbf{D}$ is

$$\mathbf{D} = \begin{bmatrix} 0 & \frac{M}{K-1} & \cdots & \frac{M}{K-1} & \frac{M}{K-1} \\ \frac{M}{K-1} & 0 & \cdots & \frac{M}{K-1} & \frac{M}{K-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{M}{K-1} & \frac{M}{K-1} & \cdots & 0 & \frac{M}{K-1} \\ \frac{M}{K-1} & \frac{M}{K-1} & \cdots & \frac{M}{K-1} & 0 \end{bmatrix}. \tag{39}$$

Here, we separate the analysis into two cases.

Case 1: M is divisible by $K-1$. Denote

$$\mathbf{s}_{\oplus} = \begin{bmatrix} \mathbf{s}_{1,2} + \mathbf{s}_{2,1} \\ \mathbf{s}_{1,3} + \mathbf{s}_{3,1} \\ \vdots \\ \mathbf{s}_{i,j} + \mathbf{s}_{j,i} \\ \vdots \\ \mathbf{s}_{K-1,K} + \mathbf{s}_{K,K-1} \end{bmatrix} \tag{40}$$

as the network-coded messages expected to obtain at the relay, where each $\mathbf{s}_{i,j}$ is a $\frac{M}{K-1} \times 1$ vector.

We design the projection matrix $\mathbf{P}$ by the method of *Theorem 3*. Each $\frac{M}{K-1} \times N$ submatrix $\mathbf{P}_{i,j}$ is designed as

$$\mathbf{P}_{i,j}^T \subseteq \textbf{Null} \left[\mathbf{H}_{1,r} \ \cdots \ \mathbf{H}_{i-1,r} \ \mathbf{H}_{i+1,r} \ \cdots \ \mathbf{H}_{j-1,r} \ \mathbf{H}_{j,r} \ \cdots \ \mathbf{H}_{K,r}\right]^T. \tag{41}$$

Then we design the $M \times M$ precoding matrix $\mathbf{V}_i$ for each source node by the method of *Theorem 2*. Each pair of $M \times \frac{M}{K-1}$ precoding matrices is designed as

$$\begin{bmatrix} \mathbf{V}_{i,j} \\ \mathbf{V}_{j,i} \end{bmatrix} \subseteq \textbf{Null} \ \left[\mathbf{P}\mathbf{H}_{i,r} \ \ -\mathbf{P}\mathbf{H}_{j,r}\right]. \tag{42}$$

Similar to (31), we can obtain the direction of the aligned signals of signal pair (1, 2) as

$$\mathbf{B}_{1,2} = \mathbf{P}\mathbf{H}_{1,r}\mathbf{V}_{1,2} = \mathbf{P}\mathbf{H}_{3,r}\mathbf{V}_{3,1} = \begin{bmatrix} \alpha_{1,2}^1 & 0 & \cdots & 0 \\ 0 & \alpha_{1,2}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{1,2}^{d_{1,2}} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{43}$$

where $\alpha_{1,2}^l$ $(1 \leq l \leq d_{1,2})$ is a constant and $\mathbf{B}_{1,2}$ is a $\frac{KM}{2} \times d_{1,2}$ matrix.

Plugging (41) and (42) into (16), we can obtain the signals after projection as

$$\hat{\mathbf{y}}_r = \boldsymbol{\alpha}\mathbf{s}_{\oplus} + \mathbf{P}\mathbf{n}_r \tag{44}$$

where $\boldsymbol{\alpha} = [\alpha_{1,2}^{1}\ \alpha_{1,2}^{2}\ \cdots\ \alpha_{1,2}^{d_{1,2}}\ \cdots\ \alpha_{i,j}^{1}\ \cdots\ \alpha_{i,j}^{d_{i,j}}\ \cdots\ \alpha_{K-1,K}^{1}\ \cdots\ \alpha_{K-1,K}^{d_{K-1,K}}]$. Then, the network-coded symbol vector can be easily estimated from (44).

During the BC phase, we use the method of interference nulling to design the precoding matrix $\mathbf{U}$. We can write $\mathbf{U}$ as follows.

$$\mathbf{U} = \left[\ \mathbf{U}_1\ \mathbf{U}_2\ \cdots\ \mathbf{U}_{\frac{K(K-1)}{2}}\ \right], \tag{45}$$

where each $\mathbf{U}_i$ is an $N \times \frac{M}{K-1}$ matrix and

$$\begin{aligned} \mathbf{U}_1 &\subseteq \textbf{Null}\ \left[\mathbf{G}_{r,3}^T\ \mathbf{G}_{r,4}^T\ \cdots\ \mathbf{G}_{r,K}^T\right]^T \\ \mathbf{U}_2 &\subseteq \textbf{Null}\ \left[\mathbf{G}_{r,2}^T\ \mathbf{G}_{r,4}^T\ \cdots\ \mathbf{G}_{r,K}^T\right]^T \\ &\vdots \\ &\vdots \\ \mathbf{U}_{\frac{K(K-1)}{2}} &\subseteq \textbf{Null}\ \left[\mathbf{G}_{r,1}^T\ \mathbf{G}_{r,2}^T\ \cdots\ \mathbf{G}_{r,K-2}^T\right]^T \end{aligned} \tag{46}$$

We can see that $\left[\mathbf{G}_{r,1}^T\ \cdots\ \mathbf{G}_{r,s-1}^T\ \mathbf{G}_{r,s+1}^T\ \cdots\ \mathbf{G}_{r,t-1}^T\ \mathbf{G}_{r,t+1}^T\ \cdots\ \mathbf{G}_{r,K}^T\right]^T$ is n $(K-2)M \times N$ matrix. The matrix $\mathbf{U}_i$ ($N \times \frac{M}{K-1}$) exists if and only if $N - (K-2)M \geq \frac{M}{K-1}$, or equivalently $N \geq \frac{(K^2-3K+3)M}{K-1}$. Hence, we can apply GSA-based transmission scheme when M is divisible by $K-1$ and $N \geq \frac{(K^2-3K+3)M}{K-1}$ to achieve the DoF upper bound KM.

Case 2: M is not divisible by $K-1$. In this case, we use the idea of the symbol extension [18] together with GSA to prove the achievability of the DoF upper bound KM. We consider the $(K-1)$-symbol extension of the channel model, where the channel coefficients do not necessarily vary over time. The received signal at relay can be written as

$$\begin{aligned} \mathbf{y}_r &= \begin{bmatrix} \mathbf{y}_r(1) \\ \mathbf{y}_r(2) \\ \vdots \\ \mathbf{y}_r(K-1) \end{bmatrix} = \begin{bmatrix} \mathbf{H}(1) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}(2) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H}(K-1) \end{bmatrix} \begin{bmatrix} \mathbf{x}(1) \\ \mathbf{x}(2) \\ \vdots \\ \mathbf{x}(K-1) \end{bmatrix} + \begin{bmatrix} \mathbf{n}_r(1) \\ \mathbf{n}_r(2) \\ \vdots \\ \mathbf{n}_r(K-1) \end{bmatrix} \\ &= \mathbf{H}^{\S}\mathbf{x}^{\S} + \mathbf{n}_r^{\S}. \end{aligned} \tag{47}$$

where $\mathbf{y}_r(t)$, $\mathbf{H}(t)$, $\mathbf{x}(t)$ and $\mathbf{n}_r(t)$ denote the t-th time slot of received signals, channel matrices, transmitted signals and noise, $\mathbf{H}^{\S}$ denotes the equivalent channel matrix, $\mathbf{x}^{\S}$ denotes the equivalent transmitted signals, and $\mathbf{n}_r^{\S}$ denotes the equivalent noise.

Note that $\mathbf{H}^{\S}$ is a $(K-1)N \times (K-1)KM$ matrix. The system model is equivalent to the K-user MIMO Y channel with each source node equipped with $(K-1)M$ antennas and the relay equipped with $(K-1)N$ antennas. It turns to be **Case 1** and we can then apply GSA to achieve the DoF $(K-1)KM$ over $(K-1)$ channel uses. This implies that the DoF of KM per channel use is achievable in the original K-user MIMO Y channel. The antenna constraint can be written as

$$(K-1)N - (K-2)(K-1)M \geq \frac{(K-1)M}{K-1}$$

or equivalently,

$$N \geq \frac{(K^2-3K+3)M}{K-1}. \tag{48}$$

The above analysis shows that the generalized signal alignment based transmission scheme can achieve the DoF of KM when $N \geq \frac{(K^2-3K+3)M}{K-1}$ in K-user MIMO Y channel. On the other hand, it is clear to see that under the condition $N \geq \frac{(K^2-3K+3)M}{K-1} \geq \frac{KM}{2}$, KM is also the the DoF upper bound of the K-user MIMO Y channel. Thus, the proof of the theorem completes. ■

Remark 2: Since $K \geq 3$, we have $K^2-3K+3 \leq K^2-2K$. Thus, the achievable region $\frac{N}{M} \geq \frac{K^2-2K}{K-1}$ of the DoF upper bound in [14] is only a subset of ours.

Theorem 5: The DoF of $\frac{K(K-1)N}{K^2-3K+3}$ is achievable for the K-user MIMO Y channel with generalized signal alignment when $\frac{K}{2} \leq \frac{N}{M} < \frac{(K^2-3K+3)}{K-1}$.

Proof: When $\frac{K}{2} \leq \frac{N}{M} < \frac{K^2-3K+3}{K-1}$, the number of antennas at the relay, N, no longer satisfies *Theorem 3*. To tackle this problem, we use the method of antenna deactivation [14]. Let each source node only use $\frac{(K-1)N}{K^2-3K+3} \triangleq \tilde{M}$ antennas. Then this leads to $\frac{N}{\tilde{M}} = \frac{(K^2-3K+3)}{K-1}$. From *Theorem 4*, the DoF of $K\tilde{M}$, i.e. $\frac{K(K-1)N}{K^2-3K+3}$, is achievable. If $\tilde{M}$ is not an integer, we can utilize the symbol extension method similar to the **case 2** in *Theorem 4*. ■

Fig. 3 and Fig. 4 plot the total DoF of the K-user MIMO Y channel achieved by the proposed GSA for $K=4$ and $K \geq 4$, respectively, in comparison to the existing results [19] and [14]. It is seen that our result offers the new cases of antenna configuration $\frac{K^2-3K+3}{K-1} < \frac{N}{M} \leq \frac{K^2-2K}{K-1}$ where the DoF upper bound KM is tight.

B. Multi-pair MIMO two-way relay channel

The multi-pair MIMO two-way relay channel consists of $\frac{K}{2}$ pairs of source nodes, each source node equipped with M antennas, and one relay node, equipped with N antennas. The two source

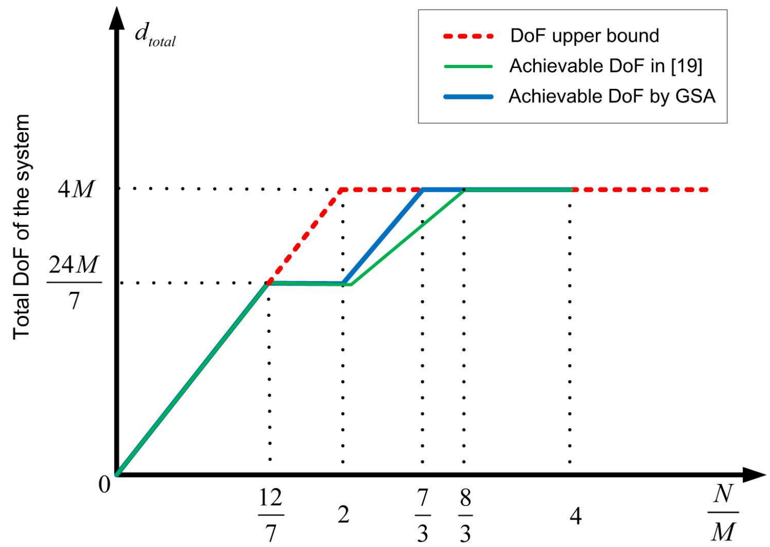

Fig. 3. Achievable DoF for 4-user MIMO Y channel.

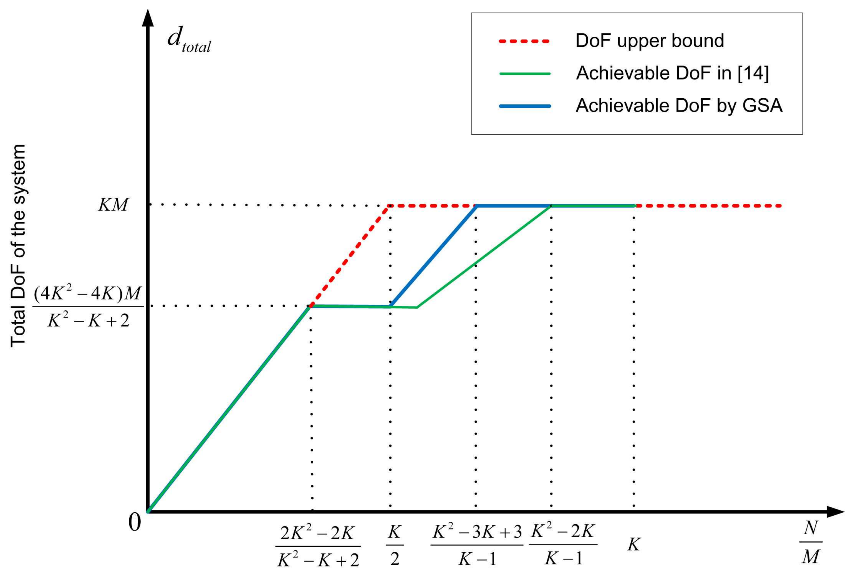

Fig. 4. Achievable DoF for K-user MIMO Y channel.

nodes, denoted as i and $K+1-i$ exchange messages with each other with the help of the relay. Previously, the authors in [11] show that the DoF upper bound $2N$ is achievable when $N \leq \frac{2KM}{K+2}$ and the DoF upper bound KM is achievable when $N \geq KM$. Our result is given in the following theorem.

Theorem 6: The DoF upper bound of KM is achievable for the multi-pair MIMO two-way relay channel when $\frac{N}{M} \geq (K-1)$.

Proof: The data switch matrix $\mathbf{D}$ for the multi-pair MIMO two-way relay channel is

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & \cdots & 0 & M \\ 0 & 0 & \cdots & M & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & M & \cdots & 0 & 0 \\ M & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{49}$$

From *Theorem 3* and (49), it can be shown that the necessary condition for the existence of the projection matrix $\mathbf{P}$ is

$$\begin{aligned} N &\geq (K-2)M + \max\{d_{i,\bar{i}} \mid \forall i \leq \frac{K}{2}\} \\ &\geq (K-2)M + M \\ &= (K-1)M, \end{aligned} \tag{50}$$

where the source node $\bar{i} = K + 1 - i$ is the pair of source node i.

We design the projection matrix $\mathbf{P}$ by the method of *Theorem 3*. Each $M \times N$ submatrix $\mathbf{P}_{i,j}$ is designed as

$$\mathbf{P}_{i,\bar{i}}^T \subseteq \textbf{Null} \left[\mathbf{H}_{1,r} \ \cdots \ \mathbf{H}_{i-1,r} \ \mathbf{H}_{i+1,r} \ \cdots \ \mathbf{H}_{\bar{i}-1,r} \ \mathbf{H}_{\bar{i}+1,r} \ \cdots \ \mathbf{H}_{K,r}\right]^T. \tag{51}$$

Then we design the $M \times M$ precoding matrix $\mathbf{V}_i$ for each source node by the method of *Theorem 2*. Each source pair of source precoding matrices is designed as

$$\begin{bmatrix} \mathbf{V}_{i,\bar{i}} \\ \mathbf{V}_{\bar{i},i} \end{bmatrix} \subseteq \textbf{Null} \left[\mathbf{P}\mathbf{H}_{i,r} \ \ -\mathbf{P}\mathbf{H}_{\bar{i},r}\right]. \tag{52}$$

Plugging (51) and (52) into (16), we can obtain the signals after projection as

$$\hat{\mathbf{y}}_r = \boldsymbol{\alpha}\mathbf{s}_{\oplus} + \mathbf{P}\mathbf{n}_r \tag{53}$$

where $\boldsymbol{\alpha} = [\alpha_{1,K}^1 \ \alpha_{1,K}^2 \ \cdots \ \alpha_{1,K}^M \ \cdots \ \alpha_{i,\bar{i}}^1 \ \cdots \ \alpha_{i,\bar{i}}^M \ \cdots \ \alpha_{\frac{K}{2},\frac{K}{2}+1}^1 \ \cdots \ \alpha_{\frac{K}{2},\frac{K}{2}+1}^M]$ and $\mathbf{s}_{\oplus} = [\mathbf{s}_{1,K}^T + \mathbf{s}_{K,1}^T, \mathbf{s}_{2,K-1}^T + \mathbf{s}_{K-1,2}^T, \cdots, \mathbf{s}_{i,\bar{i}}^T + \mathbf{s}_{\bar{i},i}^T, \cdots, \mathbf{s}_{\frac{K}{2},\frac{K}{2}+1}^T + \mathbf{s}_{\frac{K}{2}+1,\frac{K}{2}}^T]^T$. Then, the network-coded symbol vector can be easily estimated from (53).

During the BC phase, we use similar interference nulling method as in the K-user MIMO Y channel to design the precoding matrix **U**. We can write **U** as follows.

$$\mathbf{U} = \left[\ \mathbf{U}_1 \ \mathbf{U}_2 \ \cdots \ \mathbf{U}_{\frac{K}{2}} \ \right] \tag{54}$$

where each $\mathbf{U}_i$ is an $N \times M$ matrix and

$$\begin{aligned}
\mathbf{U}_1 &\subseteq \textbf{Null}\left[\mathbf{G}_{r,2}^T\ \mathbf{G}_{r,3}^T\ \cdots\ \mathbf{G}_{r,K-1}^T\right]^T \\
\mathbf{U}_2 &\subseteq \textbf{Null}\left[\mathbf{G}_{r,1}^T\ \mathbf{G}_{r,3}^T\ \cdots\ \mathbf{G}_{r,K-2}^T\ \mathbf{G}_{r,K}^T\right]^T \\
&\vdots \\
&\vdots \\
\mathbf{U}_{\frac{K}{2}} &\subseteq \textbf{Null}\left[\mathbf{G}_{r,1}^T\ \mathbf{G}_{r,2}^T\ \cdots\ \mathbf{G}_{r,\frac{K}{2}-1}^T\ \mathbf{G}_{r,\frac{K}{2}+2}^T\ \cdots\ \mathbf{G}_{K}^T\right]^T
\end{aligned} \tag{55}$$

We can see that $\left[\mathbf{G}_{r,1}^T\ \cdots\ \mathbf{G}_{r,s-1}^T\ \mathbf{G}_{r,s+1}^T\ \cdots\ \mathbf{G}_{r,t-1}^T\ \mathbf{G}_{r,t+1}^T\ \cdots\ \mathbf{G}_{r,K}^T\right]^T$ is a $(K-2)M \times N$ matrix. $\mathbf{U}_i$ ($N \times M$) exists if and only if

$$N - (K-2)M \geq M \tag{56}$$

or equivalently,

$$N \geq (K-1)M. \tag{57}$$

The rest of the proof is similar to that in the proof of *Theorem 4*. ■

Theorem 7: The DoF of $\frac{KN}{K-1}$ is achievable for the multi-pair MIMO two-way relay channel with generalized signal alignment when $\frac{K}{2} \leq \frac{N}{M} < K-1$.

Proof: When $\frac{K}{2} \leq \frac{N}{M} < K-1$, the number of antennas at the relay, N, no longer satisfies *Theorem 3*. Similar to the proof of *Theorem 5*, we use the antenna deactivation. Let each source node only use $\frac{N}{K-1} \triangleq \tilde{M}$ antennas. Then this leads to $\frac{N}{\tilde{M}} = K-1$. From *Theorem 6*, the DoF of $K\tilde{M}$, i.e. $\frac{KN}{K-1}$, is achievable. ■

Fig. 5 plots the total DoF of the multi-pair MIMO two-way relay channel achieved by the proposed GSA in comparison to the existing result [11]. It is seen that our result offers the new cases of antenna configuration $K-1 < \frac{N}{M} \leq K$ where the DoF upper bound KM is tight.

C. Generalized MIMO two-way X relay channel

The generalized MIMO two-way X relay channel consists of two groups of source nodes of size $\frac{K}{2}$, each equipped with M antennas, and one relay node, equipped with N antennas. Each source node in one group, denoted as $i = 1,\ 2,\ \cdots,\ \frac{K}{2}$, exchanges independent messages with every source node in the other group, denoted as $i = \frac{K}{2}+1,\ \frac{K}{2}+2,\ \cdots,\ K$, with the help of the relay. In the special case of $K = 4$, i.e. MIMO two-way X relay channel, the work in [9]

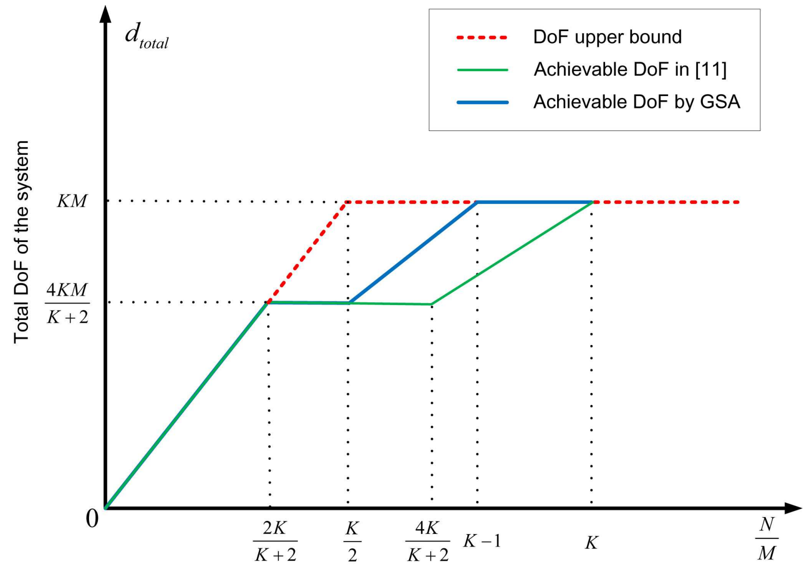

Fig. 5. Achievable DoF for multi-pair MIMO two-way relay channel.

showed that the DoF of $2N$ is achievable when $N \leq \lfloor \frac{8M}{5} \rfloor$; the work in [15] showed that the DoF of $4M$ is achievable when $N \geq \lceil \frac{5M}{2} \rceil$. Our result is given in the following theorem.

Theorem 8: The DoF upper bound of KM is achievable for the generalized MIMO two-way X relay channel when $\frac{N}{M} \geq \frac{K^2-2K+2}{K}$.

Proof: In the generalized MIMO two-way X relay channel, the data switch matrix $\mathbf{D}$ is

$$\mathbf{D} = \begin{bmatrix} 0 & \cdots & 0 & d_{1,\frac{K}{2}+1} & \cdots & d_{1,K} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & d_{\frac{K}{2},\frac{K}{2}+1} & \cdots & d_{\frac{K}{2},K} \\ d_{\frac{K}{2}+1,1} & \cdots & d_{\frac{K}{2}+1,\frac{K}{2}} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ d_{K,1} & \cdots & d_{K,\frac{K}{2}} & 0 & \cdots & 0 \end{bmatrix}. \tag{58}$$

From **Assumption 1** in the Section III-A, we have

$$d_i = \begin{cases} \sum_{j=\frac{K}{2}+1}^{K} d_{i,j} = M, & 1 \leq i \leq \frac{K}{2}; \\ d_i = \sum_{j=1}^{\frac{K}{2}} d_{i,j} = M, & \frac{K}{2}+1 \leq i \leq K. \end{cases}$$

It is obvious that

$$\max\{d_{i,j}\} \geq \frac{M}{\frac{K}{2}} = \frac{2M}{K}. \tag{59}$$

where the equality holds if $d_{i,j} = \frac{2M}{K}$, for $\forall i, j$ with $[\mathbf{D}]_{i,j} \neq 0$.

From *Theorem 3* and (59), it can be shown that the necessary condition of the existence of the projection matrix $\mathbf{P}$ with *Theorem 3* is

$$\begin{aligned} N &\geq (K-2)M + \max\{d_{i,j} \mid \forall i, j\} \\ &\geq (K-2)M + \frac{2M}{K} \\ &= \frac{(K^2 - 2K + 2)M}{K}, \end{aligned} \tag{60}$$

which is equivalent to $\frac{N}{M} \geq \frac{K^2-2K+2}{K}$.

Next, we show how to achieve the DoF upper bound when $\frac{N}{M} = \frac{K^2-2K+2}{K}$. We let $\{d_{i,j} \mid 1 \leq i \leq \frac{K}{2}, \frac{K}{2}+1 \leq j \leq K$ or $1 \leq j \leq \frac{K}{2}, \frac{K}{2}+1 \leq i \leq K\}$ be $\frac{2M}{K}$. The data switch matrix $\mathbf{D}$ is

$$\mathbf{D} = \begin{bmatrix} 0 & \cdots & 0 & \frac{2M}{K} & \cdots & \frac{2M}{K} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \frac{2M}{K} & \cdots & \frac{2M}{K} \\ \frac{2M}{K} & \cdots & \frac{2M}{K} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{2M}{K} & \cdots & \frac{2M}{K} & 0 & \cdots & 0 \end{bmatrix}. \tag{61}$$

Here, we separate the analysis into two cases.

Case 1: M is divisible by $\frac{K}{2}$. Denote

$$\mathbf{s}_{\oplus} = [\mathbf{s}_{1,2}^T + \mathbf{s}_{2,1}^T, \mathbf{s}_{1,3}^T + \mathbf{s}_{3,1}^T, \cdots, \mathbf{s}_{i,j}^T + \mathbf{s}_{j,i}^T, \cdots, \mathbf{s}_{K-1,K}^T + \mathbf{s}_{K,K-1}^T]^T \tag{62}$$

as the network-coded messages expected to obtain at the relay, where each $\mathbf{s}_{i,j}$ is a $d_{i,j} \times 1$ vector.

We design the projection matrix $\mathbf{P}$ by the method of *Theorem 3*. Each $\frac{2M}{K} \times N$ submatrix $\mathbf{P}_{i,j}$ is designed as

$$\mathbf{P}_{i,j}^T \subseteq \textbf{Null} \left[\mathbf{H}_{1,r} \ \cdots \ \mathbf{H}_{i-1,r} \ \mathbf{H}_{i+1,r} \ \cdots \ \mathbf{H}_{j-1,r} \ \mathbf{H}_{j,r} \ \cdots \ \mathbf{H}_{K,r}\right]^T. \tag{63}$$

Then we design the $M \times M$ precoding matrix $\mathbf{V}_i$ for each source node by the method of *Theorem 2*. Each pair of $M \times \frac{2M}{K}$ precoding matrices is designed as

$$\begin{bmatrix} \mathbf{V}_{i,j} \\ \mathbf{V}_{j,i} \end{bmatrix} \subseteq \textbf{Null} \ [\mathbf{PH}_{i,r} \ \ -\mathbf{PH}_{j,r}]. \tag{64}$$

Plugging (63) and (64) into (16), we can obtain the signals after projection as

$$\hat{\mathbf{y}}_r = \boldsymbol{\alpha}\mathbf{s}_{\oplus} + \mathbf{P}\mathbf{n}_r \tag{65}$$

where $\boldsymbol{\alpha} = [\alpha_{1,2}^1 \ \alpha_{1,2}^2 \ \cdots \ \alpha_{1,2}^{d_{1,2}} \ \cdots \ \alpha_{i,j}^1 \ \cdots \ \alpha_{i,j}^{d_{i,j}} \ \cdots \ \alpha_{K-1,K}^1 \ \cdots \ \alpha_{K-1,K}^{d_{K-1,K}}]$. Then, the network-coded symbol vector can be easily estimated from (65).

During the BC phase, we use the method of interference nulling to design the precoding matrix **U**. We can write **U** as follows.

$$\mathbf{U} = \left[\ \mathbf{U}_1 \ \mathbf{U}_2 \ \cdots \ \mathbf{U}_{\frac{K^2}{4}} \ \right] \tag{66}$$

where each $\mathbf{U}_i$ is an $N \times \frac{2M}{K}$ matrix and

$$\begin{aligned}
\mathbf{U}_1 &\subseteq \textbf{Null} \left[\mathbf{G}_{r,2}^T \ \mathbf{G}_{r,3}^T \ \cdots \ \mathbf{G}_{r,\frac{K}{2}}^T \ \mathbf{G}_{r,\frac{K}{2}+2}^T \ \cdots \ \mathbf{G}_{r,K}^T\right]^T \\
\mathbf{U}_2 &\subseteq \textbf{Null} \left[\mathbf{G}_{r,2}^T \ \mathbf{G}_{r,3}^T \ \cdots \ \mathbf{G}_{r,\frac{K}{2}+1}^T \ \mathbf{G}_{r,\frac{K}{2}+3}^T \ \cdots \ \mathbf{G}_{r,K}^T\right]^T \\
&\vdots \\
&\vdots \\
\mathbf{U}_{\frac{K^2}{4}} &\subseteq \textbf{Null} \left[\mathbf{G}_{r,1}^T \ \mathbf{G}_{r,2}^T \ \cdots \ \mathbf{G}_{r,\frac{K}{2}-1}^T \ \mathbf{G}_{r,\frac{K}{2}+1}^T \ \cdots \ \mathbf{G}_{r,K-1}^T\right]^T
\end{aligned} \tag{67}$$

We can see that $\left[\mathbf{G}_{r,1}^T \ \cdots \ \mathbf{G}_{r,s-1}^T \ \mathbf{G}_{r,s+1}^T \ \cdots \ \mathbf{G}_{r,t-1}^T \ \mathbf{G}_{r,t+1}^T \ \cdots \ \mathbf{G}_{r,K}^T\right]^T$ is a $(K-2)M \times N$ matrix. $\mathbf{U}_i$ ($N \times \frac{2M}{K}$) exists if and only if $N-(K-2)M \geq \frac{2M}{K}$, or equivalently $N \geq \frac{(K^2-2K+2)M}{K}$. Hence, we can apply GSA-based transmission scheme when M is a multiple of $\frac{K}{2}$ to achieve the DoF upper bound KM.

Case 2: M **is not divisible by** $\frac{K}{2}$**.** We can utilize the ($\frac{K}{2}$)-symbol extension. The proof is similar to the **Case 2** of *Theorem 4*. We omit the detail proof here. The rest of the proof is similar to that of *Theorem 4*. ∎

Theorem 9: The DoF of $\frac{K^2N}{K^2-2K+2}$ is achievable for the generalized MIMO two-way X relay channel with generalized signal alignment when $\frac{K}{2} \leq \frac{N}{M} \leq \frac{(K^2-2K+2)M}{K}$.

Proof: When $\frac{K}{2} \leq \frac{N}{M} < \frac{K^2-2K+2}{K}$, the number of antennas at the relay N no longer satisfies *Theorem 3*. Therefore, we use the method of antenna deactivation [14]. Let each source node only use $\frac{KN}{K^2-2K+2} \triangleq \tilde{M}$ antennas. Then this leads to $\frac{N}{\tilde{M}} = \frac{(K^2-2K+2)M}{K}$. From *Theorem 8*, the DoF of $K\tilde{M}$, i.e. $\frac{K^2N}{K^2-2K+2}$, is achievable. ∎

Fig. 6 shows the achievable total DoF of the generalized MIMO two-way X relay channel using different schemes.

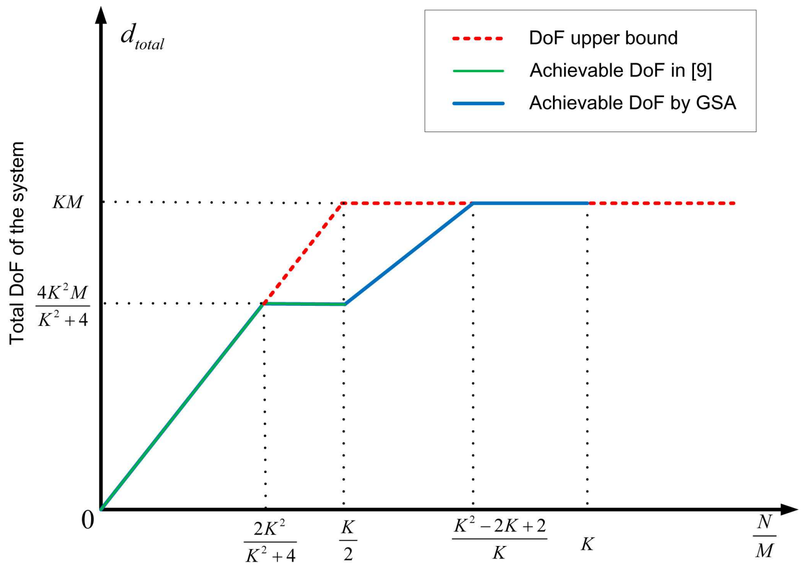

Fig. 6. Achievable DoF for generalized MIMO two-way X relay channel.

D. L-cluster K-user MIMO multiway relay channel

The DoF analysis of another popular channel, L-cluster K-user MIMO multiway relay channel, is first analyzed in [11]. It consists of L clusters of K source nodes, yielding $\tilde{K} = LK$ total number of source nodes, and one relay node. Each source node, equipped with M antennas, exchanges independent messages with each source node in the same cluster with the help of the relay, equipped with N antennas. The $\tilde{K} \times \tilde{K}$ data switch matrix $\mathbf{D}$ is

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}^{\S} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{\S} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D}^{\S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{D}^{\S} \end{bmatrix}, \tag{68}$$

where $\mathbf{D}^{\S}$ is a $K \times K$ matrix and $\mathbf{D}^{\S}$ is

$$\mathbf{D}^{\S} = \begin{bmatrix} 0 & d_{1,2} & d_{1,3} & \cdots & d_{1,K} \\ d_{2,1} & 0 & d_{2,3} & \cdots & d_{2,K} \\ d_{3,1} & d_{3,2} & 0 & \cdots & d_{3,K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{K,1} & d_{K,2} & d_{K,3} & \cdots & 0 \end{bmatrix}. \tag{69}$$

Theorem 10: The DoF upper bound of $\tilde{K}M$ is achievable for the L-cluster K-user MIMO multiway relay channel when $\frac{N}{M} \geq \frac{(K-1)(\tilde{K}-2)+1}{K-1}$.

Proof: Similar to (36) and (37), we should let all the nonzero entries of the matrix $\mathbf{D}$ in (68) be $\frac{K^2-2K+2}{K}$ to obtain the minimum number of antenna N at the relay with fixed M. The data switch matrix $\mathbf{D}^{\S}$ in each cluster is

$$\mathbf{D}^{\S} = \begin{bmatrix} 0 & \frac{M}{K-1} & \cdots & \frac{M}{K-1} & \frac{M}{K-1} \\ \frac{M}{K-1} & 0 & \cdots & \frac{M}{K-1} & \frac{M}{K-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{M}{K-1} & \frac{M}{K-1} & \cdots & 0 & \frac{M}{K-1} \\ \frac{M}{K-1} & \frac{M}{K-1} & \cdots & \frac{M}{K-1} & 0 \end{bmatrix}. \tag{70}$$

To satisfy the necessary condition in *Theorem 3*, we can derive the antenna constraint to achieve the DoF upper bound of $\tilde{K}M$ as

$$N - (\tilde{K}-2)M \geq \frac{M}{K-1} \tag{71}$$

or equivalently

$$N \geq \frac{(K-1)(\tilde{K}-2)+1}{K-1}M. \tag{72}$$

Clearly, this achievable region enlarges the one $N > (LK-1)M$ in [16] and the one $N > LKM$ in [11]. ■

Theorem 11: The DoF of $\frac{\tilde{K}(K-1)N}{(K-1)(\tilde{K}-2)+1}$ is achievable for the L-cluster K-user MIMO multiway relay channel with generalized signal alignment when $\frac{K}{2} \leq \frac{N}{M} \leq \frac{(K-1)(\tilde{K}-2)+1}{K-1}$.

Proof: When $\frac{K}{2} \leq \frac{N}{M} < \frac{(K-1)(\tilde{K}-2)+1}{K-1}$, the number of the antennas at relay N no longer satisfies *Theorem 3*. Therefore, we use the method of antenna deactivation [14]. Let each source node only use $\frac{(K-1)N}{(K-1)(\tilde{K}-2)+1} \triangleq \tilde{M}$ antennas. Then this leads to $\frac{N}{\tilde{M}} = \frac{(K-1)(\tilde{K}-2)+1}{K-1}$. From *Theorem 10*, the DoF of $\tilde{K}\tilde{M}$, i.e. $\frac{\tilde{K}(K-1)N}{(K-1)(\tilde{K}-2)+1}$, is achievable. ■

V. Numerical Results

In this section, we provide numerical results to show the sum rate performance of the proposed scheme for K-user MIMO Y channel, generalized MIMO two-way X relay channel and multi-pair MIMO two-way relay channel. The channel between each source node and the relay node is modeled as Rayleigh distribution with unit variance and it is independent for different node. The numerical results are illustrated with the ratio of the total transmitted signal power to the noise variance at each receive antenna and the total throughput of the channel. Each result is averaged over 10000 independent channel realizations.

We now explain how we compute the sum rate when applying the GSA transmission scheme. GSA transmission scheme can be used in both amplify-and-forward (AF) and decode-and-forward (DF) strategy. From (6) and (8), when we apply AF strategy, we can obtain

$$\begin{aligned}\mathbf{y}_i &= \mathbf{G}_{r,i}\mathbf{U}(\mathbf{s}_{\oplus} + \mathbf{A}\mathbf{n}_r) + \mathbf{n}_i \\ &= \mathbf{G}_{r,i}\mathbf{U}\mathbf{s}_{\oplus} + \mathbf{G}_{r,i}\mathbf{U}\mathbf{A}\mathbf{n}_r + \mathbf{n}_i \\ &= \underbrace{\mathbf{G}_{r,i}\big[\sum \mathbf{U}_i(\mathbf{s}_{s,t} + \mathbf{s}_{t,s})\big]}_{Signal} + \underbrace{\mathbf{G}_{r,i}\mathbf{U}\mathbf{A}\mathbf{n}_r + \mathbf{n}_i}_{Noise} \\ &= \tilde{\mathbf{G}}_{r,i}\tilde{\mathbf{s}}_i + \tilde{\mathbf{n}}_i. \end{aligned} \tag{73}$$

Let R_i denote the achievable sum rate in bits per channel use from all the other nodes to source node i. From (73), we have

$$R_i = \log_2\left[\det(\mathbf{I} + (\varepsilon(\tilde{\mathbf{n}}_i\tilde{\mathbf{n}}_i^H))^{-1}\tilde{\mathbf{G}}_{r,i}\varepsilon[\tilde{\mathbf{s}}_i\tilde{\mathbf{s}}_i^H]\tilde{\mathbf{G}}_{r,i}^H)\right]. \tag{74}$$

Then total sum rate of the system is given by

$$R = \sum_{i=1}^{K} R_i. \tag{75}$$

In Fig. 7, we plot the total sum rate performance of the proposed generalized signal alignment framework for the K-user MIMO Y channel at $\frac{N}{M} = \frac{K^2-3K+3}{K-1}$. In the figure, SNR denotes the total transmitted signal power from all the K source nodes to the noise variance at relay, i.e. SNR $= KP$. From Fig. 7, we observe that the increasing speed of sum-rate (the increase in bps/Hz for every 3dB in SNR) matches with the theoretical DoF KM very well when SNR is high enough.

In Fig. 8, we plot the sum rate performance of the proposed generalized signal alignment framework for the generalized MIMO two-way X relay channel when $K = 6$ and $N = \lceil\frac{(K^2-2K+2)M}{K}\rceil$. We observe that the increasing speed of sum-rate matches with the theoretical DoF KM very well when SNR is high enough.

In Fig. 9, we plot the sum rate performance of the proposed generalized signal alignment framework for the mutli-pair MIMO two-way relay channel when $K = 6$. It is observed that the increasing speed of sum-rate matches with the theoretical DoF KM very well with $N = (K-1)M$ when SNR is high enough.

The above numerical results validate that our proposed GSA transmission scheme is feasible and effective.

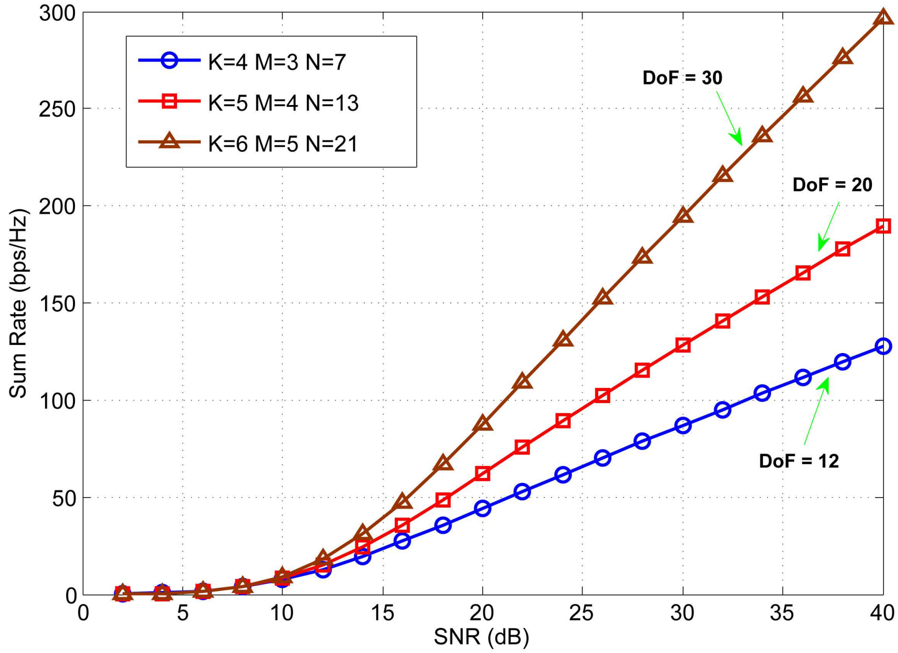

Fig. 7. The average sum rate for the K-user MIMO Y channel under generalized signal alignment with $M = K - 1$.

VI. Conclusion

In this paper, we have analyzed the achievability of the DoF upper bound when $N \geq 2M$ for multi-user MIMO two-way relay channels. In our proposed generalized signal alignment, the projection matrix at the relay and the precoding matrices at the source nodes are designed jointly so that the signals to be exchanged between each source node pair are aligned on a projected subspace at the relay. We showed that GSA is feasible when $N \geq (K-2)M + \max\{d_{i,j} \mid \forall i,j\}$, where $d_{i,j}$ is the number of independent data streams transmitted from source node i to source node j, if any. Our GSA-based analysis reveals new antenna configurations where the DoF upper bound is tight for several special cases of multi-user MIMO two-way relay channels, including the K-user MIMO Y channel, the multi-pair MIMO two-way relay channel, the generalized MIMO two-way X relay channel, and the L-cluster K-user MIMO multiway relay channel.

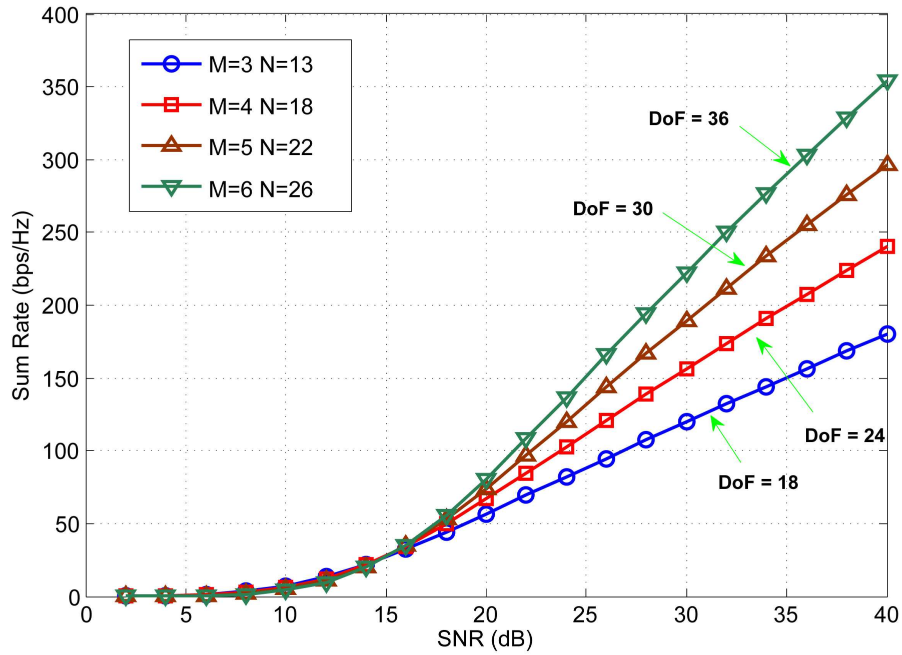

Fig. 8. The average sum rate for the generalized MIMO two-way X relay channel under generalized signal alignment with $K = 6$.

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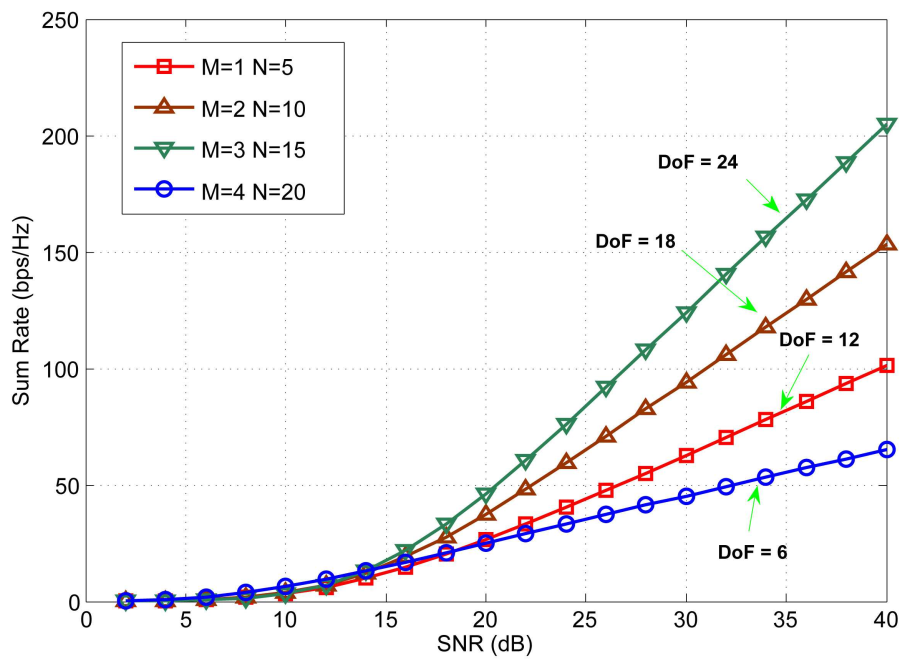

Fig. 9. The average sum rate for the multi-pair MIMO two-way relay channel under generalized signal alignment with $K = 6$.

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