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ACTIVE CONTROL OF SOUND FOR COMPOSITE REGIONS*

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Abstract. We present a methodology for the active control of time-harmonic wave fields, e.g., acoustic disturbances, in composite regions. This methodology extends our previous approach developed for the case of arcwise connected regions. The overall objective is to eliminate the effect of all outside field sources on a given domain of interest, i.e., to shield this domain. In this context, active shielding means introducing additional field sources, called active controls, that generate the annihilating signal and cancel out the unwanted component of the field. As such, the problem of active shielding can be interpreted as a special inverse source problem for the governing differential equation or system. For a composite domain, not only do the controls prevent interference from all exterior sources, but they can also enforce a predetermined communication pattern between the individual subdomains (as many as desired). In other words, they either allow the subdomains to communicate freely with one another or otherwise have them shielded from their peers. In the paper, we obtain a general solution for the composite active shielding problem and show that it reduces to solving a collection of auxiliary problems for arcwise connected domains. The general solution is constructed in two stages. Namely, if a particular subdomain is not allowed to hear another subdomain, then the supplementary controls are employed first. They communicate the required data prior to building the final set of controls. The general solution can be obtained with only the knowledge of the acoustic signals propagating through the boundaries of the subdomains. No knowledge of the field sources is required, nor is any knowledge of the properties of the medium needed.

Key words. active shielding, noise control, inverse source problem, time-harmonic acoustic fields, composite domain, communication pattern, the Helmholtz equation, generalized Calderon's potentials, exact volumetric cancellation, general solution, incoming and outgoing waves, wave split

AMS subject classifications. 35J05, 35C15, 31C99, 47G30, 35B37

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1. Introduction.

Active shielding and control of noise is a very rich field with a variety of applications. In the most general terms, exercising active control means introducing additional sources of sound, called controls, to facilitate a specific change in the overall acoustic field. In particular, the desired change may imply canceling all or part of the field on a given region. Referring the reader to other, more detailed sources for a comprehensive review (see [17, 8, 24]), we mention several representative publications in the area. Research by Elliott, Stothers, and Nelson [7] focused on the minimization of noise at pointwise locations. Wright and Vuksanovic expanded the field to include directional noise cancellation in [30, 31]. A large portion of the research done today has been motivated by the airline industry and its desire to control unwanted engine noise in the cabin during flight. There are various methods of dealing with in-flight noise. Damping structural vibrations is one approach to attenuating low frequencies. This is done by placing actuators and sensors throughout the cabin at optimized locations. Kincaid, Laba, and Padula worked extensively on this problem [12, 11]. A comprehensive account of the area, along with many additional references, can be found in [4]. Another method involves placing a series of microphones and

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speakers throughout the cabin and uses acoustic excitation to cancel unwanted noise. Passive techniques such as sound insulation are more effective in dealing with high frequencies. Van der Auweraer et al. tackled the problem of aircraft noise in [26] by using a combination of both methods.

In [9], Fuller and von Flotow present an overview of current common practices in active noise control. One of the most popular algorithms used today in the control of noise is based on a least mean squares (LMS) method. It is employed to tune the control filter to reduce unwanted noise near the sensors and was first introduced by Burgess in [1] and by Widrow, Shur, and Shaffer in [29]. This algorithm was later improved upon by Cabell and Fuller in [2]. While LMS methods offer good results near the sensors in small-scale applications such as mobile phones, they do not allow for the exact volumetric cancellation of noise desired in an airline cabin.

In the current paper, we introduce and study a new formulation of active noise control problem. Namely, the overall region of space to be protected from noise is assumed to be composed of a number of simple, i.e., arcwise connected, (sub)domains. The standard part of the formulation involves shielding the overall domain, i.e., the union of all subdomains, from the unwanted noise. In addition, the individual subdomains are selectively allowed to either communicate freely with one another according to a predetermined pattern or else be shielded from their peers. In doing so, no reciprocity is assumed; i.e., for a given pair of subdomains one may be allowed to hear the other, but not vice versa.

The method of analysis used in this paper builds upon the previous research done by Lončarić, Ryaben'kii, and Tsynkov in [13] and by Tsynkov in [25] for the case of a single arcwise connected domain, and subsequently extended in [14, 15, 16] by investigating various optimization formulations. The approach of [13] allows for the exact volumetric cancellation of time-harmonic noise in a given region. In other words, this region is shielded from the unwanted sound that comes from the outside. The shielding is achieved by first splitting the total acoustic field into the incoming and outgoing components. This can be done unambiguously using only the knowledge of the field and its normal derivative measured at the boundary of the region to be protected. Subsequently, the unwanted incoming component of the field is canceled by additional sources that are insensitive to the outgoing component. Other methods, such as those employed by Nelson and Elliott in [17], require that the noise to be canceled be measured at the boundary by itself, and be distinguished from other components of the acoustic field ahead of time. This restriction does not exist in the methodology presented herein. Moreover, our methodology requires knowledge of neither the volumetric properties of the medium nor the location and strength of the noise sources. Decomposition of the overall sound field into incoming and outgoing components, as well as design of the antinoise sources, are accomplished by applying Calderon's potentials and projections [3]; see also [23]. This is a very convenient and powerful apparatus that allows one to describe all appropriate control sources in closed form. In the simplest case of constant coefficients, the Calderon operators can be obtained using boundary integrals of classical potential theory.

There are two types of control sources that can be explored, volumetric and surface. In [13], it is determined that the general solution for volumetric controls $g = g(x)$ is given by

$$g(x) = -Lw$$

outside of the region to be shielded, where w is a special auxiliary function which must satisfy the Sommerfeld radiation condition at infinity, as well as coincide with the acoustic field u and its normal derivative $\frac{\partial u}{\partial n}$ at the boundary. Here $L = \Delta + k^2 I$ denotes the Helmholtz operator. Since these are fairly loose restrictions, volumetric controls define a very broad class of solutions to the problem.

Surface controls are concentrated at the boundary. They are given by

$$g^{(surf)} = -\left[\frac{\partial w}{\partial n} - \frac{\partial u}{\partial n}\right]_\Gamma \delta(\Gamma) - \frac{\partial}{\partial n}([w-u]_\Gamma \delta(\Gamma)),$$

where the auxiliary function w is additionally required to satisfy the homogeneous Helmholtz equation, $Lw = 0$, outside the boundary but is no longer required to satisfy any boundary conditions at Γ. The general solution to the surface control problem is discussed in [25]. It is to be noted that surface controls have the same fundamental properties as volumetric controls. A universal framework for both volumetric and surface controls is built by Ryaben'kii and Utyuzhnikov in the recent paper [22]; it treats the governing equation for the field in an operator form.

We should also emphasize that the continuous formulation is not practical for implementation. Any realistic implementation would consist only of a finite number of sensors (microphones) and actuators (speakers). This will lead to a discretization of the problem on a grid. Discrete active shielding problems were analyzed, and the corresponding solutions obtained in [14, 15, 16, 25], as well as more recently in [22]. The finite-difference analysis of [14, 15, 16, 22, 25] uses the constructs developed previously in the works by Ryaben'kii [18] and by Veizman and Ryaben'kii [27, 28].

Specific objective of the current paper. We will extend the methodology of [13] to the case of composite regions. This will allow two or more separate subregions to be fully protected from the influence of outside sources. Moreover, according to a predetermined communication pattern, each individual subregion may or may not be allowed to hear any other subregion. As in [13], only the total acoustic field and its normal derivative specified at the boundaries will be needed for the exact volumetric cancellation of the outside noise, as well as for the realization of a given communication pattern. It will not be necessary to distinguish the "adverse," i.e., unwanted, part of the acoustic field from its "friendly," i.e., wanted, part as this is done automatically by the control system. The methodology will provide a closed form general solution for the controls, including the case of an inhomogeneous medium.

2. Two regions.

In this section, the formulation for two separate domains will be discussed. In other words, we will distinguish between the two given disjoint regions and the rest of the space. Let it be noted, however, that the forthcoming methodology could be presented in a more general framework. The rest of the space outside of the two given regions can be treated as a third region on equal terms with the first two. This formulation lends itself more naturally to surface controls separating the three regions. A rigorous analysis of this approach for the finite-difference setting can be found in the recent paper [21], which, in turn, builds upon [18]. We, however, choose a simpler form of presentation in order to make it more accessible for applications. Accordingly, the focus will be on volumetric controls, which will allow for more flexibility in their construction.

2.1. Formulation.

Let Ω_1 and Ω_2 be given, where $\Omega_i \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$ is either bounded or unbounded. For simplicity we will first assume that Ω_1 and Ω_2 are two separate bounded regions of $\mathbb{R}^n$ (see Figure 2.1), and such that $\text{dist}(\Omega_1, \Omega_2) \geq \epsilon > 0$. Let Γ_1 and Γ_2 be the boundaries of Ω_1 and Ω_2, respectively. Consider the time-

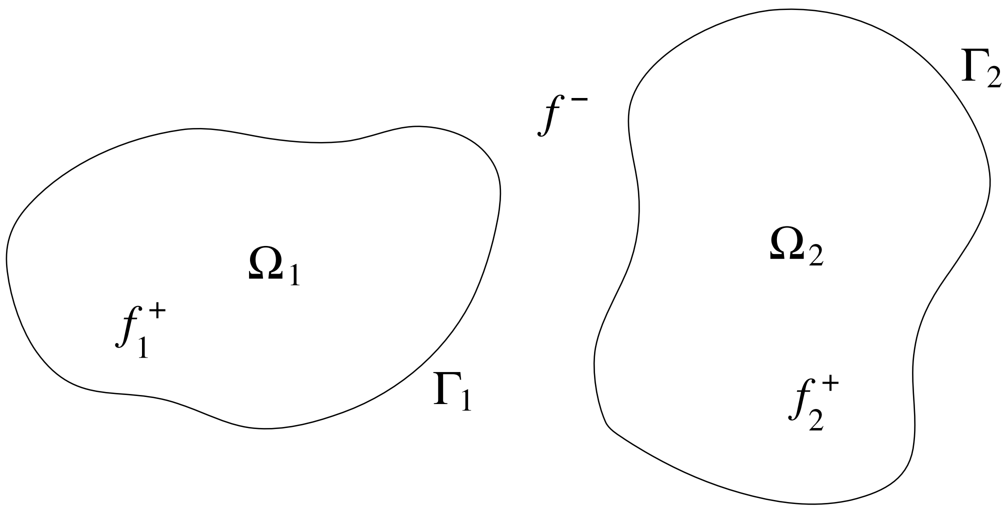

Fig. 2.1. *Two domains.*

harmonic acoustic field $u = u(x)$ governed by the inhomogeneous Helmholtz equation:

$$Lu \equiv \Delta u + k^2 u = f = f_1^+ + f_2^+ + f^-, \tag{2.1}$$

where for the sources we have $\text{supp} f_1^+ \subset \Omega_1$, $\text{supp} f_2^+ \subset \Omega_2$, and $\text{supp} f^- \subset \mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2)$. The overall acoustic field can be represented as

$$u = u_1^+ + u_2^+ + u^-, \tag{2.2a}$$

where

$$Lu_1^+ = f_1^+, \tag{2.2b}$$

$$Lu_2^+ = f_2^+, \tag{2.2c}$$

and

$$Lu^- = f^-. \tag{2.2d}$$

Our first goal is to eliminate all sound from the exterior sources f^- inside Ω_1 and Ω_2 while allowing sound from the sources f_1^+ and f_2^+ to propagate freely between Ω_1 and Ω_2. This is to be achieved by introducing a new control source g. After that, the total acoustic field $\tilde{u}$ will be governed by

$$L\tilde{u} = f_1^+ + f_2^+ + f^- + g.$$

We would like to choose the controls g to guarantee

$$\tilde{u}|_{x \in \Omega_1} = (u_1^+ + u_2^+)|_{x \in \Omega_1}$$

and

$$\tilde{u}|_{x \in \Omega_2} = (u_1^+ + u_2^+)|_{x \in \Omega_2}.$$

In other words, the field after the control inside either Ω_1 or Ω_2 should contain no contribution from the sources f^-. Notice that $g = -f^-$ is a solution to the problem, but it can be very difficult to implement and also requires previous knowledge of the sources f^-. Therefore, other, less expensive, solutions that do not require extensive knowledge of the exterior sources are preferable.

Our second goal is to selectively eliminate the sound that propagates between the regions Ω_1 and Ω_2. This is to be done in addition to the cancellation of the common exterior sound. For example, Ω_1 may be allowed to hear Ω_2, but not vice versa.

Note that the problem of active noise control as formulated above is, in fact, a problem of enabling a desired change in the solution of a given differential equation by appropriately modifying its source terms, i.e., by adding new sources. Consequently, it can be interpreted as an inverse source problem for the corresponding differential equation. Inverse source problems have been extensively studied in the literature, both from the standpoint of physics/engineering (see, e.g., [6, 5]), as well as from the standpoint of mathematics (see, e.g., [10]).

2.2. General solution. Let us first recall that in order to guarantee uniqueness of the solution to the Helmholtz equation (2.1) on unbounded regions, we must require that this solution satisfy the Sommerfeld radiation condition at infinity:

$$\frac{\partial v(x)}{\partial |x|} + ikv(x) = o(|x|^{-1/2}), \quad x \in \mathbb{R}^2, \tag{2.3a}$$

or

$$\frac{\partial v(x)}{\partial |x|} + ikv(x) = o(|x|^{-1}), \quad x \in \mathbb{R}^3. \tag{2.3b}$$

In particular, for any sufficiently smooth function $v = v(x)$ that satisfies the Sommerfeld condition we get

$$v(x) = \int_{\mathbb{R}^n} G(x-y) Lv(y) dy, \tag{2.4}$$

where

$$Lv = \Delta v + k^2 v$$

is the Helmholtz operator and $G = G(x)$ is its fundamental solution on $\mathbb{R}^n$. For $\mathbb{R}^2$, the fundamental solution is given by

$$G(x) = -\frac{1}{4i} H_0^{(2)}(k|x|), \tag{2.5}$$

where $H_0^{(2)}(z)$ is the Hankel function of the second kind defined by means of the Bessel functions $J_0(z)$ and $Y_0(z)$ as $H_0^{(2)}(z) = J_0(z) - iY_0(z)$. For $\mathbb{R}^3$, we have

$$G(x) = -\frac{1}{4\pi} \frac{e^{-ik|x|}}{|x|}. \tag{2.6}$$

Note that the fundamental solutions (2.5) and (2.6) satisfy the Sommerfeld radiation condition at infinity (2.3a) and (2.3b), respectively.

2.2.1. Straightforward cancellation. Let $u = u(x)$ be the overall acoustic field (see (2.1)) and n be the exterior normal to the boundary, and introduce an auxiliary function $w = w(x)$ such that

$$w\big|_{\Gamma_1\cup\Gamma_2} = u\big|_{\Gamma_1\cup\Gamma_2}$$

and

$$\frac{\partial w}{\partial n}\Big|_{\Gamma_1\cup\Gamma_2} = \frac{\partial u}{\partial n}\Big|_{\Gamma_1\cup\Gamma_2}$$

(recall that $\text{dist}(\Gamma_1, \Gamma_2) \geq \epsilon > 0$). We also require that $w(x)$ satisfies the Sommerfeld condition (2.3a) or (2.3b). Next, we define the control sources as follows:

$$g(x) = \begin{cases} -Lw, & x \in \{\mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2)\}, \\ 0, & x \in (\Omega_1 \cup \Omega_2). \end{cases} \tag{2.7}$$

To analyze properties of the controls (2.7), we must determine their output $v = v(x)$ for $x \in \mathbb{R}^n$. Using (2.4), we get[1]

$$v(x) = \int_{\mathbb{R}^n} Ggdy = -\int_{\mathbb{R}^n\backslash(\Omega_1\cup\Omega_2)} GLwdy$$
$$= -\left(w(x) - \int_{\Omega_1} GLwdy - \int_{\Omega_2} GLwdy\right),$$

where the individual integrals on the right-hand side are computed by integrating over Ω_1 and Ω_2 and are completely independent. Yet we emphasize that even though the computation of $v(x)$ can be reduced to integration over Ω_1 and Ω_2, the shape of $w(x)$ inside these two domains will not affect the output $v(x)$ since the original controls g are defined outside of $\Omega_1 \cup \Omega_2$; see formula (2.7).

Let us examine the individual terms. By Green's theorem, for $x \in \Omega_1$ we obtain

$$w(x) - \int_{\Omega_1} GLwdy = \int_{\Gamma_1} \left(w\frac{\partial G}{\partial n} - \frac{\partial w}{\partial n}G\right) ds_y$$
$$= \int_{\Gamma_1} \left(u\frac{\partial G}{\partial n} - \frac{\partial u}{\partial n}G\right) ds_y$$
$$= u^-(x) + u_2^+(x), \qquad x \in \Omega_1,$$

where n is the normal exterior to Γ_1 and $u^-(x)$ and $u_2^+(x)$ are defined by (2.2d) and (2.2c), respectively. This expression yields the entire incoming component of the field for the domain Ω_1. Next, we need to see what the contribution of $-\int_{\Omega_1} GLwdy$ will be outside of Ω_1. Introduce a smooth auxiliary function $w_1(x)$ such that $w_1(x) = w(x)$ on Ω_1 and $w_1(x)$ is compactly supported on a small neighborhood of Ω_1. Then, for

[1] All integrals hereafter are of the convolution type, as in formula (2.4).

$x \in \mathbb{R}^n \backslash \Omega_1$ we have

$$\begin{aligned}
-\int_{\Omega_1} GLwdy &= -\int_{\Omega_1} GLw_1dy \\
&= -\int_{\Omega_1} GLw_1dy + w_1 - w_1 \\
&= \int_{\mathbb{R}^n \backslash \Omega_1} GLw_1dy - w_1 \\
&= \int_{\Gamma_1} \left(w_1 \frac{\partial G}{\partial n} - \frac{\partial w_1}{\partial n} G \right) ds_y \\
&= -u_1^+(x), \qquad x \in \mathbb{R}^n \backslash \Omega_1,
\end{aligned}$$

where the third equality in the chain is obtained with the help of formula (2.4) applied to $w_1(x)$. Therefore, we can write

$$-\int_{\Omega_1} GLwdy = \begin{cases} -u_1^+, & x \in \mathbb{R}^n \backslash \Omega_1, \\ -w + u^- + u_2^+, & x \in \Omega_1. \end{cases}$$

We also have a similar output from Ω_2 given by

$$-\int_{\Omega_2} GLwdy = \begin{cases} -u_2^+, & x \in \mathbb{R}^n \backslash \Omega_2, \\ -w + u^- + u_1^+, & x \in \Omega_2. \end{cases}$$

Altogether, the full output of the controls $g(x)$ of (2.7) is as follows:

$$\begin{aligned}
v(x) &= -\left(w - \int_{\Omega_1} GLwdy - \int_{\Omega_2} GLwdy \right) \\
&= \begin{cases} -(u^- + u_2^+) + u_2^+ = -u^-, & x \in \Omega_1, \\ -(u^- + u_1^+) + u_1^+ = -u^-, & x \in \Omega_2, \\ -(w - u_1^+ - u_2^+), & x \in \mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2). \end{cases}
\end{aligned}$$

Consequently, these controls enable the cancellation of sound due to the exterior sources f^- on the domains Ω_1 and Ω_2 regardless of the specific choice of the auxiliary function w. The output of the controls outside $\Omega_1 \cup \Omega_2$ is given by $u_1^+ + u_2^+ - w$. It duplicates the acoustic field generated inside the two regions with the correction $-w$.

Let us elaborate a little further on the structure of the control output $v(x)$. Assume that $x \in \Omega_1$. Then,

$$\begin{aligned}
v(x) &= -\,w(x) + \int_{\Omega_1} GLwdy + \int_{\Omega_2} GLwdy \\
&= -\,w + \underbrace{w - (u^- + u_2^+)}_{\text{contribution due to } \Omega_1} + \underbrace{u_2^+}_{\text{due to } \Omega_2} \\
&= -\,u^-(x), \;\; x \in \Omega_1,
\end{aligned}$$

where $-(u^- + u_2^+)$ from the second term above renders cancellation of the entire incoming wave for Ω_1, and u_2^+ is the interior sound from Ω_2 duplicated by the controls. The same is true for Ω_2. Hence we conclude that the controls double the output of the sources interior to a region on the way out and then halve it as it comes into the other region. As such, the overall acoustic field after the control is given by

$$u = u_1^+ + u_2^+ + u^- + v$$
$$= \begin{cases} u_1^+ + u_2^+, & x \in \Omega_1, \\ u_1^+ + u_2^+, & x \in \Omega_2, \\ -w + u^- + 2u_1^+ + 2u_2^+, & x \in \mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2), \end{cases} \tag{2.8}$$

allowing the domains Ω_1 and Ω_2 to communicate freely with each other without interference from outside sources.

2.2.2. Selective cancellation. Now suppose that we would like Ω_1 to hear Ω_2 without outside interference, but we do not allow Ω_2 to hear anything from outside its boundary, including Ω_1. To achieve this we must elaborate further on how the split between the incoming and outgoing waves works. Consider just one domain Ω_1 with the boundary Γ_1 and again choose the auxiliary function $w_1 = w_1(x)$. Let

$$w_1\big|_{\Gamma_1} = u\big|_{\Gamma_1} \tag{2.9a}$$

and

$$\frac{\partial w_1}{\partial n}\Big|_{\Gamma_1} = \frac{\partial u}{\partial n}\Big|_{\Gamma_1}, \tag{2.9b}$$

where $u = u_1^- + u_1^+$ is the total acoustic field and $u_1^- = u^- + u_2^+$ is the acoustic field generated outside of Ω_1. Then the surface integral gives us

$$\int_{\Gamma_1} \left(w_1 \frac{\partial G}{\partial n} - \frac{\partial w_1}{\partial n} G \right) ds_y = \begin{cases} u_1^-, & x \in \Omega_1, \\ -u_1^+, & x \in \mathbb{R}^n \backslash \Omega_1. \end{cases}$$

With respect to the domain Ω_1, the field u_1^- is incoming, and u_1^+ is outgoing. Assuming that $w_1(x)$ also satisfies the appropriate Sommerfeld radiation condition (2.3a) or (2.3b), the surface integral can be replaced by the volumetric integral, so that for $x \in \Omega_1$ we have

$$\int_{\Gamma_1} \left(w_1 \frac{\partial G}{\partial n} - \frac{\partial w_1}{\partial n} G \right) ds_y = w_1 - \int_{\Omega_1} G L w_1 dy\Big|_{x \in \Omega_1}$$
$$= \int_{\mathbb{R}^n \backslash \Omega_1} G(x-y) L w_1(y) dy\Big|_{x \in \Omega_1}.$$

This is precisely why we would choose the controls as $g_1(x) = -Lw_1|_{\mathbb{R}^n \backslash \Omega_1}$ if we were to completely eliminate all of the outside sound on Ω_1—because they produce $-u_1^-$ on Ω_1. At the same time, on the complementary domain $\mathbb{R}^n \backslash \Omega_1$ the output of the controls $g_1(x)$ is the duplicate of the outgoing field u_1^+ corrected by $-w_1$:

$$\begin{aligned}\int_{\mathbb{R}^n} G(x-y)g_1(y)dy &= -\int_{\mathbb{R}^n\backslash\Omega_1} G(x-y)Lw_1(y)dy \\ &= w_1 - \int_{\mathbb{R}^n\backslash\Omega_1} GLw_1 dy - w_1 \\ &= -\int_{\Gamma_1}\left(w_1\frac{\partial G}{\partial n} - \frac{\partial w_1}{\partial n}G\right)ds_y - w_1 = u_1^+ - w_1.\end{aligned} \tag{2.10}$$

Having described individual controls g_1 for a single domain Ω_1, we are now ready to construct the controls so that Ω_1 will hear Ω_2 without outside interference, but Ω_2 will not hear anything from outside its boundary, including Ω_1. The procedure will consist of two stages. At the first stage, we will use the controls $g_1(x)$ as a supplementary tool. Namely, choose an auxiliary function $w_1(x)$ that satisfies conditions (2.9a) and (2.9b), as well as the Sommerfeld condition at infinity. In addition, require that w_1 be compactly supported near Ω_1, in particular, that $w_1(x) = 0$ near Ω_2. This is clearly possible since the distance between the subdomains Ω_1 and Ω_2 is positive. Then, build the supplementary controls

$$g_1(x) = -Lw_1|_{\mathbb{R}^n\backslash\Omega_1}, \qquad g_1(x) = 0|_{\Omega_1}. \tag{2.11}$$

According to formula (2.10), the output of these controls on $\mathbb{R}^n\backslash\Omega_1$ is

$$v_1 = \int_{\mathbb{R}^n} Gg_1 dy = u_1^+ - w_1, \qquad x \in \mathbb{R}^n\backslash\Omega_1, \tag{2.12}$$

and since w_1 is compactly supported near Ω_1, we have $v_1 = u_1^+$ near Ω_2.

At the second stage of building the controls, we begin as usual with our auxiliary function w. It is still required that w satisfy the Sommerfeld condition at infinity, while on Γ_1 we still impose the same boundary conditions (2.9a) and (2.9b):

$$w\big|_{\Gamma_1} = u\big|_{\Gamma_1}$$

and

$$\frac{\partial w}{\partial n}\bigg|_{\Gamma_1} = \frac{\partial u}{\partial n}\bigg|_{\Gamma_1},$$

where u is the given total acoustic field. The difference is in the boundary conditions on Γ_2. Here it is required that

$$w\big|_{\Gamma_2} = (u+v_1)\big|_{\Gamma_2} \equiv (u+u_1^+)\big|_{\Gamma_2}$$

and

$$\frac{\partial w}{\partial n}\bigg|_{\Gamma_2} = \frac{\partial(u+v_1)}{\partial n}\bigg|_{\Gamma_2} \equiv \frac{\partial(u+u_1^+)}{\partial n}\bigg|_{\Gamma_2},$$

where v_1 was obtained at the first stage; see (2.12). Then, defining the controls as

$$g(x) = -Lw|_{\mathbb{R}^n\backslash(\Omega_1\cup\Omega_2)}, \qquad g(x) = 0|_{(\Omega_1\cup\Omega_2)}$$

yields the output

$$v(x) = -\left(w - \int_{\Omega_1} GLwdy - \int_{\Omega_2} GLwdy\right)$$

$$(2.13)\qquad = \begin{cases} -(u^- + u_2^+) + u_2^+ = -u^-, & x \in \Omega_1, \\ -(u^- + 2u_1^+) + u_1^+ = -(u^- + u_1^+), & x \in \Omega_2, \\ -(w - u_1^+ - u_2^+), & x \in \mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2). \end{cases}$$

Therefore, we see that Ω_1 hears Ω_2 without outside interference, but Ω_2 does not hear anything from outside its boundary, including Ω_1.

2.2.3. Proofs. We will now prove that what we have obtained is, in fact, a general solution for the controls with the prescribed properties. That is, we will prove that our method of construction gives all possible controls.

THEOREM 2.1. *Suppose that* $\Omega_1 \subset \mathbb{R}^n$ *and* $\Omega_2 \subset \mathbb{R}^n$ *are two disjoint regions:* $\text{dist}(\Omega_1, \Omega_2) \geq \epsilon > 0$, *with the boundaries* $\partial\Omega_1 = \Gamma_1$ *and* $\partial\Omega_2 = \Gamma_2$. *Assume that the total acoustic field in* $\mathbb{R}^n$ *is governed by* $Lu \equiv \Delta u + k^2 u = f = f_1^+ + f_2^+ + f^-$, *where the sources* f *are located according to* $\text{supp} f_1^+ \subset \Omega_1$, $\text{supp} f_2^+ \subset \Omega_2$, *and* $\text{supp} f^- \subset \mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2)$. *Let the overall acoustic field* u *be represented as* $u = u_1^+ + u_2^+ + u^-$.

Let a control source $g = g(x)$ *be added to the other sources* $f(x)$ *such that the overall field* $\tilde{u}$ *governed by* $L\tilde{u} = f_1^+ + f_2^+ + f^- + g$ *satisfies*

$$(2.14)\qquad \tilde{u} = \begin{cases} u_1^+ + u_2^+, & x \in \Omega_1, \\ u_1^+ + u_2^+, & x \in \Omega_2. \end{cases}$$

Then the general solution for the desired control is given by

$$(2.15)\qquad g = -Lw|_{\mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2)}, \qquad g = 0|_{(\Omega_1 \cup \Omega_2)},$$

where w *satisfies the Sommerfeld condition* (2.3a) *or* (2.3b) *at infinity, as well as the interface conditions*

$$(2.16a)\qquad w\Big|_{\Gamma_1 \cup \Gamma_2} = u\Big|_{\Gamma_1 \cup \Gamma_2}$$

and

$$(2.16b)\qquad \frac{\partial w}{\partial n}\Big|_{\Gamma_1 \cup \Gamma_2} = \frac{\partial u}{\partial n}\Big|_{\Gamma_1 \cup \Gamma_2}.$$

Proof. We need to prove that any control g given by (2.15) is an appropriate control and, conversely, that any appropriate control g can be obtained by using a suitable auxiliary function w. Suppose we have a function $w(x)$ that satisfies (2.16a), (2.16b), and the Sommerfeld condition at infinity. Then, according to formula (2.8), the corresponding control g given by formula (2.15) yields the desired properties by eliminating u^- on $\Omega_1 \cup \Omega_2$.

Conversely, suppose that a control g achieves the desired cancellation; see formula (2.14). Then, substituting $\tilde{u} = \tilde{u}(x)$ into the equation $L\tilde{u} = f_1^+ + f_2^+ + f^- + g$, we immediately obtain that $g(x) = 0$ for $x \in (\Omega_1 \cup \Omega_2)$. In other words, $\text{supp}\, g \subset \mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2)$. Consequently, the output v of the control g is as follows:

$$v(x) = \int_{\mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2)} Ggdy = \begin{cases} -u^-, & x \in \Omega_1, \\ -u^-, & x \in \Omega_2. \end{cases}$$

Consider the equation $-Lw = g - f_1^+ - f_2^+$, where f_1^+ and f_2^+ are the sound sources from Ω_1 and Ω_2, respectively. Its solution, subject to the Sommerfeld condition at infinity (2.3a) or (2.3b), satisfies

$$w = -v + u_1^+ + u_2^+$$
$$= \begin{cases} u^- + u_1^+ + u_2^+ = u, & x \in \Omega_1, \\ u^- + u_1^+ + u_2^+ = u, & x \in \Omega_2. \end{cases}$$

Since $w(x)$ is at least C^1 smooth on $\mathbb{R}^n$, we can claim that it satisfies relations (2.16a) and (2.16b). Therefore, the control $g(x)$ can be obtained by formula (2.15), since $\operatorname{supp} f_1^+ \subset \Omega_1$ and $\operatorname{supp} f_2^+ \subset \Omega_2$. □

THEOREM 2.2. *Suppose that* $\Omega_1 \subset \mathbb{R}^n$ *and* $\Omega_2 \subset \mathbb{R}^n$ *are two disjoint regions:* $\operatorname{dist}(\Omega_1, \Omega_2) \geq \epsilon > 0$, *with the boundaries* $\partial\Omega_1 = \Gamma_1$ *and* $\partial\Omega_2 = \Gamma_2$. *Assume that the total acoustic field in* $\mathbb{R}^n$ *is governed by* $Lu \equiv \Delta u + k^2 u = f = f_1^+ + f_2^+ + f^-$, *where the sources* f *are located according to* $\operatorname{supp} f_1^+ \subset \Omega_1$, $\operatorname{supp} f_2^+ \subset \Omega_2$, *and* $\operatorname{supp} f^- \subset \mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2)$. *Let the overall acoustic field* u *be represented as* $u = u_1^+ + u_2^+ + u^-$.

Let a control source $g = g(x)$ *be added to the other sources* $f(x)$ *such that the overall field* $\tilde{u}$ *governed by* $L\tilde{u} = f_1^+ + f_2^+ + f^- + g$ *satisfies*

$$\tilde{u} = \begin{cases} u_1^+ + u_2^+, & x \in \Omega_1, \\ u_2^+, & x \in \Omega_2. \end{cases} \tag{2.17}$$

Then the general solution for the desired control is given by

$$g = -Lw|_{\mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2)}, \qquad g = 0|_{(\Omega_1 \cup \Omega_2)}, \tag{2.18}$$

where $w = w(x)$ *satisfies the Sommerfeld condition* (2.3a) *or* (2.3b) *at infinity and the following interface conditions:*

$$w\big|_{\Gamma_1} = u\big|_{\Gamma_1}, \quad w\big|_{\Gamma_2} = (u + u_1^+)\big|_{\Gamma_2} \tag{2.19a}$$

and

$$\frac{\partial w}{\partial n}\Big|_{\Gamma_1} = \frac{\partial u}{\partial n}\Big|_{\Gamma_1}, \quad \frac{\partial w}{\partial n}\Big|_{\Gamma_2} = \frac{\partial (u + u_1^+)}{\partial n}\Big|_{\Gamma_2}. \tag{2.19b}$$

The function u_1^+ *on* Γ_2 *can be obtained as the output* v_1 *given by formula* (2.12) *of the supplementary controls* g_1 *of* (2.11).

Theorem 2.2 essentially implies that the controls (2.18) are obtained by means of a predictor-corrector procedure. The predictor stage consists of computing v_1 of (2.12) as the output of the control g_1 of (2.11), whereas the corrector stage consists of obtaining the overall composite controls $g(x)$ with the help of the auxiliary function $w(x)$ defined via (2.19).

Proof. We need to prove that any control g given by (2.18) is an appropriate control and, conversely, that any appropriate control g can be obtained by using a suitable auxiliary function w. Suppose we have a function $w(x)$ that satisfies (2.19a), (2.19b), and the Sommerfeld condition at infinity. Then, according to formula (2.13), the corresponding control given by (2.18) provides the desired properties eliminating u^- on $\Omega_1 \cup \Omega_2$ and additionally eliminating u_1^+ on Ω_2.

Conversely, suppose that a control g achieves the desired cancellation; see formula (2.17). Then, $\operatorname{supp} g \in \mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2)$, and the output of the control, v, is as follows:

$$v(x) = \int_{\mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2)} G g dy = \begin{cases} -u^-, & x \in \Omega_1, \\ -u^- - u_1^+, & x \in \Omega_2. \end{cases}$$

Consider the equation $-Lw = g - f_1^+ - f_2^+$. Its solution, subject to the Sommerfeld condition at infinity (2.3a) or (2.3b), satisfies

$$\begin{aligned} w &= -v + u_1^+ + u_2^+ \\ &= \begin{cases} u^- + u_1^+ + u_2^+ = u, & x \in \Omega_1, \\ u^- + 2u_1^+ + u_2^+ = u + u_1^+, & x \in \Omega_2. \end{cases} \end{aligned}$$

Since $w(x)$ is at least C^1 smooth on $\mathbb{R}^n$, it satisfies relations (2.19a) and (2.19b). Therefore, the control $g(x)$ can be obtained by formula (2.18) since $\operatorname{supp} f_1^+ \subset \Omega_1$ and $\operatorname{supp} f_2^+ \subset \Omega_2$. $\square$

3. Multiple regions.

3.1. Formulation. Let $\Omega_1, \Omega_2, \ldots, \Omega_N$ be given, where $\Omega_i \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$ is either bounded or unbounded. For simplicity we will assume that $\Omega_1, \Omega_2, \ldots, \Omega_N$ are separate bounded regions of $\mathbb{R}^n$. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ be the boundaries of $\Omega_1, \Omega_2, \ldots, \Omega_N$ respectively. Consider the time-harmonic acoustic field u governed by the inhomogeneous Helmholtz equation:

$$Lu \equiv \Delta u + k^2 u = f = f_1^+ + f_2^+ + \cdots + f_N^+ + f^-,$$

where the sources are $\operatorname{supp} f_1^+ \subset \Omega_1$, $\operatorname{supp} f_2^+ \subset \Omega_2, \ldots, \operatorname{supp} f_n^+ \subset \Omega_N$, and $\operatorname{supp} f^- \subset \mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N)$. Therefore, the overall acoustic field can be represented as

$$u = u_1^+ + u_2^+ + \cdots + u_N^+ + u^-,$$

where

$$\begin{aligned} Lu_1^+ &= f_1^+, \\ Lu_2^+ &= f_2^+, \\ &\cdots \\ Lu_N^+ &= f_N^+, \end{aligned}$$

and

$$Lu^- = f^-.$$

Our goal is to eliminate all sound from the sources f^- inside $\Omega_1, \Omega_2, \ldots, \Omega_N$, while allowing sound from the sources $f_1^+, f_2^+, \ldots, f_N^+$ to propagate between $\Omega_1, \Omega_2, \ldots, \Omega_N$ as we see fit. That is, we wish to selectively eliminate unwanted sound from various regions while leaving other regions free to receive predetermined communications. This is done as before by introducing a new control source g. Therefore, the total acoustic field is now governed by the modified equation

$$L\tilde{u} = f_1^+ + f_2^+ + \cdots + f_N^+ + f^- + g.$$

3.2. General solution.

3.2.1. Straightforward cancellation. We will first demonstrate how to eliminate all sound in $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N$ that originates from $\mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N)$. As before, we introduce an auxiliary function $w = w(x)$, which satisfies the Sommerfeld condition (2.3a) or (2.3b) at infinity, and is such that

$$w\big|_{\Gamma_i} = u\big|_{\Gamma_i}$$

and

$$\frac{\partial w}{\partial n}\Big|_{\Gamma_i} = \frac{\partial u}{\partial n}\Big|_{\Gamma_i}$$

for all $i = 1, \dots, N$.

Next, we define the control sources as (cf. formula (2.7))

$$g(x) = \begin{cases} -Lw, & x \in \{\mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N)\}, \\ 0, & x \in (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N), \end{cases}$$

and see that their output $v = v(x)$, $x \in \mathbb{R}^n$, is given by

$$\begin{aligned} v(x) &= \int_{\mathbb{R}^n} Ggdy = -\int_{\mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N)} GLwdy \\ &= -\left(w(x) - \int_{\Omega_1} GLwdy - \int_{\Omega_2} GLwdy - \cdots - \int_{\Omega_N} GLwdy \right), \end{aligned}$$

where the individual integrals are computed by integrating over $\Omega_1, \Omega_2, \dots, \Omega_N$ and are completely independent. Again, all integrals are convolutions, as in section 2.

Let us examine the individual terms. By Green's theorem, for $x \in \Omega_i$, where $i \in \{1, 2, \dots, N\}$, we obtain

$$\begin{aligned} w(x) - \int_{\Omega_i} GLwdy &= \int_{\Gamma_i} \left(w\frac{\partial G}{\partial n} - \frac{\partial w}{\partial n} G \right) ds_y \\ &= \int_{\Gamma_i} \left(u\frac{\partial G}{\partial n} - \frac{\partial u}{\partial n} G \right) ds_y \\ &= u^-(x) + \sum_{\substack{j=1,2,\dots,N \\ j \neq i}} u_j^+(x), \qquad x \in \Omega_i. \end{aligned}$$

This is the entire incoming component for the domain Ω_i. Now the effect of the integral $-\int_{\Omega_i} GLwdy$ outside of Ω_i must be examined. To do this, we introduce a smooth auxiliary function $w_i(x)$ such that $w_i(x) = w(x)$ on Ω_i and $w_i(x)$ is compactly supported on a small neighborhood of Ω_i. Consequently, for $x \in \mathbb{R}^n \backslash \Omega_i$ we have

$$
\begin{aligned}
-\int_{\Omega_i} GLwdy &= -\int_{\Omega_i} GLw_i dy \\
&= -\int_{\Omega_i} GLw_i dy + w_i - w_i \\
&= \int_{\mathbb{R}^n \backslash \Omega_i} GLw_i dy - w_i \\
&= \int_{\Gamma_i} \left(w_i \frac{\partial G}{\partial n} - \frac{\partial w_i}{\partial n} G \right) ds_y \\
&= -u_i^+(x), \quad x \in \mathbb{R}^n \backslash \Omega_i.
\end{aligned}
$$

Therefore we can write

$$
-\int_{\Omega_i} GLwdy = \begin{cases} -u_i^+, & x \in \mathbb{R}^n \backslash \Omega_i, \\ -w + u^- + \displaystyle\sum_{\substack{j=1,2,\ldots,N, \\ j \neq i}} u_j^+, & x \in \Omega_i. \end{cases}
$$

Altogether, the full output of the controls $g(x)$ is as follows:

$$
\begin{aligned}
v(x) &= -\left(w - \int_{\Omega_i} GLwdy - \sum_{j \neq i} \int_{\Omega_j} GLwdy \right) \\
&= \begin{cases} -u^-, & x \in \Omega_i, \ i = 1, 2, \ldots, N, \\ -\left(w - \displaystyle\sum_{j=1,2,\ldots,N} u_j^+ \right), & x \in \mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N). \end{cases}
\end{aligned} \tag{3.1}
$$

Consequently, these controls enable the cancellation of sound due to the exterior sources on the domains $\Omega_1, \Omega_2, \ldots, \Omega_N$ regardless of the specific choice of the auxiliary function w. The output of the controls outside $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N$ is given by $\sum_{j=1,2,\ldots,N} u_j^+ - w$. It basically duplicates the acoustic field generated inside the regions with the correction $-w$. More specifically, for any given Ω_i the controls double the output of the sources interior to a region on the way out and then halve the result as it comes into another region. As such, the overall acoustic field is given by

$$
\begin{aligned}
u &= u_1^+ + u_2^+ + \cdots + u_N^+ + u^- + v \\
&= \begin{cases} u_1^+ + u_2^+ + \cdots + u_N^+, & x \in \Omega_i, \ i = 1, 2, \ldots, N, \\ -w + u^- + 2u_1^+ + 2u_2^+ + \cdots + 2u_N^+, & x \in \mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N), \end{cases}
\end{aligned}
$$

which means that the subdomains $\Omega_1, \Omega_2, \ldots, \Omega_N$ can communicate freely with each other without interference from outside sources.

THEOREM 3.1. *Suppose that* $\Omega_1, \Omega_2, \ldots, \Omega_N$ *are given, where* $\Omega_i \subseteq \mathbb{R}^n$ *are disjoint regions:* $\text{dist}(\Omega_i, \Omega_j) \geq \epsilon > 0$ *if* $i \neq j$, *with the boundaries* $\partial\Omega_1 = \Gamma_1, \partial\Omega_2 =$

$\Gamma_2, \ldots, \partial\Omega_n = \Gamma_N$. *Assume that the total acoustic field in* $\mathbb{R}^n$ *is governed by* $Lu \equiv \Delta u + k^2 u = f = f_1^+ + f_2^+ + \cdots + f_N^+ + f^-$, *where the sources are located according to* $\text{supp} f_1^+ \subset \Omega_1$, $\text{supp} f_2^+ \subset \Omega_2, \ldots, \text{supp} f_n^+ \subset \Omega_N$, *and* $\text{supp} f^- \subset \mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N)$. *Let the overall acoustic field* u *be represented as* $u = u_1^+ + u_2^+ + \cdots + u_N^+ + u^-$.

Let a control source $g = g(x)$ *be added to the other sources* $f(x)$ *such that the overall field* $\tilde{u}$ *governed by* $L\tilde{u} = f_1^+ + f_2^+ + \cdots + f_N^+ + f^- + g$ *satisfies*

$$\tilde{u} = \sum_{j=1,2,\ldots,N} u_j^+, x \in \Omega_i, \ i = 1, 2, \ldots, N. \tag{3.2}$$

Then the general solution for the desired control is given by

$$g = -Lw|_{\mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N)}, \qquad g = 0|_{(\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N)}, \tag{3.3}$$

where $w = w(x)$ *satisfies the Sommerfeld condition at infinity and the following interface conditions:*

$$w\Big|_{\Gamma_i} = u\Big|_{\Gamma_i} \tag{3.4a}$$

and

$$\frac{\partial w}{\partial n}\Big|_{\Gamma_i} = \frac{\partial u}{\partial n}\Big|_{\Gamma_i} \tag{3.4b}$$

for all $i = 1, \ldots, N$.

Proof. We need to prove that any control g given by (3.3) is an appropriate control and, conversely, that any appropriate control g can be obtained by using a suitable auxiliary function w. Suppose we have a function $w(x)$ that satisfies (3.4a), (3.4b), and the Sommerfeld condition (2.3a) or (2.3b) at infinity. Then, formula (3.1) implies that the corresponding control (3.3) provides the desired properties eliminating the exterior sound u^- on $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N$.

Conversely, suppose a control g achieves the desired cancellation, so that equality (3.2) holds. Then, clearly, $g(x) = 0$ for $x \in \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N$. Consequently, the output v of the control g is as follows:

$$v(x) = \int_{\mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2 \cup \ldots \Omega_N)} G g dy = -u^-, \quad x \in \Omega_i, \ i = 1, \ldots N.$$

Consider the equation $-Lw = g - f_1^+ - f_2^+ - \cdots - f_N^+$. Its solution, subject to the Sommerfeld condition at infinity (2.3a) or (2.3b), satisfies

$$\begin{aligned} w &= -v + u_1^+ + u_2^+ + \cdots + u_N^+ \\ &= u, \quad x \in \Omega_i, \ i = 1, 2, \ldots, N. \end{aligned}$$

Since $w(x)$ is at least C^1 smooth on $\mathbb{R}^n$, it satisfies relations (3.4a) and (3.4b). Therefore the control $g(x)$ can be obtained by formula (3.3) applied to this particular $w(x)$, since $\text{supp} f_1^+ \subset \Omega_1$, $\text{supp} f_2^+ \subset \Omega_2, \ldots, \text{supp} f_N^+ \subset \Omega_N$. □

3.2.2. Selective cancellation. Now suppose that in each subdomain Ω_i, we would like to eliminate all outside interference and, in addition, selectively eliminate sound from some other subdomains. It will be helpful to formulate a convenient way of keeping track of communications between the subdomains. For that purpose, let us introduce an $N \times N$ matrix $\mathbf{M}$, such that each row i corresponds to a region Ω_i, and the entry (0 or 1) in each column is used to determine whether this Ω_i hears a region corresponding to that column or not. In other words, if the entry at the intersection of row i and column j is 0, then Ω_i hears Ω_j. If this entry is 1, then it does not. Obviously the diagonal of $\mathbf{M}$ is filled with zeros since the regions hear themselves. So, in the case of Theorem 2.1 we have

$$\mathbf{M} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and for Theorem 2.2 we get

$$\mathbf{M} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Notice that no reciprocity in the communication pattern is assumed; i.e., the matrix $\mathbf{M}$ is not necessarily symmetric.

For a given matrix $\mathbf{M}$ that corresponds to a specific communication pattern between the regions $\Omega_1, \Omega_2, \ldots, \Omega_N$, we will now build the auxiliary function $w(x)$ and the controls $g(x)$ as before, i.e., in two stages. At the first stage, we take the auxiliary functions $w_i(x)$ for all Ω_i, $i = 1, \ldots, N$, that satisfy

$$w_i\big|_{\Gamma_i} = u\big|_{\Gamma_i}$$

and

$$\frac{\partial w_i}{\partial n}\Big|_{\Gamma_i} = \frac{\partial u}{\partial n}\Big|_{\Gamma_i},$$

as well as the Sommerfeld condition at infinity. We also require that each w_i be compactly supported near the corresponding Ω_i. Then, we build the supplementary controls:

$$g_i(x) = -Lw_i|_{\mathbb{R}^n \backslash \Omega_i}, \qquad g_i(x) = 0|_{\Omega_i}.$$

According to formula (2.10) applied to a given subdomain Ω_i, the output of these controls on $\mathbb{R}^n \backslash \Omega_i$ is

$$v_i = \int_{\mathbb{R}^n} G g_i dy = u_i^+ - w_i, \quad x \in \mathbb{R}^n \backslash \Omega_i, \tag{3.5}$$

and since w_i is taken compactly supported near Ω_i, we have $v_i = u_i^+$ near Ω_j, where $j = 1, 2, \ldots, N$ and $j \neq i$.

At the second stage, we start with introducing the auxiliary function $w(x)$, which satisfies the Sommerfeld radiation condition (2.3a) or (2.3b) at infinity. In addition, on each Γ_i we require that

$$w\big|_{\Gamma_i} = (u + \mathbf{e}_i^T \mathbf{M}\mathbf{v})\big|_{\Gamma_i}$$

and

$$\frac{\partial w}{\partial n}\Big|_{\Gamma_i} = \frac{\partial (u + \mathbf{e}_i^T \mathbf{M}\mathbf{v})}{\partial n}\Big|_{\Gamma_i}.$$

In these formulae, $\mathbf{e}_i$ is a vector with its ith component equal to 1 and all other components equal to 0, and

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix},$$

where each v_i is obtained at the first stage with the help of the supplementary controls $g_i(x)$ according to formula (3.5).

Next, we define the control sources $g(x)$ as

$$g(x) = -Lw|_{\mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N)}, \qquad g(x) = 0|_{(\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N)}.$$

Their output $v = v(x)$, $x \in \mathbb{R}^n$, is given by

$$\begin{aligned} v(x) &= \int_{\mathbb{R}^n} Ggdy \\ &= - \int_{\mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N)} GLwdy \\ &= - \left(w(x) - \int_{\Omega_1} GLwdy - \int_{\Omega_2} GLwdy - \cdots - \int_{\Omega_N} GLwdy \right), \\ &= \begin{cases} -(u^- + \mathbf{e}_i^T \mathbf{M} \mathbf{v}), & x \in \Omega_i, \\ -(w - u_1^+ - u_2^+ - \cdots - u_N^+), & x \in \mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N), \end{cases} \end{aligned} \tag{3.6}$$

which obviously enables the desired cancellation.

We now prove that this is in fact the general solution for the controls with the prescribed properties, i.e., that we obtain all possible controls.

THEOREM 3.2. *Suppose that* $\Omega_1, \Omega_2, \ldots, \Omega_N$ *are given, where* $\Omega_i \subseteq \mathbb{R}^n$ *are disjoint regions:* $\mathrm{dist}(\Omega_i, \Omega_j) \geq \epsilon > 0$ *if* $i \neq j$, *with the boundaries* $\partial\Omega_1 = \Gamma_1, \partial\Omega_2 = \Gamma_2, \ldots, \partial\Omega_n = \Gamma_N$. *Assume that the total acoustic field in* $\mathbb{R}^n$ *is governed by* $Lu \equiv \Delta u + k^2 u = f = f_1^+ + f_2^+ + \cdots + f_N^+ + f^-$, *where the sources are located according to* $\mathrm{supp} f_1^+ \subset \Omega_1$, $\mathrm{supp} f_2^+ \subset \Omega_2, \ldots, \mathrm{supp} f_n^+ \subset \Omega_N$, *and* $\mathrm{supp} f^- \subset \mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N)$. *Let the overall acoustic field* u *be represented as* $u = u_1^+ + u_2^+ + \cdots + u_N^+ + u^-$.

Let a control source $g = g(x)$ *be added to the other sources* $f(x)$ *such that the overall field* $\tilde{u}$ *governed by* $L\tilde{u} = f_1^+ + f_2^+ + \cdots + f_N^+ + f^- + g$ *satisfies*

$$\tilde{u} = \mathbf{e}_i^T (\mathbf{1} - \mathbf{M}) \mathbf{u}, \;\; x \in \Omega_i, \;\; i = 1, 2, \ldots, N, \tag{3.7}$$

where $\mathbf{1}$ *is an* $N \times N$ *matrix with all entries equal to* 1, *and*

$$\mathbf{u} = \begin{pmatrix} u_1^+ \\ u_2^+ \\ \vdots \\ u_N^+ \end{pmatrix}.$$

Then, the general solution for the desired control is given by

$$g = -Lw|_{\mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N)}, \qquad g = 0|_{(\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N)}, \tag{3.8}$$

where $w = w(x)$ satisfies the Sommerfeld condition at infinity and the following interface conditions:

$$w\big|_{\Gamma_i} = (u + \mathbf{e}_i^T \mathbf{M}\mathbf{v})\big|_{\Gamma_i} \tag{3.9a}$$

and

$$\frac{\partial w}{\partial n}\Big|_{\Gamma_i} = \frac{\partial (u + \mathbf{e}_i^T \mathbf{M}\mathbf{v})}{\partial n}\Big|_{\Gamma_i} \tag{3.9b}$$

for all $i = 1, \dots, N$. Note that if $\mathbf{M} = \mathbf{0}$, then (3.7) *reduces to* (3.2), *and the current theorem becomes the same as Theorem* 3.1.

Similarly to Theorem 2.2, Theorem 3.2 implies that the controls (3.8) are built using a predictor-corrector procedure. The predictor stage consists of computing $\mathbf{v}$ of (3.5), whereas the corrector stage consists of obtaining the overall composite controls $g(x)$ by means of the auxiliary function $w(x)$ defined via (3.9).

Proof. We need to prove that any control g given by (3.8) is an appropriate control and, conversely, that any appropriate control g can be obtained by using a suitable auxiliary function w. Suppose we have a function $w(x)$ that satisfies (3.9a), (3.9b), and the Sommerfeld condition at infinity. Then, according to formula (3.6), the corresponding control (3.8) provides the desired properties as it eliminates u^- on $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N$ and selectively allows the sound to propagate between the subdomains following a predetermined pattern $\mathbf{M}$.

Conversely, suppose that a control g achieves the desired cancellation; see formula (3.7). Then, $g(x) = 0$ for $x \in \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N$, and the output v of the control g is

$$\begin{aligned} v(x) &= \int_{\mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N)} G g dy \\ &= -(u^- + \mathbf{e}_i^T \mathbf{M}\mathbf{v}),\ x \in \Omega_i,\ i = 1, 2, \dots, N. \end{aligned}$$

Consider the equation $-Lw = g - f_1^+ - f_2^+ - \cdots - f_N^+$. Its solution, subject to the Sommerfeld condition at infinity (2.3a) or (2.3b), satisfies

$$\begin{aligned} w &= -v + u_1^+ + u_2^+ + \cdots + u_N^+ \\ &= \mathbf{e}_i^T (\mathbf{1} - \mathbf{M})\mathbf{u},\ x \in \Omega_i,\ i = 1, 2, \dots, N. \end{aligned}$$

Since $w(x)$ is at least C^1 smooth on $\mathbb{R}^n$, it satisfies the interface conditions (3.9a) and (3.9b). Therefore, the control $g(x)$ can be obtained by formula (3.8) applied to this $w(x)$, because $\mathrm{supp} f_1^+ \subset \Omega_1$, $\mathrm{supp} f_2^+ \subset \Omega_2, \dots, \mathrm{supp} f_N^+ \subset \Omega_N$. ∎

4. Generalized Calderon's potentials. We will now show how the split between $u_1^+, u_2^+, \dots, u_N^+$, and u^- can be conveniently described in terms of the generalized potentials and boundary projection operators of Calderon's type. For more detail, the reader is referred to the work of Lončarić, Ryaben'kii, and Tsynkov [13].

Consider some function $u(x)$ that satisfies $Lu = 0$, where $x \in \Omega_i$ for a given i. Then the Green's formula yields

$$u(x) = \int_{\Gamma_i} \left(u \frac{\partial G}{\partial n} - \frac{\partial u}{\partial n} G \right) ds_y, \quad x \in \Omega_i. \tag{4.1}$$

Note than the direction of the normal n is fixed to always point outward from a given domain Ω_i. A generalized potential of Calderon's type with vector density $\xi_{\Gamma_i} = (\xi_0, \xi_1)$ specified on Γ_i is defined by the following formula:

$$P_{\Omega_i}\xi_{\Gamma_i}(x) = \int_{\Gamma_i} \left(\xi_0 \frac{\partial G}{\partial n} - \xi_1 G \right) ds_y, \quad x \in \Omega_i, \tag{4.2}$$

which is similar to (4.1) except that we do not require ahead of time that ξ_0 and ξ_1 in (4.2) be the boundary values of some function u that solves $Lu = 0$ on Ω_i and its normal derivative. With the help of (4.2), formula (4.1) can be rewritten as

$$u = P_{\Omega_i} \left(u, \frac{\partial u}{\partial n} \right) \Big|_{\Gamma_i}, \; x \in \Omega_i.$$

Next, for any sufficiently smooth function v specified on Ω_i, we define its vector trace on Γ_i as

$$Tr_i \, v = \left(v, \frac{\partial v}{\partial n} \right) \Big|_{\Gamma_i} \tag{4.3}$$

and then introduce the boundary operator as P_{Γ_i} as a combination of the potential P_{Ω_i} of (4.2) and trace Tr_i of (4.3):

$$P_{\Gamma_i}\xi_{\Gamma_i} = Tr_i \, P_{\Omega_i}\xi_{\Gamma_i}. \tag{4.4}$$

Note that the operator P_{Γ_i} is a projection, $P_{\Gamma_i}^2 = P_{\Gamma_i}$.

The previous construction can easily be changed from the use of surface integrals to that of volume integrals. Given a vector density $\xi_{\Gamma_i} = (\xi_0, \xi_1)$, we take a sufficiently smooth auxiliary function $w(x)$ that is compactly supported near Γ_i and such that

$$Tr_i \, w = \xi_{\Gamma_i}. \tag{4.5}$$

Then, the potential (4.2) can be redefined as follows:

$$\begin{aligned} P_{\Omega_i}\xi_{\Gamma_i}(x) &= w(x) - \int_{\Omega_i} GLw dy \\ &= \int_{\mathbb{R}^n \setminus \Omega_i} GLw dy, \; x \in \Omega_i. \end{aligned} \tag{4.6}$$

Note that $P_{\Omega_i}\xi_{\Gamma_i}(x)$ of (4.6) does not depend on the specific choice of $w(x)$ as long as condition (4.5) is satisfied. We can also define the exterior potential, $Q_{\mathbb{R}^n \setminus \Omega_i}\xi_{\Gamma_i}(x)$, $x \in \mathbb{R}^n \setminus \Omega_i$, for the complementary domain $\mathbb{R}^n \setminus \Omega_i$ as

$$\begin{aligned} Q_{\mathbb{R}^n \setminus \Omega_i}\xi_{\Gamma_i}(x) &= w(x) - \int_{\mathbb{R}^n \setminus \Omega_i} GLw dy \\ &= \int_{\Omega_i} GLw dy, \; x \in \mathbb{R}^n \setminus \Omega_i. \end{aligned} \tag{4.7}$$

The exterior projection operator Q_{Γ_i} will be given by

$$Q_{\Gamma_i}\xi_{\Gamma_i} = Tr_i \, Q_{\mathbb{R}^n \setminus \Omega_i}\xi_{\Gamma_i}. \tag{4.8}$$

Combining (4.2), (4.6), and (4.7), we obtain a scalar function defined on both Ω_i and $\mathbb{R}^n\backslash\Omega_i$:

$$\int_{\Gamma_i}\left(\xi_0\frac{\partial G}{\partial n}-\xi_1 G\right)ds_y=\begin{cases}P_{\Omega_i}\xi_{\Gamma_i}(x), & x\in\Omega_i,\\ -Q_{\mathbb{R}^n\backslash\Omega_i}\xi_{\Gamma_i}(x), & x\in\mathbb{R}^n\backslash\Omega_i.\end{cases} \tag{4.9}$$

As has already been seen, we can calculate each branch of (4.9) using volumetric integrals instead of surface integrals.

Now let $u=u_i^+ + u_i^-$, where u_i^+ originates inside its corresponding Ω_i and u_i^- originates from outside of Ω_i. That is, $u_i^- = u^- + \sum_{j\neq i} u_i^+$ is the entire incoming component for Ω_i. Also denote $\xi_{\Gamma_i}=(u,\frac{\partial u}{\partial n})|_{\Gamma_i}$ and

$$\xi_{\Gamma_i}^+=\left(u_i^+,\frac{\partial u_i^+}{\partial n}\right)\Big|_{\Gamma_i},$$

$$\xi_{\Gamma_i}^-=\left(u_i^-,\frac{\partial u_i^-}{\partial n}\right)\Big|_{\Gamma_i}.$$

According to formula (4.9) and definitions of the projections (4.4) and (4.8), we then have

$$\begin{aligned}P_{\Gamma_i}\xi_{\Gamma_i}&=\xi_{\Gamma_i}^-,\\ Q_{\Gamma_i}\xi_{\Gamma_i}&=\xi_{\Gamma_i}^+.\end{aligned} \tag{4.10}$$

Hence the sum of the two projections is the identity $P_{\Gamma_i}+Q_{\Gamma_i}=I$. Formula (4.10) renders the wave split. The space Ξ_{Γ_i} of all two-dimensional vector functions ξ_{Γ_i} is split into a direct sum of two subspaces: $\Xi_{\Gamma_i}=\Xi_{\Gamma_i}^+\oplus\Xi_{\Gamma_i}^-$, where $\Xi_{\Gamma_i}^-=ImP_{\Gamma_i}\equiv KerQ_{\Gamma_i}$ contains traces of all incoming waves and $\Xi_{\Gamma_i}^+=ImQ_{\Gamma_i}\equiv KerP_{\Gamma_i}$ contains traces of all outgoing waves. The split is done only on the boundary, and no knowledge of the wave sources is needed. Any function ξ_{Γ_i} is represented as $\xi_{\Gamma_i}^-+\xi_{\Gamma_i}^+$, where $\xi_{\Gamma_i}^-$ can be extended to Ω_i and $\xi_{\Gamma_i}^+$ can be extended to $\mathbb{R}^n\backslash\Omega_i$, as solutions of the homogeneous equation $Lu=0$. The extensions are given by the incoming and outgoing branches of the potential:

$$P_{\Omega_i}\xi_{\Gamma_i}=P_{\Omega_i}\xi_{\Gamma_i}^-=u_i^-,\quad x\in\Omega_i,$$

and

$$Q_{\mathbb{R}^n\backslash\Omega_i}\xi_{\Gamma_i}=Q_{\mathbb{R}^n\backslash\Omega_i}\xi_{\Gamma_i}^+=u_i^+,\quad x\in\mathbb{R}^n\backslash\Omega_i,$$

respectively. If a given ξ_{Γ_i} satisfies the boundary equation with projection

$$P_{\Gamma_i}\xi_{\Gamma_i}=\xi_{\Gamma_i}, \tag{4.11}$$

then this function is the trace of some u_i^-. That is, it is extendible to Ω_i as a solution of $Lu=0$. In other words, those and only those ξ_{Γ_i} that are traces of solutions to the homogeneous equation $Lu=0$ on Ω_i satisfy the Calderon boundary equation (4.11). A reciprocal result holds for $Q_{\Gamma_i}\xi_{\Gamma_i}=\xi_{\Gamma_i}$.

Having defined the potentials and projections for individual domains Ω_i, we will now extend the definitions to the entire composite domain $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N$. Denote $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_N$, and let ξ_Γ be a two-dimensional vector function on this composite boundary. The interior branch of the potential with the density ξ_Γ is defined similarly to (4.6):

$$P_\Omega \xi_\Gamma(x) = \int_{\mathbb{R}^n \backslash \Omega} GLw dy, \quad x \in \Omega,$$

where $w = w(x)$ is an auxiliary function that satisfies the interface conditions

$$Tr\, w = \xi_\Gamma \iff \left\{ Tr_i\, w = \xi_\Gamma\big|_{\Gamma_i},\ i = 1, 2, \ldots, N \right\}$$

and the appropriate Sommerfeld condition (2.3a) or (2.3b) at infinity. Other than that, $w(x)$ may be arbitrary. Likewise, the exterior branch of the potential is given by

$$Q_{\mathbb{R}^n \backslash \Omega} \xi_\Gamma(x) = \int_\Omega GLw dy, \quad x \in \mathbb{R}^n \backslash \Omega.$$

Using definition (4.7), for the exterior region $\mathbb{R}^n \backslash \Omega$ we can write

$$Q_{\mathbb{R}^n \backslash \Omega} \xi_\Gamma(x) = \sum_{i=1}^N Q_{\mathbb{R}^n \backslash \Omega_i} \xi_{\Gamma_i}(x) = \sum_{i=1}^N u_i^+(x), \quad x \in \mathbb{R}^n \backslash \Omega, \tag{4.12}$$

whereas for the interior of Ω_i, $i = 1, 2, \ldots, N$, we have according to (4.6) and (4.7)

$$\begin{aligned} P_\Omega \xi_\Gamma(x) &= P_{\Omega_i} \xi_{\Gamma_i}(x) - \sum_{\substack{j=1 \\ j \neq i}}^N Q_{\mathbb{R}^n \backslash \Omega_j} \xi_{\Gamma_j}(x) \\ &= u_i^- - \sum_{\substack{j=1 \\ j \neq i}}^N u_j^+(x) = u^-(x), \quad x \in \Omega_i. \end{aligned} \tag{4.13}$$

In formula (4.13), u_i^- denotes the entire incoming field with respect to the domain Ω_i. In other words, u_i^- is composed of u^- and u_j^+ from all Ω_j except $j = i$.

The projections for composite domains are defined as traces of the potentials:

$$\begin{aligned} P_\Gamma \xi_\Gamma &= Tr\, P_\Omega \xi_\Gamma, \\ Q_\Gamma \xi_\Gamma &= Tr\, Q_{\mathbb{R}^n \backslash \Omega} \xi_\Gamma. \end{aligned} \tag{4.14}$$

They possess the same properties as the projections built previously for individual subdomains. Namely, $P_\Gamma + Q_\Gamma = I$, and the projections render the wave split at the interface Γ into incoming waves $\xi_\Gamma^- = P_\Gamma \xi_\Gamma$ and outgoing waves $\xi_\Gamma^+ = Q_\Gamma \xi_\Gamma$.

Now that we have defined the potentials and projections for individual subdomains and for the composite domain, we can once again obtain the controls for composite domains. First, we will investigate the simple case of fully eliminating the exterior noise inside $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N$, i.e., eliminating the entire incoming component of the acoustic field with respect to Ω. Our control function $g = g(x)$ is defined as

$$g(x) = -Lw|_{\mathbb{R}^n\backslash\Omega}, \qquad g(x) = 0|_{\Omega}, \tag{4.15}$$

giving the output $v = v(x)$ in the form

$$v(x) = \begin{cases} -P_\Omega \xi_\Gamma(x), & x \in \Omega, \\ -w(x) + Q_{\mathbb{R}^n\backslash\Omega}\xi_\Gamma(x), & x \in \mathbb{R}^n\backslash\Omega. \end{cases} \tag{4.16}$$

Hence we achieve the desired cancellation on Ω, because for $x \in \Omega$ according to (4.13) we have $v(x) = -P_\Omega \xi_\Gamma(x) = -u^-(x)$. As for the exterior region $\mathbb{R}^n\backslash\Omega$, formulae (4.12), (4.13), and (4.16) indicate that the controls basically duplicate the output of a given Ω_i and subsequently halve it as it enters another subdomain Ω_j.

Next, we will explore the operator interpretation of the selective cancellation for individual subdomains. As before, assume that the $N \times N$ communication matrix $\mathbf{M}$ is given that determines which regions are allowed to hear one another. If the entry m_{ij} of this matrix at the intersection of row i and column j is equal to zero, then Ω_i hears Ω_j; otherwise, if $m_{ij} = 1$, then Ω_i does not hear Ω_j. In doing so, no reciprocity is assumed; i.e., the matrix $\mathbf{M}$ is not necessarily symmetric. At the first stage of building the selective controls, we will modify the boundary trace ξ_Γ with the help of the matrix $\mathbf{M}$.

Let $u = u(x)$ be the overall acoustic field from all original sources, and let $\xi_\Gamma = Tr\, u$. Denote $\xi_{\Gamma_i} = \xi_\Gamma\big|_{\Gamma_i}$ and introduce

$$\tilde{\xi}_\Gamma \overset{\text{def}}{=} \left\{ \tilde{\xi}_{\Gamma_i},\ i = 1, 2, \dots, N \,\middle|\, \tilde{\xi}_{\Gamma_i} = \xi_{\Gamma_i} + \sum_{\substack{j=1 \\ m_{ij}=1}}^{N} Tr_i\, Q_{\mathbb{R}^n\backslash\Omega_j}\xi_{\Gamma_j} \right\}. \tag{4.17}$$

At the second stage, we obtain the controls $\tilde{g}$ according to the same formula (4.15) as we used previously, but substituting a different auxiliary function $\tilde{w} = \tilde{w}(x)$. In addition to the appropriate Sommerfeld condition (2.3a) or (2.3b) at infinity, this new auxiliary function is supposed to satisfy an alternative interface condition at Γ:

$$Tr\, \tilde{w} = \tilde{\xi}_\Gamma, \tag{4.18}$$

where $\tilde{\xi}_\Gamma$ is defined by formula (4.17). The output of the control sources $\tilde{g}(x)$ on the domain $\Omega = \Omega_1 \cup \ldots \cup \Omega_N$ is given by the potential

$$\begin{aligned} v(x) = &- P_\Omega \tilde{\xi}_\Gamma(x) = -P_{\Omega_i}\tilde{\xi}_{\Gamma_i}(x) + \sum_{\substack{j=1 \\ j\neq i}}^{N} Q_{\mathbb{R}^n\backslash\Omega_j}\tilde{\xi}_{\Gamma_j} \\ = &- P_{\Omega_i}\xi_{\Gamma_i}(x) - \sum_{\substack{j=1 \\ m_{ij}=1}}^{N} Q_{\mathbb{R}^n\backslash\Omega_j}\xi_{\Gamma_j} + \sum_{\substack{j=1 \\ j\neq i}}^{N} Q_{\mathbb{R}^n\backslash\Omega_j}\tilde{\xi}_{\Gamma_j} \\ = &- u^- - \sum_{\substack{j=1 \\ j\neq i}}^{N} u_j^+ - \sum_{\substack{j=1 \\ m_{ij}=1}}^{N} u_j^+ + \sum_{\substack{j=1 \\ j\neq i}}^{N} u_j^+ \\ = &- u^- - \sum_{\substack{j=1 \\ m_{ij}=1}}^{N} u_j^+, \quad x \in \Omega_i, \end{aligned}$$

where we have taken into account that $P_{\Omega_i} Tr_i Q_{\mathbb{R}^n\setminus\Omega_j} \xi_{\Gamma_j} = Q_{\mathbb{R}^n\setminus\Omega_j} \xi_{\Gamma_j}$ for $x \in \Omega_i$ if $i \neq j$. Consequently, the overall field on Ω after applying the control $\tilde{g}$ is given by

$$\begin{aligned}\tilde{u}(x) &= u(x) + v(x)\\ &= u^-(x) + \sum_{j=1}^{N} u_j^+(x) - u^-(x) - \sum_{\substack{j=1\\ m_{ij}=1}}^{N} u_j^+(x)\\ &= \sum_{\substack{j=1\\ m_{ij}=0}}^{N} u_j^+(x), \quad x \in \Omega_i.\end{aligned}$$

In other words, the unwanted exterior noise $u^-(x)$ gets canceled out on all Ω_i, $i = 1, \ldots, N$, as before. Moreover, the sound field on a given Ω_i contains only the contributions from those Ω_j for which $m_{ij} = 0$, i.e., from those regions that Ω_i is allowed to hear. This is precisely the type of selective cancellation that we strived to achieve. Note also that even though we did not formulate the results in this section as theorems, it is clear that they are equivalent to the theorems of section 3.

5. A more realistic formulation. As of yet, we have only used the Calderon potentials and projections of section 4 to recast the results of section 3 in a more convenient yet equivalent operator form. However, the operator framework introduced in section 4 will also allow us to analyze a more elaborate formulation of the problem compared to that from section 3.

Instead of the Helmholtz equation (2.1), consider a general variable coefficient differential (or operator) equation

$$Lu = f, \tag{5.1}$$

where both the unknown solution $u = u(x)$ and the given right-hand side $f = f(x)$ are defined on some domain Ω_0 that may, but does not have to, coincide with the entire space $\mathbb{R}^n$. In the context of acoustics, (5.1) may, for example, govern the propagation of sound through a nonhomogeneous medium, where the propagation speed depends on the location.

A very important consideration is to define the solvability class for (5.1) on Ω_0. In most generic terms, let us require that $u \in U$, where U is a certain linear subspace of the space of all sufficiently smooth functions on Ω_0. We will assume that the solution $u = u(x)$ of (5.1) exists and is unique in U, provided that the right-hand side f belongs to another appropriate class F. Note that in the context of sections 2, 3, and 4, we had $\Omega_0 = \mathbb{R}^n$ and the class U was defined by the Sommerfeld condition (2.3a) or (2.3b) at infinity.

Since for any $f \in F$ there is a unique solution $u \in U$ of (5.1), we can introduce the inverse operator $G : F \longmapsto U$ that provides the solution for a given right-hand side:

$$u = Gf, \quad u \in U, \quad f \in F. \tag{5.2}$$

Note that previously (in the context of constant coefficients) the operator G was introduced by means of the convolution (2.4) with the fundamental solution (2.5) or (2.6). For variable coefficients, and/or when the domain Ω_0 is smaller than the entire space $\mathbb{R}^n$, the apparatus of fundamental solutions does not apply. Yet the inverse operator G of (5.2) is well defined. In practice, it can be computed; i.e., problem (5.1) subject to the condition $u \in U$ can be discretized on Ω_0 and solved numerically.

Another very important consideration is the structure of the boundary trace that corresponds to the new operator L of (5.1). For the Laplace and Helmholtz operators, the vector traces on Γ are defined as traces of the solution itself and of the normal derivative; see formula (4.3). In the general theory of Calderon's operators (see [19]), the traces are constructed to guarantee a key property of the potentials (4.6), (4.7) and projections (4.4), (4.8), namely, their independence of the auxiliary function $w(x)$ as long as it has the correct trace, i.e., as long as the interface condition (4.5) is satisfied. For the second order variable coefficient operators L that have the form

$$Lv = \nabla(p\nabla v) + \{\text{lower order terms}\}, \quad p = p(x), \tag{5.3}$$

the Neumann data reduce to the standard normal derivative, and, consequently, the previous definition of the trace (see (4.3)) applies with no change. Hereafter, we will assume for simplicity that this is the case. This assumption does not entail a considerable loss of generality because operators (5.3) cover many important applications.

Having introduced the operator equation (5.1), defined the inverse (5.2), and identified the boundary trace Tr (4.3), we can extend all the operator constructions of section 4 in a straightforward manner, as done in [13] for a single domain. The only thing that will change is that every time a volumetric convolution with the fundamental solution appears in an equation, it ought to be replaced by the operator G of (5.2) applied to the corresponding source function. This way, we define the generalized Calderon potentials (cf. formulae (4.6) and (4.7))

$$P_{\Omega_i}\xi_{\Gamma_i}(x) = G\Big\{Lw\big|_{\mathbb{R}^n\setminus\Omega_i}\Big\}, \quad x \in \Omega_i, \tag{5.4}$$

$$Q_{\mathbb{R}^n\setminus\Omega_i}\xi_{\Gamma_i}(x) = G\Big\{Lw\big|_{\Omega_i}\Big\}, \quad x \in \mathbb{R}^n\setminus\Omega_i, \tag{5.5}$$

and the boundary projection operators (cf. formulae (4.4) and (4.8))

$$P_{\Gamma_i}\xi_{\Gamma_i} = Tr_i P_{\Omega_i}\xi_{\Gamma_i}, \tag{5.6}$$

$$Q_{\Gamma_i}\xi_{\Gamma_i} = Tr_i Q_{\mathbb{R}^n\setminus\Omega_i}\xi_{\Gamma_i} \tag{5.7}$$

for all $i = 1, 2, \ldots, N$. Combined operators for the composite domain $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N$ are also introduced similarly to section 4, according to formulae (4.12), (4.13), and (4.14), where the individual operators are now given by (5.4)–(5.7).

The fundamental properties of the projections (5.6) and (5.7) are the same as before. Namely, the function $u \in U$ is a solution to the homogeneous equation $Lu = 0$ on the domain Ω_i if and only if its boundary trace $\xi_{\Gamma_i} = Tr_i u$ satisfies the boundary equation with projection,

$$P_{\Gamma_i}\xi_{\Gamma_i} = \xi_{\Gamma_i}. \tag{5.8}$$

Similarly, the function $u \in U$ is a solution to the homogeneous equation $Lu = 0$ on the complementary domain $\Omega_0 \setminus \Omega_i$ if and only if its boundary trace $\xi_{\Gamma_i} = Tr_i u$ satisfies the boundary equation with projection,

$$Q_{\Gamma_i}\xi_{\Gamma_i} = \xi_{\Gamma_i}. \tag{5.9}$$

Accordingly, if the solutions to (5.1) are interpreted as waves, then one can say that the boundary equations with projections (5.8) and (5.9) render the wave split into incoming and outgoing with respect to a given Ω_i. If $u \in U$ and $Tr_i u = \xi_{\Gamma_i}$, then

$$\xi_{\Gamma_i} = P_{\Gamma_i}\xi_{\Gamma_i} + Q_{\Gamma_i}\xi_{\Gamma_i} \stackrel{\text{def}}{=} \xi^-_{\Gamma_i} + \xi^+_{\Gamma_i},$$

where the component $\xi^-_{\Gamma_i}$ is the trace of the incoming field due to the sources outside Ω_i,

$$\xi^-_{\Gamma_i} = Tr_i u_i^-, \quad Lu_i^- = 0 \ \text{ for } \ x \in \Omega_i,$$

and the component $\xi^+_{\Gamma_i}$ is the trace of the outgoing field due to the sources inside Ω_i,

$$\xi^+_{\Gamma_i} = Tr_i u_i^+, \quad Lu_i^+ = 0 \ \text{ for } \ x \in \mathbb{R}^n \setminus \Omega_i.$$

In doing so, the entire space $\Xi_{\Gamma_i} = \left\{\xi_{\Gamma_i}\right\}$ can be represented as a direct sum of the traces of incoming waves and those of the outgoing waves:

$$\Xi_{\Gamma_i} = \Xi^-_{\Gamma_i} \oplus \Xi^+_{\Gamma_i}.$$

The exact same results automatically extend to the operators built for the composite domain $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N$ as well.

Moreover, all the conclusions of sections 3 and 4 regarding the active control sources are also preserved. Namely, to cancel out the unwanted exterior sound u^- on $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N$, we build the controls according to formula (4.15):

$$g(x) = -Lw|_{\mathbb{R}^n\setminus\Omega}, \qquad g(x) = 0|_{\Omega}, \tag{5.10}$$

where the auxiliary function $w = w(x)$ satisfies $w \in U$ and $Trw = Tru$, and $u = u(x)$ is the overall acoustic field. We emphasize that in order to obtain the controls $g(x)$ of (5.10), we only need to know Tru at the boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_N$. Moreover, the coefficients of the operator L, i.e., the properties of the medium, only need to be known outside Ω on the region where the auxiliary function $w(x) \neq 0$. This region can be a narrow layer outside Γ right next to it. This conclusion seems counterintuitive at first glance, because the controls $g(x)$ of (5.10) are supposed to eliminate the unwanted component of the field inside Ω, and yet it seems that the properties of the medium do not need to be known. The explanation, however, is quite simple. Both the unwanted noise $u^-(x)$ and the output of the controls $v(x) = Gg$ propagate across one and the same medium, and to achieve cancellation we do not necessarily need to know what this medium is inside Ω. Equivalently, one can think that the entire incoming component $u^-(x)$ is canceled by the controls (5.10) right at the entry to Ω so that it does not propagate any further; see [13, sections 4.2 and 4.3].

Active controls $\tilde{g}(x)$ that will render the selective cancellation of sound on the system of subdomains Ω_i, $i = 1, 2, \ldots, N$, according to a predetermined communication pattern $\mathbf{M}$ are also obtained with the help of formula (5.10). The only difference is that as before, the application of this formula requires a preliminary stage. At this preliminary stage, we construct a modified boundary trace $\tilde{\xi}_\Gamma$ according to formula (4.17), where the operators $Q_{\mathbb{R}^n\setminus\Omega_j}$ are defined by (5.5). At the final stage, we take an auxiliary function $\tilde{w} = \tilde{w}(x)$ that satisfies $\tilde{w} \in U$ and $Tr\tilde{w} = \tilde{\xi}_\Gamma$ and substitute it into (5.10), thus obtaining the desired selective controls $\tilde{g}(x)$.

For more detail on the theory of generalized Calderon potentials and projections, as well as their efficient computation by means of the method of difference potentials, we refer the reader to the monograph [19].

6. Conclusions. We have introduced and studied the problem of active control of sound for composite regions. This problem is, in fact, a particular inverse source problem for the differential equation (or system) that governs the sound field. Allowing for composite domains is a key innovation proposed here as compared to our previous work on the subject (see [13] and related references). We obtained a closed form general solution for the control sources. This solution allows all individual subdomains to either communicate freely with one another or else be shielded from their peers. In doing so, no reciprocity is assumed; i.e., for a given pair of subdomains one may be allowed to hear the other but not necessarily vice versa.

If the controls in the composite case are built exactly as in the previously analyzed case of simple, i.e., arcwise connected, domains, then the communications between all subdomains is allowed. In other words, by default all subdomains hear one another. If, however, a particular subdomain is not allowed to hear another given subdomain, then the supplementary controls are employed prior to building the final set of controls. The role of the supplementary controls (one can call it the predictor stage) is to communicate the specific acoustic output of the domain not to be heard to the domain that is not allowed to hear it. Subsequently, the final controls (corrector stage) use these data to render the desired sound cancellation.

Moreover, the general solution requires no information on the original acoustic sources and can be constructed based solely on the knowledge of the field quantities at the boundaries of the subdomains. In practice, those quantities can be obtained by measurements. In doing so, the methodology guarantees the exact volumetric cancellation of the unwanted noise, as opposed to many other techniques available in the literature that would only provide for a pointwise or directional cancellation, and would not even offer an approach to selective cancellation on composite domains.

The problem is solved for a general formulation that allows the propagation of sound across a medium with variable characteristics. In doing so, to cancel out the outside sound on a given domain, no actual knowledge of the medium properties on this domain is required. The explanation of this seemingly counterintuitive behavior is simple—both the original sound and the output of the controls propagate across one and the same medium, and for building the control sources we do not necessarily need to know what this medium is.

It is also important to mention that for every subdomain there is a component of the acoustic field to be canceled out and another component to be left unaffected. Yet the quantities at the boundary that need to be measured in order to build the control system can pertain to the overall field rather than only to its unwanted component, and the methodology will automatically distinguish between the two. Of course, the locations and shapes of the subdomains need to be known ahead of time.

Finally, it is clear that in the context of implementation, obtaining the continuous data, as well as providing a continuous excitation (control sources), along the interface Γ is not practical. Instead, the problem needs to be discretized so that only finite arrays of individual sensors (microphones) and actuators (loudspeakers) are used. A powerful apparatus for the analysis of discrete active shielding problems is provided by the method of difference potentials [19]. This method offers a comprehensive finite-difference theory, which is fully analogous to the continuous theory of Calderon's operators [3, 23] and in many instances even goes beyond it. As mentioned in section 1, discrete active controls have been built, and their properties established, for various settings; see [14, 15, 16, 18, 22, 25, 27, 28]. In particular, the case of a composite region in the discrete framework is analyzed in [21]. A brief account of the method of difference potentials, along with the analysis of discrete active shielding problems, can be found in [20, Chapter 14].

REFERENCES

[1] J. C. Burgess, *Active adaptive sound control in a duct: A computer simulation*, J. Acoust. Soc. Amer., 70 (1981), pp. 715–726.
[2] R. H. Cabell and C. R. Fuller, *Active control of periodic disturbances using principal component LMS: Theory and experiment*, in Proceedings of the 3rd AST/HSR Interior Noise Workshop, Part I: Sessions A, B, and C, NASA Langley Research Center, Hampton, VA, 1998.
[3] A. P. Calderon, *Boundary-value problems for elliptic equations*, in Proceedings of the Soviet-American Conference on Partial Differential Equations, Fizmatgiz, Novosibirsk, Moscow, 1963, pp. 303–304.
[4] R. L. Clark, W. R. Saunders, and G. P. Gibbs, *Adaptive Structures: Dynamics and Control*, John Wiley and Sons, New York, 1998.
[5] A. J. Devaney and G. C. Sherman, *Nonuniqueness in inverse source and scattering problems*, IEEE Trans. Antennas and Propagation, 30 (1982), pp. 1034–1037; see also pp. 1037–1042.
[6] A. J. Devaney and E. Wolf, *Radiating and nonradiating classical current distributions and the fields they generate*, Phys. Rev. D, 8 (1973), pp. 1044–1047.
[7] S. J. Elliott, I. M. Stothers, and P. A. Nelson, *A multiple error LMS algorithm and its application to the active control of sound and vibration*, in IEEE Trans. Acoust., Speech, Signal Processing, ASSP-35 (1987), pp. 1423–1434.
[8] C. R. Fuller, S. J. Elliott, and P. A. Nelson, *Active Control of Vibration*, Academic Press, London, 1996.
[9] C. R. Fuller and A. H. von Flotow, *Active control of sound and vibration*, IEEE Control Syst. Mag., 15 (1995), pp. 9–19.
[10] V. Isakov, *Inverse Source Problems*, Mathematical Surveys Monogr. 34, American Mathematical Society, Providence, RI, 1990.
[11] R. K. Kincaid and K. Laba, *Reactive TABU search and sensor selection in active structural control problems*, J. Heuristics, 4 (1998), pp. 199–220.
[12] R. K. Kincaid, K. Laba, and S. L. Padula, *Quelling cabin noise in turboprop aircraft via active control*, J. Comb. Optim., 1 (1997), pp. 229–250.
[13] J. Lončarić, V. S. Ryaben'kii, and S. V. Tsynkov, *Active shielding and control of noise*, SIAM J. Appl. Math., 62 (2001), pp. 563–596.
[14] J. Lončarić and S. V. Tsynkov, *Optimization of acoustic source strength in the problems of active noise control*, SIAM J. Appl. Math., 63 (2003), pp. 1141–1183.
[15] J. Lončarić and S. V. Tsynkov, *Optimization of power in the problems of active control of sound*, Math. Comput. Simulation, 65 (2004), pp. 323–335.
[16] J. Lončarić and S. V. Tsynkov, *Quadratic optimization in the problems of active control of sound*, Appl. Numer. Math., 52 (2005), pp. 381–400.
[17] P. A. Nelson and S. J. Elliott, *Active Control of Sound*, Academic Press, San Diego, 1999.
[18] V. S. Ryaben'kii, *A difference screening problem*, Funct. Anal. Appl., 29 (1995), pp. 70–71.
[19] V. S. Ryaben'kii, *Method of Difference Potentials and Its Applications*, Springer Ser. Comput. Math. 30, Springer-Verlag, Berlin, 2002.
[20] V. S. Ryaben'kii and S. V. Tsynkov, *A Theoretical Introduction to Numerical Analysis*, Chapman & Hall/CRC, Boca Raton, FL, 2007.
[21] V. S. Ryaben'kii, S. V. Tsynkov, and S. V. Utyuzhnikov, *Inverse source problem and active shielding for composite domains*, Appl. Math. Lett., 20 (2007), pp. 511–516.
[22] V. S. Ryaben'kii and S. V. Utyuzhnikov, *Differential and finite-difference problems of active shielding*, Appl. Numer. Math., 57 (2007), pp. 374–382.
[23] R. T. Seeley, *Singular integrals and boundary value problems*, Amer. J. Math., 88 (1966), pp. 781–809.
[24] M. O. Tokhi and S. M. Veres, eds., *Active Sound and Vibration Control: Theory and Applications*, IEE Control Ser. 62, The Institution of Electrical Engineers, London, 2002.
[25] S. V. Tsynkov, *On the definition of surface potentials for finite-difference operators*, J. Sci. Comput., 18 (2003), pp. 155–189.
[26] H. Van der Auweraer, M. Iadevaia, U. Emborg, and M. Gustavsson, *Derivation of experimental vibro-acoustical models for ANC configuration design*, AIAA Paper 97-1618, in Proceedings of the 3rd AIAA/CEAS Aeroacoustics Conference, Atlanta, GA, 1997, pp. 222–234.
[27] R. I. Veizman and V. S. Ryaben'kii, *Difference problems of screening and simulation*, Dokl. Akad. Nauk, 354 (1997), pp. 151–154.
[28] R. I. Veizman and V. S. Ryaben'kii, *Difference simulation problems*, Trans. Moscow Math. Soc., 58 (1997), pp. 239–248.

[29] B. Widrow, D. Shur, and S. Shaffer, *On adaptive inverse control*, in Proceedings of the 15th IEEE Asilomar Conference on Circuits, Systems and Computers, Pacific Grove, CA, 1981, pp. 185–189.

[30] S. E. Wright and B. Vuksanovic, *Active control of environmental noise*, J. Sound Vibration, 190 (1996), pp. 565–585.

[31] S. E. Wright and B. Vuksanovic, *Active control of environmental noise*, II: *Non-compact acoustic sources*, J. Sound Vibration, 202 (1997), pp. 313–359.