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On the Structure of A_0 in Graded Principal Ideal Domains

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Abstract

In Caenepeel and Oystaeyen's [3] discussion of Bauer groups and the cohomology of graded rings, the topic of commutative rings arises. Specifically, certain results arise regarding graded principal ideal domains and graded Dedekind domains. Fossum and Foxby in [3] classify the graded fields. Motivated by this and Caenepeel and Oystaeyen's discussion, this paper will delve into the structure of graded principal ideal domains and specifically analyze the structure of the ring A_0 within the graded ring $A = \bigoplus_{n \in Z} A_n$.

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1. Preliminaries

Definition. A ring R is said to be a *graded ring of type* Z (integers) if there is a family of additive subgroups $\{R_n \mid n \in Z\}$ such that $R = \bigoplus_{n \in Z} R_n$ and

$R_i R_j \subseteq R_{i+j}$ for $i, j \in Z$.

Note. Although for any abelian group G, it is very well possible to study gradations of type G, we will only consider gradations of type Z in this paper. Also, all rings (graded and otherwise) in this paper are assumed to be commutative with a multiplicative identity.

Definition. If R is a graded ring, an R-module M is said to be a *graded* R*-module* if there exists a family $\{M_n \mid n \in Z\}$ of additive subgroups of M with $M = \bigoplus_{n \in Z} M_n$ and $R_i M_j \subseteq M_{i+j}$ for $i, j \in Z$.

Definition. The elements of $\bigcup_{n \in Z} R_n$ and $\bigcup_{n \in Z} M_n$ are called *homogeneous elements* of R and M, respectively.

Notation. If $m \neq 0$, $m \in M_i$, then m is called a homogeneous element of degree i, we write $deg\ m = i$.

Definition. An ideal of the graded ring $R = \bigoplus_{n \in Z} R_n$ is a graded (homogeneous) ideal of R if and only if it is generated by homogeneous elements.

Definition. Let $R = \bigoplus_{n \in Z} R_n$ be a graded ring and M a graded R-module. A submodule N of M is a *graded submodule* if $N = \bigoplus_{n \in Z} N \cap M_n$ or, equivalently, for any $x \in N$ the homogeneous components of x are again in N.

Definition. Let $R = \bigoplus_{n \in Z} R_n$ be a graded ring. If $R_i = 0$ for $i < 0$, then R is said to be *positively graded*, and if $R_i = 0$ for $i > 0$, then R is said to be *negatively graded*.

Definition. $R = \bigoplus_{n \in Z} R_n$ is said to be a *graded principal ideal domain* if and only if every graded ideal of R is generated by a single homogeneous element.

Definition. Let D be an integral domain with quotient field K. A *fractional ideal* of D is a nonzero D-submodule I of K such that $aI \subseteq D$ for some nonzero $a \in D$.

Note. An asterisk (*) preceding a term or property denotes its graded analog. For example, *ideal = graded ideal, *module = graded module, *PID = graded principal ideal domain, etc.

Definition. An integral domain D is a *Dedekind domain* if and only if every ideal of D is projective.

Note. An equivalent definition of a Dedekind domain is as follows: An integral domain D is a Dedekind domain if and only if the set fractional ideals of D form a group under multiplication.

Definition. Let D be an integral domain with quotient field K. A *fractional ideal* of D is a nonzero D-submodule I of K such that $aI \subseteq D$ for some nonzero $a \in D$.

2. On the Structure of A_0 in Graded Principal Ideal Domains

We shall assume in the following that A is a graded ring with $A = \bigoplus_{n\in Z} A_n$

From Fossum and Foxby [2, Theorem 2.2] we have that if A has only the trivial graded ideals, then A_0 is a field and either $A = A_0$ or $A \cong A_0[x, x^{-1}]$ (where x is an indeterminant of degree d).

Lemma 1. *Suppose M, N are R-modules with S any multiplicative subset of R, and suppose the sequence*

$$M_1 \xrightarrow{\psi_1} M_2 \xrightarrow{\psi_2} M_3$$

is exact. Then the sequence

$$S^{-1}M_1 \xrightarrow{\overline{\psi}_1} S^{-1}M_2 \xrightarrow{\overline{\psi}_2} S^{-1}M_3$$

$$\left(\overline{\psi}_1\left(\frac{m_1}{s}\right) = \frac{\psi_1(m_1)}{s},\ \overline{\psi}_2\left(\frac{m_2}{s}\right) = \frac{\psi_2(m_2)}{s}\right)$$

is exact.

Proof. It suffices to show $Ker\,(\overline{\psi}_2) = Im\,(\overline{\psi}_1)$. First we will show $Ker\,(\overline{\psi}_2) \subseteq Im\,(\overline{\psi}_1)$.

$$\begin{aligned} \frac{m_2}{s} \in Ker\,(\overline{\psi}_2) \quad &\Rightarrow \overline{\psi}_2\left(\frac{m_2}{s}\right) = \frac{\psi_2(m_2)}{s} = \frac{0}{1} \\ &\Rightarrow \bar{s}\,\psi_2(m_2) = 0,\ \bar{s} \in s \\ &\Rightarrow \psi_2(\bar{s}m_2) = 0 \\ &\Rightarrow \bar{s}m_2 \in Ker\,(\psi_2) = Im\,(\psi_1) \\ &\Rightarrow \psi_1(m_1) = \bar{s}m_2,\ m_1 \in M_1 \end{aligned}$$

$$\Rightarrow \overline{\psi}\left(\frac{m_1}{\bar{s}s}\right) = \frac{\psi_1(m_1)}{\bar{s}s} = \frac{\bar{s}m_2}{\bar{s}s} = \frac{m_2}{s}$$

$$\Rightarrow \frac{m_2}{s} \in Im\,(\overline{\psi}_1).$$

Now we will show $Im\,(\overline{\psi}_1) \subseteq Ker\,(\overline{\psi}_2)$.

$$\frac{m_2}{s} \in Im\,(\overline{\psi}_1) \quad \Rightarrow \overline{\psi}_1\left(\frac{m_1}{s}\right) = \frac{\psi_1(m_1)}{\bar{s}} = \frac{m_2}{s},\ \frac{m_1}{\bar{s}} \in S^{-1}M$$

$$\Rightarrow t(s\psi_1(m_1) - \bar{s}m_2) = 0,\ t \in s$$

$$\Rightarrow ts\psi_1(m_1) = t\bar{s}m_2$$

$$\Rightarrow \psi_1(tsm_1) = t\bar{s}m_2$$

$$\Rightarrow t\bar{s}m_2 \in Im\,(\psi_1) = Ker\,(\psi_2)$$

$$\Rightarrow \psi_2(t\bar{s}m_2) = 0$$

$$\Rightarrow t\bar{s}\,\psi_2(m_2) = 0$$

■

Lemma 2. *Let* M, N *be* R*-modules. Then*

$$M \xrightarrow{\psi} N \to 0$$

is an exact sequence if and only if

$$S^{-1}M \xrightarrow{\overline{\psi}} S^{-1}N \to 0$$

$\left(\overline{\psi}\left(\frac{m}{s}\right) = \frac{\psi(m)}{s}\right)$ *is an exact sequence for every multiplicative set* $S = R - P$, *where* P *is a prime ideal of* R.

Proof. ($\Rightarrow$) See Lemma 1.

Conversely, suppose the sequence

$$S^{-1}M \xrightarrow{\overline{\psi}} S^{-1}N \to 0$$

is exact for every $S = R - P$, where P is a prime ideal of R. Now consider the exact sequence

$$M \xrightarrow{\psi} N \xrightarrow{\pi} N/Im\,(\psi) \to 0,$$

where π is the canonical epimorphism. So the sequence

$$S^{-1}M \to S^{-1}N \to S^{-1}(N/Im\,(\psi)) \to 0$$

is exact for every $S = R - P$, where P is a prime ideal of R. Since $\overline{\psi}$ is surjective, we have

$$* \qquad S^{-1}(N/Im(\psi)) = 0$$

for every $S = R - P$, where P is a prime ideal of R.

Suppose $N/Im(\psi)$ is nontrivial. Hence there exists $n + Im(\psi) \in N/Im(\psi)$ such that

$$n + Im(\psi) \neq 0 + Im(\psi).$$

Consider the ideal Ann $(n + Im(\psi)) \subseteq R$ defined by

$$Ann(n + Im(\psi)) = \{r \in R \mid r(n + Im(\psi)) = 0 + Im(\psi)\}.$$

Since $n + Im(\psi) \neq 0 + Im(\psi)$, $1 \notin Ann(n + Im(\psi))$. Hence $Ann(n + Im(\psi)) \neq R$. So $Ann(n + Im(\psi))$ is contained in some maximal (prime) ideal M of R. Now let $S = R - M$. From * we have

$$S^{-1}(N/Im(\psi)) = 0.$$

Thus

$$\frac{n + Im(\psi)}{1} = \frac{0 + Im(\psi)}{1}.$$

But this implies there exists some $r \in s$ such that

$$r(n + Im(\psi)) = 0 + Im(\psi).$$

This contradicts the fact $S \cap Ann(n + Im(\psi)) = \emptyset$. Therefore, $N/Im(\psi) = 0$. ■

Now let $S \subset R$ where R is a commutative ring and S a multiplicative subset of R, and let M, N be R-modules. Consider the map

$$Hom_R(M, N) \to Hom_{S^{-1}R}(S^{-1}M, S^{-1}N)$$

that maps f to $S^{-1}f$.

This induces a map

$$\psi : S^{-1}(Hom_R(M, N)) \to Hom_{S^{-1}R}(S^{-1}M, S^{-1}N).$$

Lemma 3. *If M is a finitely generated R-module with R Noetherian, then the map ψ (above) is an isomorphism.*

Proof. Suppose M is a finitely generated R-module with R Noetherian.

If $M = R$, then $Hom_R(R, N) \cong N$ and $Hom_{S^{-1}R}(S^{-1}R, S^{-1}N) \cong S^{-1}N$.

Now with the above observation and noting that $S^{-1}N \cong S^{-1}Hom_R(R, N)$ we have

$$S^{-1}Hom_R(R, N) \cong Hom_{S^{-1}R}(S^{-1}R, S^{-1}N).$$

Similarly, if $M = R^n$ $(n \geq 1)$, then

$$Hom_R(R^n, N) \cong N^n$$

and

$$S^{-1}Hom_R(R^n,N)\cong Hom_{S^{-1}R}(S^{-1}R^n,S^{-1}N). \qquad (1)$$

Since M is finitely generated R-module and R is Noetherian, we can (with the obvious maps) create the exact sequence

$$R^m \to R^n \to M \to 0.$$

This yields the exact sequence

$$0\to Hom_R(M,N)\to Hom_R(R^n,N)\to Hom_R(R^m,N).$$

Thus

$$0\to S^{-1}Hom_R(M,N)\to S^{-1}Hom_R(R^n,N)\to S^{-1}Hom_R(R^m,N) \qquad (2)$$

is an exact sequence.

But $R^m \to R^n \to M \to 0$ being exact also yields the exact sequence

$$S^{-1}R^m \to S^{-1}R^n \to S^{-1}M \to 0.$$

Therefore,

$$0\to Hom_{S^{-1}R}(S^{-1}M,S^{-1}N)\to Hom_{S^{-1}R}(S^{-1}R^n,S^{-1}N)\to Hom_{S^{-1}R}(S^{-1}R^m,S^{-1}N)$$

(3) is an exact sequence.

Consider the commutative diagram

$$\begin{array}{llll} 0\to & S^{-1}Hom_R(M,N) & \to S^{-1}Hom_R(R^n,N)\to & S^{-1}Hom_R(R^m,N) \\ & \downarrow \psi & \downarrow & \downarrow \\ 0\to & Hom_{S^{-1}R}(S^{-1}M,S^{-1}N) & \to Hom_{S^{-1}R}(S^{-1}R^n,N)\to & Hom_{S^{-1}R}(S^{-1}R^m,N) \end{array}$$

with the rows being the exact sequences (2) and (3), respectively. Since the last two vertical maps are the isomorphisms from (1), ψ must be an isomorphism. ■

Theorem 1. *If A_0 is a Noetherian integral domain, then $S^{-1}A_0$ is a PID for all $S = A_0 - P$ (P a prime ideal of A_0) if and only if A_0 is a Dedekind domain.*

Proof. Suppose A_0 is a Dedekind domain. Thus $S^{-1}A_0$ is a local Dedekind domain. Let M be the unique maximal ideal of $S^{-1}A_0$. If $M = 0$, then $S^{-1}A_0$ is a field and is therefore a PID.

If $M \neq 0$ then M is inveritible. So $\exists$ an ideal M^{-1} of $S^{-1}A_0$ such that $MM^{-1} = S^{-1}A_0$. So we can find

$$\frac{a_1}{s_1},\frac{a_2}{s_2},\dots,\frac{a_n}{s_n}\in M \quad and \quad \frac{b_1}{\bar{s}_1},\frac{b_2}{\bar{s}_2},\dots,\frac{b_n}{\bar{s}_n}\in M^{-1}$$

such that

$$\frac{a_1b_1}{s_1\bar{s}_1}+\cdots+\frac{a_nb_n}{s_n\bar{s}_n}=\frac{1}{1}$$

Since $S^{-1}A_0$ is a local ring, one of $\frac{a_1b_1}{s_1\bar{s}_1},\cdots,\frac{a_nb_n}{s_n\bar{s}_n}$ is a unit. WLOG suppose $\frac{a_1b_1}{s_1\bar{s}_1}$ is a unit. So $\exists\left(\frac{a_1b_1}{s_1\bar{s}_1}\right)^{-1}\in S^{-1}A_0$ such that $\left(\frac{a_1b_1}{s_1\bar{s}_1}\right)\left(\frac{a_1b_1}{s_1\bar{s}_1}\right)^{-1}=\frac{1}{1}$.

CLAIM: $M=\left(\frac{a_1}{s_1}\right)$.

Clearly $\left(\frac{a_1}{s_1}\right)\subseteq M$.

Now let $\frac{a}{s}\in M$. So

$$\frac{a}{s}=\frac{a}{s}\left(\frac{1}{1}\right)=\frac{a}{s}\left[\left(\frac{a_1b_1}{s_1\bar{s}_1}\right)\cdot\left(\frac{a_1b_1}{s_1\bar{s}_1}\right)^{-1}\right]=\frac{a_1}{s_1}\left[\left(\frac{ab_1}{s\bar{s}_1}\right)\cdot\left(\frac{a_1b_1}{s_1\bar{s}_1}\right)^{-1}\right]\in\left(\frac{a_1}{s_1}\right)$$

Thus the unique maximal ideal M of $S^{-1}A_0$ is generated by $\frac{a_1}{s_1}$ (i.e. $M=\left(\frac{a_1}{s_1}\right)$)

CLAIM: $\bigcap_{n=1}^{\infty}M^n=0$.

First we note $\frac{a}{s}\in\bigcap_{n=1}^{\infty}M^n$ if and only if $\left(\frac{a_1}{s_1}\right)^n\mid\frac{a}{s}$ in $S^{-1}A_0$ for $n=1,2,...$

Now let $\frac{a}{s}\in\bigcap_{n=1}^{\infty}M^n$. So by the above note, $\frac{a_1}{s_1}\mid\frac{a}{s}$. Hence $\frac{a}{s}=\frac{a_1}{s_1}\left(\frac{b}{s}\right)$, where $\frac{b}{s}\in S^{-1}A_0$. Thus

$$\left(\frac{a_1}{s_1}\right)^n\mid\frac{b}{n}\quad for\quad n=1,2,...$$

So $\frac{b}{s}\in\bigcap_{n=1}^{\infty}M^n$.

This implies

$$\frac{a_1}{s_1}\left(\bigcap_{n=1}^{\infty} M^n\right) = \bigcap_{n=1}^{\infty} M^n.$$

Thus

$$M\left(\bigcap_{n=1}^{\infty} M^n\right) = \bigcap_{n=1}^{\infty} M^n.$$

But $S^{-1}A_0$ is a Dedekind domain and is therefore Noetherian. So $\bigcap_{n=1}^{\infty} M^n$ is a finitely generated $S^{-1}A_0$ module.

So by Nakayama's Theorem (1955) [which states that if J is an ideal in the Jacobson radical of a ring R and B is a submodule of a finitely generated R-module A with $A = JA + B$ then $A = B$] we have

$$\bigcap_{n=1}^{\infty} M^n = 0.$$

This implies if $\frac{a}{s} \in S^{-1}A_0$ and $\frac{a}{s} \neq \frac{0}{1}$, then $\exists\ n_0 \geq 0$ such that $\frac{a}{s} \in M^{n_0}$ and $\frac{a}{s} \notin M^{n_0+1}$ (where $M^0 = S^{-1}A_0$). Therefore,

$$\left(\frac{a_1}{s_1}\right)^{n_0} \text{ dvides } \frac{a}{s} \ \ but \ \left(\frac{a_1}{s_1}\right)^{n_0+1} \text{ does not divide } \frac{a}{s}.$$

So $\frac{a}{s} = \left(\frac{a_1}{s_1}\right)^{n_0} \frac{b}{\bar{s}}$ for some $\frac{b}{\bar{s}_0} \in S^{-1}A_0$. Then $\frac{a_1}{s_1}$ does not divide $\frac{b}{\bar{s}}$. Thus $\frac{b}{\bar{s}} \notin \left(\frac{a_1}{s_1}\right) = M$. Since M is the unique maximal ideal of $S^{-1}A_0$, $\frac{b}{\bar{s}}$ must be a unit. Hence

$$\left(\left(\frac{a_1}{s_1}\right)^{n_0}\right) = \left(\frac{a}{s}\right).$$

Hence the only principal ideals of $S^{-1}A_0$ are

$$S^{-1}A_0 = \left(\left(\frac{a_1}{s_1}\right)^0\right) > \left(\left(\frac{a_1}{s_1}\right)^1\right) > \left(\left(\frac{a_1}{s_1}\right)^2\right) > \left(\left(\frac{a_1}{s_1}\right)^3\right) > \cdots > (0).$$

Now let I be any ideal of $S^{-1}A_0$. If $I = 0$, then I is principal. If $I \neq 0$, then $\left(\frac{a_1}{s_1}\right)^n \in I$ for some $n \geq 0$. Choose n_0 to be the smallest such n. If $n_0 = 0$, then $I = S^{-1}A_0 = \frac{1}{1}$.

Now suppose $n_0 > 0$. Clearly, $\left(\left(\frac{a_1}{s_1}\right)^{n_0}\right) \subseteq I$.

Choose $\frac{a}{s} \in I$, where $\frac{a}{s} \neq \frac{0}{1}$. Then by the above,

$$\left(\frac{a}{s}\right) = \left(\left(\frac{a_1}{s_1}\right)^m\right)$$

for some $m \geq 0$. Thus $\left(\frac{a_1}{s_1}\right)^m \in I$.

By our choice of n_0, $m \geq n_0$. So $\left(\frac{a}{s}\right) = \left(\left(\frac{a_1}{s_1}\right)^m\right) \subseteq \left(\left(\frac{a_1}{s_1}\right)^{n_0}\right)$. Thus $\frac{a}{s} \in \left(\left(\frac{a_1}{s_1}\right)^{n_0}\right)$. So $I = \left(\left(\frac{a_1}{s_1}\right)^{n_0}\right)$. Hence all ideals of $S^{-1}A_0$ are principal.

Conversely, suppose $S^{-1}A_0$ is a PID. Hence $S^{-1}A_0$ is a Dedekind domain. Let I be any ideal of A_0, and let M and N be A_0 modules with $g : M \to N$ an A_0-module epimorphism. Thus $S^{-1}I$ is an ideal of $S^{-1}A_0$ with $S^{-1}M$ and $S^{-1}N$ being $S^{-1}A_0$ modules. Also, the $S^{-1}A_0$ -module homomorphism

$$\overline{g} : S^{-1}M \to S^{-1}N$$

defined by $\overline{g}\left(\frac{m}{s}\right) = \frac{g(m)}{s}$ is clearly an epimorphism. Since $S^{-1}A_0$ is a Dedekind domain, $S^{-1}I$ is a projective $S^{-1}A_0$-module. Therefore, the sequence

$$Hom_{S^{-1}A_0}(S^{-1}I, S^{-1}M) \xrightarrow{\overline{\psi}} Hom_{S^{-1}A_0}(S^{-1}I, S^{-1}N) \to 0$$

is exact (where $\overline{\psi}(\overline{h}) = \overline{g} \circ \overline{h}$). From Lemma 3 we have

$$S^{-1}Hom_{A_0}(I, M) \cong Hom_{S^{-1}A_0}(S^{-1}I, S^{-1}M)$$

and

$$S^{-1}Hom_{A_0}(I, N) \cong Hom_{S^{-1}A_0}(S^{-1}I, S^{-1}N).$$

So the sequence

$$S^{-1}Hom_{A_0}(I, M) \xrightarrow{\overline{\psi}} S^{-1}Hom_{A_0}(I, N) \to 0$$

is exact $\left(where\ \overline{\psi}\left(\frac{f}{s}\right) = \frac{\psi(f)}{s} \right)$.

Therefore, by Lemma 2, the sequence

$$Hom_{A_0}(I, M) \xrightarrow{\psi} Hom_{A_0}(I, N) \to 0$$

is exact (where $\psi(h) = g \circ h$). Thus I is a projective A_0-module, and A_0 is therefore a Dedekind domain. ■

Theorem 2. *If A is a *PID, then A_0 is a Dedekind domain.*

Proof. See Propositions 1.2, 1.3 in [3], and Theorem 1. ■

Theorem 3. *Let D be a Dedekind domain and let I be a nonzero fractional ideal of D. Then the graded ring $\bigoplus_{i \in Z} I^i$ $(I^0 = D)$ is a *PID if and only if for every nonzero ideal J_0 of D, $I^n J_0$ is a principal fractional ideal of D for some $n \in Z$ Z.*

Proof. Let J be any nonzero graded ideal of the graded ring $\bigoplus_{i \in Z} I^i$. So

$$J = \bigoplus_{i \in Z} J_i$$

where $J_i \subseteq I^i$ for $i \in Z$.

Then for any $n \in Z$ Z,

$$I^{-n} J_n \subseteq J_0.$$

But

$$I^n (I^{-n} J_n) = J_n.$$

This means J_n is generated by elements of J_0. Thus

$$J = \bigoplus_{i \in Z} J_i = \bigoplus_{i \in Z} I^i J_0$$

where J_0 is an ideal of D.

But since $I^n J_0$ is a principal fractional ideal of D for some $n \in Z$, $I^n J_0 = Dr$ for some $r \in I^n J_0$.

So

$$J = \bigoplus_{i \in Z} J_i = \left(\bigoplus_{i \in Z} I^i \right) r$$

Therefore, J is a principal ideal of $\underset{i \in Z}{\oplus} I^i$.

Conversely, suppose $\underset{i \in Z}{\oplus} I^i$ is a *PID and let J_0 be any ideal of D. So $\underset{i \in Z}{\oplus} I^i\, J_0$ is a graded ideal of $\underset{i \in Z}{\oplus} I^i$. Since $\underset{i \in Z}{\oplus} I^i$ is a *PID,

$$\underset{i \in Z}{\oplus} I^i J_0 = \left(\underset{i \in Z}{\oplus} I^i \right) r$$

for some $r \in I^n J_0$.

Therefore, $I^n J_0 = Dr$. So $I^n J_0$ is a principal fractional ideal of D. ■

Proposition 1. *If A is a *PID, then each A_n is isomorphic to some fractional ideal of the Dedekind domain A_0.*

Proof. Since A is a *PID, by Theorem 2 we have that A_0 is a Dedekind domain. Now choose $A_n \neq 0$ and consider the field of fractions $S^{-1}A_0 (S = A_0 - (0))$. Therefore, by Proposition 1.6 of [3] we have $dim_{S^{-1}A_0}(S^{-1}A_n) = 1$. Consequently, $S^{-1}A_n \cong S^{-1}A_0$.

Now consider the injective map $\psi : A_n \to S^{-1}A_n$ defined by $\psi(a_n) = \frac{a_n}{1}$. Thus A_n is isomorphic to a nonzero finitely generated A_0-submodule of $S^{-1}A_0$. Hence A_n is isomorphic to some fractional ideal of A_0. ■

Proposition 2. *If A is a *PID with A_0 a PID, then any $A_n \neq 0$ is a free A_0-module with the rank of A_n over A_0 being 1.*

Proof. Choose $A_n \neq 0$. From Lemma 1 we have that A_n is a finitely generated A_0-module. Also, since A is an integral domain, A_n is a torsion-free A_0-module with A_0 a PID. Therefore, A_n is a free A_0-module.

Now let $S = A_0 - (0)$. By Proposition 1, $S^{-1}A$ is a *PID with $S^{-1}A_0$ a field. Consequently, by Proposition 1.6 of [3],

$$dim_{S^{-1}A_0}(S^{-1}A_n) = 1.$$

Hence the rank of A_n over A_0 is 1. ■

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