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FGT-injective dimensions of Π-coherent rings and almost excellent extension

YUEMING XIANG

College of Mathematics and Computer Science, Hunan Normal University,
Changsha 410006, China
E-mail: xymls999@126.com

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Abstract. We study, in this article, the FGT-injective dimensions of Π-coherent rings. If R is right Π-coherent, and $\mathcal{TI}$ (resp. $\mathcal{TF}$) stands for the class of FGT-injective (resp. FGT-flat) R-modules ($n \geq 0$), we show that the following are equivalent:

(1) $\mathrm{FGT}-\mathrm{Id}_R(R) \leq n$;
(2) If $0 \to M \to F^0 \to F^1 \to \cdots$ is a right $\mathcal{TF}$-resolution of left R-module M, then the sequence is exact at F^k for $k \geq n-1$;
(3) For every flat right R-module F, there is an exact sequence $0 \to F \to A^0 \to A^1 \to \cdots \to A^n \to 0$ with each $A^i \in \mathcal{TI}$;
(4) For every injective left R-module A, there is an exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to A \to 0$ with each $F_i \in \mathcal{TF}$;
(5) If $\cdots \to I_1 \to I_0 \to M \to 0$ is a minimal left $\mathcal{TI}$-resolution of a right R-module M, then the sequence is exact at I_k for $k \geq n-1$.

Further, we characterize such homological dimension in terms of $\mathcal{TI}$-syzygy and $\mathcal{TF}$-cosyzygy of modules. Finally, we consider almost excellent extensions of rings. These extend the corresponding results in [10] as well.

Keywords. Π-coherent ring; FGT-injective dimension; resolution; almost excellent extension.

1. Introduction

Throughout this article, R is an associative ring with identity and all modules are unitary. ${}_R\mathcal{M}(\mathcal{M}_R)$ stands for the category of all left (right) R-modules. Let M and N be R-modules. $\mathrm{Hom}(M, N)$ (resp. $\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ (resp. $\mathrm{Ext}^n_R(M, N)$), and similarly $M \otimes N$ (resp. $\mathrm{Tor}_n(M, N)$) denotes $M \otimes_R N$ (resp. $\mathrm{Tor}^R_n(M, N)$). The dual module $M^* = \mathrm{Hom}(M, R)$. The character module M^+ is defined by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ and $\sigma_M \colon M \to M^{++}$ denotes the evaluation map.

We first recall some known notions and facts needed in the sequel.

1.1 Π-*coherent rings*

A ring R is said to be *right* Π-*coherent* if every finitely generated torsionless right R-module is finitely presented (see [1]). This concept was called strongly coherent rings in [9]. It is well known that right noetherian rings ⇒ right Π-coherent rings ⇒ right coherent rings. Π-coherent rings have been extensively studied by many authors (see [1–3, 9, 10, 14]).

1.2 FGT-*injective dimensions*

The *right* FGT-*injective dimension of a right R-module M* (see [2]), denoted by $\mathrm{FGT}-\mathrm{Id}(M)$, is defined as the least nonnegative integer n such that $\mathrm{Ext}^{n+1}(T, M) = 0$ for any finitely generated torsionless right R-module T. If no such n exist, then $\mathrm{FGT}-\mathrm{Id}(M) = \infty$. Set $\mathrm{FGT}-I\cdot\dim(R) = \sup\{\mathrm{FGT}-\mathrm{Id}(M): M \in \mathcal{M}_R\}$ and call $\mathrm{FGT}-I\cdot\dim(R)$ the *right* FGT-*injective dimension of R*. It has been proven that $\mathrm{FGT}-I\cdot\dim(R) = 0$ if and only if R is right Π-coherent and left semihereditary ring. A right R-module M is called FGT-*injective* if $\mathrm{Ext}^1(T, M) = 0$ for any finitely generated torsionless right R-module T (i.e., $\mathrm{FGT}-\mathrm{Id}(M) = 0$). A left R-module F is called FGT-*flat* if $\mathrm{Tor}_1(T, F) = 0$ for any finitely generated torsionless right R-module T (see [3]). A left R-module M is FGT-flat if and only if M^+ is FGT-injective by the standard isomorphism $\mathrm{Ext}^1(T, M^+) \cong \mathrm{Tor}_1(M, T)^+$ for any finitely generated torsionless right R-module T. In what follows, we write $\mathcal{TI}$ for the class of all FGT-injective R-modules, and $\mathcal{TF}$ stands for the class of all FGT-flat R-modules.

1.3 *(Pre)covers and (Pre)envelopes*

Let $\mathcal{C}$ be the class of R-modules. For an R-module M, a homomorphism $g: C \to M, C \in \mathcal{C}$ is called a $\mathcal{C}$-cover of M if the following hold: (1) For any homomorphism $g': C' \to M$ with $C' \in \mathcal{C}$, there exists a homomorphism $f: C' \to C$ with $g' = gf$. (2) If f is an endomorphism of C with $gf = g$, then f must be an automorphism. If (1) holds but (2) may not, $g: C \to M$ is called a $\mathcal{C}$-precover. Dually we have the definition of a $\mathcal{C}$-(pre)envelope. $\mathcal{C}$-cover and $\mathcal{C}$-envelope may not exist in general, but if they exist, they are unique up to isomorphism. A homomorphism $g: M \to C$ with $C \in \mathcal{C}$ is said to be a $\mathcal{C}$-envelope with the unique mapping property (see [4]) if for any homomorphism $g': M \to C'$ with $C' \in \mathcal{C}$, there is a unique homomorphism $f: C \to C'$ such that $fg = g'$. Dually, we have the definition of $\mathcal{C}$-cover with the unique mapping property.

1.4 *Resolutions*

Let M be a left R-module. A *right $\mathcal{TF}$-resolution of M* is a complex (need not be exact) $0 \to M \to F^0 \to F^1 \to \cdots$ with each $F^i \in \mathcal{TF}$. Write

$$L^0 = M,\ L^1 = \mathrm{Coker}(M \to F^0),\ L^i = \mathrm{Coker}(F^{i-2} \to F^{i-1}) \text{ for } i \geq 2.$$

where $M \to F^0, L^1 \to F^1, L^i \to F^i$ for $i \geq 2$ are $\mathcal{TF}$-preenvelopes. The n-th cokernel $L^n (n \geq 0)$ is called the n-th $\mathcal{TF}$-cosyzygy of M.

A *right $\mathcal{TI}$-resolution of M* is a complex (is exact since injective R-module is FGT-injective) $0 \to M \to I^0 \to I^1 \to \cdots$ with each $I^i \in \mathcal{TI}$. Write

$$C^0 = M,\ C^1 = \mathrm{Coker}(M \to I^0),\ C^i = \mathrm{Coker}(I^{i-2} \to I^{i-1}) \text{ for } i \geq 2$$

where $M \to I^0, C^1 \to I^1, C^i \to I^i$ for $i \geq 2$ are $\mathcal{TI}$-preenvelopes. The n-th cokernel $C^n (n \geq 0)$ is called the n-th $\mathcal{TI}$-cosyzygy of M.

A *left $\mathcal{TI}$-resolution of M* is a complex $\cdots \to I_1 \to I_0 \to M \to 0$ with each $I_i \in \mathcal{TI}$. Write

$$K_0 = M,\quad K_1 = \mathrm{Ker}(I_0 \to M),\quad K_i = \mathrm{Ker}(I_{i-1} \to I_{i-2}) \text{ for } i \geq 2$$

where $I_0 \to M, I_1 \to K_1, I_i \to K_i$ for $i \geq 2$ are $\mathcal{TI}$-precovers. The n-th Kernel $K_n (n \geq 0)$ is called the n-th $\mathcal{TI}$-syzygy of M.

A left $\mathcal{TI}$-resolution $\cdots \to I_1 \to I_0 \to M \to 0$ is called *minimal* if every $I_i \to K_i$ is $\mathcal{TI}$-cover for any $i \geq 0$.

It is shown in [10] that if R is right Π-coherent, then every right R-module has a $\mathcal{TI}$-cover and every left R-module has a $\mathcal{TF}$-preenvelope, so every right R-module has a minimal left $\mathcal{TI}$-resolution and every left R-module has a right $\mathcal{TF}$-resolution.

1.5 *Balance*

The concept of balance follows from [6]. Let $\mathfrak{C}$, $\mathfrak{D}$ and $\mathfrak{E}$ be categories of modules and $T\colon \mathfrak{C} \times \mathfrak{D} \to \mathfrak{E}$ be an additive functor contravariant in the first variable and covariant in the second. Let $\mathcal{I}$ and $\mathcal{F}$ be the classes of modules of $\mathfrak{C}$ and $\mathfrak{D}$ respectively. Then T is said to be *right balance* by $\mathcal{I} \times \mathcal{F}$ if for each module M of $\mathfrak{C}$, there is a $T(-, \mathcal{F})$ exact complex $\cdots \to I_1 \to I_0 \to M \to 0$ with each $I_i \in \mathcal{I}$, and for each module N of $\mathfrak{D}$, there is a $T(\mathcal{I}, -)$ exact complex $0 \to N \to F^0 \to F^1 \to \cdots$ with $F^i \in \mathcal{F}$. Similarly, we have the definition of *left balance*. If T is covariant in both variables, then we would postulate the existence of complexes $\cdots \to I_1 \to I_0 \to M \to 0$ and $\cdots \to F_1 \to F_0 \to N \to 0$ or $0 \to M \to I^0 \to I^1 \to \cdots$ and $0 \to N \to F^0 \to F^1 \to \cdots$ to define the left or right balance functors relative to $\mathcal{I} \times \mathcal{F}$, respectively.

1.6 *Almost excellent extensions*

A ring S is said to be an *almost excellent extension* of a ring R, if the following conditions are satisfied:

(1) S is right R-projective, that is, if M_S is a submodule of N_S and M_R is a direct summand of N_R, then M_S is a direct summand of N_S.
(2) S is a finite normalizing extension of R, that is, R and S have the same identity and there are elements $s_1, \ldots, s_n \in S$ such that $S = Rs_1 + \cdots + Rs_n$ and $Rs_i = s_i R$ for all $i = 1, \ldots, n$.
(3) ${}_RS$ is flat and S_R is projective.

S is an *excellent extension* of R if S is an almost excellent extension of R and ${}_RS$ and S_R are free R-module with a common basis $\{s_1, \ldots, s_n\}$. Examples include finite matrix rings and the crossed product $R * G$ where G is a finite group with $|G|^{-1} \in R$. The concept of almost excellent extensions was introduced in [16] as a non-trivial generalization of excellent extensions.

In [7], Enochs and Jenda investigated the global dimension of a left Noetherian ring using the left injective resolutions of left R-modules. Mao recently generalized their results to left coherent rings (see [11]). In §2, we study the FGT-injective dimensions of modules and rings in terms of left $\mathcal{TI}$-resolutions and right $\mathcal{TF}$-resolutions of modules. Let R be right Π-coherent. We prove that the following are equivalent:

(1) $\mathrm{FGT} - \mathrm{Id}(R) \leq n$.
(2) If $0 \to M \to F^0 \to F^1 \to \cdots$ is a right $\mathcal{TF}$-resolution of left R-module M, then the sequence is exact at F^k for $k \geq n - 1$, where $F^{-1} = M$.
(3) For every flat right R-module F, there is an exact sequence $0 \to F \to A^0 \to A^1 \to \cdots \to A^n \to 0$ with each $A^i \in \mathcal{TI}$.

(4) For every injective left R-module A, there is an exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to A \to 0$ with each $F_i \in \mathcal{TF}$.
(5) If $\cdots \to I_1 \to I_0 \to M \to 0$ is a minimal left $\mathcal{TI}$-resolution of a right R-module M, then the sequence is exact at I_k for $k \geq n-1$, where $I_{-1} = M$.

Next it is shown that $\mathrm{FGT} - I \cdot \dim(R) \leq n (n \geq 1)$ if and only if every n-th $\mathcal{TI}$-syzygy of a right R-module is FGT-injective if and only if every $(n-1)$-th $\mathcal{TI}$-syzygy of a right R-module has a $\mathcal{TI}$-cover which is a monomorphism if and only if every n-th $\mathcal{TF}$-cosyzygy of a left R-module is FGT-flat if and only if every $(n-1)$-th $\mathcal{TF}$-cosyzygy of a left R-module has a $\mathcal{TF}$-envelope which is an epimorphism. Moreover, $\mathrm{FGT} - I \cdot \dim(R) \leq n (n \geq 2)$ if and only if every $(n-2)$-th $\mathcal{TI}$-syzygy in a minimal left $\mathcal{TI}$-resolution of a right R-module has a $\mathcal{TI}$-cover with the unique mapping property. Some known results in [9] are obtained as immediate consequences of our results.

In §3, we also study the FGT-injective, FGT-flat dimensions and Π-coherent rings under almost excellent extensions of rings. If S is an almost excellent extension of a right Π-coherent ring R, it is verified that

(1) If M_S is right S-module then $\mathrm{FGT} - \mathrm{Id}(M_S) = \mathrm{FGT} - \mathrm{Id}(M_R) = \mathrm{FGT} - \mathrm{Id}(\mathrm{Hom}_R(S, M)_S)$;
(2) If ${}_SM$ is a left S-module, then $\mathrm{FGT} - \mathrm{fd}({}_SM) = \mathrm{FGT} - \mathrm{fd}({}_RM) = \mathrm{FGT} - \mathrm{fd}({}_S(S \otimes_R M))$;
(3) $\mathrm{FGT} - I \cdot \dim(S) \leq \mathrm{FGT} - I \cdot \dim(R)$, the equality holds if $\mathrm{FGT} - I \cdot \dim(R) < \infty$;
(4) S is right Π-coherent.

2. FGT-injective dimensions

We start by the following lemmas.

Lemma 2.1. *Let R be a ring and M a right R-module. There is an exact sequence $0 \to M \to I \to N \to 0$ with I FGT-injective and $\mathrm{Ext}^1(N, I') = 0$ for all FGT-injective right R-modules I'. Moreover, $\mathrm{Tor}_1(N, F) = 0$ for all FGT-flat left R-modules F.*

Proof. In view of Theorem 4.1.6 of [8] and Corollary 3.5 of [12], any right R-module M has a special $\mathcal{TI}$-preenvelope $f\colon M \to I$, that is, there is an exact sequence $0 \to M \to I \to N \to 0$, where I is FGT-injective and $\mathrm{Ext}^1(N, I') = 0$ for all FGT-injective right R-modules I'.

For any FGT-flat left R-module F, F^+ is FGT-injective by Lemma 3.1 of [10]. Thus $(\mathrm{Tor}_1(N, F))^+ \simeq \mathrm{Ext}^1(N, F^+) = 0$, hence $\mathrm{Tor}_1(N, F) = 0$. □

Lemma 2.2. *Let R be a right Π-coherent ring and M a right R-module. Then $\mathrm{FGT} - \mathrm{Id}_R(M) \leq n$ $(n \geq 0)$ if and only if for every left $\mathcal{TI}$-resolution $\cdots \to I_n \to I_{n-1} \to \cdots \to I_1 \to I_0 \to N \to 0$ of any right R-module N, $\mathrm{Hom}(M, I_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism, where K_n is the n-th $\mathcal{TI}$-syzygy of N.*

Proof. The proof is modeled on that of Lemma 2.1 of [11]. We proceed by induction on n. For $n \geq 1$, by Lemma 2.1, there is an exact sequence $0 \to M \to I \to N \to 0$, where I is FGT-injective and $\mathrm{Ext}^1(N, I') = 0$ for all FGT-injective right R-modules I'. Then we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Hom}(I, I_n) & \to & \mathrm{Hom}(I, K_n) & \to 0 & \\ \downarrow & & \downarrow & & \\ \mathrm{Hom}(M, I_n) & \to & \mathrm{Hom}(M, K_n) & & . \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

Since $I_n \to K_n$ is a $\mathcal{TI}$-precover of K_n, the first arrow is exact. In addition, the first column is exact since $\mathrm{Ext}^1(N, I_n) = 0$. Furthermore, there is commutative diagram

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathrm{Hom}(N, K_n) & \to & \mathrm{Hom}(N, I_{n-1}) & \to & \mathrm{Hom}(N, K_{n-1}) & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathrm{Hom}(I, K_n) & \to & \mathrm{Hom}(I, I_{n-1}) & \to & \mathrm{Hom}(I, K_{n-1}) & \to 0 \quad . \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathrm{Hom}(M, K_n) & \to & \mathrm{Hom}(M, I_{n-1}) & \to & \mathrm{Hom}(M, K_{n-1}) & \\ & & & \downarrow & & & \\ & & & 0 & & & \end{array}$$

$\mathrm{FGT}-\mathrm{Id}(M) \leq n$ if and only if $\mathrm{FGT}-\mathrm{Id}(N) \leq n-1$ by Corollary 5.5.6 of [3] if and only if $\mathrm{Hom}(N, I_{n-1}) \to \mathrm{Hom}(N, K_{n-1})$ is an epimorphism by induction if and only if $\mathrm{Hom}(I, K_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism by the second diagram if and only if $\mathrm{Hom}(M, I_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism by the first diagram.

For $n = 0$, let $K_0 = M$. Then $\mathrm{Hom}(M, I_0) \to \mathrm{Hom}(M, M)$ is epic means $\mathrm{Hom}(I, M) \to \mathrm{Hom}(M, M)$ is epic. Thus $0 \to M \to I \to N \to 0$ splits and hence M is FGT-injective. Conversely, if M is FGT-injective, then it is clear that $\mathrm{Hom}(M, I_0) \to \mathrm{Hom}(M, K_0)$ is epic. □

Lemma 2.3. *If R is right Π-coherent, then $-\otimes-$ on $\mathcal{M}_R \times_R \mathcal{M}$ is right balanced by $\mathcal{TI} \times \mathcal{TF}$.*

Proof. Let M be any left R-module. By Theorem 3.4 of [10], there is a right $\mathcal{TF}$-resolution $0 \to M \to F^0 \to F^1 \to \cdots$. For any FGT-injective right R-module N, N^+ is FGT-flat in terms of Lemma 3.2 of [10]. Thus we have an exact sequence

$$\cdots \to \mathrm{Hom}(F^1, N^+) \to \mathrm{Hom}(F^0, N^+) \to \mathrm{Hom}(M, N^+) \to 0.$$

Hence

$$\cdots \to (N \otimes F^1)^+ \to (N \otimes F^0)^+ \to (N \otimes M)^+ \to 0$$

is exact. Then $0 \to N \otimes M \to N \otimes F^0 \to N \otimes F^1 \to \cdots$ is exact. In addition, by Lemma 2.1, right $\mathcal{TI}$-resolution $0 \to G \to I^0 \to I^1 \to \cdots$ of any right R-module G is exact, so the sequence $0 \to G \otimes F \to I^0 \otimes F \to I^1 \otimes F \to \cdots$ is exact for any $F \in \mathcal{TF}$ by Lemma 2.1 again, as desired. □

Remark 2.4.

(1) $\mathrm{Tor}^n(-,-)$ denotes the n-th right derived functor of $-\otimes-$ with respect to the pair $\mathcal{TI} \times \mathcal{TF}$. If R is a right Π-coherent ring, for any right R-module M and left R-module N, $\mathrm{Tor}^n(M, N)$ can be computed using either right $\mathcal{TF}$-resolution of N or right $\mathcal{TI}$-resolution of M by Lemma 2.3.

(2) By the proof of Lemma 2.1, every right R-module has a $\mathcal{TI}$-preenvelope over a right Π-coherent ring. Thus every right R-module has a right $\mathcal{TI}$-resolution. So $\mathrm{Hom}(-,-)$ is left balanced on $\mathcal{M}_R \times \mathcal{M}_R$ by $\mathcal{TI} \times \mathcal{TI}$ if R is a right Π-coherent ring. Let $\mathrm{Ext}_n(-,-)$ be the n-th left derived functor of $\mathrm{Hom}(-,-)$ with respect to the pair $\mathcal{TI} \times \mathcal{TI}$. Then, for two right R-modules M and N, $\mathrm{Ext}_n(M,N)$ can be computed using a right $\mathcal{TI}$-resolution of M or a left $\mathcal{TI}$-resolution of N.

We are now in a position to prove the following theorem.

Theorem 2.5. *If R is right Π-coherent and $n \geq 0$, then the following are equivalent*:

(1) $\mathrm{FGT}-\mathrm{Id}(R) \leq n$.
(2) *If $0 \to M \to F^0 \to F^1 \to \cdots$ is a right $\mathcal{TF}$-resolution of left R-module M, then the sequence is exact at F^k for $k \geq n-1$, where $F^{-1} = M$.*
(3) *For every flat right R-module F, there is an exact sequence $0 \to F \to A^0 \to A^1 \to \cdots \to A^n \to 0$ with each $A^i \in \mathcal{TI}$.*
(4) *For every injective left R-module A, there is an exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to A \to 0$ with each $F_i \in \mathcal{TF}$.*
(5) *If $\cdots \to I_1 \to I_0 \to M \to 0$ is a left $\mathcal{TI}$-resolution of a right R-module M, then the sequence is exact at I_k for $k \geq n-1$, where $I_{-1} = M$.*

Proof.
(3) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2). By Remark 2.4(1), the right derived functor $\mathrm{Tor}^n(R,M)$ can be computed using either right $\mathcal{TF}$-resolution of M or right $\mathcal{TI}$-resolution of R.

If $n \geq 2$, we have the exact sequence $0 \to R \to A^0 \to \cdots \to A^n \to 0$ with $A^i \in \mathcal{TI}$, so $\mathrm{Tor}^k(R,M) = 0$ for $k \geq n-1$. Computing using $0 \to M \to F^0 \to F^1 \to \cdots$ in (2), we see that the sequence $\cdots \to R \otimes F^{n-2} \to R \otimes F^{n-1} \to R \otimes F^n \to \cdots$ is exact at $R \otimes F^k$ for $k \geq n-1$, so $0 \to M \to F^0 \to F^1 \to \cdots$ is exact at F^k for $k \geq n-1$.

If $n = 1$, $0 \to R \to A^0 \to A^1 \to 0$ is exact, where A^i is FGT-injective. So $\mathrm{Tor}^1(R,M) = 0$ as above, $F^0 \to F^1 \to F^2$ is exact and $R \otimes M \to \mathrm{Tor}^0(R,M)$ is epic. Computing the latter morphism using $0 \to M \to F^0 \to F^1$, we have $M \to F^0 \to F^1 \to \cdots$ is exact.

If $n = 0$, then R is FGT-injective as a right R-module. But the balance of $-\otimes-$ then gives $0 \to R \otimes M \to R \otimes F^0 \to R \otimes F^1 \to \cdots$ is exact. Thus $0 \to M \to F^0 \to F^1 \to \cdots$ is exact.

(2) $\Rightarrow$ (3). Let $0 \to M \to F^0 \to F^1 \to \cdots$ be a right $\mathcal{TF}$-resolution of a finitely presented left R-module M. By assumption, the sequence is exact at F^k for $k \geq n-1$. Let $0 \to F \to A^0 \to A^1 \to \cdots$ be exact with F flat and each A^i FGT-injective. If $n \geq 2$, we get $\mathrm{Tor}^k(F,M) = 0$ for $k \geq n-1$ since F is flat. Computing using $0 \to F \to A^0 \to A^1 \to \cdots$, $A^{n-2} \otimes M \to A^{n-1} \otimes M \to A^n \otimes M \to A^{n+1} \otimes M$ is exact. By Lemma 8.4.23 of [5], $K = \mathrm{Ker}(A^n \to A^{n+1})$ is a pure submodule of A^n, hence K is also FGT-injective by Lemma 3.2(4) of [10]. Then $0 \to F \to A^0 \to A^1 \to \cdots \to A^{n-1} \to K \to 0$ gives the desired exact sequence.

If $n = 1$, then $M \to F^0 \to F^1 \to \cdots$ is exact. Thus $\mathrm{Tor}^k(F,M) = 0$ for $k \geq 1$ and $F \otimes M \to \mathrm{Tor}^0(F,M)$ is epic. So $F \otimes M \to A^0 \otimes M \to A^1 \otimes M \to A^2 \otimes M$ is exact. By Lemma 8.4.23 of [5] again, we get the exact sequence $0 \to F \to A^0 \to K \to 0$ with $K = \mathrm{Ker}(A^1 \to A^2)$ FGT-injective.

If $n = 0$, then $0 \to M \to F^0 \to F^1 \to \cdots$ is exact, so $\mathrm{Tor}^k(F, M) = 0$ for $k \geq 0$ and $F \otimes M \to \mathrm{Tor}^0(F, M)$ is an isomorphism. This gives $0 \to F \otimes M \to A^0 \otimes M \to A^1 \otimes M$ is exact, which implies F is a pure submodule of A^0, hence F is FGT-injective.

(5) $\Rightarrow$ (1). By assumption, $I_n \to I_{n-1} \to I_{n-2}$ is exact at I_{n-1}. Thus $I_n \to K_n$ is epic, where $K_n = \mathrm{Ker}(I_{n-1} \to I_{n-2})$. Hence $\mathrm{Hom}(R, I_n) \to \mathrm{Hom}(R, K_n)$ is epic. By Lemma 2.2, $\mathrm{FGT} - \mathrm{Id}(R) \leq n$.

(1) $\Rightarrow$ (5). If $n \geq 2$, let $0 \to R \to A^0 \to \cdots \to A^n \to 0$ be a right $\mathcal{TI}$-resolution of a right R-module M. Then $\mathrm{Ext}_k(R, M) = 0$ for $k \geq n-1$. By Remark 2.4(2), we can compute $\mathrm{Ext}_k(R, M) = 0$ using a left $\mathcal{TI}$-resolution of M $\cdots \to I_1 \to I_0 \to M \to 0$, so $\cdots \to \mathrm{Hom}(R, I_n) \to \mathrm{Hom}(R, I_{n-1}) \to \cdots \to \mathrm{Hom}(R, I_1) \to \mathrm{Hom}(R, I_0) \to \mathrm{Hom}(R, M) \to 0$ is exact at $\mathrm{Hom}(R, I_k)$ for $k \geq n-1$. Hence $\cdots \to I_1 \to I_0 \to M \to 0$ is exact at I_k for $k \geq n-1$.

If $n = 1$, then there is an exact sequence $0 \to R \to A^0 \to A^1 \to 0$ with $A^i \in \mathcal{TI}$. So $0 \to \mathrm{Hom}(A^1, M) \to \mathrm{Hom}(A^0, M) \to \mathrm{Hom}(R, M)$ is exact. Thus $\mathrm{Ext}_k(R, M) = 0$ for $k \geq 1$ and $\mathrm{Ext}_0(R, M) \to \mathrm{Hom}(R, M)$ is monomorphism. But computing $\mathrm{Ext}_0(R, M)$ using a left $\mathcal{TI}$-resolution of M, we see that $I_1 \to I_0 \to M$ is exact at I_0, so $\cdots \to I_1 \to I_0 \to M \to 0$ is exact at I_k for $k \geq 0$.

If $n = 0$, then R is FGT-injective as a right R-module. So every $\mathcal{TI}$-precover is epic, and hence $\cdots \to I_1 \to I_0 \to M \to 0$ is exact.

(1) $\Leftrightarrow$ (4) follows from Theorem 5.6.17 of [3]. □

Choose $n = 0$ in Theorem 2.5. We immediately deduce the following corollary.

COROLLARY 2.6

If R is right Π*-coherent, then the following are equivalent*:

(1) *R is right FGT-injective.*
(2) *Every flat right R-module is FGT-injective.*
(3) *Every injective left R-module is FGT-flat.*
(4) *Every right R-module has an epic* $\mathcal{TI}$*-precover.*
(5) *Every left R-module has a monic* $\mathcal{TF}$*-preenvelope.*

Lemma 2.7. *Let R be a right* Π*-coherent ring and M a right R-module. If* $\mathrm{Ext}^1(I, M) = 0$ *for all FGT-injective right R-modules I, then M has a* $\mathcal{TI}$*-cover* $L \to M$ *with L injective.*

Proof. In view of Theorem 3.4 of [10], M has a $\mathcal{TI}$-cover $f\colon L \to M$. Consider the exact sequence $0 \to L \xrightarrow{i} E \to L' \to 0$ with E injective. By Lemma 3.2(5) of [10], L' is FGT-injective. Thus $\mathrm{Hom}(E, M) \to \mathrm{Hom}(L, M) \to 0$ is exact since $\mathrm{Ext}^1(L', M) = 0$, and hence there is $g \in \mathrm{Hom}(E, M)$ such that $f = gi$. Then there exists $h\colon E \to L$ such that $g = fh$ since $f\colon L \to M$ is a $\mathcal{TI}$-cover of M. So $f = fhi$, implies hi is isomorphism. Therefore, L is injective. □

Theorem 2.8. *If R is right* Π*-coherent and* $n \geq 1$, *then the following are equivalent*:

(1) $\mathrm{FGT} - I \cdot \dim(R) \leq n$.
(2) *Every n-th* $\mathcal{TI}$*-syzygy of a right R-module is FGT-injective.*
(3) *Every* $(n-1)$*-th* $\mathcal{TI}$*-syzygy of a right R-module has a* $\mathcal{TI}$*-cover which is a monomorphism.*

Moreover, if $n \geq 2$, *then the above conditions are equivalent to*:

(4) *Every* $(n-2)$*-th* $\mathcal{TI}$*-syzygy in a minimal left* $\mathcal{TI}$*-resolution of a right* R*-module has a* $\mathcal{TI}$*-cover with the unique mapping property.*

Proof. (1) $\Rightarrow$ (2). Let K_n be the n-th $\mathcal{TI}$-syzygy of a right R-module. Then FGT $-$ Id$(K_n) \leq n$. So $\mathrm{Hom}(K_n, I_n) \to \mathrm{Hom}(K_n, K_n)$ is an epimorphism by Lemma 2.2, hence K_n is FGT-injective.

(2) $\Rightarrow$ (3). Let $f\colon I_{n-1} \to K_{n-1}$ be $\mathcal{TI}$-precover of the $(n-1)$-th $\mathcal{TI}$-syzygy K_{n-1}, and $K_n = \mathrm{Ker}(f)$. Then we have exact sequence $0 \to K_n \to I_{n-1} \to \mathrm{im}(f) \to 0$. By assumption, K_n is FGT-injective, so is $\mathrm{im}(f)$ in terms of Lemma 3.2(5) of [10]. Thus the inclusion $\mathrm{im}(f) \to K_{n-1}$ is a $\mathcal{TI}$-cover which is a monomorphism.

(3) $\Rightarrow$ (2). Let $\cdots \to I_n \to I_{n-1} \to \cdots \to I_1 \to I_0 \to N \to 0$ be any left $\mathcal{TI}$-resolution of a right R-module N and $K_n = \mathrm{Ker}(I_{n-1} \to I_{n-2})$, $K_{n-1} = \mathrm{Ker}(I_{n-2} \to I_{n-3})$. K_{n-1} has a monic $\mathcal{TI}$-cover $I \to K_{n-1}$ by assumption. Thus $K_n \oplus I \simeq I_{n-1}$ in terms of Lemma 8.6.3 of [5]. So K_n is FGT-injective by Proposition 5.5.3 of [3].

(2) $\Rightarrow$ (1). Let M be a right R-module. For a left $\mathcal{TI}$-resolution $\cdots \to I_n \to I_{n-1} \to \cdots \to I_1 \to I_0 \to N \to 0$ of a right R-module N, $I_n \to K_n$ is a split epimorphism since K_n is FGT-injective. Thus $\mathrm{Hom}(M, I_n) \to \mathrm{Hom}(M, K_n)$ is epic, hence FGT$-$Id$(M) \leq n$ by Lemma 2.2. Then FGT $- I \cdot \dim(R) \leq n$.

(3) $\Rightarrow$ (4). Let $\cdots \to I_{n-3} \to I_{n-4} \to \cdots \to I_1 \to I_0 \to M \to 0$ be a minimal $\mathcal{TI}$-resolution of a right R-module M with $K_{n-2} = \mathrm{Ker}(I_{n-3} \to I_{n-4})$. By assumption, $K_{n-1} = \mathrm{Ker}(I_{n-2} \to I_{n-3})$ has a monic $\mathcal{TI}$-cover $i\colon I_{n-1} \to K_{n-1}$. Note $\mathrm{Ext}^1(I, K_{n-1}) = 0$ for all FGT-injective right R-modules I by Wakamatsu's lemma. Thus I_{n-1} is injective by Lemma 2.7. But K_{n-1} has no nonzero injective submodule by Corollary 1.2.8 of [15]. Thus $I_{n-1} = 0$, and hence $\mathrm{Hom}(I, K_{n-1}) = \mathrm{Hom}(I, I_{n-1}) = 0$ for any FGT-injective right R-module I. So we have exact sequence $0 \to \mathrm{Hom}(I, I_{n-2}) \to \mathrm{Hom}(I, K_{n-2}) \to 0$ for any FGT-injective right R-module I, as desired.

(4) $\Rightarrow$ (2). Let $\cdots \to I_n \to I_{n-1} \to \cdots \to I_1 \to I_0 \to M \to 0$ be a $\mathcal{TI}$-resolution of a right R-module M with $K_n = \mathrm{Ker}(I_{n-1} \to I_{n-2})$. By assumption, M has a minimal $\mathcal{TI}$-resolution of the form $0 \to I'_{n-2} \to I'_{n-3} \to \cdots \to I'_1 \to I'_0 \to M \to 0$. In view of Corollary 8.6.4 of [5], $K_n \oplus I_{n-2} \oplus I'_{n-3} \oplus \cdots \cong I_{n-1} \oplus I'_{n-2} \oplus I_{n-3} \oplus \cdots$. Thus K_n is FGT-injective by Proposition 5.5.3 of [3] again. □

Next we shall consider the $\mathcal{TF}$-cosyzygy version of Theorem 2.8. We first give some lemmas needed in the proof of theorem.

Lemma 2.9. *Let* R *be a right* Π*-coherent ring. If* $\varphi\colon M \to F$ *is a* $\mathcal{TF}$*-preenvelope of a left* R*-module* M, *then* $\varphi^+\colon F^+ \to M^+$ *is a* $\mathcal{TI}$*-precover of* M^+.

Proof. Since F is FGT-flat, F^+ is FGT-injective right R-module in view of Lemma 3.1 of [10]. For any homomorphism $g\colon D \to M^+$ with $D \in \mathcal{TI}$, we have $g^+\colon M^{++} \to D^+$, hence $g^+\sigma_M\colon M \to D^+$, where $\sigma_M\colon M \to M^{++}$ is an evaluation map. By Lemma 3.2 of [10], D^+ is FGT-flat since R is right Π-coherent. Thus there exists $f\colon F \to D^+$ such that $f\varphi = g^+\sigma_M$. Whence $\sigma_M^+ g^{++} = \varphi^+ f^+$. Since $g^{++}\sigma_D = \sigma_{M^+} g$, let $f^+\sigma_D\colon D \to F^+$, note $\sigma_M^+\sigma_{M^+} = I_{M^+}$. Then $\varphi^+(f^+\sigma_D) = \sigma_M^+ g^{++}\sigma_D = \sigma_M^+\sigma_{M^+} g = g$. Therefore $\varphi^+\colon F^+ \to M^+$ is a $\mathcal{TI}$-precover. □

By Lemma 2.9, if $0 \to N \to F^0 \to F^1 \to \cdots$ is a right $\mathcal{TF}$-resolution of a left R-module N, then $\cdots \to (F^1)^+ \to (F^0)^+ \to (N)^+ \to 0$ is $\mathcal{TI}$-resolution of right R-module N^+.

Lemma 2.10. *If R is right Π-coherent and $M \subset N$ is a pure submodule of the right R-module N, then* $\mathrm{FGT}-\mathrm{Id}(M) \leq \mathrm{FGT}-\mathrm{Id}(N)$.

Proof. We suppose that $\mathrm{FGT}-\mathrm{Id}(N) = n < \infty$. Then $\mathrm{Ext}^{\mathrm{n}+1}(T, N) = 0$ for any finitely generated torsionless right R-module T. Since R is right Π-coherent, T is finitely presented. So there is an exact sequence $0 \to S \to F_n \to \cdots \to F_0 \to T \to 0$ with S finitely generated right R-module and $F_0, \ldots, F_n$ finitely generated free right R-modules. Since $\mathrm{Ext}^{n+1}(T, N) = 0$, the sequence $\mathrm{Hom}(F_n, N) \to \mathrm{Hom}(S, N) \to 0$ is exact. But since $M \subset N$ is pure, an homomorphism $S \to M$ has an extension $F_n \to N$ if and only if there is an extension $F_n \to M$. Hence $\mathrm{Hom}(F_n, M) \to \mathrm{Hom}(S, M) \to 0$ is exact. This means that $\mathrm{Ext}^{n+1}(T, M) = 0$ for any finitely generated torsionless right R-module T. Therefore, $\mathrm{FGT}-\mathrm{Id}(M) \leq n$. □

The following lemma is dual to Corollary 8.6.4 of [5].

Lemma 2.11. *Let* $n \geq 0$, $0 \to N \to F^0 \to F^1 \to \cdots F^{n-1} \to F^n \to \cdots$ *and* $0 \to N \to G^0 \to G^1 \to \cdots G^{n-1} \to G^n \to \cdots$ *be right $\mathcal{TF}$-resolutions of N. If* $L = \mathrm{Coker}(F^{n-1} \to F^n)$ *and* $L' = \mathrm{Coker}(G^{n-1} \to G^n)$, *then* $L \oplus G^n \oplus F^{n-1} \oplus \cdots \cong L' \oplus F^n \oplus G^{n-1} \oplus \cdots$.

Theorem 2.12. *If R is right Π-coherent and $n \geq 1$, then the following are equivalent*:

(1) $\mathrm{FGT}-I \cdot \dim(R) \leq n$.
(2) *Every n-th $\mathcal{TF}$-cosyzygy of a left R-module is* FGT-*flat.*
(3) *Every $(n-1)$-th $\mathcal{TF}$-cosyzygy of a left R-module has a $\mathcal{TF}$-envelope which is an epimorphism.*
Moreover, the above conditions hold if R satisfies that
(4) *Every $(n-2)$-th $\mathcal{TF}$-cosyzygy of a left R-module has a $\mathcal{TF}$-envelope with the unique mapping property.*

Proof. (1) ⇒ (2). Let $0 \to N \to F^0 \to F^1 \to \cdots$ be a right $\mathcal{TF}$-resolution of a left R-module N, and let $L^n = \mathrm{Coker}(F^{n-2} \to F^{n-1})$ be n-th $\mathcal{TF}$-cosyzygy. By Lemma 2.9, $\cdots \to (F^1)^+ \to (F^0)^+ \to (N)^+ \to 0$ is $\mathcal{TI}$-resolution of right R-module N^+. Then $(L^n)^+ = \mathrm{Ker}((F^{n-1})^+ \to (F^{n-2})^+)$ is FGT-injective by Theorem 2.8. Hence L^n is FGT-flat by Lemma 3.1 of [10].

(2) ⇒ (3). Let $f\colon L^{n-1} \to F^{n-1}$ be a $\mathcal{TF}$-preenvelope of L^{n-1}. Then we have an exact sequence $0 \to \mathrm{im}(f) \to F^{n-1} \to L^n \to 0$. By assumption, L^n is FGT-flat. Then $\mathrm{im}(f)$ is FGT-flat by Lemma 3.1(3) of [10]. Thus $L^{n-1} \to \mathrm{im}(f)$ is a $\mathcal{TF}$-envelope which is epic.

(3) ⇒ (1). Let M be a right R-module. Then left R-module M^+ has a right $\mathcal{TF}$-resolution $0 \to M^+ \to F^0 \to F^1 \to \cdots$, by assumption, $(n-1)$-th $\mathcal{TF}$-cosyzygy has an epic $\mathcal{TF}$-envelope. In view of Lemma 2.9, $(n-1)$-th $\mathcal{TI}$-syzygy of M^{++} has a monic $\mathcal{TI}$-precover, and hence a monic $\mathcal{TI}$-cover. By the proof of Theorem 2.8, $\mathrm{FGT}-\mathrm{Id}(M^{++}) \leq n$. Since M is a pure submodule of M^{++}, by Lemma 2.10, $\mathrm{FGT}-\mathrm{Id}(M) \leq \mathrm{FGT}-\mathrm{Id}(M^{++}) \leq n$. Therefore, $\mathrm{FGT}-I \cdot \dim(R) \leq n$.

(4) ⇒ (2). Let $0 \to N \to F^0 \to F^1 \to \cdots \to F^{n-2} \to F^{n-1}$ be a right $\mathcal{TF}$-resolution of a left R-module N with $L^n = \mathrm{Coker}(F^{n-2} \to F^{n-1})$. By assumption, N has a right $\mathcal{TF}$-resolution of the form $0 \to N \to G^0 \to G^1 \to \cdots G^{n-3} \to G^{n-2} \to 0$. In view of Lemma 2.11, $L^n \oplus F^{n-2} \oplus G^{n-3} \oplus \cdots \cong F^{n-1} \oplus G^{n-2} \oplus F^{n-3} \oplus \cdots$. Therefore, L^n is FGT-flat. □

Remark 2.13. Let $R = \mathbb{F}[x_1, x_2, \ldots, x_n, x_{n+1}]$, the ring of polynomials in $n + 1$ indeterminates over a field $\mathbb{F}$. Then R is a commutative Noetherian ring, and hence it is a Π-coherent ring. It is clear that weak global dimension of R is equal to $n + 1$. By Theorem 5.5.11 of [3], $\mathrm{FGT} - I \cdot \dim(R) = n$. Thus for any R-module M and $n \geq 1$, every $(n - 1)$-th $\mathcal{TI}$-syzygy of M has a $\mathcal{TI}$-cover which is a monomorphism and every $(n - 1)$th $\mathcal{TF}$-cosyzygy of M has a $\mathcal{TF}$-envelope which is an epimorphism. Moreover, if $n \geq 2$, every $(n-2)$-th $\mathcal{TI}$-syzygy in a minimal left $\mathcal{TI}$-resolution of M has a $\mathcal{TI}$-cover with the unique mapping property.

COROLLARY 2.14

The following are equivalent for a right Π*-coherent ring*:

(1) $\mathrm{FGT} - I \cdot \dim(R) \leq 1$.
(2) *Every right* R*-module has a monic* $\mathcal{TI}$*-cover.*
(3) *Every left* R*-module has an epic* $\mathcal{TF}$*-envelope.*

Proof. Let $n = 1$ in Theorems 2.8 and 2.12. □

COROLLARY 2.15

The following statements are equivalent for a right Π*-coherent ring*:

(1) $\mathrm{FGT} - I \cdot \dim(R) \leq 2$.
(2) *Every right* R*-module has a* $\mathcal{TI}$*-cover with the unique mapping property.*
Moreover, the above conditions hold if R *satisfies that*
(3) *Every left* R*-module has a* $\mathcal{TF}$*-envelope with the unique mapping property.*

Proof. Let $n = 2$ in Theorems 2.8 and 2.12. □

3. Almost excellent extension

In this section, we shall consider the FGT-injective, FGT-flat dimensions and Π-coherent rings under almost excellent extensions of rings.

Lemma 3.1. *Let* S *be an almost excellent extension of a ring* R*. If* M *is a finitely generated torsionless right* R*-module and one of the following conditions holds, then* $(M \otimes_R S)_S$ *is finitely generated torsionless.*

(1) R *is a left and right* Π*-coherent ring.*
(2) S *is a finitely presented left* R*-module.*

Proof.

(1) As M is finitely generated torsionless, there is an exact sequence $0 \to M \to M^{**}$, and hence the sequence $0 \to M \otimes_R S \to M^{**} \otimes_R S$ is exact. Since R is left and right Π-coherent, M and M^* are finitely presented. By Lemma 3.2.4 of [5], $M^{**} \otimes_R S \cong \mathrm{Hom}_S(S \otimes_R M^*, S) \cong \mathrm{Hom}_S(\mathrm{Hom}_S(M \otimes_R S, S), S)$. Therefore, $M \otimes_R S$ is a finitely generated torsionless right S-module.
(2) If M is finitely generated torsionless, then there is an exact sequence $0 \to M \to \prod R$, there induces right S-modules exact sequence $0 \to M \otimes_R S \to \prod R \otimes_R S$. Since S is a finitely presented left R-module, by Theorem 3.2.22 of [5], $\prod R \otimes_R S \cong \prod S$. Then $M \otimes_R S$ is a finitely generated torsionless right S-module. □

PROPOSITION 3.2

Let S be an almost excellent extension of a left and right Π*-coherent ring R and* M_S $({}_SM)$ *a right (left) S-module. Then the following hold.*

(1) *If* M_S *is finitely generated torsionless if and only if* M_R *is finitely generated torsionless.*
(2) M_S *is* FGT*-injective if and only if* M_R *is* FGT*-injective if and only if* $(\mathrm{Hom}_R(S, M))_S$ *is* FGT*-injective.*
(3) ${}_SM$ *is* FGT*-flat if and only if* ${}_RM$ *is* FGT*-flat if and only if* ${}_S(S \otimes_R M)$ *is* FGT*-flat.*

Proof.
(1)($\Rightarrow$) is trivial.

($\Leftarrow$). If M_R is finitely generated torsionless, by Lemma 3.1, then $(M \otimes_R S)_S$ is finitely generated torsionless. In view of Lemma 1.1 of [17], M_S is isomorphic to a direct summand of $(M \otimes_R S)_S$, so M_S is finitely generated torsionless.

(2) Assume that M_S is FGT-injective. Let T be a finitely generated torsionless right R-module. As ${}_RS$ is flat, by Theorem 11.65 of [13], $\mathrm{Ext}_R^1(T, M) \cong \mathrm{Ext}_S^1(T \otimes_R S, M)$. In view of Lemma 3.1, $T \otimes_R S$ is a finitely generated torsionless right S-module, so $\mathrm{Ext}_S^1(T \otimes_R S, M) = 0$, and hence $\mathrm{Ext}_R^1(T, M) = 0$. Therefore, M_R is FGT-injective.

Assume that M_R is FGT-injective. Let N_S be a finitely generated torsionless right S-module. By (1), N_R is a finitely generated torsionless right R-module, so $\mathrm{Ext}_R^1(N, M) = 0$. Thus $\mathrm{Ext}_S^1(N \otimes_R S, M) \cong \mathrm{Ext}_R^1(N, M) = 0$, and hence $\mathrm{Ext}_S^1(N, M) = 0$ since N_S is isomorphic to a direct summand of $(N \otimes_R S)_S$. So M_S is FGT-injective.

Finally, since M_S is isomorphic to a direct summand of $(\mathrm{Hom}_R(S, M))_S$ by Lemma 1.1 of [17], in view of the isomorphism $\mathrm{Ext}_R^1(T \otimes_S S, M) \cong \mathrm{Ext}_S^1(T, \mathrm{Hom}_R(S, M))$, M_R is FGT-injective $\Rightarrow$ $(\mathrm{Hom}_R(S, M))_S$ is FGT-injective $\Rightarrow$ M_S is FGT-injective, as desired.

(3) ${}_SM$ is FGT-flat if and only if M_S^+ is FGT-injective if and only if M_R^+ is FGT-injective by (2) if and only if ${}_RM$ is FGT-flat.

${}_SM$ is FGT-flat if and only if M_S^+ is FGT-injective if and only if $(\mathrm{Hom}_R(S, M^+))_S$ is FGT-injective by (2) if and only if ${}_S(S \otimes_R M)$ is FGT-flat by the isomorphism $\mathrm{Hom}_R(S, M^+) \cong ({}_S(S \otimes_R M))^+$. We complete the proof. □

Recall that a left R-module M has *left* FGT*-flat dimension at most n* [3], denoted $\mathrm{FGT}-\mathrm{fd}(M) \le n$, if $\mathrm{Tor}_{n+1}(T, M) = 0$ for any finitely generated torsionless right R-module T.

COROLLARY 3.3

Let S be an almost excellent extension of a left and right Π*-coherent ring R and* $M_S({}_SM)$ *a right (left) S-module. Then*:

(1) $\mathrm{FGT}-\mathrm{Id}(M_S) = \mathrm{FGT}-\mathrm{Id}(M_R) = \mathrm{FGT}-\mathrm{Id}((\mathrm{Hom}_R(S, M))_S)$.
(2) $\mathrm{FGT}-\mathrm{fd}({}_SM) = \mathrm{FGT}-\mathrm{fd}({}_RM) = \mathrm{FGT}-\mathrm{fd}({}_S(S \otimes_R M))$.

Proof. The proof is similar to that of (2) and (3) in Proposition 3.2. □

COROLLARY 3.4

Let S be an almost excellent extension of a left and right Π*-coherent ring R. Then*:

(1) $\mathrm{FGT}-I\cdot\dim(S) \le \mathrm{FGT}-I\cdot\dim(R)$, *the equality holds if* $\mathrm{FGT}-I\cdot\dim(R) < \infty$.

(2) *If* $\mathrm{FGT} - I \cdot \dim(R) < \infty$, *then R is right* Π*-coherent and left semihereditary if and only if S is right* Π*-coherent and left semihereditary.*

Proof.

(1) The proof is similar to that of Corollary 1.5 of [17].
(2) Note that $\mathrm{FGT} - I \cdot \dim(R) = 0$ if and only if R is right Π-coherent and left semihereditary ring. □

Theorem 3.5. *If S be an almost excellent extension of a right* Π*-coherent ring* R, *then S is right* Π*-coherent.*

Proof. Let M_S be a finitely generated torsionless S-module. It is enough to prove that M_S is finitely presented. Note that M_R is a finitely generated torsionless R-module, and hence M_R is finitely presented since R is right Π-coherent. Then there is an exact sequence $0 \to K \to F \to M \to 0$ of right R-modules with F finitely generated free and K finitely generated. Moreover, we have the following right S-module exact sequence

$$0 \to K \otimes_R S \to F \otimes_R S \to M \otimes_R S \to 0.$$

Note that $(K \otimes_R S)_S$ is finitely generated and $(F \otimes_R S)_S$ is finitely generated free, then $(M \otimes_R S)_S$ is finitely presented. But M_S is isomorphic to a direct summand of $(M \otimes_R S)_S$ by Lemma 1.1 of [17], so M_S is finitely presented. □

We do not know if the converse holds for an almost excellent extension. However, we have the following result which have been shown in [10]. Here we give a different method.

PROPOSITION 3.6

Let S be an excellent extension of a ring R. Then R is right Π*-coherent if and only if S is right* Π*-coherent.*

Proof.
($\Rightarrow$). It follows by Theorem 3.5.

($\Leftarrow$). Now let M be any finitely generated torsionless R-module. There is a right R-module exact sequence $0 \to K \to F \xrightarrow{f} M \to 0$ with F finitely generated free and $K = \mathrm{Ker}(f)$. Since ${}_R S$ is free, there induces a right S-modules exact sequence

$$0 \to K \otimes_R S \to F \otimes_R S \to M \otimes_R S \to 0.$$

By Lemma 3.1(2), $(M \otimes_R S)_S$ is finitely generated torsionless, and hence it is finitely presented since S is right Π-coherent. Then there is a right S-modules exact sequence $0 \to K' \to F' \to M \otimes_R S \to 0$, where F' is finitely generated free and K' is finitely generated. By Schaunel lemma, $(K \otimes_R S)_S$ is finitely generated. Let $\{a_1, \ldots, a_m\}$ be a generating set of $(K \otimes_R S)_S$. But $S = s_1 R + \cdots + s_n R$, then $\{a_i s_j | 1 \leq i \leq m, 1 \leq j \leq n\}$ is a generating set of $(K \otimes_R S)_R$. Note that ${}_R S = R^n$, then $K^n = (K \otimes_R S)_R$ is finitely generated, and so is K. Therefore, M_R is finitely presented. □

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