On the Symmetric Capacity of the K-User Symmetric Cyclic Gaussian Interference Channel

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***Abstract*—The capacity region of Gaussian interference channel in the weak interference regime is an open problem. Recently, Etkin, Tse and Wang derived an outer bound for the two-user Gaussian interference channel and proved that a simple Han-Kobayashi signaling scheme can achieve within one bit of the capacity region for all values of channel parameters. In this paper, we extend their result to the K-user symmetric cyclic Gaussian interference channel. Our result shows that both the Etkin, Tse and Wang's upper bound and their one-bit achievability result on the symmetric rate continue to hold for the symmetric rate of the K-user symmetric cyclic channel using the same Han-Kobayashi strategy.**

I. Introduction

An interference channel is a communication network consisted of several mutually interfering transmitter-receiver pairs. Each transmitter communicates with its intended receiver while causing interference to other receivers. The interference channel is a useful model for many practical systems, such as the wireless network. However, the capacity region of the interference channel is not known in most cases, even for the two-user Gaussian case.

The largest known achievable rate region for the two-user interference channel is derived by Han and Kobayashi [1] using a fairly complex characterization involving both common-private power splitting and time sharing techniques. Recently, Chong et. al. [2] obtained the same rate region in a simpler form. The capacity region for the Gaussian interference channel in the strong interference can be shown to be a rectangular or a pentagon region [1], [3], [4]. However, the capacity region in the weak interference remains open.

The Han-Kobayashi scheme has recently been shown to be sum capacity achieving in a very weak interference regime [5]–[7] and to be within one bit of the capacity region in general [8]. In particular, Etkin, Tse and Wang showed in [8] that for a two-user Gaussian interference channel, by keeping the interference-to-noise-ratio (INR) of the private message to be as close to 1 as possible in the Han-Kobayashi scheme, one can achieve within one bit of the capacity region. They also found a new outer bound for the capacity region using a genie-aided technique.

In practical systems, an interference channel can have more than two transmitter-receiver pairs. However, the generalization of Etkin, Tse and Wang's result to interference channels with more than two users has proved difficult. First, it appears that Han-Kobayashi superposition coding is no longer

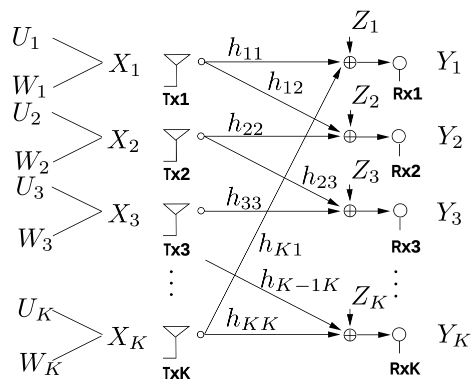

Fig. 1. Cyclic Gaussian interference channel

adequate. Interference alignment types of coding scheme [9] [10] need to be used. Second, even within the Han-Kobayashi framework, when more than two receivers are involved, multiple common messages at each transmitter may be needed, making the optimization of the resulting rate region expression difficult. Recently, Bresler, Parekh and Tse studied the many-to-one and one-to-many interference channels in [10], where a lattice coding technique is used to manage the receiver side interference.

In this paper, we focus on a simpler model in which each receiver suffers only one interference from its neighboring transmitter, thereby avoiding the aforementioned issues. We show that there exists a specific subclass of symmetric cyclic K-user interference channels for which the technique of [8] and [11] can be easily used. We extend the one-bit result on the symmetric rate of the two-user Gaussian interference channel in [8] and [11] to the K-user symmetric cyclic scenario. We show that the achievable symmetric rate in this case is also within one bit of the symmetric capacity.

II. Cyclic Gaussian Interference Channel And Its Achievable Symmetric Rate

The K-user cyclic Gaussian interference channel (as depicted in Fig. 1) has K transmitter-receiver pairs. Each transmitter tries to communicate with its intended receiver while causing interference to only one neighboring receiver. Each receiver receives a signal intended for it and an interference signal from the neighboring sender plus an additive white Gaussian noise (AWGN). As shown in Fig. 1, $X_1, X_2, \cdots X_K$

and $Y_1, Y_2, \cdots Y_K$ are the input and output signals, respetively, and $Z_i \sim \mathcal{CN}(0, \sigma_i^2)$ is the Gaussian noise at receiver i. The input-output model can be written as

$$\begin{aligned} Y_1 &= h_{11}X_1 + h_{K1}X_K + Z_1 \\ Y_2 &= h_{22}X_2 + h_{12}X_1 + Z_2 \\ &\vdots \\ Y_K &= h_{KK}X_K + h_{K-1,K}X_{K-1} + Z_K \end{aligned} \quad (1)$$

where h_{ij} is the channel gain from transmitter i to receiver j.

In this paper, we only deal with the symmetric case, where all the direct links from transmitters to receivers have the same gain of $\sqrt{g_d}$ and all the cross links $\sqrt{g_c}$, i.e.,

$$\begin{aligned} h_{11} &= h_{22} = \cdots = h_{KK} = \sqrt{g_d} \\ h_{12} &= h_{23} = \cdots = h_{K1} = \sqrt{g_c}. \end{aligned}$$

In addition, we focus on the weak interference regime, where $g_c \leq g_d$. Further, we assume that all the noises at the receiver side have the same power level σ^2, and all the input signals are independent with the same power constraint P, i.e.,

$$E\{|X_i|^2\} \leq P, \quad 1 \leq i \leq K. \quad (2)$$

For convenience, we define

$$\mathsf{SNR} = \frac{g_d P}{\sigma^2}; \qquad \mathsf{INR} = \frac{g_c P}{\sigma^2}. \quad (3)$$

The symmetric capacity of the K-user interference channel is defined as

$$C_{sym} = \begin{cases} \text{maximize} & \min\{R_1, R_2, \cdots, R_K\} \\ \text{subject to} & (R_1, R_2, \cdots, R_K) \in \mathcal{R} \end{cases} \quad (4)$$

where $\mathcal{R}$ is the capacity region of the K-user interference channel. For the symmetric interference channel, $C_{sym} = \frac{1}{K}C_{sum}$, where C_{sum} is the sum capacity of the K-user symmetric interference channel. Therefore, for the symmetric interference channel, the symmetric capacity problem is equivalent to the sum capacity problem.

The largest known achievable rate region for the two-user interference channel is derived using the Han-Kobayashi common-private power splitting scheme. For the K-user interference channel, the Han-Kobayashi scheme can be easily generalized as follows: each input signal X_i is split into two parts (as shown in Fig. 1): W_i and U_i, where W_i is the common message that can be decoded by all the receivers who receive W_i, and U_i represents the private message that is decodeable only at the intended receiver. The common message and the private message follow the power constraint $P_w + P_u = P$.

Etkin, Tse and Wang proved in [8], [11] that for a two-user Gaussian weak interference channel, by simply keeping the interference power of the private message at the interfered receiver to be as close to the noise power as possible, i.e., $g_c P_u/\sigma^2 = \min(1, \mathsf{INR})$, one can achieve within one bit of the symmetric capacity for all ranges of SNR and INR. Inspired by this insight, we extend their one-bit result to the symmetric capacity of the K-user cyclic channel.

Theorem 1: For the K-user symmetric cyclic Gaussian weak interference channel, when $\mathsf{INR} \geq 1$, the following rate is achievable

$$R_{sym} = \min\left\{\log(1 + \mathsf{INR} + \frac{\mathsf{SNR}}{\mathsf{INR}}) - 1, \right. \\ \left. \frac{1}{2}\log(1 + \mathsf{INR} + \mathsf{SNR}) + \frac{1}{2}\log(2 + \frac{\mathsf{SNR}}{\mathsf{INR}}) - 1\right\} \quad (5)$$

using a Han-Kobayashi scheme with private message power set to $P_u = \sigma^2/g_c$. When $\mathsf{INR} < 1$, using the same Han-Kobayashi scheme with $P_u = P$ achieves the following rate:

$$R_{sym} = \log\left(1 + \frac{\mathsf{SNR}}{1 + \mathsf{INR}}\right). \quad (6)$$

Proof: This result is a natural extension of Han and Kobayashi's strategy where the achievable rate region is characterized by intersecting the the capacity regions of multiple access channels (MAC) involving common and private messages. For example, the private and common rates at Y_1 is characterized by $\mathcal{R}(U_1, W_1, W_K)$. In general, $\mathcal{R}(U_i, W_i, W_{i-1})$ denotes the capacity region of the multiple access channel with inputs (U_i, W_i, W_{i-1}) and output Y_i.

The capacity region of the i^{th} multiple access channel $\mathcal{R}(U_i, W_i, W_{i-1})$ can be written as:

$$\begin{aligned} S_i &\leq I(Y_i; U_i|W_i, W_{i-1}) \\ T_i &\leq I(Y_i; W_i|U_i, W_{i-1}) \\ T_{i-1} &\leq I(Y_i; W_{i-1}|U_i, W_i) \\ S_i + T_i &\leq I(Y_i; U_i W_i|W_{i-1}) \\ S_i + T_{i-1} &\leq I(Y_i; U_i W_{i-1}|W_i) \\ T_i + T_{i-1} &\leq I(Y_i; W_i W_{i-1}|U_i) \\ S_i + T_i + T_{i-1} &\leq I(Y_i; U_i W_i W_{i-1}). \end{aligned} \quad (7)$$

where i goes from 1 to K (when $i = 1$, $i - 1 \equiv K$), S_i and T_i are the achievable rates of private message U_i and common message W_i respectively.

At the receiver side, common messages (W_i, W_{i-1}) are decoded first with U_i treated as noise. Then, W_i and W_{i-1} are subtracted, and the private message U_i is decoded next. As a result, the achievable rate for the private message U_i is given by

$$S_i = I(Y_i; U_i|W_i, W_{i-1}).$$

Substituting the private message rate S_i into the capacity region $\mathcal{R}(U_i, W_i, W_{i-1})$ in (7), we have

$$\begin{aligned} T_i &\leq I(Y_i; W_i|W_{i-1}) \\ T_{i-1} &\leq I(Y_i; W_{i-1}|W_i) \\ T_i + T_{i-1} &\leq I(Y_i; W_i W_{i-1}). \end{aligned} \quad (8)$$

Assume that same common-private power splitting ratio is adopted at different transmitters, i.e., both U_i and W_i have the same power for all the transmitters. Then, by symmetry, the mutual information expressions in (7) do not depend on i. Now define

$$\begin{aligned}R_u &= I(Y_i;U_i|W_i,W_{i-1})\\ R_{wd} &= I(Y_i;W_i|W_{i-1})\\ R_{wc} &= I(Y_i;W_{i-1}|W_i)\\ R_{mac} &= I(Y_i;W_iW_{i-1})\end{aligned}$$

for $i=1,2,\cdots,K$. By the chain rule of mutual information and the fact that $g_c \le g_d$, we have

$$R_{wc} \le R_{wd} \le R_{mac}. \tag{9}$$

Using the fact that $R_{wc} \le R_{wd}$, the achievable rate region of the common messages in (8) can be further simplified as:

$$\begin{aligned}T_i &\le R_{wc}\\ T_i + T_{i-1} &\le R_{mac}\end{aligned} \tag{10}$$

where $i=1,2,\cdots,K$.

Inspecting the above formula, we conclude that when $R_{mac} \ge 2R_{wc}$,

$$R_{sum} \le KR_u + KR_{wc} \tag{11}$$

with equality achieved by $T_i = R_{wc}, i=1,2,\cdots,K$. When $2R_{wc} \ge R_{mac} \ge R_{wc}$,

$$R_{sum} \le KR_u + \frac{K}{2}R_{mac} \tag{12}$$

with equality achieved by $T_i = R_{mac}/2, i=1,2,\cdots,K$. Combining (11) and (12), we obtain the following achievable sum rate for the K-user symmetric cyclic Gaussian interference channel:

$$R_{sum} = KR_u + \min\left\{KR_{wc}, \frac{K}{2}R_{mac}\right\}. \tag{13}$$

Now, when $\mathsf{INR} \ge 1$, we set the private message power to be the same as the noise power at the receiver, i.e. set $P_u = \sigma^2/g_c$. In this case, we have

$$\begin{aligned}R_u &= \log\left(1+\frac{\mathsf{SNR}}{2\mathsf{INR}}\right)\\ R_{wc} &= \log\left(1+\frac{\mathsf{INR}(\mathsf{INR}-1)}{\mathsf{SNR}+2\mathsf{INR}}\right)\\ R_{mac} &= \log\left(1+\frac{(\mathsf{SNR}+\mathsf{INR})(\mathsf{INR}-1)}{\mathsf{SNR}+2\mathsf{INR}}\right).\end{aligned}$$

When $\mathsf{INR} < 1$, we set $P_u = P$ and $P_w = 0$ to obtain

$$\begin{aligned}R_u &= \log\left(1+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right)\\ R_{wc} &= R_{mac} = 0\end{aligned}$$

The proof of Theorem 1 is completed by substituting the above R_u, R_{wc} and R_{mac} into (13) and noting that $R_{sym} = R_{sum}/K$. ∎

III. Outer Bounds For The Symmetric Capacity

Theorem 2: For the K-user symmetric cyclic Gaussian weak interference channel shown in Fig. 1, the symmetric capacity is upper bounded by

$$R_{ub} = \min\left\{\log(1+\mathsf{INR}+\frac{\mathsf{SNR}}{1+\mathsf{INR}}),\right.\\ \left.\frac{1}{2}\log(1+\mathsf{SNR})+\frac{1}{2}\log(1+\frac{\mathsf{SNR}}{1+\mathsf{INR}})\right\}. \tag{14}$$

In the weak interference regime of $\mathsf{INR} \le \mathsf{SNR}$, we have

$$R_{ub} - R_{sym} < 1. \tag{15}$$

Note: the upper bound in (14) is the same as the upper bound derived for the two-user symmetric Gaussian interference channel in [8] and [11].

Proof: For a block of length n, from Fano's inequality, we have

$$\begin{aligned}&n(R_1+R_2)\\ &\le I(x_1^n;y_1^n)+I(x_2^n;y_2^n)+n\epsilon_n\\ &\le I(x_1^n;y_1^nx_K^n)+I(x_2^n;y_2^n)+n\epsilon_n\\ &\overset{(a)}{=} I(x_1^n;y_1^n|x_K^n)+I(x_2^n;y_2^n)+n\epsilon_n\\ &= h(y_2^n)-h(y_2^n|x_2^n)+h(y_1^n|x_K^n)-h(y_1^n|x_1^nx_K^n)+n\epsilon_n\\ &= h(y_2^n)-h(z_1^n)+h(\sqrt{g_d}x_1^n+z_1^n)\\ &\quad -h(\sqrt{g_c}x_1^n+z_2^n)+n\epsilon_n\\ &\overset{(b)}{\le} n\log(1+\mathsf{SNR})+n\log(1+\frac{\mathsf{SNR}}{1+\mathsf{INR}})+n\epsilon_n\\ &= nR_{ub1}+n\epsilon_n\end{aligned}$$

where x_k^n and y_k^n are the input sequence and the output sequence of user k and the term ϵ_n diminishes when the block length n goes to infinity. Note that, equality (a) comes from the fact that x_1^n and x_K^n are independent. The last inequality (b) comes form [8].

Proceeding in the same way for the other users, we obtain the following inequalities:

$$\begin{aligned}R_1+R_2 &\le R_{ub1}\\ R_2+R_3 &\le R_{ub1}\\ &\vdots\\ R_K+R_1 &\le R_{ub1}\end{aligned}$$

Adding up all the inequalities above, we obtain an upper bound for the sum capacity:

$$C_{sum} \le \frac{K}{2}R_{ub1}$$

Therefore,

$$\begin{aligned}C_{sym} &= \frac{1}{K}C_{sum}\\ &\le \frac{1}{2}\log(1+\mathsf{SNR})+\frac{1}{2}\log\left(1+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right)\end{aligned} \tag{16}$$

gives an upper bound for the symmetric capacity.

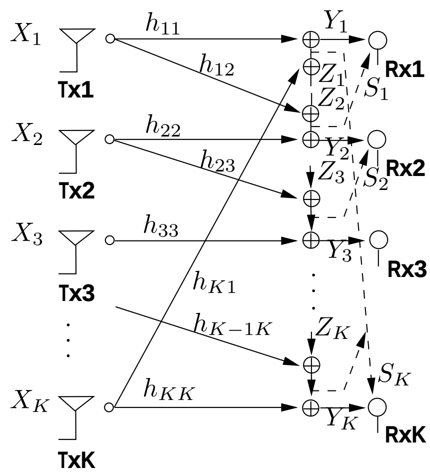

Fig. 2. Genie-aided cyclic Gaussian interference channel

To obtain the other upper bound for the symmetric capacity, we define the following genies:

$$\begin{aligned} s_1^n &= \sqrt{g_c}x_1^n + z_2^n \\ s_2^n &= \sqrt{g_c}x_2^n + z_3^n \\ &\vdots \\ s_K^n &= \sqrt{g_c}x_K^n + z_1^n \end{aligned}$$

Consider the genie-aided interference channel as shown in in Fig. 2. Each genie s_k^n is provided at receiver k. This extra piece of information at the receiver side guarantees that the sum rate upper bound of this genie-aided channel is also the an upper bound of the original channel.

Again, starting from the Fano's inequality,

$$\begin{aligned} &n(R_1 + R_2 + \cdots + R_K) \\ \leq\; & I(x_1^n; y_1^n) + I(x_2^n; y_2^n) + \cdots + I(x_K^n; y_K^n) + n\epsilon_n \\ \leq\; & I(x_1^n; y_1^n s_1^n) + I(x_2^n; y_2^n s_2^n) + \cdots + I(x_K^n; y_K^n s_K^n) + n\epsilon_n \\ =\; & I(x_1^n; s_1^n) + I(x_1^n; y_1^n | s_1^n) + I(x_2^n; s_2^n) + I(x_2^n; y_2^n | s_2^n) \\ & + \cdots + I(x_K^n; s_K^n) + I(x_K^n; y_K^n | s_K^n) + n\epsilon_n \\ =\; & h(s_1^n) - h(s_1^n | x_1^n) + h(y_1^n | s_1^n) - h(y_1^n | x_1^n s_1^n) + \\ & h(s_2^n) - h(s_2^n | x_2^n) + h(y_2^n | s_2^n) - h(y_2^n | x_2^n s_2^n) + \\ & \vdots \\ & h(s_K^n) - h(s_K^n | x_K^n) + h(y_K^n | s_K^n) - h(y_K^n | x_K^n s_K^n) + n\epsilon_n \\ =\; & h(s_1^n) - h(z_2^n) + h(y_1^n | s_1^n) - h(s_K^n) + \\ & h(s_2^n) - h(z_3^n) + h(y_2^n | s_2^n) - h(s_1^n) + \\ & \vdots \\ & h(s_K^n) - h(z_1^n) + h(y_K^n | s_K^n) - h(s_{K-1}^n) + n\epsilon_n \\ \stackrel{(a)}{\leq}\; & \sum_{i=1}^{n}\sum_{k=1}^{K} \{h(y_{ki}|s_{ki}) - h(z_{ki})\} + n\epsilon_n \\ \stackrel{(b)}{\leq}\; & nK \log\left(1 + \mathsf{SNR} + \frac{\mathsf{SNR}}{1+\mathsf{INR}}\right) + n\epsilon_n \\ =\; & nKR_{ub2} + n\epsilon_n \end{aligned}$$

where inequality (a) comes from the following facts:

$$\begin{aligned} h(z_k^n) &= \sum_{i=1}^{n} h(z_{ki}) \\ h(y_k^n | s_k^n) &\leq \sum_{i=1}^{n} h(y_{ki}|s_k^n) \leq \sum_{i=1}^{n} h(y_{ki}|s_{ki}) \end{aligned}$$

and (b) comes directly from [8, Theorem 1].

As n goes to infinity, ϵ_n vanishes. As a result, the symmetric capacity is upper bounded by

$$C_{sym} \leq R_{ub2} = \log\left(1 + \mathsf{SNR} + \frac{\mathsf{SNR}}{1+\mathsf{INR}}\right). \tag{17}$$

Finally, the symmetric upper bound is obtained by combining (16) and (17).

As we can see, both the achievable symmetric rate in Theorem 1 and the upper bound in Theorem 2 are exactly the same as those obtained for the two-user interference channel in [8] and [11]. As a result, the one-bit result continues to hold. ∎

IV. Conclusion

In this paper, we investigate the achievable symmetric rate and the capacity outer bound for the K-user symmetric cyclic Gaussian interference channel. Our analysis shows that by using the same signaling strategy as proposed by Etkin, Tse and Wang [8] [11] for the two-user Gaussian interference channel, the symmetric capacity of the K-user symmetric cyclic interference channel can be achieved within one bit.

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