ON FREYD'S GENERATING HYPOTHESIS

MARK HOVEY

Abstract. We revisit Freyd's generating hypothesis in stable homotopy theory. We derive new equivalent forms of the generating hypothesis and some new consequences of it. A surprising one is that I, the Brown-Comenetz dual of the sphere and the source of many counterexamples in stable homotopy, is the cofiber of a self map of a wedge of spheres. We also show that a consequence of the generating hypothesis, that the homotopy of a finite spectrum that is not a wedge of spheres can never be finitely generated as a module over $\pi_* S$, is in fact true for many finite torsion spectra.

Introduction

Freyd's generating hypothesis [4] is perhaps the most important question in stable homotopy theory. A precise statement of it follows.

Conjecture A (Freyd's generating hypothesis)**.** *If X and Y are finite spectra, and S is the sphere spectrum, then the natural map*

$$[X, Y] \to \operatorname{Hom}_{\pi_* S}(\pi_* X, \pi_* Y)$$

is a monomorphism.

If we fix Y (perhaps not finite) and allow X to vary, we get a special case of the generating hypothesis which I will refer to as **Freyd's generating hypothesis with target** Y. Here $[X, Y]$ denotes maps from X to Y in the stable homotopy category, and $\pi_* X = [S, X]_*$ denotes the homotopy groups of X. In practice, we implicitly assume that we are actually in the p-local stable homotopy category for some fixed integer prime p.

Freyd proves that the generating hypothesis actually implies that the map

$$[X, Y] \to \operatorname{Hom}_{\pi_* S}(\pi_* X, \pi_* Y)$$

is an isomorphism for all finite spectra X, Y. Kahn derived other consequences of the truth or falsity of the generating hypothesis in a series of papers, including [11, 12, 13].

Devinatz and Hopkins [3] have a program for proving the generating hypothesis with target S using chromatic technology. This approach generalizes Devinatz' work in [1], where he proves that if $f\colon X \to S$ has $\pi_* f = 0$, and p is odd, then the composite $X \to S \to L_1 S$ is null. The program depends on the truth of either the telescope conjecture (currently believed to be likely false) or a weak form of the chromatic splitting conjecture and several other conjectures.

In this paper, we prove the following theorem. Let X_p denote the p-completion of a spectrum X.

Theorem B. *Let Y be a finite spectrum. The following are equivalent*:

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(1) *Freyd's generating hypothesis with target Y;*
(2) *$\pi_* Y_p$ is an injective $\pi_* S$-module;*
(3) *$\pi_* Y_p$ is an injective $\pi_* S_p$-module;*
(4) *The natural map*

$$[X, Y_p] \to \operatorname{Hom}_{\pi_* S}(\pi_* X, \pi_* Y_p)$$

is an isomorphism for all spectra X.

We also prove the following theorem. Recall that the Spanier-Whitehead dual DX of X is defined by $DX = F(X, S)$, the spectrum of maps from X to S.

Theorem C. *Suppose Freyd's generating hypothesis with target S holds. Let R be a finite ring spectrum that is Spanier-Whitehead self-dual, in the sense that DR is a suspension of R. Then $\pi_*(R_p)$ is injective as a left R_*-module and as a left module over itself. In particular, the natural map*

$$[X, R_p] \to \operatorname{Hom}_{R_*}(R_* X, \pi_* R_p)$$

is an isomorphism for all X.

For example, this theorem means that, if the generating hypothesis with target S holds, then $\pi_* M(p^n)$ is a self-injective ring for $p > 2$ and n arbitrary or for $p = 2$ and $n > 1$.

Freyd [4, Theorem 9.9] proved that the generating hypothesis is equivalent to $\pi_* Y$ being an injective $\pi_* S$-module for all finite torsion spectra Y. T. Y. Lin [16] showed that $\pi_* Y$ is not an injective $\pi_* S$-module if Y is not torsion, but did not realize that completion would solve this problem if the generating hypothesis is true. Our approach is different from Freyd's, and yields a more precise result. In addition, Freyd does not mention part (4) of Theorem B, which focuses attention on maps from infinite spectra X to Y_p. Infinite spectra X that might be worth studying in this context include the rational Eilenberg-MacLane spectrum $H\mathbb{Q}$ and $\Sigma^\infty BG_+$, the classifying space of a finite group, where the Segal conjecture tells us $[X, S_p]$.

Of course, even if the generating hypothesis is false, $\pi_* Y_p$ has some injective hull J_Y as a $\pi_* S$-module, so one can attempt to study the map $\pi_* Y_p \to J_Y$. We show in this paper that $\pi_* S_p \to J_S$ is a split monomorphism of abelian groups in degree 0, for example.

The methods of this paper may also be helpful in investigating the genreating hypothesis in other stable homotopy categories. Lockridge [17] has investigated this question; he shows that the generating hypothesis holds in the unbounded derived category $\mathcal{D}(R)$ of a commutative ring R if and only if R is von Neumann regular, for example.

Theorem B has a number of corollaries. Perhaps the most surprising of them is the following. Let IY denote the Brown-Comenetz dual of Y, so that

$$[X, IY] = \operatorname{Hom}_{\mathbb{Z}_{(p)}}(\pi_*(X \wedge Y), \mathbb{Q}/\mathbb{Z}_{(p)})$$

for all X.

Corollary D. *Let Y be a finite spectrum. Freyd's generating hypothesis with target Y holds if and only if $\pi_*(IY)$ is a flat $\pi_* S$-module. In particular, this implies that the natural map*

$$\pi_*(IY) \otimes_{\pi_* S} \pi_* X \to \pi_*(IY \wedge X)$$

is an isomorphism for all X. Furthermore, in this case $\pi_*(IY)$ has projective dimension 1 as a π_*S-module and, if Freyd's generating hypothesis with target S holds as well, there is a cofiber sequence

$$\Sigma^{-1}IY \xrightarrow{\delta} W \to W \to IY$$

for which W is a coproduct of spheres of varying dimension and δ is a phantom map.

Note that the map δ cannot be 0, for then IY would be a coproduct of spheres itself. Since $IY = DY \wedge I$ is BP-acyclic, this is impossible unless $IY = 0$, which is false.

On the other hand, there are several reasons to think that δ should be 0, and so the generating hypothesis should be false. For example, δ is a map from a BP-acyclic spectrum to a coproduct of spheres, and one might expect that a coproduct of spheres would be BP-local. Any bounded below coproduct of spheres is BP-local, as is any suspension spectrum [6], but we do not know about arbitrary coproducts of spheres. Similarly, one might think that there are no phantom maps to W_p, which should also lead to a disproof of the generating hypothesis. Again, however, the fact that the spheres in W occur in infinitely many dimensions makes us unable to prove this.

But we can also give some weak evidence that the generating hypothesis might be true. One of the most simple corollaries of the generating hypothesis is that π_*Y is not a finitely generated π_*S-module for any finite Y except for finite coproducts of spheres.

Theorem E. *Suppose Y is a finite spectrum of type n, for some $n > 0$, and suppose the map $\pi_*Y \to \pi_*L_nY$ is nonzero. This hypothesis holds, for example, if Y is a ring spectrum or a μ-spectrum in the sense of* [9, Definition 4.8]. *Then π_*Y is not a finitely generated π_*S-module.*

This theorem applies in particular to the ring spectrum $DX \wedge X$ for any finite torsion spectrum X, and to the generalized Moore spectra $M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ for large enough values of the exponents (see [2]). The telescope conjecture [19] (which is true if $n = 1$) would imply that every finite torsion spectrum satisfies the hypotheses of Theorem E, but even if the telescope conjecture fails, the author would be astounded if there were any nonzero finite spectra of type n for which the map $\pi_*Y \to \pi_*L_nY$ is zero. It is just that current techniques do not seem to be sufficient to prove this.

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1. Proof of Theorem B

We begin with a basic result about injective π_*S-modules.

Lemma 1.1. *Suppose E is a spectrum such that π_*E is an injective π_*S-module. Then the natural map*

$$[X, E] \to \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*E)$$

is an isomorphism for all spectra X.

This lemma shows that condition (2) of Theorem B implies condition (4). We note that Lemma 1.1 holds, with the proof given below, in any monogenic stable homotopy category in the sense of [8].

Proof. Since π_*E is injective, the functor $\mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*E)$ is exact. The Brown representability theorem then implies that there is a spectrum J and a natural isomorphism

$$[X, J] \cong \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*E)$$

for all X. Taking $X = S$ tells us that there is an isomorphism $r\colon \pi_*J \cong \pi_*E$ of π_*S-modules, but not that the natural isomorphism takes $f \in [X, J]$ to f_*. In fact, if $1 \in [J, J]$ corresponds to $\alpha\colon \pi_*J \to \pi_*E$, then naturality tells us that $f \in [X, J]$ corresponds to $\alpha \circ f_*$. In particular, there is a map $f\colon E \to J$ such that $\alpha \circ f_*$ is the isomorphism r. Hence α is a split epimorphism. On the other hand, if $x \in \pi_nJ \cong [S^n, J]$ is a homotopy class of J, then x corresponds to $\alpha \circ x_* \in \mathrm{Hom}_{\pi_*S}(\pi_*S^n, \pi_*J)$, which is multiplication by the class $\alpha(x)$. Thus $\alpha(x)$ can't be 0, and so α is a monomorphism as well. Thus f_* is an isomorphism as well, and so $E \cong J$. The lemma follows. □

The following proposition is the heart of the argument proving Theorem B.

Proposition 1.2. *Suppose Y is a spectrum such that there are no nonzero phantom maps to Y. Then Freyd's generating hypothesis with target Y holds if and only if π_*Y is an injective π_*S-module.*

This proposition will also hold in any monogenic stable homotopy category.

Proof. The "if" half of this proposition follows immediately from Lemma 1.1 and does not require the assumption about phantom maps. For the "only if" half, assume

$$[X, Y] \to \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*Y)$$

is injective for all finite X. Let J denote the injective hull of π_*Y as a π_*S-module. Then Brown representability and Lemma 1.1 imply that there is a spectrum I with $\pi_*I = J$ and such that the natural map

$$[X, I] \to \mathrm{Hom}_{\pi_*S}(\pi_*X, J)$$

is an isomorphism for all X. In particular, there is a map $Y \to I$ corresponding to the inclusion $\pi_*Y \to J$. Consider the commutative diagram below.

$$\begin{array}{ccc} [X, Y] & \longrightarrow & [X, I] \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*Y) & \longrightarrow & \mathrm{Hom}_{\pi_*S}(\pi_*X, J) \end{array}$$

The left-hand vertical map is injective for all finite X, and the bottom horizontal map is always injective. It follows that the top horizontal map is injective for all finite X, and hence that the fiber $F \to Y$ of the map $Y \to I$ is a phantom map. Since there are no nonzero phantom maps to Y, we see that Y is a retract of I. Hence π_*Y is a retract of J, and so is an injective π_*S-module. □

To apply Proposition 1.2, we need to know the relationship between the generating hypothesis with target Y and the generating hypothesis with target Y_p. Since the fiber of the map $Y \to Y_p$ is rational, we need the following proposition.

Proposition 1.3. *Suppose V is a graded rational vector space. Then there is a unique π_*S-module structure on V extending the abelian group structure, and V is an injective π_*S-module with this structure.*

This proposition will hold if π_*S is replaced by any ring R such that $R/\mathfrak{p} \cong \mathbb{Z}_{(p)}$, where $\mathfrak{p}$ is the ideal of p-torsion elements.

Proof. Let $\mathfrak{p}$ denote the ideal of p-torsion elements in π_*S. Then, since V is torsion-free, the only way to make π_*S act on V is through the homomorphism $\pi_*S \to \pi_*S/\mathfrak{p} \cong \mathbb{Z}_{(p)}$.

Now suppose $f\colon M \to V$ is a map of π_*S-modules and $i\colon M \to N$ is an inclusion of π_*S-modules. Let $\operatorname{Tor}(M)$ denote the p-torsion in M, which is a π_*S-submodule. Then f factors through $\overline{f}\colon M/\operatorname{Tor}(M) \to V$, and furthermore i induces an inclusion $\overline{i}\colon M/\operatorname{Tor}(M) \to N/\operatorname{Tor}(N)$. Since V is an injective abelian group, there is then a map $\overline{g}\colon N/\operatorname{Tor}(N) \to V$ of abelian groups extending $\overline{f}$. But $\overline{g}$ is in fact a map of π_*S-modules, since $N/\operatorname{Tor}(N)$ and V are torsion-free so $\mathfrak{p}$ acts trivially. Hence

$$N \to N/\operatorname{Tor}(N) \xrightarrow{\overline{g}} V$$

gives us the desired extension of f. □

We then have the following proposition.

Proposition 1.4. *Suppose Y is a finite spectrum. Then Freyd's generating hypothesis with target Y holds if and only if Freyd's generating hypothesis with target Y_p holds.*

Proof. Let C denote the cofiber of the map $Y \to Y_p$, so that C is rational. Recall that the fiber $\Sigma^{-1}C \to Y$ is a phantom map (see [18, Theorem 9.5], for example). Then we have the commutative diagram below.

$$\begin{array}{ccccc} [X,Y] & \longrightarrow & [X,Y_p] & \longrightarrow & [X,C] \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Hom}_{\pi_*S}(\pi_*X,\pi_*Y) & \longrightarrow & \operatorname{Hom}_{\pi_*S}(\pi_*X,\pi_*Y_p) & \longrightarrow & \operatorname{Hom}_{\pi_*S}(\pi_*X,\pi_*C) \end{array}$$

The top row of this diagram is short exact for finite X, since $\Sigma^{-1}C \to Y$ is phantom. The bottom row is left exact for all X. The right-hand vertical map is an isomorphism. A simple diagram chase now shows that the left-hand vertical map is a monomorphism for finite X if and only if the middle vertical map is a monomorphism for finite X. □

Analysis of the proof of Proposition 1.4 in fact gives us the following proposition, independent of the generating hypothesis.

Proposition 1.5. *Suppose X and Y are finite spectra. Then the natural map*

$$\operatorname{Ext}^n_{\pi_*S}(\pi_*X,\pi_*Y) \to \operatorname{Ext}^n_{\pi_*S}(\pi_*X,\pi_*Y_p)$$

is an isomorphism for all $n \geq 1$.

Proof. In the commutative diagram used in the proof of Proposition 1.4, we pointed out that the bottom row

$$\operatorname{Hom}_{\pi_*S}(\pi_*X,\pi_*Y) \to \operatorname{Hom}_{\pi_*S}(\pi_*X,\pi_*Y_p) \to \operatorname{Hom}_{\pi_*S}(\pi_*X,\pi_*C)$$

was left exact. But in fact a diagram chase shows that it is exact, since the right-hand vertical map is an isomorphism. From this and the fact that $\pi_* C$ is an injective $\pi_* S$ module, the proposition follows. □

The last ingredient we need for Theorem B is the simple proof that conditions (2) and (3) are equivalent.

Lemma 1.6. *Let Y be a finite spectrum. Then $\pi_* Y_p$ is an injective $\pi_* S$-module if and only if it is an injective $\pi_* S_p$-module.*

Proof. Since $\pi_* S_p$ is a flat $\pi_* S$-module, the forgetful functor from $\pi_* S_p$-modules to $\pi_* S$-modules preserves injectives. Conversely, assume $\pi_* Y_p$ is an injective $\pi_* S$-module. To see that $\pi_* Y_p$ is injective over $\pi_* S_p$, we use Baer's criterion, which tells us we need only check that, given an ideal $\mathfrak{a}$ of $\pi_* S_p$ and a map $f\colon \mathfrak{a} \to \pi_* Y_p$, there is an extension $\pi_* S_p \to \pi_* Y_p$ of $\pi_* S_p$-modules. Let $\mathfrak{b} = \mathfrak{a} \cap \pi_* S$. Then $\mathfrak{b}$ is an ideal of $\pi_* S$, so we have an extension $\pi_* S \to \pi_* Y_p$ of $\pi_* S$-modules. This gives a map $\pi_* S_p = \pi_* S \otimes \mathbb{Z}_{(p)} \to \pi_* Y_p$ of $\pi_* S_p$-modules. When restricted to $\mathfrak{b}$, this map extends f. But since $\mathfrak{a} = \mathfrak{b} \otimes \mathbb{Z}_{(p)}$, it follows that it is an extension of f on $\mathfrak{a}$ as well. □

We can now prove Theorem B.

Proof of Theorem B. Suppose Freyd's generating hypothesis with target Y holds. Then Proposition 1.4 implies that the generating hypothesis with target Y_p holds. Since there are no nonzero phantom maps to Y_p, Proposition 1.2 tells us that $\pi_* Y_p$ is an injective $\pi_* S$-module. Thus condition (1) implies condition (2).

Lemma 1.6 tells us that conditions (2) and (3) are equivalent, and Lemma 1.1 tells us that condition (2) implies condition (4). Condition (4) obviously implies that Freyd's generating hypothesis holds with target Y_p, and then Proposition 1.4 implies that it holds with target Y as well. □

2. Brown-Comenetz duality

In this section, we investigate the consequences of the generating hypothesis for Brown-Comenetz duals of finite spectra, proving Corollary D.

Proof of Corollary D. Suppose Y is finite. Then $Y_p = I^2 Y$, as is well-known. Hence $\pi_* Y_p = \operatorname{Hom}_{\mathbb{Z}_{(p)}}(\pi_*(IY), \mathbb{Q}/\mathbb{Z}_{(p)})$. Now apply Lambek's theorem [15, Theorem 4.9] to conclude that $\pi_* Y_p$ is injective if and only if $\pi_*(IY)$ is flat. Once $\pi_*(IY)$ is flat, then the map

$$\pi_*(IY) \otimes_{\pi_* S} \pi_* X \to \pi_*(IY \wedge X)$$

is a natural transformation of homology theories that is an isomorphism when $X = S$, so is always an isomorphism.

Now Lemma 2 of [10] implies that, over a countable ring like $\pi_* S$, any flat module has projective dimension ≤ 1. Since $\pi_* S$ is a local ring, projectives are free [14] (we actually need the graded case of this result, which has been recently written up in [5, Appendix A]). Thus, if $\pi_*(IY)$ had projective dimension 0, it would be free. From that it is easy to conclude that IY is a coproduct of spheres. But, since $IY = DY \wedge I$ is BP-acyclic (see [8, Lemma B.11]), IY would have to be trivial. Since this is false, the projective dimension of IY is 1.

Thus there is a short exact sequence

$$0 \to F_1 \to F_0 \to \pi_* IY \to 0$$

of $\pi_* S$-modules, where F_1 and F_0 are free. In fact, by tensoring over $\mathbb{Z}_{(p)}$ with $\mathbb{Q}$, we see that F_1 and F_0 are isomorphic. By choosing generators, we can find a coproduct of spheres W with $\pi_* W \cong F_1 \cong F_0$. By looking at the image of the generators in homotopy, we can find maps

$$W \xrightarrow{f} W \xrightarrow{g} IY$$

such that the induced maps on homotopy give our original free resolution of $\pi_* IY$. In fact, this sequence is a cofiber sequence. Indeed, the composite gf is null, so there is an induced map from the cofiber of f to IY, which one can easily see induces an isomorphism on homotopy.

Now, given Freyd's generating hypothesis with target S, we claim that the map $\Sigma^{-1} IY \xrightarrow{\delta} W$ that is the fiber of f is phantom. Indeed, if F is finite, and $h\colon F \to \Sigma^{-1} IY$ is a map, then δh must factor through some a map $h'\colon F \to W'$ for some finite subcoproduct of spheres h'. If δh is nonzero, then h' is nonzero, and so, by Freyd's generating hypothesis with target S, must induce a nontrivial map on homotopy. Then δh also induces a nontrivial map on homotopy, as does the trivial map $f\delta h$. This contradiction implies δ is phantom. □

Corollary D has some interesting consequences. Suppose that Freyd's generating hypothesis with target S holds, so that we have the cofiber sequence

$$\Sigma^{-1} I \xrightarrow{\delta} W \xrightarrow{f} W \to I$$

where W is a coproduct of spheres and δ is a phantom map. Then $E_* f$ is a monomorphism for all E, and is an isomorphism for the many E for which $E_* I = 0$, such as all BP-module spectra and I itself. In fact, $E^* f$ is an isomorphism for all BP-module spectra and all harmonic spectra E, since I is BP-acyclic and dissonant.

On the other hand, suppose E is one of the many spectra for which $[E, S]_* = 0$, such as $I, H\mathbb{F}_p$, $K(n)$ for $n > 0$, or any dissonant spectrum. Then any map $E \to W$ goes to 0 in the corresponding product P of spheres, and hence factors through the fiber of $W \to P$. Since this is a phantom map, $[E, W]_*$ consists entirely of phantom maps, which necessarily go to 0 in $[E, I]_*$. Hence $[E, f]_*$ is in fact surjective in this case. One might think that this happens because $[E, W]_* = 0$, but in fact $[E, W]_* = 0$ if and only if $E = 0$, since if $[E, W]_* = 0$ then $[E, I]_* = 0$, and so $E = 0$.

Another corollary is the following.

Corollary 2.1. *Suppose Freyd's generating hypothesis with target S holds. Then there is a product J of suspensions of I such that $S_p \vee J \cong J$.*

Proof. Apply the functor $F(-, I)$ to the cofiber sequence

$$\Sigma^{-1} I \xrightarrow{\delta} W \to W \to I$$

to get a cofiber sequence

$$F(I, I) = S_p \to J = F(W, I) \to J = F(W, I) \to F(\Sigma^{-1} I, I) = \Sigma S_p.$$

On homotopy, the last map takes a map $\alpha\colon \Sigma^i W \to I$ into the composite $\delta \circ \alpha$, which is necessarily 0 since there are no phantom maps to I. On the other hand, because $\pi_* S_p$ is an injective $\pi_* S$-module, any map into S_p that is trivial on homotopy is in fact trivial. Hence the cofiber sequence above splits, giving the corollary. □

3. Other consequences of the generating hypothesis

In this section, we use Theorem B to draw some further consequences of the generating hypothesis, including Theorem C, with which we begin.

Lemma 3.1. *Suppose R is a ring spectrum and M is an R-module spectrum such that M_* is injective as a left R_*-module. Then the natural map*

$$M^*X \to \operatorname{Hom}_{R_*}(R_*X, M_*)$$

is an isomorphism for all X.

Proof. The natural map in question takes $f\colon X \to M$ to the map $\mu_* \circ R_*f$, where $\mu\colon R \wedge M \to M$ is the structure map of M. Because M_* is injective, the functor $\operatorname{Hom}_{R_*}(R_*(-), M_*)$ is a cohomology theory. Since the natural transformation in question is an isomorphism when $X = S$, it is an isomorphism for all X. □

Proof of Theorem C. Suppose the generating hypothesis with target S holds, and suppose that R is a finite ring spectrum that is Spanier-Whitehead self-dual. By Corollary 2.1, S_p is a retract of a product J of suspensions of I. By smashing with R, which commutes with products since R is finite, we find that R_p is a retract, as an R-module, of a product of suspensions of $I \wedge R = F(DR, I)$. Since R is Spanier-Whitehead self dual, R_p is a retract, as an R-module, of a product of suspensions of IR. But $\pi_*(IR) = \operatorname{Hom}_{\mathbb{Z}_{(p)}}(R_*, \mathbb{Q}/\mathbb{Z}_{(p)})$ is an injective R_*-module [15, Corollary 3.6B]. It follows that π_*R_p, as a retract of a product of injective R_*-modules, is an injective R_*-module. The same proof used in Lemma 1.6 implies that π_*R_p is also injective as a left module over itself. Lemma 3.1 completes the proof. □

Let R be a finite Spanier-Whitehead self-dual ring spectrum as in Theorem C, and suppose the generating hypothesis holds for both S and R. Then, on the one hand, we have the isomorphism

$$[X, R_p] \cong \operatorname{Hom}_{\pi_*S}(\pi_*X, \pi_*R_p) \cong \operatorname{Hom}_{R_*}(R_* \otimes_{\pi_*S} \pi_*X, \pi_*R_p),$$

and on the other hand, we have the isomorphism

$$[X, R_p] \cong \operatorname{Hom}_{R_*}(R_*X, \pi_*R_p).$$

These isomorphisms are related by the map

$$\sigma_X\colon\ R_* \otimes_{\pi_*S} \pi_*X \to R_*X,$$

and one might be tempted to think that σ_X has to be an isomorphism, and so R_* has to be flat over π_*S. However, all we actually know, under the generating hypothesis for S and R, is that $\operatorname{Hom}_{R_*}(\sigma_X, \pi_*R_p)$ is an isomorphism. Thus we can only conclude that

$$\operatorname{Hom}_{R_*}(K_X, \pi_*R_p) = \operatorname{Hom}_{R_*}(C_X, \pi_*R_p) = 0$$

for all X, where K_X and C_X are the kernel and cokernel of σ_X.

Now we give a more precise version of Freyd's "faithful implies full" result [4, Proposition 9.7].

Corollary 3.2. *Suppose Freyd's generating hypothesis with target Y holds. Then the natural map*

$$[X, Y] \to \operatorname{Hom}_{\pi_*S}(\pi_*X, \pi_*Y)$$

is an isomorphism for all finite X.

Proof. We again let C denote the cofiber of $Y \to Y_p$, so that we get the same commutative diagram used in the proof of Proposition 1.4.

$$\begin{array}{ccccc} [X,Y] & \longrightarrow & [X,Y_p] & \longrightarrow & [X,C] \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Hom}_{\pi_*S}(\pi_*X,\pi_*Y) & \longrightarrow & \operatorname{Hom}_{\pi_*S}(\pi_*X,\pi_*Y_p) & \longrightarrow & \operatorname{Hom}_{\pi_*S}(\pi_*X,\pi_*C) \end{array}$$

The top row of this diagram is exact for all finite X since $\Sigma^{-1}C \to Y$ is phantom. The bottom row is also exact, as pointed out in Proposition 1.5. The right-hand vertical map is an isomorphism by Proposition 1.3, and the middle vertical map is an isomorphism by Theorem B. A simple diagram chase shows the left-hand vertical map is an isomorphism for all finite X. □

Proposition 1.5 immediately gives us the following corollary of Theorem B.

Corollary 3.3. *If Freyd's generating hypothesis with target Y holds, then*

$$\operatorname{Ext}^n_{\pi_*S}(\pi_*X,\pi_*Y) = 0$$

for all finite X.

Using the well-known fact that $\mathbb{Z}_p/\mathbb{Z}_{(p)}$ is a rational vector space, we get the following consequence of the generating hypothesis.

Corollary 3.4. *If Freyd's generating hypothesis with target Y holds, then*

$$0 \to \pi_*Y \to \pi_*Y_p \to \pi_*Y_p/\pi_*Y \to 0$$

is an injective resolution of π_*Y in the category of π_*S-modules. In particular, if Y is a finite spectrum of type 0, then π_*Y has injective dimension 1.

4. Injective π_*S-modules

Theorem B focusses attention on injective π_*S-modules; in this section we prove a few facts about them. Without assuming Freyd's generating hypothesis, we still know that π_*Y has some injective hull J_Y. We cannot say very much about J_Y, but we can say a little.

Proposition 4.1. *The map $\pi_*S \to \pi_*S_p$ is an essential extension of π_*S-modules.*

Hence, whatever J_S is, at least it contains π_*S_p.

Proof. The only elements in π_*S_p not in π_*S are elements in $\pi_0 S_p \cong \mathbb{Z}_p$. Choose a nonzero $x \in \mathbb{Z}_p$, and suppose p^n divides x but p^{n+1} does not, so that x is congruent to an integer of the form $kp^n \in \pi_0 S$ modulo p^{n+1}, where k is a unit. Now choose an element $\alpha \in \pi_*S$ of order p^{n+1}, which can be done in the image of the J homomorphism. Then $\alpha x = kp^n x$, which is a nontrivial element of π_*S. Therefore, $(x) \cap \pi_*S$ is nonzero, completing the proof. is a $y \in \pi_0 S$ such that $x - y$ is a multiple of p in $\mathbb{Z}_p$. □

In fact, we know a little more about J.

Proposition 4.2. *Let J denote the injective hull of π_*S_p as a π_*S_p-module. The inclusion $\mathbb{Z}_p \to J_0$ is a split monomorphism of abelian groups.*

Proof. We will prove that $\mathbb{Z}_p \to J_0$ is a pure monomorphism. That is, we will show that if we have an equation $x = p^n y$ for $x \in \mathbb{Z}_p$ and $y \in J_0$, then in fact we have an equation $x = p^n z$ for some $z \in \mathbb{Z}_p$. Indeed, we may as well assume $x = p^k$, so that we have $p^k\alpha = p^n \alpha y$ for all $\alpha \in \pi_* S_p$. But then, if $n > k$, we may take α to be an element of exact order p^n and conclude that $p^k \alpha = 0$. This contradiction shows that $\mathbb{Z}_p \to J_0$ is pure.

Since $\mathbb{Z}_p$ is a pure injective abelian group, the proposition follows. □

Since $\pi_* S_p \to J$ is an essential extension, for every element $y \in J$, there is an element $\alpha_y \in \pi_* S_p$ with $\alpha_y y \in \pi_* S_p$. We can thus look for an element x in $\pi_* S_p$ of lowest possible degree such that $x = \gamma y$ for some $y \in J \setminus \pi_* S_p$. Proposition 4.2 tells us the degree of x must be positive. Our knowledge of $\pi_* S$ is sufficient to rule out some possibilities for the pair (x, γ), but insufficient, as far as the author knows, to say anything systematic.

We point that there is one more injective $\pi_* S$-module known, besides the rational ones and, conjecturally, $\pi_* S_p$.

Proposition 4.3. *Let I denote the Brown-Comenetz dual of S. Then $\pi_* I$ is the injective hull of $\mathbb{F}_p$ as a $\pi_* S$-module.*

The same argument as in the proof below shows that $\pi_* I L_n S$ is an injective $\pi_* L_n S$-module for any n.

Proof. We have $\pi_* I = \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\pi_* S, \mathbb{Q}/\mathbb{Z}_{(p)})$. Since $\mathbb{Q}/\mathbb{Z}_{(p)}$ is an injective abelian group, $\pi_* I$ is injective by a standard result about injective modules [15, Corollary 3.6B]. The obvious inclusion $\mathbb{F}_p \to \pi_0 I \to \pi_* I$ is obviously a map of $\pi_* S$-modules. The action of $\pi_* S$ on $\pi_* I$ is given by

$$\mu\colon\ \pi_k S \otimes \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\pi_n S, \mathbb{Q}/\mathbb{Z}_{(p)}) \to \mathrm{Hom}(\pi_{n-k} S, \mathbb{Q}/\mathbb{Z}_{(p)})$$

where $\mu(x \otimes f)(y) = f(xy)$. In particular, if f is a nontrivial element of $\pi_{-n} I = \mathrm{Hom}(\pi_n S, \mathbb{Q}/\mathbb{Z}_{(p)})$, then there is an $x \in \pi_n S$ such that $f(x)$ is a nonzero element of $\mathbb{F}_p \subseteq \mathbb{Q}/\mathbb{Z}_{(p)}$. It follows that $\mu(x \otimes f)$ is a nonzero element of $\mathbb{F}_p$, and therefore that $\mathbb{F}_p \to \pi_* I$ is an essential extension of $\pi_* S$-modules. □

Note that it is tempting to conclude from Proposition 4.3 that $\mathbb{F}_p$ has injective dimension 1 as a $\pi_* S$-module. This is wrong, however. The cokernel of $\mathbb{F}_p \to \pi_* I$ is isomorphic as a graded abelian group to $\pi_* I$, but not as a $\pi_* S$-module.

5. Infinitely generated homotopy

It was G. Whitehead who realized that the generating hypothesis implies that the homotopy of a finite complex Y is not finitely generated over $\pi_* S$ unless Y is a finite coproduct of spheres [4, Proposition 9.5]. The proof of this fact is so easy we recall it here. Suppose Y has finitely generated homotopy, so that we have a cofiber sequence

$$F \xrightarrow{f} W \xrightarrow{g} Y \xrightarrow{h} \Sigma F$$

where W is a finite wedge of spheres and $\pi_*(g)$ is onto. Then $\pi_* h = 0$, so, by the generating hypothesis, $h = 0$. Hence Y is a retract of W, so $\pi_* Y$ is projective, and hence free. Thus Y is itself a wedge of spheres.

Don Kahn [11] has shown that, for any finite spectrum Y, it is possible to attach two cells (one if Y is not torsion) to Y to get a new complex Z with $\pi_* Z$

not finitely generated. Thus there are many finite spectra whose homotopy is not finitely generated.

We can use the existence of v_n self-maps to prove Theorem E, which, we recall, says that if X is a finite type n spectrum with $n > 1$ such that the map $\pi_* X \to \pi_* L_n X$ is nonzero, then $\pi_* X$ is not finitely generated.

Proof of Theorem E. By the nilpotence theorem [7], X has a non-nilpotent self-map v of positive degree. This map can be taken to have Adams-Novikov filtration 0; see, for example, [9, Theorem 4.6]. Let $\mathrm{AN}(\alpha)$ denote the Adams-Novikov filtration of an element $\alpha \in \pi_* X$. We need to choose an element $\beta \in \pi_* X$ such that $\lim_{k \Rightarrow \infty} \mathrm{AN}(v^k \beta)$ is minimal. Unfortunately, to do this we need to know that there exists a β such that $\lim_{k \Rightarrow \infty} \mathrm{AN}(v^k \beta)$ is finite. To see this, note that if this limit is not finite, then the analogous limit for the $E(n)$-Adams filtration is also infinite, since $E(n)$ is a BP-module spectrum. But the $E(n)$-based Adams-Novikov spectral sequence for $L_n X$ converges strongly and has a horizontal vanishing line at the E_∞ term by [9, Proposition 6.5]. Hence the image of $v^k \beta$ in $\pi_* L_n X$ must be zero; since v acts as a unit on $L_n X$, we conclude that β maps to 0 in $\pi_* L_n X$. Therefore, if we choose a β that does not map to 0 in $\pi_* L_n X$, we will have $\lim_{k \Rightarrow \infty} \mathrm{AN}(v^k \beta)$ finite.

So now we have chosen a β such that $\lim_{k \Rightarrow \infty} \mathrm{AN}(v^k \beta)$ is minimal. Choose a generating set $\{x_i\}$ for $\pi_* X$ as a $\pi_* S$-module, and write

$$\beta = x_1 \circ \alpha_1 + \cdots + x_r \circ \alpha_r$$

for some $\alpha_j \in \pi_* S$. Then for large k, we have

$$v^k \circ \beta = v^k \circ x_i \circ \alpha_i + \cdots + v^k \circ x_r \circ \alpha_r,$$

and $v^k \beta$ will have the least Adams-Novikov filtration among all the $v^k x_i$. This implies that there must be an i with α_i nonzero such that the Adams-Novikov filtration of $v^k x_i$ is the same as that of $v^k \beta$. Hence α_i has Adams-Novikov filtration 0, so is in $\pi_0 S$. We conclude that the degree of x_i is the same as the degree of β. By reeating the argument on $v^j \beta$, we see that there must be a generator of $\pi_* X$ in the degree of $v^j \beta$ for all $k \geq 0$. Thus $\pi_* X$ is not finitely generated.

Now, the statement of Theorem E included the claim that the theorem holds when X is a μ-spectrum. This follows because if X is a μ-spectrum, then there is a unit $\eta \colon S \to X$ and a multiplication $\mu \colon X \wedge X \to X$ such that $\mu \circ (\eta \wedge 1)$ is the identity. In particular, if η went to 0 in $\pi_* L_n X$, then $L_n X$ itself would be zero, which is false since X is type n. □

6. The generating hypothesis and thick subcategories

One difficulty that the generating hypothesis has always posed is that knowing the generating hypothesis with target Y does not seem to say very much about the generating hypothesis with other targets. Freyd's work does imply, however, that if the generating hypothesis with target Y is true for all finite torsion spectra Y, then it is true for all finite Y (this can be obtained from the proof of Theorem 9.9 of [4]). In this section we extend Freyd's result to finite spectra of type at least n.

Proposition 6.1. *Suppose X is a type n finite spectrum for some n, with v_n self-map v. Let X/v^k denote the cofiber of v^k, and consider the cofiber sequence*

$$Z \xrightarrow{\delta} X \to \prod_{k \geq 1} X/v^k.$$

Then δ is a phantom map.

Proof. Suppose first that $n=0$, so that $v=p$. If F is a finite spectrum, the group $[F,X]$ is finitely generated abelian, and therefore any $f\colon F\to X$ is not divisible by p^k for large enough k. Hence the image of f in $[F,X/p^k]$ is nonzero for large enough k. Thus

$$[F,X]\to[F,\prod X/p^k]$$

is a monomorphism, and so δ is phantom.

Now suppose $n\geq 1$, so that the map v has some positive degree d (see [7]). Let F be a finite spectrum, and suppose $f\colon F\to X$ is a nontrivial map. We claim that the composite $F\to X\to X/v^k$ is nontrivial for some k. Indeed, if not, then f factors through $\Sigma^{dk}X$ for all k. For k large enough, every cell of F will be in lower degree than all the cells of $\Sigma^{dk}X$, and so $[F,\Sigma^{dk}X]=0$ and $f=0$. Thus

$$[F,X]\to[F,\prod X/v^k]$$

is a monomorphism, and so δ is phantom. □

Corollary 6.2. *Suppose X is a type n finite spectrum for some n, with v_n self-map v. Then X_p is a retract of $\prod_{k\geq 1} X/v^k$.*

Proof. Recall that completion is really Bousfield localization $L_{M(p)}$. The space $\prod X/v^k$ is already $L_{M(p)}$-local, since each X/v^k is so. Hence we have a cofiber sequence

$$L_{M(p)}Z\xrightarrow{L_{M(p)}\delta}L_{M(p)}X\to\prod_{k\geq 1}X/v^k.$$

The map $L_{M(p)}\delta$ is determined by its restriction to Z, which is phantom by Proposition 6.1. Since there are no phantom maps to X_p, we conclude that $L_{M(p)}\delta=0$, giving us the desired splitting. □

Corollary 6.3. *Fix $n\geq 0$. The generating hypothesis with target Y is true for all finite spectra Y if and only if it is true for all finite Y of type at least n.*

Corollary 6.3 is the closest we can come to showing that the collection of all Y for which the generating hypothesis with target Y is true is a thick subcategory.

Proof. It is enough to show that, if the generating hypothesis with target Y is true for all finite Y of type at least k, then the generating hypothesis with target Y is true for all finite Y of type at least $k-1$. Suppose X has type $k-1$. Choose a v_{k-1} self-map v of X. By Corollary 6.2, X_p is a summand in $\prod X/v^k$. Each X/v^k has type k, and so $\pi_* * X/v^k$ is an injective π_*S-module, by Theorem B. It follows that π_*X_p is an injective π_*S-module, and so the generating hypothesis with target X_p is true. But then Proposition 1.4 implies that the generating hypothesis with target X is true. □

Another interesting corollary of Proposition 6.1 is the following. Let $\mathcal{C}_n$ denote the thick subcategory of finite spectra whose type is at least n.

Corollary 6.4. *The subcategory $\mathcal{C}_n$ generates and cogenerates the category of finite spectra.*

This means that, given a nonzero map $f\colon X\to Y$ of finite spectra, there are maps $g\colon Z\to X$ and $h\colon Y\to W$ with $Z,W\in\mathcal{C}_n$ and $f\circ g$ and $h\circ f$ both nonzero. This corollary was proved by Freyd [4, Proposition 6.8] in the case $n=1$.

Proof. Let $f\colon X \to Y$ be a nonzero map. Suppose Y is of type k. Then it follows from Proposition 6.1 that there is a Z of type $k+1$, namely Y/v^r for large r, and a map $h\colon Y \to Z$ such that hf is nonzero. We can then proceed by induction to see that $\mathcal{C}_n$ cogenerates the category of finite spectra.

Given this, consider the Spanier-Whitehead dual Df of f. There is a spectrum V of type at least n and a map $k\colon DX \to V$ such that $k \circ Df$ is nonzero. Dualizing, we see that see that $f \circ Dk$ is nonzero, and DV also has type at least n. □

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Department of Mathematics, Wesleyan University, Middletown, CT 06459

E-mail address: hovey@member.ams.org