OPTIMAL KRONECKER PRODUCT APPROXIMATION OF BLOCK TOEPLITZ MATRICES

JULIE KAMM* AND JAMES G. NAGY†

Abstract. This paper considers the problem of finding $n \times n$ matrices A_k and B_k that minimize $||T - \sum A_k \otimes B_k||_F$, where $\otimes$ denotes Kronecker product, and T is a banded $n \times n$ block Toeplitz matrix with banded $n \times n$ Toeplitz blocks. It is shown that the optimal A_k and B_k are banded Toeplitz matrices, and an efficient algorithm for computing the approximation is provided. An image restoration problem from the Hubble Space Telescope is used to illustrate the effectiveness of an approximate SVD preconditioner constructed from the Kronecker product decomposition.

Key words. block Toeplitz matrix, conjugate gradient method, Kronecker product, image restoration, preconditioning, singular value decomposition

AMS subject classifications: 65F20, 65F30.

1. Introduction. A Toeplitz matrix is characterized by the property that its entries are constant on each diagonal. Toeplitz and block Toeplitz matrices arise naturally in many signal and image processing applications; see, for example, Bunch [4] and Jain [17] and the references therein. In image restoration [21], for instance, one needs to solve large, possibly ill-conditioned linear systems in which the coefficient matrix is a banded block Toeplitz matrix with banded Toeplitz blocks (BTTB).

Iterative algorithms, such as conjugate gradients (CG), are typically recommended for large BTTB systems. Matrix-vector multiplications can be done efficiently using fast Fourier transforms [14]. In addition, convergence can be accelerated by preconditioning with block circulant matrices with circulant blocks (BCCB). A circulant matrix is a Toeplitz matrix in which each column (row) can be obtained by a circular shift of the previous column (row), and a BCCB matrix is a natural extension of this structure to two dimensions; c.f. Davis [10].

Circulant and BCCB approximations are used extensively in signal and image processing applications, both in direct methods which solve problems in the "Fourier domain" [1, 17, 21], and as preconditioners [7]. The *optimal circulant preconditioner* introduced by Chan [8] finds the closest circulant matrix in the Frobenius norm. Chan and Olkin [9] extend this to the block case; that is, a BCCB matrix C is computed to minimize

$$||T - C||_F \,.$$

BCCB approximations work well for certain kinds of BTTB matrices [7], especially if the unknown solution is almost periodic. If this is not the case, however, the performance of BCCB preconditioners can degrade [20]. Moreover, Serra-Capizzano and Tyrtyshnikov [6] have shown recently that it may not be possible to construct a BCCB preconditioner that results in superlinear convergence of CG.

Here we consider an alternative approach: optimal Kronecker product approximations. A Kronecker product $A \otimes B$ is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}$$

* Raytheon Systems Company, Dallas, TX 75266 (email: j-kamm@ti.com)

† Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322 (nagy@mathcs.emory.edu).

In particular, we consider the problem of finding matrices A_k, B_k to minimize

$$||T - \sum_{k=1}^{s} A_k \otimes B_k||_F \,, \tag{1.1}$$

where T is an $n^2 \times n^2$ banded BTTB matrix, and A_k, B_k are $n \times n$ banded Toeplitz matrices. A general approach for constructing such an optimal approximation was proposed by Van Loan and Pitsianis [25] (see also Pitsianis [23]). Their approach, which we describe in more detail in Section 2, requires computing principal singular values and vectors of an $n^2 \times n^2$ matrix related to T.

An alternative approach for computing a Kronecker product approximation $T \approx A \otimes B$ for certain deconvolution problems was proposed by Thirumalai [24]. A similar approach for banded BTTB matrices was considered by Nagy [22]. As opposed to the method of Van Loan and Pitsianis, the schemes described in [22, 24] require computing principal singular values and vectors of an array having dimension at most $n \times n$, and thus can be substantially less expensive. Moreover, Kamm and Nagy [20] show how these approximations can be used to efficiently construct approximate SVD preconditioners.

Numerical examples in [20, 22, 24] indicate that this more efficient approach can lead to preconditioners that perform better than BCCB approximations. However, theoretical results establishing optimality of the approximations, such as in equation (1.1), were not given. In this paper, we provide these results. In particular, we show that some modifications to the method proposed in [22, 24] are needed to obtain an approximation of the form (1.1). Our theoretical results lead to an efficient algorithm for computing Kronecker product approximations of banded BTTB matrices.

This paper is organized as follows. Some notation is defined, and a brief review of the method proposed by Van Loan and Pitsianis is provided in Section 2. In Section 3 we show how to exploit the banded BTTB structure to obtain an efficient scheme for computing terms in the Kronecker product decomposition. A numerical example from image restoration is given in Section 4.

2. Preliminaries and Notation.

In this section we establish some notation to be used throughout the paper, and describe some previous work on Kronecker product approximations. To simplify notation, we assume T is an $n \times n$ block matrix with $n \times n$ blocks.

2.1. Banded BTTB Matrices.

We assume that the matrix T is a block banded Toeplitz matrix with banded Toeplitz blocks (BTTB), so it can be uniquely determined by a single column $\mathbf{t}$ which contains all of the non-zero values in T; that is, some central column. It will be useful to define an $n \times n$ array P as $\mathbf{t} = \text{vec}(P^T)$, where the **vec** operator transforms matrices into vectors by stacking columns as follows:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \quad \Leftrightarrow \quad \text{vec}(A) = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix} .$$

Suppose further that the entry of P corresponding to the diagonal of T is known[1]. For example, suppose that

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}, \tag{2.1}$$

where the diagonal of T is located at $(i, j) = (2, 3)$. Then $\mathbf{t} = \text{vec}(P^T)$ is the sixth column of T, and we write

$$T = \text{toep2}[\mathbf{t}, 2, 3] = \left[\begin{array}{ccc|ccc|ccc} p_{23} & p_{22} & p_{21} & p_{13} & p_{12} & p_{11} & 0 & 0 & 0 \\ 0 & p_{23} & p_{22} & 0 & p_{13} & p_{12} & 0 & 0 & 0 \\ 0 & 0 & p_{23} & 0 & 0 & p_{13} & 0 & 0 & 0 \\ \hline p_{33} & p_{32} & p_{31} & p_{23} & p_{22} & p_{21} & p_{13} & p_{12} & p_{11} \\ 0 & p_{33} & p_{32} & 0 & p_{23} & p_{22} & 0 & p_{13} & p_{12} \\ 0 & 0 & p_{33} & 0 & 0 & p_{23} & 0 & 0 & p_{13} \\ \hline 0 & 0 & 0 & p_{33} & p_{32} & p_{31} & p_{23} & p_{22} & p_{21} \\ 0 & 0 & 0 & 0 & p_{33} & p_{32} & 0 & p_{23} & p_{22} \\ 0 & 0 & 0 & 0 & 0 & p_{33} & 0 & 0 & p_{23} \end{array}\right]. \tag{2.2}$$

In general, if the diagonal of T is p_{ij}, then the upper and lower block bandwidths of T are $i-1$ and $n-i$, respectively. The upper and lower bandwidths of each Toeplitz block are $j-1$ and $n-j$, respectively.

In a similar manner, the notation $X = \text{toep}(\mathbf{x}, i)$ is used to represent a banded point Toeplitz matrix X constructed from the vector $\mathbf{x}$, where x_i corresponds to the diagonal entry. For example, if the second component of the vector $\mathbf{x} = [\, x_1 \; x_2 \; x_3 \; x_4 \,]^T$ corresponds to the diagonal element of a banded Toeplitz matrix X, then

$$X = \text{toep}(\mathbf{x}, 2) = \begin{bmatrix} x_2 & x_1 & 0 & 0 \\ x_3 & x_2 & x_1 & 0 \\ x_4 & x_3 & x_2 & x_1 \\ 0 & x_4 & x_3 & x_2 \end{bmatrix}.$$

2.2. Kronecker Product Approximations. In this subsection we review the work of Van Loan and Pitsianis. We require the following properties of Kronecker products:

- $(A \otimes B)^T = A^T \otimes B^T$,
- $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$,
- If U_1 and U_2 are orthogonal matrices, then $U_1 \otimes U_2$ is also orthogonal,
- $(A \otimes B)\mathbf{x} = \text{vec}(BXA^T)$, $\text{vec}(X) = \mathbf{x}$,

A more complete discussion and additional properties of Kronecker products can be found in Horn and Johnson [16] and Graham [13].

Van Loan and Pitsianis [25] (see also, Pitsianis [23]) propose a general technique for an approximation involving Kronecker products where $||T - \sum_k (A_k \otimes B_k)||_F$ is minimized. By defining the transformation to *tilde space* of a block matrix T,

$$T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix},$$

[1] In image restoration, P is often referred to as a "point spread function", and the diagonal entry is the location of the "point source". See Section 4 for more details.

as

$$\tilde{T} = \text{tilde}(T) = \begin{bmatrix} \text{vec}(T_{11})^T \\ \vdots \\ \text{vec}(T_{n1})^T \\ \vdots \\ \text{vec}(T_{1n})^T \\ \vdots \\ \text{vec}(T_{nn})^T \end{bmatrix},$$

it is shown in [23, 25] that

$$||T - \sum_{k=1}^{s}(A_k \otimes B_k)||_F = ||\tilde{T} - \sum_{k=1}^{s}(\tilde{\mathbf{a}}_k \tilde{\mathbf{b}}_k^T)||_F,$$

where $\tilde{\mathbf{a}}_k = \text{vec}(A_k)$ and $\tilde{\mathbf{b}}_k = \text{vec}(B_k)$. Thus, the Kronecker product approximation problem is reduced to a rank-s approximation problem. Given the SVD of $\tilde{T}$, $\tilde{T} = \sum_{k=1}^{r} \tilde{\sigma}_k \tilde{\mathbf{u}}_k \tilde{\mathbf{v}}_k^T$, $\text{rank}(\tilde{T}) = r$, it is well known [12] that the rank-s approximation $\tilde{T}_s$, $s \leq r$, which minimizes $||\tilde{T} - \tilde{T}_s||_F$ is $\tilde{T}_s = \sum_{k=1}^{s} \tilde{\sigma}_k \tilde{\mathbf{u}}_k \tilde{\mathbf{v}}_k^T$. Choosing $\tilde{\mathbf{a}}_k = \sqrt{\tilde{\sigma}_k}\tilde{\mathbf{u}}_k$, $\tilde{\mathbf{b}}_k = \sqrt{\tilde{\sigma}_k}\tilde{\mathbf{v}}_k$ minimizes $||\tilde{T} - \sum_{k=1}^{s} \tilde{\mathbf{a}}_k \tilde{\mathbf{b}}_k^T||_F$ over all rank-s approximations, and thus one can construct an approximation $\hat{T} = \sum_{k=1}^{s}(A_k \otimes B_k)$ which minimizes $||T - \hat{T}||_F$.

This general technique requires computing the largest s singular triplets of an $n^2 \times n^2$ matrix, which may be expensive for large n. Thirumalai [24] and Nagy [22] show that a Kronecker product approximation of a banded BTTB matrix T can be found by computing the largest s singular triplets of the $n \times n$ array P. However, this method does not find the Kronecker product which minimizes the Frobenius norm approximation problem in equation (1.1). In the next section we show that if T is a banded BTTB matrix, then this optimal approximation can be computed from an SVD of a weighted version of the $n \times n$ array P.

3. BTTB Optimal Kronecker Product Approximation.

Recall that the Van Loan and Pitsianis approach minimizes $||T - \sum_{k=1}^{s}(A_k \otimes B_k)||_F$ for a general (unstructured) matrix T, by minimizing $||\tilde{T} - \sum_{k=1}^{s}(\tilde{\mathbf{a}}_k \tilde{\mathbf{b}}_k^T)||_F$. If it is assumed that A_k and B_k are banded Toeplitz matrices, then the array P associated with the central column of T can be weighted and used to construct an approximation which minimizes $||\tilde{T} - \sum_{k=1}^{s}(\tilde{\mathbf{a}}_k \tilde{\mathbf{b}}_k^T)||_F$.

THEOREM 3.1. *Let T be the $n^2 \times n^2$ banded BTTB matrix constructed from P, where p_{ij} is the diagonal element of T (therefore, the upper and lower block bandwidths of T are $i-1$ and $n-i$, and the upper and lower bandwidths of each Toeplitz block are $j-1$ and $n-j$). Further, let A_k be an $n \times n$ banded Toeplitz matrix with upper bandwidth $i-1$ and lower bandwidth $n-i$, and let B_k be an $n \times n$ banded Toeplitz matrix with upper bandwidth $j-1$ and lower bandwidth $n-j$. Define $\mathbf{a}_k$ and $\mathbf{b}_k$ such that $A_k = toep(\mathbf{a}_k, i)$ and $B_k = toep(\mathbf{b}_k, j)$, and define*

$$\begin{aligned} \tilde{T} &= tilde(T), \\ \tilde{\mathbf{a}}_k &= vec(A_k), \\ \tilde{\mathbf{b}}_k &= vec(B_k), \\ W_a &= diag(\sqrt{n-i+1}, \sqrt{n-i+2}, \ldots, \sqrt{n-1}, \sqrt{n}, \sqrt{n-1}, \ldots, \sqrt{i+1}, \sqrt{i}), \end{aligned}$$

$$W_b = diag(\sqrt{n-j+1}, \sqrt{n-j+2}, \ldots, \sqrt{n-1}, \sqrt{n}, \sqrt{n-1}, \ldots, \sqrt{j+1}, \sqrt{j}),$$
$$P_w = W_a P W_b.$$

Then for $s \leq r = rank(P)$,

$$||\tilde{T} - \sum_{k=1}^{s} \tilde{\mathbf{a}}_k \tilde{\mathbf{b}}_k^T||_F = ||P_w - \sum_{k=1}^{s} (W_a \mathbf{a}_k)(W_b \mathbf{b}_k)^T||_F.$$

Proof. See Section 3.1. □

Therefore, if A_k and B_k are constrained to be banded Toeplitz matrices, then $||T - \sum_{k=1}^{s}(A_k \otimes B_k)||_F$ can be minimized by finding $\mathbf{a}_k$, $\mathbf{b}_k$ which minimize $||P_w - \sum_{k=1}^{s}(W_a\mathbf{a}_k)(W_b\mathbf{b}_k)^T||_F$. This is a rank-s approximation problem, involving a matrix of relatively small dimension, which can be constructed using the SVD of P_w. Noting that W_a and W_b are diagonal matrices which do not need to be formed explicitly, the construction of $\hat{T} = \sum_{k=1}^{s} A_k \otimes B_k$ which minimizes $||T - \hat{T}||_F$, where A_k and B_k are banded Toeplitz matrices, can be computed as follows:

- Define the weight vectors w_a and w_b based on the (i,j) location (in P) of the diagonal entry of T:

$$\mathbf{w}_a = \begin{bmatrix} \sqrt{n-i+1} & \cdots & \sqrt{n-1} & \sqrt{n} & \sqrt{n-1} & \cdots & \sqrt{i} \end{bmatrix}^T,$$
$$\mathbf{w}_b = \begin{bmatrix} \sqrt{n-j+1} & \cdots & \sqrt{n-1} & \sqrt{n} & \sqrt{n-1} & \cdots & \sqrt{j} \end{bmatrix}^T,$$

- Calculate $P_w = (\mathbf{w}_a \mathbf{w}_b^T). * P$ and its SVD $P_w = \sum_{k=1}^{r} \sigma_k \mathbf{u}_k \mathbf{v}_k^T$, where ".*" denotes point-wise multiplication.
- Calculate

$$\begin{aligned} \mathbf{a}_k &= (\sqrt{\sigma_k}\mathbf{u}_k)./\mathbf{w}_a, \\ A_k &= \text{toep}(\mathbf{a}_k, i), \\ \mathbf{b}_k &= (\sqrt{\sigma_k}\mathbf{v}_k)./\mathbf{w}_b, \\ B_k &= \text{toep}(\mathbf{b}_k, j), \end{aligned}$$

for $k = 1, \ldots, s$, $s \leq r$, where "./" denotes point-wise division.

The proof of Theorem 3.1 is based on observing that $\tilde{T}$ has at most n unique rows and n unique columns, which consist precisely of the rows and columns of P. This observation will become clear in the following subsection.

3.1. Proof of Theorem 3.1. To prove Theorem 3.1, we first observe that if a matrix has one row which is a scalar multiple of another row, then a rotator can be constructed to zero out one of these rows, i.e.,

$$Q \begin{bmatrix} \alpha \mathbf{x}^T \\ \mathbf{x}^T \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{\sqrt{\alpha^2+1}} & \frac{1}{\sqrt{\alpha^2+1}} \\ \frac{-1}{\sqrt{\alpha^2+1}} & \frac{\alpha}{\sqrt{\alpha^2+1}} \end{bmatrix} \begin{bmatrix} \alpha \mathbf{x}^T \\ \mathbf{x}^T \end{bmatrix} = \begin{bmatrix} \sqrt{\alpha^2+1}\mathbf{x}^T \\ \mathbf{0}^T \end{bmatrix}. \tag{3.1}$$

If this is extended to the case where more than two rows are repeated, then a simple induction proof can be used to establish the following lemma.

LEMMA 3.2. *Suppose an $n \times n$ matrix X has k identical rows:*

$$X = \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_1^T \\ \vdots \\ \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_{n-k+1}^T \end{bmatrix}.$$

Then a sequence of $k-1$ orthogonal plane rotators $Q_1, Q_2, \ldots, Q_{k-1}$ can be constructed such that

$$QX = Q_{k-1}Q_{k-2}\cdots Q_1 X = \begin{bmatrix} \sqrt{k}\mathbf{x}_1^T \\ \mathbf{0}^T \\ \vdots \\ \mathbf{0}^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_{n-k+1}^T \end{bmatrix},$$

thereby zeroing out all the duplicate rows.

It is easily seen that this result can be applied to the columns of a matrix as well, using the transpose of the plane rotators defined in Lemma 3.2.

LEMMA 3.3. *Suppose an $n \times n$ matrix X contains k identical columns:*

$$X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_1 & \cdots & \mathbf{x}_1 & \mathbf{x}_2 & \cdots \mathbf{x}_{n-k+1} \end{bmatrix}.$$

Then an orthogonal matrix Q can be constructed from a series of plane rotators such that

$$XQ^T = \begin{bmatrix} \sqrt{k}\mathbf{x}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{x}_2 & \cdots & \mathbf{x}_{n-k+1} \end{bmatrix}.$$

The above results illustrate the case where the first occurrence of a row (column) is modified to zero out the remaining occurrences. However, this is for notational convenience only. By appropriately constructing the plane rotators, any one of the duplicate rows (columns) may be selected for modification, and the remaining rows (columns) zeroed out. These rotators can now be applied to the matrix $\tilde{T}$.

LEMMA 3.4. *Let T be the $n^2 \times n^2$ banded* BTTB *matrix constructed from P, where p_{ij} is the diagonal entry of T. In other words, $T = toep2[vec(P^T), i, j]$. Further, define*

$$\begin{aligned} \tilde{T} &= tilde(T), \\ W_a &= diag(\sqrt{n-i+1}, \sqrt{n-i+2}, \ldots, \sqrt{n-1}, \sqrt{n}, \sqrt{n-1}, \ldots, \sqrt{i+1}, \sqrt{i}), \\ W_b &= diag(\sqrt{n-j+1}, \sqrt{n-j+2}, \ldots, \sqrt{n-1}, \sqrt{n}, \sqrt{n-1}, \ldots, \sqrt{j+1}, \sqrt{j}). \end{aligned}$$

Then orthogonal matrices Q_1 and Q_2 can be constructed such that

$$Q_1 \tilde{T} Q_2^T = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & W_a P W_b & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Proof. By definition,

$$T = \begin{bmatrix} T_i & & T_1 & & 0 \\ & \ddots & & \ddots & \\ T_n & & T_i & & T_1 \\ & \ddots & & \ddots & \\ 0 & & T_n & & T_i \end{bmatrix}.$$

Defining $\tilde{\mathbf{t}}_i^T = \text{vec}(T_i)^T$, and representing $\tilde{T}$ using the $n \times n^2$ submatrices $\tilde{T}_i$,

$$\tilde{T} = \begin{bmatrix} \tilde{T}_1 \\ \vdots \\ \tilde{T}_i \\ \vdots \\ \tilde{T}_n \end{bmatrix},$$

it is clear that $\tilde{T}$ contains only n unique rows, which are $\tilde{\mathbf{t}}_1^T, \ldots, \tilde{\mathbf{t}}_n^T$, and that the i^{th} submatrix, $\tilde{T}_i$ contains all the unique rows, i.e.,

$$\tilde{T}_i = \begin{bmatrix} \tilde{\mathbf{t}}_1^T \\ \tilde{\mathbf{t}}_2^T \\ \vdots \\ \tilde{\mathbf{t}}_n^T \end{bmatrix}.$$

Furthermore, it can be seen that there are $n-i+1$ occurrences of $\tilde{\mathbf{t}}_1^T$, ..., $n-1$ occurrences of $\tilde{\mathbf{t}}_{i-1}^T$, n occurrences of $\tilde{\mathbf{t}}_i^T$, $n-1$ occurrences of $\tilde{\mathbf{t}}_{i+1}^T$, ..., and i occurrences of $\tilde{\mathbf{t}}_n^T$. Therefore, a sequence of orthogonal plane rotators can be constructed to zero out all rows of $\tilde{T}$ except those in the submatrix $\tilde{T}_i$, i.e.,

$$Q_1\tilde{T} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ W_a\tilde{T}_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{0}^T \\ \vdots \\ \mathbf{0}^T \\ \sqrt{n-i+1}\,\tilde{\mathbf{t}}_1^T \\ \vdots \\ \sqrt{n-1}\,\tilde{\mathbf{t}}_{i-1}^T \\ \sqrt{n}\,\tilde{\mathbf{t}}_i^T \\ \sqrt{n-1}\,\tilde{\mathbf{t}}_{i+1}^T \\ \vdots \\ \sqrt{i}\,\tilde{\mathbf{t}}_n^T \\ \mathbf{0}^T \\ \vdots \\ \mathbf{0}^T \end{bmatrix}.$$

Now, partitioning $\tilde{T}_i$,

$$\tilde{T}_i = \begin{bmatrix} \tilde{T}_{i1} & \cdots & \tilde{T}_{ij} & \cdots & \tilde{T}_{in} \end{bmatrix},$$

where each $\tilde{T}_{ij}$ is an $n \times n$ submatrix, it can be seen that $\tilde{T}_i$ contains only n unique columns, which are the columns of P, $\mathbf{p}_1, \ldots, \mathbf{p}_n$, and that the j^{th} submatrix $\tilde{T}_{ij}$ contains all the unique columns, i.e.,

$$\tilde{T}_{ij} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix} = P.$$

Furthermore, the matrix $\tilde{T}_i$ contains $n-j+1$ occurrences of $\mathbf{p}_1$, …, $n-1$ occurrences of $\mathbf{p}_{j-1}$, n occurrences of $\mathbf{p}_j$, $n-1$ occurrences of $\mathbf{p}_{j+1}$, …, and j occurrences of $\mathbf{p}_n$. Therefore, a sequence of orthogonal plane rotators can be constructed such that

$$Q_1\tilde{T}Q_2^T = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & W_aPW_b & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

□

The following properties involving the **vec** and **toep2** operators are needed.

LEMMA 3.5. *Let T, $\tilde{T}$, and P be defined as in Lemma 3.4. Further, let A_k be an $n \times n$ banded Toeplitz matrix with upper bandwidth $i-1$ and lower bandwidth $n-i$, and let B_k be an $n \times n$ banded Toeplitz matrix with upper bandwidth $j-1$ and lower bandwidth $n-j$. Define $\mathbf{a}_k$ and $\mathbf{b}_k$ such that $A_k = toep(\mathbf{a}_k, i)$ and $B_k = toep(\mathbf{b}_k, j)$. Then*

1. *$vec(X) - vec(Y) = vec(X - Y)$, where X and Y are any two matrices of the same size,*

2. *toep2*$(\mathbf{x}, i, j) -$ *toep2*$(\mathbf{y}, i, j) =$ *toep2*$(\mathbf{x} - \mathbf{y}, i, j)$, *where* $\mathbf{x}$ *and* $\mathbf{y}$ *are any two vectors of the same length,*
3. *toep2*$\{$*vec*$[(\sum_{k=1}^{s} \mathbf{a}_k \mathbf{b}_k^T)^T], i, j\} = \sum_{k=1}^{s} A_k \otimes B_k$, *and*
4. *toep2*$\{$*vec*$[(P - \sum_{k=1}^{s} \mathbf{a}_k \mathbf{b}_k^T)^T], i, j\} = T - \sum_{k=1}^{s} A_k \otimes B_k$.

Proof. Properties 1 and 2 are clear from the definitions of the vec and toep2 operators. Property 3 can be seen by considering the banded Toeplitz matrices $A = \text{toep}(\mathbf{a}, i)$ and $B = \text{toep}(\mathbf{b}, j)$ and noting that the central column of $A \otimes B$ containing all the non-zero entries is

$$\text{vec}[(\mathbf{a}\mathbf{b}^T)^T] = \begin{bmatrix} a_1 b_1 \\ \vdots \\ a_1 b_n \\ \vdots \\ a_n b_1 \\ \vdots \\ a_n b_n \end{bmatrix}.$$

Therefore, property 3 holds when $k = 1$ since both sides are banded BTTB matrices constructed from the same central column, and can be extended to $k = 1, \ldots, s$ by applying property 2. Property 4 follows from properties 2 and 3. □

Using these properties, Lemma 3.4 can be extended to the matrix $\tilde{T} - \sum_k \tilde{\mathbf{a}}_k \tilde{\mathbf{b}}_k^T$.

LEMMA 3.6. *Let* T *be the* $n^2 \times n^2$ *banded* BTTB *matrix constructed from* P, *where* p_{ij} *is the diagonal entry of* T. *Further, let* A_k *be an* $n \times n$ *banded Toeplitz matrix with upper bandwidth* $i - 1$ *and lower bandwidth* $n - i$, *and let* B_k *be an* $n \times n$ *banded Toeplitz matrix with upper bandwidth* $j - 1$ *and* $n - j$. *Define* $\mathbf{a}_k$ *and* $\mathbf{b}_k$ *such that* $A_k =$ *toep*$(\mathbf{a}_k, i)$ *and* $B_k =$ *toep*$(\mathbf{b}_k, j)$, *and define* $\tilde{\mathbf{a}}_k =$ *vec*(A_k) *and* $\tilde{\mathbf{b}}_k =$ *vec*(B_k). *Let* $\tilde{T}$, W_a, *and* W_b *be defined as in Lemma 3.4. Then orthogonal matrices* Q_1 *and* Q_2 *can be constructed such that*

$$Q_1(\tilde{T} - \sum_{k=1}^{s} \tilde{\mathbf{a}}_k \tilde{\mathbf{b}}_k^T) Q_2^T = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & W_a(P - \sum_{k=1}^{s} \mathbf{a}_k \mathbf{b}_k^T) W_b & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Proof. Using Lemma 3.5,

$$T - \sum_{k=1}^{s} A_k \otimes B_k = \text{toep2}\{\text{vec}[(P - \sum_{k=1}^{s} \mathbf{a}_k \mathbf{b}_k^T)^T], i, j\}.$$

By definition of the transformation to tilde space,

$$\text{tilde}(T - \sum_{k=1}^{s} A_k \otimes B_k) = \tilde{T} - \sum_{k=1}^{s} \tilde{\mathbf{a}}_k \tilde{\mathbf{b}}_k^T.$$

Applying Lemma 3.4 to $T - \sum_{k=1}^{s} A_k \otimes B_k$ yields

$$Q_1(\tilde{T} - \sum_{k=1}^{s} \tilde{\mathbf{a}}_k \tilde{\mathbf{b}}_k^T) Q_2^T = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & W_a(P - \sum_{k=1}^{s} \mathbf{a}_k \mathbf{b}_k^T) W_b & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

□

The proof of Theorem 3.1 follows directly from Lemma 3.6 by noting that

$$\begin{aligned} ||\tilde{T} - \sum_{k=1}^{s} \tilde{\mathbf{a}}_k \tilde{\mathbf{b}}_k^T||_F &= ||Q_1(\tilde{T} - \sum_{k=1}^{s} \tilde{\mathbf{a}}_k \tilde{\mathbf{b}}_k^T) Q_2^T||_F \\ &= ||W_a(P - \sum_{k=1}^{s} \mathbf{a}_k \mathbf{b}_k^T) W_b||_F \\ &= ||P_w - \sum_{k=1}^{s} (W_a \mathbf{a}_k)(W_b \mathbf{b}_k)^T||_F. \end{aligned}$$

3.2. Further Analysis. It has been shown how to minimize $||T - \hat{T}||_F$ when the structure of $\hat{T}$ is constrained to be a sum of Kronecker products of banded Toeplitz matrices. We now show that if T is a banded BTTB matrix, then the matrix $\hat{T} = \sum_i A_i \otimes B_i$ minimizing $||T - \hat{T}||_F$ must adhere to this structure. Therefore, the approximation minimizes $||T - \hat{T}||_F$ over all matrices $\hat{T} = \sum_i A_i \otimes B_i$ when T is a banded BTTB matrix.

If T is a banded BTTB matrix, then the rows and columns of $\tilde{T}$ have a particular structure. To represent this structure, using an approach similar to Van Loan and Pitsianis [25], we define the constraint matrix $S_{n,\omega}$. Given an $n \times n$ banded Toeplitz matrix T, with upper and lower bandwidths $\omega = \begin{bmatrix} \omega_u, & \omega_l \end{bmatrix}$, $S_{n,\omega}$ is an $n^2 \times (n^2 - (\omega_u + \omega_l + 1))$ $\{-1, 0, 1\}$ matrix such that $S_{n,\omega}^T \text{vec}(T) = 0$. For example, let T be a 4×4 banded Toeplitz matrix with bandwidths $\omega_u = 2$ and $\omega_l = 1$. Then

$$T = \begin{bmatrix} t_2 & t_1 & t_0 & 0 \\ t_3 & t_2 & t_1 & t_0 \\ 0 & t_3 & t_2 & t_1 \\ 0 & 0 & t_3 & t_2 \end{bmatrix},$$

and

$$S_{4,[2,1]}^T = \left[\begin{array}{cccc|cccc|cccc|cccc}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right].$$

Note that $S_{n,\omega}^T$ clearly has full row rank. Given the matrix T in (2.2),

$$\tilde{T} = \left[\begin{array}{ccc|ccc|ccc}
p_{22} & p_{23} & 0 & p_{21} & p_{22} & p_{23} & 0 & p_{21} & p_{22} \\
p_{32} & p_{33} & 0 & p_{31} & p_{32} & p_{33} & 0 & p_{31} & p_{32} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
p_{12} & p_{13} & 0 & p_{11} & p_{12} & p_{13} & 0 & p_{11} & p_{12} \\
p_{22} & p_{23} & 0 & p_{21} & p_{22} & p_{23} & 0 & p_{21} & p_{22} \\
p_{32} & p_{33} & 0 & p_{31} & p_{32} & p_{33} & 0 & p_{31} & p_{32} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{12} & p_{13} & 0 & p_{11} & p_{12} & p_{13} & 0 & p_{11} & p_{12} \\
p_{22} & p_{23} & 0 & p_{21} & p_{22} & p_{23} & 0 & p_{21} & p_{22}
\end{array}\right],$$

$$S_{3,[1,1]}^T = \left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right],$$

and the rows and columns of $\tilde{T}$ satisfy

$$S_{3,[1,1]}^T \tilde{T}(:,i) = 0,$$
$$S_{3,[1,1]}^T \tilde{T}(i,:)^T = 0,$$

for $i = 1, \ldots, n^2$. Using the structure of $\tilde{T}$, the matrix $\hat{T} = \sum_{i=1}^k A_i \otimes B_i$ minimizing $||T - \hat{T}||_F$ must be structured such that A_i and B_i are banded Toeplitz matrices, as the following sequence of results illustrate.

Lemma 3.7. *Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ be the $n \times n$ matrix whose structure is constrained by $S_{n,\omega}^T \mathbf{a}_i = 0$, $\mathbf{a}_i \neq 0$, for $i = 1, \ldots, n$. Further, let $A = \sum_{i=1}^r \sigma_i \mathbf{u}_i \mathbf{v}_i^T$ be the* SVD *of A, where $r = rank(A)$. Then $\mathbf{u}_i$ satisfies $S_{n,\omega}^T \mathbf{u}_i = 0$, for $i = 1, \ldots, r$.*

Proof. Given the SVD of A,

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i,$$

for $i = 1, \ldots, n$, and subsequently

$$S_{n,\omega}^T A \mathbf{v}_i = \sigma_i S_{n,\omega}^T \mathbf{u}_i.$$

By definition, $S_{n,\omega}^T A = 0$ and $\sigma_i > 0$ for $i = 1, \ldots, r$. Therefore, $S_{n,\omega}^T \mathbf{u}_i = 0$ for $i = 1, \ldots, r$. ☐

Applying this result to A^T, it is clear that the right singular vectors of A satisfy $S_{n,\omega}^T \mathbf{v}_i = 0$, for $i = 1, \ldots, r$ if the rows of A are structured in the same manner.

LEMMA 3.8. *Let* $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix}$ *be the* $n \times n$ *matrix whose structure is constrained by* $S_{n,\omega}^T \mathbf{a}_i = 0$, *for* $i = 1, \ldots, n$. *Further, let* $A = \sum_{i=1}^r \sigma_i \mathbf{u}_i \mathbf{v}_i^T$ *be the* SVD *of* A, *where* $r = rank(A)$. *Then* $\mathbf{v}_i$ *satisfies* $S_{n,\omega}^T \mathbf{v}_i = 0$, *for* $i = 1, \ldots, r$.

THEOREM 3.9. *Let* T *be an* $n \times n$ *banded block Toeplitz matrix with* $n \times n$ *banded Toeplitz blocks, where the upper and lower block bandwidths of* T *are* $\omega = \begin{bmatrix} \omega_u & \omega_l \end{bmatrix}$, *and the upper and lower bandwidths of each Toeplitz block are* $\gamma = \begin{bmatrix} \gamma_u & \gamma_l \end{bmatrix}$. *Then the matrices* A_i *and* B_i *minimizing*

$$||T - \sum_{i=1}^{k} (A_i \otimes B_i)||_F,$$

for $k \leq n$, *are* $n \times n$ *banded Toeplitz matrices, where the upper and lower bandwidths of* A_i *are given by* ω, *and the upper and lower bandwidths of* B_i *are given by* γ.

Proof. Recall that

$$||T - \sum_{i=1}^{k} (A_i \otimes B_i)||_F = ||\tilde{T} - \sum_{i=1}^{k} (\tilde{\mathbf{a}}_i \tilde{\mathbf{b}}_i^T)||_F,$$

where $\text{vec}(A_i) = \tilde{\mathbf{a}}_i$ and $\text{vec}(B_i) = \tilde{\mathbf{b}}_i$. The structure of T results in $\text{rank}(\tilde{T}) = r \leq n$ and $S_{n,\omega}^T \tilde{T}(:, i) = S_{n,\gamma}^T \tilde{T}(i, :)^T = 0$, for $i = 1, \ldots, n^2$. Letting $\tilde{T} = \sum_{i=1}^r \tilde{\sigma}_i \tilde{\mathbf{u}}_i \tilde{\mathbf{v}}_i^T$ be the SVD of $\tilde{T}$, $||\tilde{T} - \sum_{i=1}^k (\tilde{\mathbf{a}}_i \tilde{\mathbf{b}}_i^T)||_F$, $k \leq r$, is minimized by $\tilde{\mathbf{a}}_i = \sqrt{\tilde{\sigma}_i} \tilde{\mathbf{u}}_i$ and $\tilde{\mathbf{b}}_i = \sqrt{\tilde{\sigma}_i} \tilde{\mathbf{v}}_i$, where $S_{n,\omega}^T \tilde{\mathbf{u}}_i = S_{n,\gamma}^T \tilde{\mathbf{v}}_i = 0$. Therefore, A_i is an $n \times n$ banded Toeplitz matrix with upper and lower bandwidths given by ω, and B_i is an $n \times n$ banded Toeplitz matrix with upper and lower bandwidths given by γ. ☐

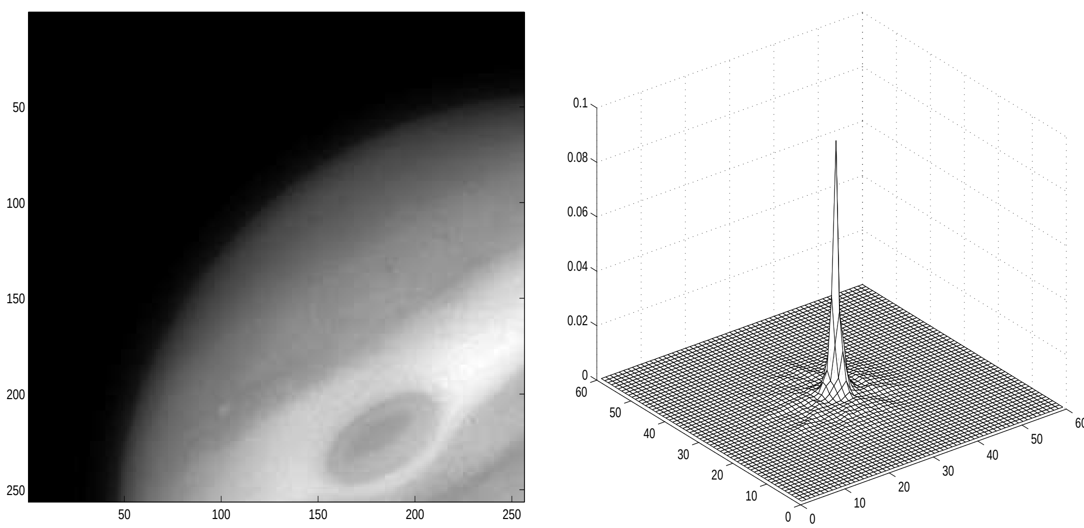
3.3. Remarks on Optimality. The approach outlined in this section results in an optimal Frobenius norm Kronecker product approximation to a banded BTTB matrix. The approximation is obtained from the principal singular components of an array $P_w = W_a P W_b$. It might be interesting to consider whether it is possible to compute approximations which are optimal in another norm. In particular, the method considered in [20, 22, 24] uses a Kronecker product approximation computed from the principal singular components of P. Unfortunately we are unable to show that this leads to an optimal norm approximation. However, there is a very close relationship between the approaches. Since W_a and W_b are full rank, well-conditioned diagonal matrices, P and P_w have the same rank. Although it is possible to establish bounds on the singular values of products of matrices (see, for example, Horn and Johnson [15]), we have not been able to determine a precise relationship between the Kronecker product approximations obtained from the two methods. However we have found through extensive numerical results that both methods give similarly good approximations. Since numerical comparisons do not provide any additional insight into the quality of the approximation, we omit such results. Instead, in the next section we provide an example from an application that motivated this work, and illustrate how a Kronecker product approximation might be used in practice. We note that further comparisons with BCCB approximations can be found in [20, 24].

4. An Image Restoration Example. In this section we consider an image restoration example, and show how the Kronecker product approximations can be used to construct an approximate SVD preconditioner. Image restoration is often modeled as a linear system:

$$\mathbf{b} = T\mathbf{x} + \mathbf{n}\,,$$

where $\mathbf{b}$ is an observed blurred, noisy image, T is a large, often ill-conditioned matrix representing the blurring phenomena, $\mathbf{n}$ is noise, and $\mathbf{x}$ is the desired true image. If the blur is assumed to be *spatially invariant*, then T is a banded BTTB matrix [1, 21]. In this case, the array P corresponding to a central column of T is called a *point spread function* (PSF).

The test data we use consists of a partial image of Jupiter taken from the Hubble Space Telescope (HST) in 1992, before the mirrors in the Wide Field Planetary Camera were fixed. The data was obtained via anonymous ftp from **`ftp.stsci.edu`**, in the directory **`pub/stsdas/testdata/restore/data/jupiter`**. Figure 4.1 shows the observed image. Also shown in Figure 4.1 is a mesh plot of the PSF, P, where the peak corresponds to the diagonal entry of T. The observed image is 256×256, so T is $65,536 \times 65,536$.



a. Observed, blurred image. b. PSF, P.

FIG. 4.1. *Observed* HST *image and point spread function.*

We mention that if T is ill-conditioned, which is often the case in image restoration, then regularization is needed to suppress noise amplification in the computed solution [21]. Although T is essentially too large to compute its condition number, certain properties of the data indicate that T is fairly well conditioned. For instance, we observe that the PSF is not very smooth (smoother PSFs typically indicate more ill-conditioned T). Another indication comes from the fact that the optimal circulant approximation of T, as well as our approximate SVD of T (to be described below) are well conditioned; specifically these approximations have condition numbers that are approximately 20.

We also mention that if the PSF can be expressed as $P = \sigma \mathbf{u}\mathbf{v}^T$ (i.e., it has rank 1), then the matrix T is separable. Using Theorem 3.1, $T = A \otimes B$, where $A = \text{toep}(\sqrt{\sigma}\mathbf{u})$ and $B = \text{toep}(\sqrt{\sigma}\mathbf{v})$. Efficient numerical methods that exploit the Kronecker product structure of T (*e.g.*, [2, 5, 11]) can then be used.

However, as can be seen from the plot of the singular values of P in Figure 4.2, for this data, P is not rank one, and so T is not separable. We therefore suggest con-

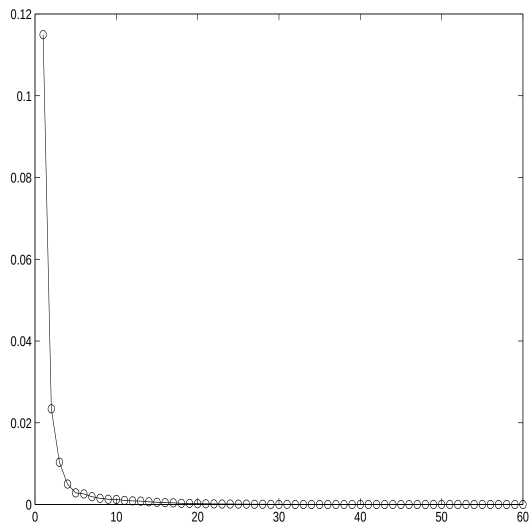


FIG. 4.2. *Singular values of the* PSF*, P.*

structing an approximate SVD to use as a preconditioner, and solve the least squares problem $T\mathbf{x} \approx \mathbf{b}$ using a conjugate gradient algorithm, such as CGLS; see Björck [3]. This preconditioning idea was proposed in [20], and can be described as follows. Given

$$T \approx \sum_{k=1}^{s} A_k \otimes B_k, \tag{4.1}$$

an SVD approximation of T can be constructed as

$$\begin{aligned} T &\approx U\Sigma V^T, \\ U &= U_A \otimes U_B, \\ V &= V_A \otimes V_B, \\ \Sigma &= \text{diag}(U^T T V) \\ &= \text{diag}(U^T(A_1 \otimes B_1 + A_2 \otimes B_2 + \cdots + A_k \otimes B_k)V), \end{aligned}$$

where $A_1 = U_A \Sigma_A V_A^T$ and $B_1 = U_B \Sigma_B V_B^T$. Note that the number of terms s only affects the setup cost of calculating Σ. For $s \geq 1$, $\Sigma = \text{diag}(U^T T V)$ clearly solves the minimization problem

$$\min_{\Sigma} ||\Sigma - U^T T V||_F = \min_{\Sigma} ||U\Sigma V^T - T||_F,$$

over all diagonal matrices Σ and therefore produces an optimal SVD approximation, given a fixed $U = U_A \otimes U_B$ and $V = V_A \otimes V_B$. This is analogous to the circulant and BCCB approximations discussed earlier, which provide an optimal eigendecomposition given a fixed set of eigenvectors (i.e., the Fourier vectors).

In our tests, we use CGLS to solve the LS problem $T\mathbf{x} \approx \mathbf{b}$ using no preconditioner, our approximate SVD preconditioner (with $s = 3$ terms in equation (4.1)) and the optimal circulant preconditioner. Although we observed that T is fairly well conditioned, we should still be cautious about noise corrupting the computed restorations. Therefore, we use the conservative stopping tolerance $||T^T\mathbf{b} - T^TT\mathbf{x}||_2/||T^T\mathbf{b}||_2 < 10^{-4}$.

Table 4.1 shows the number of iterations needed for convergence in each case, and in Figure 4.3 we plot the corresponding residuals at each iteration. The computed solutions are shown in Figure 4.4, along with the HST observed, blurred image for comparison.

TABLE 4.1
Number of CGLS *and* PCGLS *iterations needed for convergence.*

CGLS, no prec.	PCGLS, circulant prec.	PCGLS, svd prec.
43	12	4

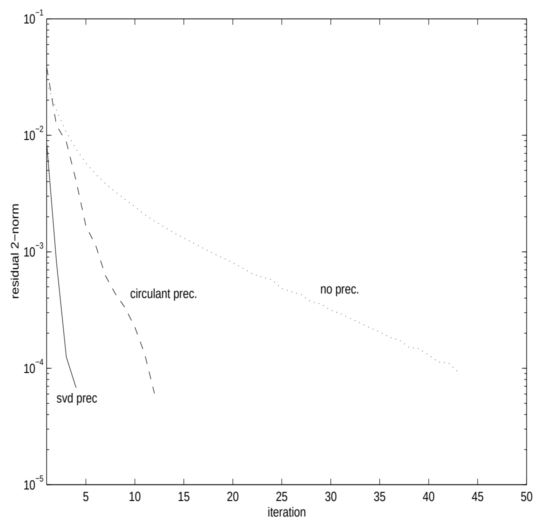
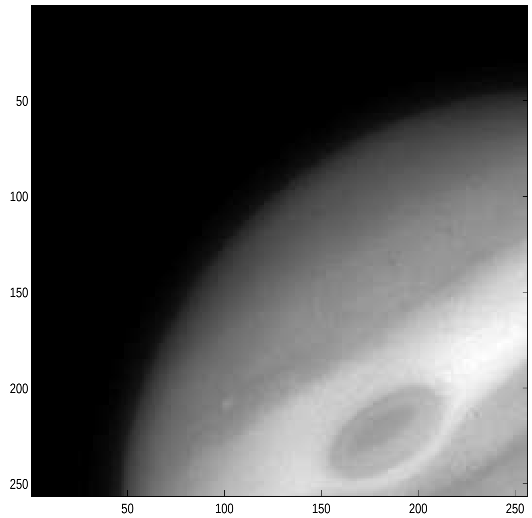
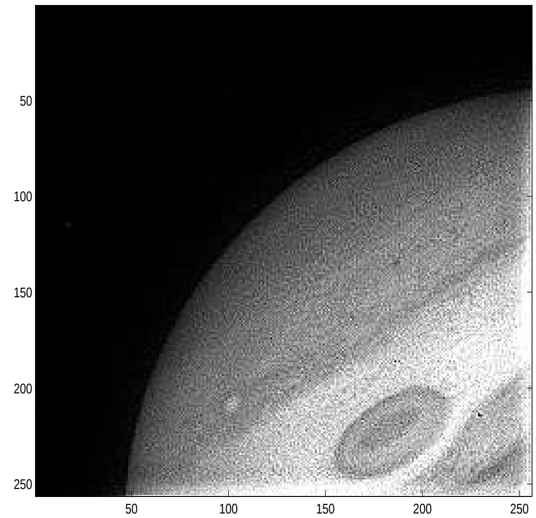


FIG. 4.3. *Plot of the residuals at each iteration.*

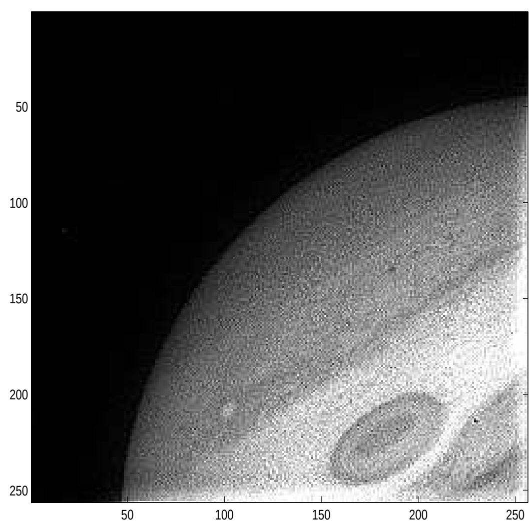
5. Concluding Remarks. Because the image and PSF used in the previous section come from actual HST data, we cannot get an analytical measure on the accuracy of the computed solutions. However, we observe from Figure 4.4 that all solutions appear to be equally good restorations of the image, and from Figure 4.3 we see that the approximate SVD preconditioner is effective at reducing the number of iterations needed to obtain the solutions. Additional numerical examples comparing the accuracy of computed solutions, as well as computational cost of BCCB and the approximation SVD preconditioner, can be found in [19, 20]. A comparison of



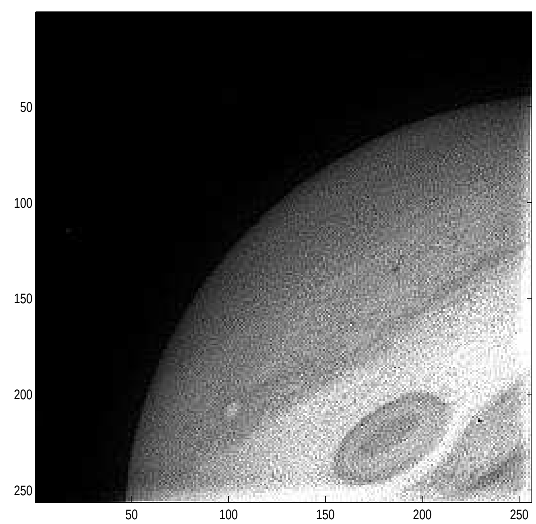
a. HST blurred image.



b. CGLS solution, 43 iterations.



c. PCGLS solution, circ. prec., 12 its.



d. PCGLS solution, svd prec., 4 its.

FIG. 4.4. *The observed image, along with computed solutions from* CGLS *and* PCGLS.

computational complexity between BCCB preconditioners and the approximate SVD preconditioner depends on many factors. For example:

- What is the dimension of P (*i.e.,* the bandwidths of T)?
- Is a Lanczos scheme used to compute SVDs of P, A_1 and B_1?
- Do we take advantage of band and Toeplitz structure when forming matrix-matrix products involving U_A, U_B, V_A, V_B and A_k, B_k, $k = 2, \cdots, s$?

- How many terms, s, do we take in the Kronecker product approximation?
- For BCCB preconditioners: is n a power of 2?

If we assume T is $n^2 \times n^2$, and $s = O(1)$, then set up and application of the approximate SVD preconditioner is *at most* $O(n^3)$. If we further assume that n is a power of 2, then the corresponding cost for BCCB preconditioners is *at least* $O(n^2 \log_2 n)$. It should be noted that the approximate SVD preconditioner does not require complex arithmetic, does not require n to be a power of 2, or any zero padding. Moreover, decomposing T into a sum of Kronecker products, whose terms are banded Toeplitz matrices, might lead to other fast algorithms (as has occurred over many years of studying displacement structure [18]). In this case, the work presented in this paper provides an algorithm for efficiently computing an optimal Kronecker product approximation.

REFERENCES

[1] H. ANDREWS AND B. HUNT, *Digital Image Restoration*, Prentice-Hall, Englewood Cliffs, NJ, 1977.

[2] E. S. ANGEL AND A. K. JAIN, *Restoration of images degraded by spatially varying pointspread functions by a conjugate gradient method*, Applied Optics, 17 (1978), pp. 2186–2190.

[3] Å. BJÖRCK, *Numerical Methods for Least Squares Problems*, SIAM, Philadelphia, PA, 1996.

[4] J. R. BUNCH, *Stability of methods for solving Toeplitz systems of equations*, SIAM J. Sci. Stat. Comp., 6 (1985), pp. 349–364.

[5] D. CALVETTI AND L. REICHEL, *Application of ADI iterative methods to the image restoration of noisy images*, SIAM J. Matrix Anal. Appl., 17 (1996), pp. 165–174.

[6] S. S. CAPIZZANO AND E. TYRTYSHNIKOV, *Any circulant-like preconditioner for multilevel Toeplitz matrices is not superlinear*, SIAM J. Matrix Anal. Appl., 21-2 (1999), pp. 431–439.

[7] R. H. CHAN AND M. K. NG, *Conjugate gradient methods for Toeplitz systems*, SIAM Review, 38 (1996), pp. 427–482.

[8] T. F. CHAN, *An optimal circulant preconditioner for Toeplitz systems*, SIAM J. Sci. Stat. Comp., 9 (1988), pp. 766–771.

[9] T. F. CHAN AND J. A. OLKIN, *Preconditioners for Toeplitz-block matrices*, Numer. Algo., 6 (1993), pp. 89–101.

[10] P. J. DAVIS, *Circulant Matrices*, Wiley, New York, 1979.

[11] L. ELDÉN AND I. SKOGLUND, *Algorithms for the regularization of ill-conditioned least squares problems with tensor product structure, and application to space-variant image restoration*, Tech. Rep. LiTH-MAT-R-82-48, Department of Mathematics, Linköping University, Sweden, 1982.

[12] G. H. GOLUB AND C. V. LOAN, *Matrix Computations, third edition*, Johns Hopkins Press, 1996.

[13] A. GRAHAM, *Kronecker Products and Matrix Calculus: with Applications*, Halsted Press, John Wiley and Sons, Inc., New York, 1981.

[14] M. HANKE AND J. G. NAGY, *Restoration of atmospherically blurred images by symmetric indefinite conjugate gradient techniques*, Inverse Problems, 12 (1996), pp. 157–173.

[15] R. A. HORN AND C. A. JOHNSON, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[16] ———, *Topics in Matrix Analysis*, Cambridge University Press, New York, 1991.

[17] A. K. JAIN, *Fundamentals of Digital Image Processing*, Prentice-Hall, Englewood Cliffs, NJ, 1989.

[18] T. KAILATH AND A. H. SAYED, *Displacement structure: Theory and applications*, SIAM Review, 37 (1995), pp. 297–386.

[19] J. KAMM, *Singular value decomposition-based methods for signal and image restoration*, PhD thesis, Southern Methodist University, Dallas, TX, 1998.

[20] J. KAMM AND J. G. NAGY, *Kronecker product and SVD approximations in image restoration*, Linear Algebra Appl., 284 (1998), pp. 177–192.

[21] R. L. LAGENDIJK AND J. BIEMOND, *Iterative Identification and Restoration of Images*, Kluwer Academic Publishers, Boston/Dordrecht/London, 1991.

[22] J. G. NAGY, *Decomposition of block Toeplitz matrices into a sum of Kronecker products with applications in image restoration*, Tech. Rep. 96-1, Department of Mathematics, Southern Methodist University, 1996.

[23] N. P. PITSIANIS, *The Kronecker Product in Approximation and Fast Transform Generation*, PhD thesis, Cornell University, Ithaca, NY, 1997.

[24] S. Thirumalai, *High performance algorithms to solve Toeplitz and block Toeplitz matrices*, PhD thesis, University of Illinois, Urbana, IL, 1996.

[25] C. F. Van Loan and N. P. Pitsianis, *Approximation with Kronecker products*, in Linear Algebra for Large Scale and Real Time Applications, M. S. Moonen and G. H. Golub, eds., Kluwer Publications, 1993, pp. 293–314.