Stability of a three-station fluid network *

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This paper studies the stability of a three-station fluid network. We show that, unlike the two-station networks in Dai and Vande Vate [18], the global stability region of our three-station network is *not* the intersection of its stability regions under static buffer priority disciplines. Thus, the "worst" or extremal disciplines are not static buffer priority disciplines. We also prove that the global stability region of our three-station network is not monotone in the service times and so, we may move a service time vector out of the global stability region by reducing the service time for a class. We introduce the *monotone* global stability region and show that a linear program (LP) related to a piecewise linear Lyapunov function characterizes this largest monotone subset of the global stability region for our three-station network. We also show that the LP proposed by Bertsimas et al. [1] does not characterize either the global stability region or even the monotone global stability region of our three-station network. Further, we demonstrate that the LP related to the linear Lyapunov function proposed by Chen and Zhang [11] does not characterize the stability region of our three-station network under a static buffer priority discipline.

Keywords: stability, fluid models, multiclass queueing networks, piecewise linear Lyapunov functions, linear Lyapunov functions, monotone global stability, static buffer priority disciplines

1. Introduction

Dai [13] introduced the notion of global stability in fluid networks and characterized the global stability regions for certain two-station re-entrant examples. A fluid network that is stable under all non-idling (work-conserving) service disciplines is said

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to be globally stable and the set of service times and arrival rates under which it is globally stable is called the global stability region. Determining the global stability region is especially important when it is difficult or impossible to implement a well-studied service discipline. In such a system, it is possible for servers to unwittingly employ a discipline under which the system is unstable even though the traffic intensity at each station is less than one. Although it is sometimes difficult to avoid such bad disciplines, we can avoid their consequences by maintaining service times that are in the global stability region. In this way, we can ensure that even under bad disciplines, the system will remain stable.

Bertsimas et al. [1] showed that, for two-station fluid networks, a linear program (LP) characterizes the global stability region. The LP characterization offers a computational test of global stability for two-station fluid networks with specified service times and arrival rates. In a recent series of papers, Dai and Vande Vate [16–18] characterized the global stability region of two-station fluid networks via a set of linear and quadratic constraints on the service times and exogenous arrival rates. These constraints generalize the usual traffic conditions and are explained by two intuitive phenomena, push starting and virtual stations.

These papers showed that, for two-station fluid networks, the global stability region is the intersection of its stability regions under the static buffer priority disciplines. Thus, the "worst" or extremal disciplines are static buffer priority disciplines. These papers also showed that, for two-station fluid networks, a piecewise linear Lyapunov function provides a sharp characterization of the global stability region. One immediate corollary of this characterization is the observation that a globally stable two-station network will remain globally stable if service times are reduced. Thus, the global stability region of two-station networks is monotone not only in the arrival rates [9], but also in the service times. Another corollary of these results is that a certain LP associated with the piecewise linear Lyapunov function gives a sharp characterization of the global stability region for two-station fluid networks.

This paper reports positive and negative findings from our efforts to extend these methods to networks with more than two stations.

We show that, unlike the two-station networks studied in Dai and Vande Vate [18], the global stability region of a certain three-station fluid network is *not* the intersection of its stability regions under the static buffer priority disciplines. Thus, the global stability region of fluid networks with more than two stations can be determined by complex disciplines, and studying static buffer priority disciplines alone may not be sufficient to determine the global stability region of fluid networks.

We further show that the global stability region is not monotone in the service times. In particular, a system that is globally stable for one set of service rates may no longer be stable for another set of *faster* service rates. For this reason it may be more practical and effective to maintain service times that are in the largest monotone subset of the global stability region, which we introduce as the *monotone global stability region*. To characterize the monotone global stability region of our three-station network, we employ a dynamic discipline in which the buffer priorities change with the state of the system. We note that nonmonotonicity of the global stability region was first demonstrated by Humes [26] for deterministic networks and later by Bramson [7] for stochastic networks. Dumas [22] showed nonmonotonicity of the stability region for a 3-station priority stochastic network.

A certain LP related to our piecewise linear Lyapunov function provides a sharp characterization of the monotone global stability region for our three-station network. Further, with this characterization, we are able to resolve a number of recent conjectures about the stability region of networks with more than two stations. First, we show that the LP proposed by Bertsimas et al. [1] does *not* reliably determine the (monotone) global stability of fluid networks with more than two stations. Second, we show that the LP proposed by Chen and Zhang [11] does *not* characterize the (monotone) stability of the fluid network under a static buffer priority discipline. Finally, we observe that push starting and virtual stations introduced in Dai and Vande Vate [17] do not explain the (monotone) global stability conditions of networks with more than two stations. In fact, not even push starts and *pseudostations*, a generalization of virtual stations introduced in Hasenbein [25], can explain the (monotone) global stability conditions for these networks.

Multiclass fluid networks and queueing networks have been used to model telecommunication networks and manufacturing systems like wafer fabrication facilities. Kumar and Seidman [29], Lu and Kumar [30], Rybko and Stolyar [32], Bramson [3,4] and Seidman [33] demonstrated that, when the underlying network is re-entrant as in models of wafer fabrication facilities, a number of non-idling disciplines can be unstable even if the traffic intensity at each station is less than one. In these unstable examples, the total number of customers in the queueing network goes to infinity with time.

Rybko and Stolyar [32] observed a connection between the stability of queueing networks and that of fluid networks. Subsequently, Dai [13], motivated by an analogous result of Dupuis and Williams [23] on semimartingale reflecting Brownian motions, proved that under some distributional assumptions, a queueing network is stable if the corresponding fluid network is. Meyn [31] and Dai [14] proved partial converses to this result and Stolyar [34], Chen [9], and Chen and Zhang [10] offered further refinements and extensions. Bramson [8] showed an example in which a queueing network operating under a specific service discipline is stable while the corresponding fluid network is unstable. Other recent work on the stability of queueing networks and fluid networks includes: Bramson [5–7], Dumas [22], Foss and Rybko [24], Winograd and Kumar [35], Kumar and Meyn [27,28], Dai and Meyn [15].

We introduce our three-station fluid network in section 2 and state our main theorems. In section 3 we explicitly construct unstable fluid solutions. These unstable solutions follow a dynamic discipline that gives different sets of buffers higher priority at different times. In section 4 we use a piecewise linear Lyapunov function to obtain sufficient conditions to ensure the global stability of the network and prove the main theorem characterizing the monotone global stability region. In section 5 we demonstrate that the LP of Bertsimas, Gamarnik and Tsitsiklis cannot determine the (monotone) global stability region of networks with more than two stations. In section 6 we review linear Lyapunov functions and show that the LP of Chen and Zhang [11] does not provide a sharp characterization of stability under static buffer priority disciplines. In section 6 we also show that the static buffer priority disciplines are not the extreme disciplines.

2. The fluid network and its stability

We begin by describing our model and setting notation. In the following, all vectors should be envisioned as column vectors and any inequalities between vectors should be interpreted componentwise.

In this paper we consider the three-station fluid network depicted in figure 1. Unless otherwise noted, all comments about fluid networks are specific to this three-station network. Fluid comes to this network at the rate of λ units per unit of time and is served at each station in turn starting with station 1. After processing at station 3, fluid returns to station 1 and is again served by each station in turn before it leaves the system. Thus, each unit of fluid is processed six times, twice at each station, before it leaves the system.

Fluid awaiting the kth processing step is called class k fluid and resides in buffer k, $k = 1, 2, \ldots, 6$. Each unit of class k fluid requires $m_k > 0$ units of service at station $\sigma(k)$. Since a single server provides all service at a station, each server must divide its time between the two classes it serves.

We let $Q_k(t)$ denote the fluid level in buffer k at time t, and $T_k(t)$, the cumulative time server $\sigma(k)$ devotes to class k in the interval $[0, t]$. Thus,

$$U_i(t) = t - \sum_{k:\sigma(k)=i} T_k(t)$$

is the cumulative idle time at station i, $i = 1, 2, 3$, in the interval $[0, t]$. The buffer levels $Q(\cdot) = (Q_k(\cdot))_{1 \leqslant k \leqslant 6}$ and the allocations $T(\cdot) = (T_k(\cdot))_{1 \leqslant k \leqslant 6}$ must satisfy

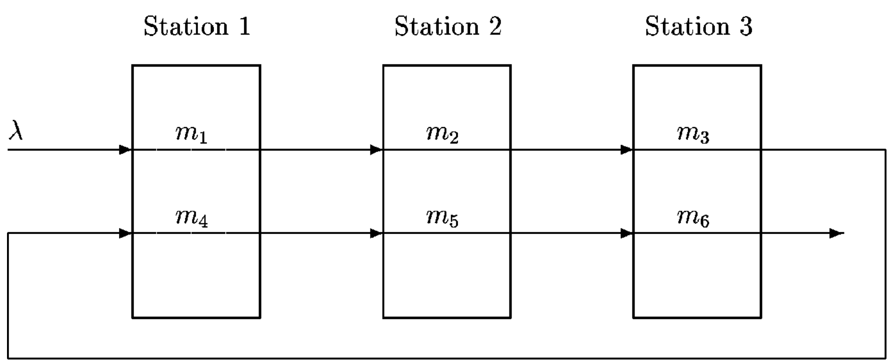

Figure 1. A three-station fluid network.

$$Q_k(t) = Q_k(0) + \mu_{k-1}T_{k-1}(t) - \mu_k T_k(t), \quad t \geqslant 0,\ k = 1, 2, \dots, 6, \tag{2.1}$$
$$Q_k(t) \geqslant 0, \quad t \geqslant 0,\ k = 1, 2, \dots, 6, \tag{2.2}$$
$$T_k(\cdot) \text{ is nondecreasing}, \quad k = 1, 2, \dots, 6, \tag{2.3}$$
$$U_i(\cdot) \text{ is nondecreasing}, \quad i = 1, 2, 3, \tag{2.4}$$

where $\mu_k = 1/m_k$ is the service rate for class k, $k = 1, 2, \dots, 6$, $\mu_0 = \lambda$ is the exogenous arrival rate and $T_0(t) = t$ models the exogenous arrival process. Notice that $\mu_k T_k(t)$ is the amount of fluid to have departed buffer k by time t.

Any solution $(Q(\cdot), T(\cdot))$ to (2.1)–(2.4) is said to be a feasible flow or fluid solution. A fluid solution $(Q(\cdot), T(\cdot))$ satisfying

$$\int_0^\infty Z_i(t)\, \mathrm{d}U_i(t) = 0, \quad i = 1, 2, 3, \tag{2.5}$$

where

$$Z_i(t) = \sum_{k:\sigma(k)=i} Q_k(t), \quad i = 1, 2, 3, \tag{2.6}$$

is said to be *non-idling* or work-conserving. Equations (2.1)–(2.5) define the fluid network under non-idling disciplines. Unless otherwise stated, we henceforth consider only non-idling fluid solutions.

For any fluid solution $(Q(\cdot), T(\cdot))$, $T(\cdot)$, and hence $Q(\cdot)$, is differentiable for almost all t in $(0, \infty)$; see, for example, Dai [13]. We say that t is a regular point for a fluid solution $(Q(\cdot), T(\cdot))$ if $T(\cdot)$ is differentiable at t. When the referenced fluid solution is clear from context, we simply call t a regular point. For a function $f: [0, \infty) \to \mathbb{R}$ that is differentiable at t, we use $\mathrm{d}f(t)/\mathrm{d}t$ or $\dot{f}(t)$ to denote the derivative of f at t. Notice that (2.5) is equivalent to the condition

$$Z_i(t) > 0 \quad \text{implies} \quad \dot{U}_i(t) = 0$$

for each regular point t, which ensures that when there is work for server i, the server must keep busy. It was shown in Dai [12,13] that each fluid limit is a fluid solution satisfying (2.1)–(2.5).

One particularly simple class of non-idling disciplines are the static buffer priority disciplines, which dictate that the server can only work on lower priority classes at a station when the requirements of higher priority classes are satisfied. Since each station in our three-station fluid network serves two classes, we can unambiguously denote the priorities by listing only the higher priority class at each station. Thus, for example, we use $\pi_{\{4,2,6\}}$, to denote the static buffer priority discipline that gives higher priority to classes 2, 4 and 6. There are eight static buffer priority disciplines associated with our three-station fluid network. They are: $\pi_{\{1,2,3\}}$, $\pi_{\{1,2,6\}}$, $\pi_{\{1,5,3\}}$, $\pi_{\{1,5,6\}}$, $\pi_{\{4,2,3\}}$, $\pi_{\{4,2,6\}}$, $\pi_{\{4,5,3\}}$, and $\pi_{\{4,5,6\}}$.

The fluid network under a static buffer priority discipline entails some equations in addition to (2.1)–(2.5). We let $\pi(i)$ denote the high priority class at station i under the static buffer priority discipline π. With this notation, our three-station fluid network under the static buffer priority discipline π requires the additional equations

$$\dot{T}_{\pi(i)}(t) = 1 \quad \text{if } Q_{\pi(i)}(t) > 0, \quad i = 1, 2, 3, \tag{2.7}$$

for each regular point t of $T(\cdot)$. These conditions simply stipulate that if fluid has accumulated in a station's higher priority buffer, the station must allocate all its effort to that buffer. Any solution $(Q(\cdot), T(\cdot))$ to (2.1)–(2.5) and (2.7) is a fluid solution under the discipline π.

Definition 2.1. The fluid network is *globally stable* if there exists a time $\delta > 0$ such that for each non-idling fluid solution $(Q(\cdot), T(\cdot))$ satisfying (2.1)–(2.5) with $|Q(0)| = 1$, $Q(t) = 0$ for all $t \geqslant \delta$, where $|\cdot|$ denotes the Euclidean norm.

Definition 2.2. The fluid network under a static buffer priority discipline π is *stable* if there exists a time $\delta > 0$ such that for each fluid solution $(Q(\cdot), T(\cdot))$ satisfying (2.1)–(2.5) and (2.7) with $|Q(0)| = 1$, $Q(t) = 0$ for $t \geqslant \delta$.

Definition 2.3. A fluid solution $(Q(\cdot), T(\cdot))$ is *unstable* if there is no $\delta > 0$ such that $Q(t) = 0$ for all $t \geqslant \delta$.

Definition 2.4. For a given $\lambda > 0$, the *global* stability region $\mathcal{D}_\infty$ of the fluid network is the set of positive service times $m = (m_k)$ for which the fluid network is globally stable. For a given $\lambda > 0$ and a static buffer priority discipline π, the *stability region* $\mathcal{D}_\pi$ of the fluid network under the discipline is the set of positive service times $m = (m_k)$ for which the fluid network under the discipline is stable.

It is well known (see, for example, Chen [9]) that all fluid solutions are unstable unless the traffic intensity or work arriving per unit time for each station is less than 1, i.e.,

$$\lambda\left(\sum_{k:\sigma(k)=i} m_k\right) < 1 \quad \text{for } i = 1, 2, 3. \tag{2.8}$$

If the above conditions hold, we say that the *usual traffic conditions* are satisfied. Thus, for any static buffer priority discipline π,

$$\mathcal{D}_\infty \subseteq \mathcal{D}_\pi \subseteq \mathcal{D}_0,$$

where

$$\mathcal{D}_0 \equiv \left\{m \in \mathbb{R}^6_+ \colon\ m > 0,\ \lambda(m_1 + m_4) < 1,\ \lambda(m_2 + m_5) < 1,\ \lambda(m_3 + m_6) < 1\right\}.$$

We show that the global stability region of the network depicted in figure 1 is not monotone. Thus, the network can be globally stable under one vector m of service times, but not be globally stable when some of the service times are reduced, i.e., not be globally stable under a service time vector $\widetilde{m} \leqslant m$.

Definition 2.5. For a given arrival rate $\lambda > 0$, the *monotone global stability region* $\mathcal{M}_\infty$ of the fluid network is the set of positive service time vectors m such that the fluid network is globally stable for all positive service time vectors $\widetilde{m} \leqslant m$.

Clearly, the monotone global stability region is contained in the global stability region. Thus,

$$\mathcal{M}_\infty \subseteq \mathcal{D}_\infty \subseteq \cap_\pi \mathcal{D}_\pi \subseteq \mathcal{D}_0, \tag{2.9}$$

where, hereafter, the intersection is over all eight static buffer priority disciplines.

To state our first theorem, we define the following system of linear constraints, which as we show in section 4 is closely related to a piecewise linear Lyapunov function for our three-station fluid network:

$$\lambda(x_1 + x_4) < x_1\mu_1, \tag{2.10}$$
$$\lambda(x_1 + x_4) < x_4\mu_4, \tag{2.11}$$
$$\lambda(x_2 + x_5) < x_2\mu_2, \tag{2.12}$$
$$\lambda(x_2 + x_5) < x_5\mu_5, \tag{2.13}$$
$$\lambda(x_3 + x_6) < x_3\mu_3, \tag{2.14}$$
$$\lambda(x_3 + x_6) < x_6\mu_6, \tag{2.15}$$
$$x_4 \leqslant x_3 + x_6, \tag{2.16}$$
$$x_5 \leqslant x_4, \tag{2.17}$$
$$x_2 + x_5 \leqslant x_1 + x_4, \tag{2.18}$$
$$x_3 + x_6 \leqslant x_2 + x_5, \tag{2.19}$$
$$x_6 \leqslant x_5. \tag{2.20}$$

Theorem 2.6. The global stability region is not monotone, i.e., $\mathcal{M}_\infty \neq \mathcal{D}_\infty$. Furthermore, for a positive service time vector m, the following are equivalent:

(a) The vector m is in the monotone global stability region $\mathcal{M}_\infty$.

(b) There exists $x = (x_1, \ldots, x_6) > 0$ satisfying (2.10)–(2.20).

(c) The vector m belongs to

$$\mathcal{D}_0 \cap \bigl\{m \in \mathbb{R}^6_+\colon\ \lambda m_2 + \lambda^2 m_4 m_6 < 1\bigr\}.$$

We leave the proof of theorem 2.6 to section 4.

The system of linear constraints (2.10)–(2.20) derived from our piecewise linear Lyapunov function provides conditions sufficient to ensure that a service time vector m is in the global stability region. In fact, we show that, together with the usual traffic conditions, the single additional condition

$$\lambda m_2 + \lambda^2 m_4 m_6 < 1 \tag{2.21}$$

is sufficient to ensure global stability.

To obtain conditions necessary for global stability, we construct unstable fluid solutions. Our second theorem, shows that when $m_4 > m_3$, the additional condition (2.21) is also necessary to ensure global stability. When $m_4 \leqslant m_3$, however, new conditions arise. First, condition (2.26) ensures that work can arrive at station 1 at least as quickly as the station processes it. Otherwise, station 1 will eventually empty and thereafter remain empty, essentially reducing the system to a two-station network. The proof of theorem 2.7, given in section 3, involves the construction of unstable fluid solutions under dynamic disciplines that give different sets of buffers higher priority at different times. When $m_4 \leqslant m_3$, condition (2.25), the strongest necessary condition we could obtain from these disciplines, is weaker than our sufficient condition (2.21). It is unclear whether or not the fluid network is globally stable when the mean service time vector m satisfies, $m \in \mathcal{D}_0$, $m_4 \leqslant m_3$ and

$$\left(\frac{\lambda m_1 m_3 + m_4 - m_3}{m_1 + m_4 - m_3}\right)\frac{\lambda m_6}{1 - \lambda m_2} < 1 \leqslant \lambda m_4 \frac{\lambda m_6}{1 - \lambda m_2}.$$

Theorem 2.7. If the service time vector of the network in figure 1 satisfies

$$m_4 > m_3, \quad \text{and} \tag{2.22}$$
$$\lambda m_2 + \lambda^2 m_4 m_6 \geqslant 1, \tag{2.23}$$

or if it satisfies

$$m_4 \leqslant m_3, \tag{2.24}$$
$$\left(\frac{\lambda m_1 m_3 + m_4 - m_3}{m_1 + m_4 - m_3}\right)\frac{\lambda m_6}{1 - \lambda m_2} \geqslant 1, \quad \text{and} \tag{2.25}$$
$$\lambda m_1 + \frac{m_4}{m_3} \geqslant 1 \tag{2.26}$$

there is an unstable (non-idling) fluid solution.

Bertsimas et al. [1] developed an LP for testing the global stability of a fluid network. For two-station fluid networks, their LP has optimal objective value 0 if and only if the network is globally stable with the given arrival and service rates. For the three-station network in figure 1, their LP is

$$\max\ \tau_1 + \tau_2 + \tau_3 \tag{2.27}$$

subject to

$$\lambda \tau_1 - \mu_1 \tau_{11} \leqslant 0, \tag{2.28}$$
$$\mu_{k-1}\tau_{k-1,\sigma(k)} - \mu_k \tau_{k,\sigma(k)} \leqslant 0, \quad k = 2, 3, \dots, 6, \tag{2.29}$$
$$\sum_{k:\sigma(k)=i} \tau_{ki} = \tau_i, \quad i = 1, 2, 3, \tag{2.30}$$

$$\sum_{k:\sigma(k)=j} \tau_{ki} \leqslant \tau_i, \quad j, i \in \{1, 2, 3\},\ j \neq i, \tag{2.31}$$

$$\lambda(\tau_1 + \tau_2 + \tau_3) - \mu_1(\tau_{11} + \tau_{12} + \tau_{13}) = 0, \tag{2.32}$$

$$\mu_{k-1}(\tau_{k-1,1} + \tau_{k-1,2} + \tau_{k-1,3}) - \mu_k(\tau_{k1} + \tau_{k2} + \tau_{k3}) = 0, \quad k = 2, \ldots, 6, \tag{2.33}$$

$$\tau_i, \tau_{ji} \geqslant 0, \quad i = 1, 2, 3,\ j = 1, \ldots, 6. \tag{2.34}$$

Theorem 2.8. The LP of Bertsimas, Gamarnik and Tsitisklis [1] does not provide a sharp characterization of (monotone) global stability for networks with more than two stations.

We prove this theorem in section 5 by demonstrating a service time vector m in the monotone global stability region $\mathcal{M}_\infty$ (with arrival rate $\lambda = 1$) for which the LP (2.27)–(2.34) of Bertsimas, Gamarnik and Tsitisklis [1] has a solution with positive objective value.

Theorem 2.9, which is proved in section 6, shows that the stability regions of all but one of the static buffer priority disciplines are defined by the usual traffic conditions. The stability region of one static buffer priority discipline, $\pi_{\{4,2,6\}}$ involves conditions more restrictive than the usual traffic conditions, but strictly contains the global stability region.

Theorem 2.9.

(a) For any static buffer priority discipline $\pi \neq \pi_{\{4,2,6\}}$, $\mathcal{D}_\pi = \mathcal{D}_0$.

(b) $\mathcal{D}_{\pi_{\{4,2,6\}}} \neq \mathcal{D}_0$.

(c) $\mathcal{D}_{\pi_{\{4,2,6\}}} \neq \mathcal{D}_\infty$.

An immediate consequence of theorem 2.9 is the following corollary. Unlike their two-station counterparts the global stability regions of fluid networks with more than two stations need not be defined by the static buffer priority disciplines.

Corollary 2.10. $\mathcal{D}_\infty \neq \bigcap_\pi \mathcal{D}_\pi$.

Chen and Zhang [11] employed linear Lyapunov functions to study the stability of a fluid network under static buffer priority disciplines. They introduced a linear program, described in lemma 6.1, that is related to the linear Lyapunov functions and showed that if this LP has strictly positive objective value, the fluid network is stable under the given discipline. Theorem 2.11 shows that the converse is not true.

Theorem 2.11. The LP of Chen and Zhang [11] need not provide a sharp characterization of stability for fluid networks under static buffer priority disciplines.

We prove this theorem in section 6 by demonstrating a service time vector m in the global stability region $\mathcal{M}_\infty$ (with arrival rate $\lambda = 1$), for which the LP of Chen and Zhang has optimal objective value 0.

3. Instability of the fluid network

To obtain conditions necessary to ensure global stability, we describe disciplines and construct unstable fluid solutions for a broad range of service times. These unstable fluid solutions explicitly demonstrate that the system is unstable over the range of service times. We offer two closely related disciplines. The first, given in part (a) of the proof, demonstrates conditions under which the fluid network is not globally stable when $m_4 > m_3$. The second, given in part (b) of the proof, provides similar conditions for the case when $m_4 \leqslant m_3$.

Proof of theorem 2.7. Part (a). We assume that the mean service vector $m > 0$ satisfies (2.22)–(2.23). We further assume that the usual traffic conditions (2.8) hold. Otherwise, any non-idling solution is unstable.

For each subset $S \subseteq \{1, 2, \ldots, 6\}$ we define

$$Q_S(t) = \sum_{i \in S} Q_i(t).$$

We construct an unstable fluid solution using a discipline under which the priorities among the classes at each station may change depending on the levels of fluid in the buffers. We set $s_0 = 0$ and let $[s_{i-1}, s_i]$, $i = 1, 2, \ldots$, be intervals in which the buffer priorities are constant. We use t_i to denote the length of the ith interval, so $t_i = s_i - s_{i-1}$. We also let d_k denote the departure rate from buffer k during a given interval.

We first note that the usual traffic conditions, along with (2.23) imply that

$$\mu_5 > \max\{\mu_4, \mu_6\} \quad \text{and} \tag{3.1}$$

$$\mu_2 < \min\{\mu_1, \mu_3\}. \tag{3.2}$$

We start at initial time s_0 and assume $Q_{\{1,2,3\}}(s_0) = 0$, $Q_{\{4,5\}}(s_0) > 0$ and $Q_6(s_0) \geqslant 0$.

Step 1. We begin by giving classes 1, 5, and 6 higher priority. We set $s_1 = \min\{t \geqslant s_0\colon\ Q_5(t) = 0, Q_6(t) = Q_6(s_0)\}$. If $Q_5(s_0) = 0$ then $s_1 = s_0$ and we go directly to step 2. Otherwise, since $\mu_6 < \mu_5$, buffer 6 begins to accumulate fluid, and thus, $d_6 = \mu_6$ in $[s_0, s_1]$. This implies that $d_3 = 0$ during this interval. We note, further, that $Q_1(s_1) = 0$ because buffer 1 has priority. So, we have that

$$\dot{Q}_{\{1,2,3\}}(t) = \lambda \quad \text{and} \quad \dot{Q}_{\{4,5,6\}}(t) = -\mu_6 \qquad \text{for } s_0 \leqslant t \leqslant s_1.$$

The above imply

$$\dot{Q}_{\{1,2,3\}}(t) + \lambda m_6 \dot{Q}_{\{4,5,6\}}(t) = 0 \qquad \text{for } s_0 \leqslant t \leqslant s_1,$$

hence,

$$Q_{\{2,3\}}(s_1) + \lambda m_6 Q_4(s_1) = \lambda m_6 Q_{\{4,5\}}(s_0). \tag{3.3}$$

Step 2. In the next period we give buffers 3, 4, and 5 higher priority. We set $s_2 = \min\{t \geqslant s_1\colon Q_3(t) + Q_4(t) = 0\}$. If $Q_3(s_1) + Q_4(s_1) = 0$ then $s_2 = s_1$ and we go directly to step 3. Otherwise, since $\mu_4 < \mu_3$, buffer 3 will empty before buffer 4. So, by our priority scheme in $[s_1, s_2]$, we must have $d_4 = \mu_4$ and $d_1 = 0$ in $[s_1, s_2]$. Also, $Q_5(s_2) = 0$ since buffer 5 has priority and $\mu_4 < \mu_5$. Thus,

$$\dot{Q}_1(t) = \lambda \quad \text{and} \quad \dot{Q}_{\{2,3,4\}}(t) = -\mu_4 \qquad \text{for } s_1 \leqslant t \leqslant s_2.$$

The above imply

$$\dot{Q}_1(t) + \lambda m_4 \dot{Q}_{\{2,3,4\}}(t) = 0 \qquad \text{for } s_1 \leqslant t \leqslant s_2,$$

hence,

$$Q_1(s_2) + \lambda m_4 Q_2(s_2) = \lambda m_4 Q_{\{2,3,4\}}(s_1). \tag{3.4}$$

Step 3. In the final period, we let buffers 1, 2, and 3 have higher priority. We set $s_3 = \min\{t \geqslant s_2\colon Q_2(t) = 0\}$. Notice that buffer 1 will empty before buffer 2 since $\mu_2 < \mu_1$. So we will have $d_2 = \mu_2$ and $d_5 = 0$ in $[s_2, s_3]$. Further, $Q_3(s_3) = 0$ since buffer 3 has high priority and $\mu_2 < \mu_3$. Thus,

$$\dot{Q}_{\{1,2\}}(t) = \lambda - \mu_2 \dot{Q}_{\{3,4,5\}}(t) = \mu_2 \qquad \text{for } s_2 \leqslant t \leqslant s_3.$$

The above imply

$$\dot{Q}_{\{3,4,5\}}(t) + \frac{\dot{Q}_{\{1,2\}}(t)}{1 - \lambda m_2} = 0 \qquad \text{for } s_2 \leqslant t \leqslant s_3,$$

hence,

$$Q_{\{4,5\}}(s_3) = \frac{Q_{\{1,2\}}(s_2)}{1 - \lambda m_2}. \tag{3.5}$$

Step 4. Now from equations (3.3)–(3.5) and the fact that $\lambda m_i < 1$ from the usual traffic conditions we have

$$\begin{aligned} Q_{\{4,5\}}(s_3) &= \frac{Q_{\{1,2\}}(s_2)}{1 - \lambda m_2} = \frac{Q_1(s_2) + Q_2(s_2)}{1 - \lambda m_2} \geqslant \frac{Q_1(s_2) + \lambda m_4 Q_2(s_2)}{1 - \lambda m_2} \\ &= \frac{\lambda m_4 Q_{\{2,3,4\}}(s_1)}{1 - \lambda m_2} = \frac{\lambda m_4 (Q_{\{2,3\}}(s_1) + Q_4(s_1))}{1 - \lambda m_2} \\ &\geqslant \frac{\lambda m_4 (Q_{\{2,3\}}(s_1) + \lambda m_6 Q_4(s_1))}{1 - \lambda m_2} = \frac{\lambda^2 m_4 m_6}{1 - \lambda m_2} Q_{\{4,5\}}(s_0). \end{aligned}$$

We remark that if either interval 1 or 2 is "null", the result still holds, by a similar (simpler) chain of inequalities.

Now, by condition (2.23) we conclude

$$Q_{\{4,5\}}(s_3) \geqslant Q_{\{4,5\}}(s_0).$$

Recalling that $Q_{\{1,2,3\}}(s_3) = 0$ under our policy, the above implies that the fluid solutions constructed under our discipline are unstable, proving that the network is not globally stable.

Part (b). Next we assume that the mean service time vector $m > 0$ satisfies (2.24)–(2.26). We begin by noting that (3.1) and (3.2) still hold under (2.24)–(2.26). We only need alter steps 2 and 4 in the proof of part (a). In particular, equations (3.3) and (3.5) continue to hold. We present the revised steps 2′ and 4′ below.

Step 2′. In this period we give buffers 3, 4 and 5 higher priority. We again set $s_2 = \min\{t \geqslant s_1\colon Q_3(t) + Q_4(t) = 0\}$. Without loss of generality, we suppose that buffer 4 drains before buffer 3, otherwise we may employ the proof used in part (a). Also, as before, if $Q_3(s_1) + Q_4(s_1) = 0$, then $s_2 = s_1$ and we go directly to step 3.

Let us denote the time at which buffer 4 empties as r (with $s_1 \leqslant r \leqslant s_2$). As before, we must have $d_4 = \mu_4$ and $d_1 = 0$ in $[s_1, r]$. Thus,

$$\dot{Q}_1(t) = \lambda \quad \text{and} \quad \dot{Q}_{\{2,3,4\}}(t) = -\mu_4 \quad \text{for } s_1 \leqslant t \leqslant r. \tag{3.6}$$

The above imply

$$\dot{Q}_1(t) = -\lambda m_4 \dot{Q}_{\{2,3,4\}}(t) \quad \text{for } s_1 \leqslant t \leqslant r$$

and this yields

$$Q_1(r) + \lambda m_4 Q_{\{2,3,4\}}(r) - \lambda m_4 Q_{\{2,3,4\}}(s_1) = 0. \tag{3.7}$$

Now during $[r, s_2]$, we have that $d_4 = d_3 = \mu_3$ and by work conservation $d_1 = \hat{d}_1 := (1/m_1)(1 - \mu_3 m_4)$. Note that $\hat{d}_1 \leqslant \lambda$ by (2.26). Thus, for this part of the interval, we have

$$\dot{Q}_1(t) = \lambda - \hat{d}_1 \quad \text{and} \quad \dot{Q}_{\{2,3,4\}}(t) = \hat{d}_1 - \mu_3 \quad \text{for } r \leqslant t \leqslant s_2.$$

The above imply

$$\dot{Q}_1(t) + \frac{\lambda - \hat{d}_1}{\mu_3 - \hat{d}_1}\dot{Q}_{\{2,3,4\}}(t) = 0 \quad \text{for } r \leqslant t \leqslant s_2,$$

and this gives

$$Q_1(s_2) - Q_1(r) + \kappa Q_2(s_2) - \kappa Q_{\{2,3,4\}}(r) = 0, \tag{3.8}$$

where we have set

$$\kappa = \frac{\lambda - \hat{d}_1}{\mu_3 - \hat{d}_1} = \frac{\lambda m_1 m_3 + m_4 - m_3}{m_1 + m_4 - m_3}.$$

Now, adding (3.7) and (3.8) and rearranging

$$Q_1(s_2) + \kappa Q_2(s_2) = \kappa Q_{\{2,3,4\}}(s_1) + (\lambda m_4 - \kappa)\big[Q_{\{2,3,4\}}(s_1) - Q_{\{2,3,4\}}(r)\big].$$

A little algebra shows that $\kappa \leqslant \lambda m_4$ and $Q_{\{2,3,4\}}(s_1) \geqslant Q_{\{2,3,4\}}(r)$ by virtue of (3.6). Thus, we have

$$Q_1(s_2) + \kappa Q_2(s_2) \geqslant \kappa Q_{\{2,3,4\}}(s_1).$$

Step $4'$.

$$\begin{aligned}Q_{\{4,5\}}(s_3) &= \frac{Q_{\{1,2\}}(s_2)}{1-\lambda m_2} = \frac{Q_1(s_2)+Q_2(s_2)}{1-\lambda m_2} \geqslant \frac{Q_1(s_2)+\kappa Q_2(s_2)}{1-\lambda m_2}\\ &\geqslant \frac{\kappa Q_{\{2,3,4\}}(s_1)}{1-\lambda m_2} = \frac{\kappa(Q_{\{2,3\}}(s_1)+Q_4(s_1))}{1-\lambda m_2}\\ &\geqslant \frac{\kappa(Q_{\{2,3\}}(s_1)+\lambda m_6 Q_4(s_1))}{1-\lambda m_2} = \frac{\kappa\lambda m_6}{1-\lambda m_2}Q_{\{4,5\}}(s_0).\end{aligned}$$

By our assumptions we can conclude

$$Q_{\{4,5\}}(s_3) \geqslant Q_{\{4,5\}}(s_0),$$

which again implies the instability of our fluid solution. □

4. Piecewise linear Lyapunov functions

In this section we prove theorem 2.6 showing that the global stability region of our three-station network is not monotone and characterizing its monotone global stability region. We first introduce the piecewise linear Lyapunov functions we use to establish conditions sufficient to ensure global stability. Given $x = (x_k) > 0$ and a fluid solution $Q(\cdot)$, let

$$f_i\bigl(x, Q(t)\bigr) = \sum_{k:\sigma(k)=i} x_k Q_k^+(t), \quad i = 1,2,3,$$

where $Q_k^+(t) = \sum_{\ell=1}^k Q_\ell(t)$. Further, let

$$f\bigl(x, Q(t)\bigr) = \max\bigl\{f_1\bigl(x,Q(t)\bigr), f_2\bigl(x,Q(t)\bigr), f_3\bigl(x,Q(t)\bigr)\bigr\}.$$

We often write $f(Q(t))$ in place of the more cumbersome $f(x, Q(t))$. Clearly, $f(Q(t))$ is a convex, piecewise linear function of $Q(t) = (Q_k(t))$.

The piecewise linear function f is said to be a Lyapunov function for the global stability of the fluid model if there exists $\varepsilon > 0$ such that for each non-idling fluid solution $(Q(\cdot), T(\cdot))$ satisfying (2.1)–(2.5),

$$\frac{\mathrm{d}f(Q(t))}{\mathrm{d}t} \leqslant -\varepsilon \tag{4.1}$$

for each time $t > 0$ that is regular for $T(\cdot)$ and $f(Q(\cdot))$ with $|Q(t)| > 0$.

Let $m > 0$ be a service time vector for which there is a piecewise linear Lyapunov function f satisfying (4.1). It follows from Dai and Weiss [19, lemma 2.2] that

$$f\bigl(Q(t)\bigr) = 0 \quad \text{for all } t \geqslant \frac{f(Q(0))}{\varepsilon},$$

or $Q(t) = 0$ for all t $\geqslant f(Q(0))/\varepsilon$. Let

$$\delta = \max\bigl\{f\bigl(Q(0)\bigr)\colon\ Q(0) \geqslant 0, \bigl|Q(0)\bigr| = 1\bigr\}/\varepsilon.$$

Clearly $\delta > 0$, and for each non-idling fluid solution $Q(\cdot)$, $Q(t) = 0$ when $t \geqslant \delta$. Thus, m is in the global stability region.

The next lemma suggests a way in which to construct piecewise linear Lyapunov functions. This type of construction was introduced by Botvich and Zamyatin [2] for a two-station network. It was independently generalized by Dai and Weiss [19], and Down and Meyn [21].

Lemma 4.1. Suppose there exists $x = (x_k) > 0$, $t_0 \geqslant 0$ and $\varepsilon > 0$ such that for each non-idling fluid solution $(Q(\cdot), T(\cdot))$ and each regular point $t > t_0$ of $T(\cdot)$, the following hold for each $i = 1, 2, 3$:

$$\frac{\mathrm{d}f_i(x, Q(t))}{dt} \leqslant -\varepsilon \quad \text{whenever } Z_i(t) > 0, \tag{4.2}$$

$$f_i\bigl(x, Q(t)\bigr) \leqslant \max\bigl\{f_j(x, Q(t))\colon\ j \in \{1,2,3\},\ j \neq i\bigr\} \quad \text{whenever } Z_i(t) = 0, \tag{4.3}$$

$$\max\bigl\{f_j\bigl(Q(t)\bigr)\colon\ j \in \{1,2,3\},\ j \neq i\bigr\} \leqslant f_i\bigl(Q(t)\bigr) \quad \text{whenever } \sum_{j \neq i} Z_j(t) = 0. \tag{4.4}$$

Then f is a piecewise linear Lyapunov function.

Proof. Let t be a regular point of f and T with $Q(t) \neq 0$. We show that (4.1) holds. Because $Q(t) \neq 0$ and (4.3)–(4.4) hold, there exists an index $i \in \{1, 2, 3\}$ such that $f_i(Q(t)) = f(Q(t))$ and $Z_i(t) > 0$. From the proof of lemma 3.2 of Dai and Weiss [19], we have

$$\frac{\mathrm{d}f(Q(t))}{\mathrm{d}t} = \frac{\mathrm{d}f_i(Q(t))}{\mathrm{d}t}.$$

Then (4.1) follows from (4.2). □

Lemma 4.2. If there is $x = (x_k) > 0$ satisfying the linear constraints (2.10)–(2.20), then there exists $\varepsilon > 0$ such that (4.2)–(4.4) hold, and hence, f is a piecewise linear Lyapunov function.

Proof. Let $t_0 = 0$ and let $x = (x_k) > 0$ satisfy (2.10)–(2.20). Define ε to be the minimum of the following 6 terms:

$$\begin{aligned} &x_1\mu_1 - \lambda(x_1 + x_4), \quad x_4\mu_4 - \lambda(x_1 + x_4),\\ &x_2\mu_2 - \lambda(x_2 + x_5), \quad x_5\mu_5 - \lambda(x_2 + x_5),\\ &x_3\mu_3 - \lambda(x_3 + x_6), \quad x_6\mu_6 - \lambda(x_3 + x_6). \end{aligned}$$

Clearly, $\varepsilon > 0$. Consider a non-idling fluid solution $(Q(\cdot), T(\cdot))$ and a time $t > 0$ that is regular for $T(\cdot)$. Observe that the amount of fluid in buffers 1 through k is

$$Q_k^+(t) = Q_k^+(0) + \lambda t - \mu_k T_k(t).$$

Hence,

$$f_1\bigl(Q(t)\bigr) = f_1(0) + (x_1 + x_4)\lambda t - x_1\mu_1 T_1(t) - x_4\mu_4 T_4(t)$$

and

$$\frac{\mathrm{d}f_1(Q(t))}{\mathrm{d}t} = \lambda(x_1 + x_4) - x_1\mu_1\dot{T}_1(t) - x_4\mu_4\dot{T}_4.$$

If $Z_1(t) > 0$, it follows from (4.3) that, since $(Q(\cdot), T(\cdot))$ is non-idling, $\dot{U}_1(t) = 0$ or $\dot{T}_1(t) + \dot{T}_4(t) = 1$. Thus, by the definition of ε,

$$\dot{f}_1(t) \leqslant -\varepsilon \quad \text{when } Z_1(t) > 0.$$

Similar analysis for $i = 2$ and $i = 3$ shows that (4.2) holds.

We next establish (4.3). When $Z_1(t) = 0$,

$$f_1\bigl(Q(t)\bigr) = x_4\bigl(Q_2(t) + Q_3(t)\bigr) \quad \text{and}$$
$$f_3\bigl(Q(t)\bigr) = x_3\bigl(Q_2(t) + Q_3(t)\bigr) + x_6\bigl(Q_2(t) + Q_3(t) + Q_5(t) + Q_6(t)\bigr),$$

and equation (2.16) ensures that $f_1(Q(t)) \leqslant f_3(Q(t))$. When $Z_2(t) = 0$,

$$f_2\bigl(Q(t)\bigr) = x_2Q_1(t) + x_5\bigl(Q_1(t) + Q_3(t) + Q_4(t)\bigr),$$
$$f_1\bigl(Q(t)\bigr) = x_1Q_1(t) + x_4\bigl(Q_1(t) + Q_3(t) + Q_4(t)\bigr),$$

and equations (2.17)–(2.18) ensure that $f_2(Q(t)) \leqslant f_1(Q(t))$. When $Z_3(t) = 0$,

$$f_3\bigl(Q(t)\bigr) = x_3\bigl(Q_1(t) + Q_2(t)\bigr) + x_6\bigl(Q_1(t) + Q_2(t) + Q_4(t) + Q_5(t)\bigr),$$
$$f_2\bigl(Q(t)\bigr) = x_2\bigl(Q_1(t) + Q_2(t)\bigr) + x_5\bigl(Q_1(t) + Q_2(t) + Q_4(t) + Q_5(t)\bigr),$$

and equations (2.19)–(2.20) ensure that $f_3(Q(t)) \leqslant f_2(Q(t))$.

Finally, we establish (4.4). When $Z_1(t) = 0$ and $Z_2(t) = 0$,

$$f_1\bigl(Q(t)\bigr) = x_4Q_3(t),$$
$$f_2\bigl(Q(t)\bigr) = x_5Q_3(t),$$
$$f_3\bigl(Q(t)\bigr) = x_3Q_3(t) + x_6\bigl(Q_3(t) + Q_6(t)\bigr).$$

Equation (2.16) ensures that $f_1(Q(t)) \leqslant f_3(Q(t))$ and equations (2.16) and (2.17) ensure that $f_2(Q(t)) \leqslant f_3(Q(t))$. The remaining cases of (4.4) can be verified similarly. □

Remark 4.3. (a) In general, condition (4.3) generates nonlinear constraints on $x = (x_k)$. However, for our network, the linear constraints arising from (4.4) imply condition (4.3) and so we have the set of linear constraints (2.10)–(2.20) associated with our piecewise linear Lyapunov function.

(b) For a d-station generalization of our fluid network in which fluid repeatedly visits all of the stations in a fixed order, there is an analogous natural set of linear constraints associated with a piecewise linear Lyapunov function. Further, it is not difficult to obtain explicit conditions in terms of the service times and arrival rate characterizing exactly when the linear constraints admit a solution x.

(c) The existence of a solution x to the system of linear constraints (2.10)–(2.20) ensures the existence of a piecewise linear Lyapunov function satisfying conditions (4.2)–(4.4). The converse, however, does not hold; see lemma 4.5.

Lemma 4.4. The linear constraints (2.10)–(2.20) admit a feasible solution $x = (x_k) > 0$ if and only if

$$\lambda(m_1 + m_4) < 1, \tag{4.5}$$

$$\lambda(m_2 + m_5) < 1, \tag{4.6}$$

$$\lambda(m_3 + m_6) < 1, \tag{4.7}$$

$$\lambda m_2 + \lambda^2 m_4 m_6 < 1. \tag{4.8}$$

Proof. Note that there exists $(x_1, \dots, x_6) > 0$ satisfying (2.10)–(2.20) if and if only there exists $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_5, \tilde{x}_6) > 0$ such that $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, 1, \tilde{x}_5, \tilde{x}_6)$ satisfies (2.10)–(2.20). Given $(x_1, \dots, x_6) > 0$ with $x_4 = 1$, let

$$y_1 = \frac{x_4}{x_1 + x_4}, \qquad y_2 = \frac{x_5}{x_2 + x_5}, \qquad y_3 = \frac{x_6}{x_3 + x_6}.$$

Then $(x_1, \dots, x_6) > 0$ with $x_4 = 1$ satisfies (2.10)–(2.20) if and only if $(y_1, y_2, y_3, x_5, x_6) > 0$ satisfies

$$\lambda m_1 < 1 - y_1, \tag{4.9}$$

$$\lambda m_4 < y_1, \tag{4.10}$$

$$\lambda m_2 < 1 - y_2, \tag{4.11}$$

$$\lambda m_5 < y_2, \tag{4.12}$$

$$\lambda m_3 < 1 - y_3, \tag{4.13}$$

$$\lambda m_6 < y_3, \tag{4.14}$$

$$y_3 \leqslant x_6, \tag{4.15}$$

$$x_5 \leqslant 1, \tag{4.16}$$

$$x_5 y_1 \leqslant y_2, \tag{4.17}$$

$$x_6 y_2 \leqslant x_5 y_3, \tag{4.18}$$

$$x_6 \leqslant x_5. \tag{4.19}$$

The existence of $(y_1, y_2, y_3, x_5, x_6) > 0$ satisfying (4.9)–(4.19) is equivalent to the existence of $(y_1, y_2, y_3, x_5) > 0$ satisfying

$$\lambda m_4 < y_1 < 1 - \lambda m_1, \tag{4.20}$$

$$\lambda m_5 < y_2 < 1 - \lambda m_2, \tag{4.21}$$

$$\lambda m_6 < y_3 < 1 - \lambda m_3, \tag{4.22}$$

$$x_5 \leqslant 1, \tag{4.23}$$

$$x_5 \leqslant \frac{y_2}{y_1}, \tag{4.24}$$

$$y_2 \leqslant x_5, \tag{4.25}$$

$$y_3 \leqslant x_5, \tag{4.26}$$

which is equivalent to the existence of (y_1, y_2, y_3) satisfying

$$\lambda m_4 < y_1 < 1 - \lambda m_1, \tag{4.27}$$
$$\lambda m_5 < y_2 < 1 - \lambda m_2, \tag{4.28}$$
$$\lambda m_6 < y_3 < 1 - \lambda m_3, \tag{4.29}$$
$$y_1 y_3 \leqslant y_2. \tag{4.30}$$

Finally, the existence of (y_1, y_2, y_3) satisfying (4.27)–(4.30) is equivalent to (4.5)–(4.8). $\square$

The following lemma establishes an alternate set of conditions sufficient to ensure global stability in our three-station fluid network.

Lemma 4.5. If

$$\lambda m_1 + \frac{m_4}{m_3} \leqslant 1, \tag{4.31}$$
$$\lambda(m_2 + m_5) < 1, \tag{4.32}$$
$$\lambda(m_3 + m_6) < 1, \tag{4.33}$$

the fluid network is globally stable.

Proof. Let $(Q(\cdot), T(\cdot))$ be a non-idling fluid solution with $|Q(0)| = 1$. Let

$$g_1(t) = m_1 Q_1(t) + m_4 \big(Q_1(t) + Q_2(t) + Q_3(t) + Q_4(t)\big)$$

be the *total* workload at station 1 at time t. It follows from (2.1) that

$$g_1(t) = g_1(0) + \lambda(m_1 + m_4)t - \big(T_1(t) + T_4(t)\big).$$

For each regular t with $Z_1(t) > 0$, by (2.5), $\dot{g}_1(t) = -(1 - \lambda(m_1 + m_4))$. Since $\lambda(m_1 + m_4) < 1$, there is positive t_0 with

$$t_0 \leqslant \frac{g_1(0)}{1 - \lambda(m_1 + m_4)} \leqslant \frac{m_1 + 4m_4}{1 - \lambda(m_1 + m_4)}$$

such that $Z_1(t_0) = 0$. Assume that (4.31) holds. We next show that $Z_1(t) = 0$ for $t \geqslant t_0$. To see this, let

$$g_2(t) = m_1 Q_1(t) + m_4 Q_4(t)$$

be the (immediate) workload at station 1. From (2.1)–(2.4),

$$g_2(t) = g_2(0) + \lambda m_1 t - T_1(t) + m_4 \mu_3 T_3(t) - T_4(t).$$

Therefore, for any regular t with $g_2(t) > 0$,

$$\dot{g}_2(t) = \lambda m_1 + m_4 \mu_3 \dot{T}_3(t) - \big(\dot{T}_1(t) + \dot{T}_4(t)\big) \leqslant \lambda m_1 + m_4 \mu_3 - 1 \leqslant 0.$$

Thus, $g_2(\cdot)$ is non-increasing. Since $g_2(t_0) = 0$, we have $g_2(t) = 0$ or equivalently $Z_1(t) = 0$ for $t \geqslant t_0$.

We now show that there is $t_1 > t_0$ such that $Z_2(t)+Z_3(t) = 0$ for each time $t \geqslant t_1$ and hence that the network is globally stable. To show that buffers at stations 2 and 3 eventually empty, we consider times $t \geqslant t_0$ and specialize the proof of lemma 4.1 to the case where $Z_1(t) = 0$ and $\dot{Q}_1(t) = \dot{Q}_4(t) = 0$. First, observe that since $Z_1(t) = 0$ for $t \geqslant t_0$, (4.2) is vacuously satisfied for $i = 1$. Similarly, (4.4) is trivially satisfied for $i = 1$. Finally, recalling that (4.4) implies (4.3) in our network, we see that we are left with the conditions

$$\frac{\mathrm{d}f_2(x, Q(t))}{\mathrm{d}t} \leqslant -\varepsilon \quad \text{whenever } Z_2(t) > 0, \tag{4.34}$$

$$\frac{\mathrm{d}f_3(x, Q(t))}{\mathrm{d}t} \leqslant -\varepsilon \quad \text{whenever } Z_3(t) > 0, \tag{4.35}$$

$$\max\bigl\{f_1\bigl(Q(t)\bigr), f_3\bigl(Q(t)\bigr)\bigr\} \leqslant f_2\bigl(Q(t)\bigr) \quad \text{whenever } Z_1(t) + Z_3(t) = 0, \tag{4.36}$$

$$\max\bigl\{f_1\bigl(Q(t)\bigr), f_2\bigl(Q(t)\bigr)\bigr\} \leqslant f_3\bigl(Q(t)\bigr) \quad \text{whenever } Z_1(t) + Z_2(t) = 0. \tag{4.37}$$

Arguments analogous to those used in the proof of lemma 4.2 show that (4.34)–(4.37) and, hence, (4.2)–(4.4) hold if there exists $(x_2, x_3, x_5, x_6) > 0$ satisfying

$$\lambda(x_2 + x_5) < \mu_2 x_2, \tag{4.38}$$

$$\lambda(x_2 + x_5) < \mu_5 x_5, \tag{4.39}$$

$$\lambda(x_3 + x_6) < \mu_3 x_3, \tag{4.40}$$

$$\lambda(x_3 + x_6) < \mu_6 x_6, \tag{4.41}$$

$$x_5 \leqslant x_3 + x_6, \tag{4.42}$$

$$x_3 + x_6 \leqslant x_2 + x_5, \tag{4.43}$$

$$x_6 \leqslant x_5. \tag{4.44}$$

Finally, arguments similar to those used in the proof of lemma 4.4 show that there exists $x > 0$ satisfying (4.38)–(4.44) if and only if the usual traffic conditions (4.32)–(4.33) at stations 2 and 3 hold. Therefore, the lemma follows from lemma 4.1. □

Remark 4.6. For two-station networks, there is $x > 0$ satisfying the linear constraints arising from our piecewise linear Lyapunov functions if and only if the network is globally stable. This is not the case for networks with more than two stations and lemma 4.5 illustrates one way in which the network can be globally stable even when the linear system (2.10)–(2.20) admits no positive solution.

We are now prepared to prove our main result, theorem 2.6, showing that the global stability region of our three-station network is not monotone in the service times and characterizing its monotone global stability region both in terms of the solvability of the linear system (2.10)–(2.20) and in terms of explicit constraints on the service times and arrival rate.

Proof of theorem 2.6. We first show that (b), the existence of a solution $x > 0$ to the linear system (2.10)–(2.20), implies (a), that $m \in \mathcal{M}_\infty$. We proved the equivalence of (b) and (c) in lemma 4.4. Then we show that (a) implies (c), thus proving the equivalence of (a), (b) and (c).

Suppose that $m > 0$ is a service time vector for which there exists an $x = (x_k) > 0$ satisfying (2.10)–(2.20). By lemma 4.2, f is a piecewise linear Lyapunov function proving that m is in the global stability region. To see that m is in the monotone global stability region, observe that for each $0 < \widetilde{m} \leqslant m$, $\tilde{\mu} = (1/\widetilde{m}_k) \geqslant \mu$ and x satisfies (2.10)–(2.20) with μ replaced by $\tilde{\mu}$. Thus, $f(x, Q(\cdot))$ is also a piecewise linear Lyapunov function proving that $\widetilde{m}$ is in the global stability region as well.

Consider a service time vector $m > 0$ such that

$$m \notin \mathcal{D}_0 \cap \{m \in \mathbb{R}^d_+ \colon \lambda m_2 + \lambda^2 m_4 m_6 < 1\}.$$

To show that (a) implies (c), it is enough to show that $m \notin \mathcal{M}_\infty$. If $m \notin \mathcal{D}_0$, then m is clearly not in the global stability region and hence not in $\mathcal{M}_\infty$. So, suppose that m is in $\mathcal{D}_0$ and $\lambda m_2 + \lambda^2 m_4 m_6 \geqslant 1$. If $m_4 > m_3$, then it follows from theorem 2.7 that m is not in the global stability region and hence not in the monotone global stability region. If $m_4 \leqslant m_3$, let

$$\widetilde{m} = (m_1, m_2, \widetilde{m}_3, m_4, m_5, m_6),$$

where $0 < \widetilde{m}_3 < m_4 \leqslant m_3$. Clearly, $\widetilde{m} \leqslant m$ and, by theorem 2.7, $\widetilde{m}$ is not in the global stability region. Therefore, m is not in the monotone global stability region of the fluid network.

Finally, we show that the global stability region $\mathcal{D}_\infty$ is not monotone. Let $\lambda = 1$ and consider the service times

$$m = (0.1, 0.85, 0.5, 0.4, 0.1, 0.4).$$

Since $\lambda m_1 + m_4/m_3 = 0.9 < 1$, it follows from lemma 4.5 that the fluid network is globally stable. Now, suppose that server 3 works faster on class 3 fluids and so the service time m_3 is reduced to $\widetilde{m}_3 = 0.1$, for example. The other service times remain unchanged. That is,

$$\widetilde{m} = (0.1, 0.85, 0.1, 0.4, 0.1, 0.4).$$

Since $\widetilde{m}_4 > \widetilde{m}_3$ and $\lambda \widetilde{m}_2 + \lambda^2 \widetilde{m}_4 \widetilde{m}_6 = 1.01 > 1$, it follows from theorem 2.7 that the network is not globally stable when the service time vector is $\widetilde{m}$. □

5. The power of the LP by Bertsimas, Gamarnik and Tsitsiklis

Based on a path decomposition approach, Bertsimas, Gamarnik and Tsitsiklis [1] proposed a linear program (LP) to determine whether a particular service time vector m is in the global stability region. They proved that for two-station networks, the LP has a solution with positive objective value if and only if the network is not globally stable. They further conjectured that the same would be true for general networks.

In this section we prove that their LP does not provide a sharp characterization of the global stability region or the monotone global stability region of the fluid network in figure 1.

Proof of theorem 2.8. When $\lambda = 1$ the service time vector

$$m = (0.5, 0.5, 0.5, 0.4, 0.01, 0.4) \tag{5.1}$$

is in $\mathcal{M}_\infty$. Therefore, the fluid network with these service times and arrival rate $\lambda = 1$ is globally stable. However, for the service time vector m, a feasible solution to the LP (2.27)–(2.34) with positive objective value is given by

$$\tau_1 = \tau_2 = \tau_3 = 10,$$

$$\begin{array}{lll} \tau_{11} = 5, & \tau_{21} = 5, & \tau_{31} = 6.25, \\ \tau_{12} = 7, & \tau_{22} = 10, & \tau_{32} = 7, \\ \tau_{13} = 3, & \tau_{23} = 0, & \tau_{33} = 1.75, \\ \tau_{41} = 5, & \tau_{51} = 0.3, & \tau_{61} = 3.75, \\ \tau_{42} = 0, & \tau_{52} = 0, & \tau_{62} = 0, \\ \tau_{43} = 7, & \tau_{53} = 0, & \tau_{63} = 8.25. \end{array}$$

□

Remark 5.1. In an earlier, unpublished version of Bertsimas et al. [1], the authors proposed a different LP for which the number of constraints grows exponentially in the number of classes in the network. It was pointed out to us that, for the service time vector m in (5.1), this LP has an optimal objective value 0, and thus correctly detects m being in the global stability region. It is an open problem whether this LP characterizes the global stability of a general fluid network or even the global stability region of our three-station fluid network.

6. Static buffer priority disciplines

Chen and Zhang [11] employed linear Lyapunov functions to study the stability of fluid networks under static buffer priority disciplines. They showed that if an LP related to their linear Lyapunov function has positive objective value, the fluid network is stable under the discipline. In this section, we show that the converse is not true. Namely, we demonstrate service times m in $\mathcal{D}_{\pi_{\{4,2,6\}}}$, the stability region of our three-station network under the discipline that gives higher priorities to classes 2, 4 and 6, for which the LP of Chen and Zhang has maximum objective value 0. Thus, their LP does not provide a sharp characterization of the stability of a priority fluid network.

For each $x = (x_k) > 0$ and fluid solution $(Q(\cdot), T(\cdot))$ under the priority discipline $\pi_{\{4,2,6\}}$ define

$$f\big(x, Q(t)\big) = \sum_{k=1}^{6} x_k Q_k(t).$$

Clearly, for fixed x, f is a linear function of $Q(t)$. We often write $f(Q(t))$ in place of the more cumbersome $f(x, Q(t))$.

If, for each fluid solution $(Q(\cdot), T(\cdot))$ under the discipline $\pi_{\{4,2,6\}}$ and regular point t such that $Q(t) \neq 0$,

$$\frac{\mathrm{d}f(Q(t))}{\mathrm{d}t} \leqslant -\varepsilon < 0, \tag{6.1}$$

then $f(Q(t)) = 0$, and hence, $Q(t) = 0$, for all $t \geqslant f(Q(0))/\varepsilon$. In this case, f is a linear Lyapunov function proving that the network is stable under the discipline $\pi_{\{4,2,6\}}$.

For each regular point t of the fluid solution $(Q(\cdot), T(\cdot))$

$$\frac{\mathrm{d}f(Q(t))}{\mathrm{d}t} = \sum_{k=1}^{6} x_k \dot{Q}_k(t) = \sum_{k=1}^{6} x_k (d_{k-1} - d_k),$$

where $d_k = \mu_k \dot{T}_k(t)$ for $k = 1, 2, \ldots, 6$ and $d_0 = \lambda$. To ensure (6.1), we impose the linear constraint

$$\sum_{k=1}^{6} x_k (d_{k-1} - d_k) + \varepsilon \leqslant 0 \tag{6.2}$$

on x for each feasible choice of $(d_1, d_2, \ldots, d_6)$.

The feasible values of $(d_1, d_2, \ldots, d_6) \geqslant 0$ depend on the fluid state $Q(t)$ in the following ways:

$$d_k = d_{k-1} \quad \text{if } Q_k(t) = 0,\ k = 1, 2, \ldots, 6, \tag{6.3}$$
$$d_1 = 0 \quad \text{if } Q_4(t) > 0, \tag{6.4}$$
$$d_5 = 0 \quad \text{if } Q_2(t) > 0, \tag{6.5}$$
$$d_3 = 0 \quad \text{if } Q_6(t) > 0, \tag{6.6}$$
$$d_1 m_1 + d_4 m_4 = 1 \quad \text{if } Z_1(t) > 0, \tag{6.7}$$
$$d_2 m_2 + d_5 m_5 = 1 \quad \text{if } Z_2(t) > 0, \tag{6.8}$$
$$d_3 m_3 + d_6 m_6 = 1 \quad \text{if } Z_3(t) > 0. \tag{6.9}$$

Equation (6.3) follows from Dai and Weiss [19, proposition 4.2]. Equations (6.4)–(6.6) follow from (2.7). Finally, equations (6.7)–(6.9) follow from (2.5). We refer to the set of all non-negative vectors $d = (d_1, d_2, \ldots, d_6)$ that satisfy (6.3)–(6.9) for some $Q(t) \geqslant 0$ as $\mathcal{T}_{\pi_{\{4,2,6\}}}$.

Lemma 6.1 is an immediate consequence of (6.2), it specializes the LP criterion of Chen and Zhang [11] to our three-station network.

Lemma 6.1. If the following LP has positive objective value:

$$\max\ \varepsilon \tag{6.10}$$

subject to:

$$\sum_{k=1}^{6} x_k \leqslant 1, \tag{6.11}$$

$$\sum_{k=1}^{6} x_k\left(d_{k-1}^s - d_k^s\right) + \varepsilon \leqslant 0 \quad \text{for each } d^s \in \mathcal{T}_{\pi_{\{4,2,6\}}}, \tag{6.12}$$

$$x = (x_k) \geqslant 0, \tag{6.13}$$

then the fluid network is stable under the static buffer priority discipline $\pi_{\{4,2,6\}}$ and so $m \in \mathcal{D}_{\pi_{\{4,2,6\}}}$.

We next show that the converse of lemma 6.1 is not true and hence that the LP of Chen and Zhang does not provide a sharp characterization of stability under static priority disciplines.

Proof of theorem 2.11. Let $\lambda = 1$ and let

$$m = (0.001,\ 0.18,\ 0.001,\ 0.9,\ 0.001,\ 0.9)$$

be the service time vector. Clearly, m satisfies the usual traffic conditions (2.8). Since

$$m_2 + m_4 m_6 = 0.99 < 1,$$

by theorem 2.6, m is in the monotone global stability region, and hence, in $\mathcal{D}_{\pi_{\{4,2,6\}}}$.

To show that there is no solution to the LP (6.10)–(6.13) with positive objective value, we demonstrate a feasible solution to the dual problem with objective value 0. The dual of (6.10)–(6.13) is:

$$\min \alpha \tag{6.14}$$

subject to:

$$\sum_{s \in \mathcal{T}_{\pi_{\{4,2,6\}}}} y_s = 1, \tag{6.15}$$

$$\sum_{s \in \mathcal{T}_{\pi_{\{4,2,6\}}}} y_s\left(d_{k-1}^s - d_k^s\right) + \alpha \geqslant 0 \quad \text{for each } k = 1, 2, \ldots, 6, \tag{6.16}$$

$$y = (y_s) \geqslant 0. \tag{6.17}$$

Table 1
Departure rates for the seven states used in our dual solution. Note that the state only lists the highest priority class at each station with positive buffer level.

Case	State	Departure rates
1	$Q_2(t) > 0,\ Q_4(t) > 0,\ Q_6(t) > 0$	$d_1 = d_3 = d_5 = 0,\ d_2 = \mu_2,\ d_4 = \mu_4,\ d_6 = \mu_6$
2	$Q_2(t) > 0,\ Q_3(t) > 0,\ Q_4(t) > 0$	$d_1 = d_5 = d_6 = 0,\ d_2 = \mu_2,\ d_3 = \mu_3,\ d_4 = \mu_4$
3	$Q_2(t) > 0,\ Q_4(t) > 0$	$d_1 = d_5 = d_6 = 0,\ d_2 = d_3 = \mu_2,\ d_4 = \mu_4$
4	$Q_4(t) > 0,\ Q_5(t) > 0,\ Q_6(t) > 0$	$d_1 = d_2 = d_3 = 0,\ d_4 = \mu_4,\ d_5 = \mu_5,\ d_6 = \mu_6$
5	$Q_4(t) > 0$	$d_1 = d_2 = d_3 = 0,\ d_4 = d_5 = d_6 = \mu_4$
6	$Q_1(t) > 0,\ Q_2(t) > 0,\ Q_6(t) > 0$	$d_3 = d_4 = d_5 = 0,\ d_1 = \mu_1,\ d_2 = \mu_2,\ d_6 = \mu_6$
7	$Q_6(t) > 0$	$d_3 = d_4 = d_5 = 0,\ d_1 = d_2 = 1,\ d_6 = \mu_6$

Our solution involves the seven states described in table 1.

Tedious algebra establishes that

$$y_6 = \frac{m_1 m_4}{1 - m_1} \approx 0.00090, \tag{6.18}$$

$$y_7 = \frac{1 - m_1 - m_4}{1 - m_1} \approx 0.09910, \tag{6.19}$$

$$y_2 = \frac{\mu_6(1 - m_2) + m_4\mu_5(m_1 - m_2)/(m_1 - 1) - 1}{\mu_6(1 - m_2\mu_3) + m_4\mu_5(\mu_5 - \mu_4)} \approx 0.00018, \tag{6.20}$$

$$y_3 = m_2 - m_2\mu_3 y_2 \approx 0.14772, \tag{6.21}$$

$$y_4 = \frac{m_1 - m_2}{1 - m_1} m_4 + m_4\mu_5 y_2 \approx 0.00013, \tag{6.22}$$

$$y_5 = m_4 - m_4\mu_5 y_2 \approx 0.73861, \tag{6.23}$$

$$y_1 = \frac{m_2 - m_1}{1 - m_1} m_4 - m_2 - (1 - m_2\mu_3)\pi_2 \approx 0.01336, \tag{6.24}$$

and $y_s = 0$ otherwise describes a feasible solution to the dual problem (6.14)–(6.17) with $\alpha = 0$ proving that there is no solution to the LP (6.10)–(6.13) with positive objective value. □

Nevertheless, linear Lyapunov functions remain a powerful tool for establishing the global stability of priority networks. In fact, we rely on this tool to prove that the stability regions of the static buffer priority disciplines do not characterize the global stability region of a network with more than two stations.

Dai and Vande Vate [18] showed that the global stability region of a two-station fluid network is determined by static buffer priority disciplines. We show that this is not the case for fluid networks with more than two stations. This helps explain why we required the dynamic disciplines used in the proof of theorem 2.6 to characterize the monotone global stability region of our three-station network.

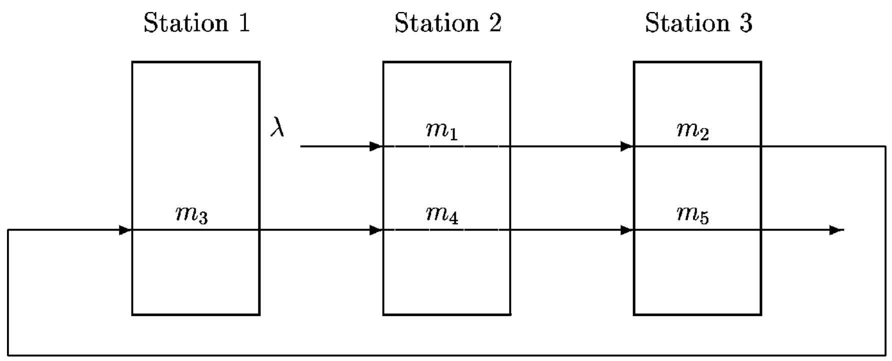

Figure 2. The five-class network obtained by deleting class 1 from the six-class fluid network.

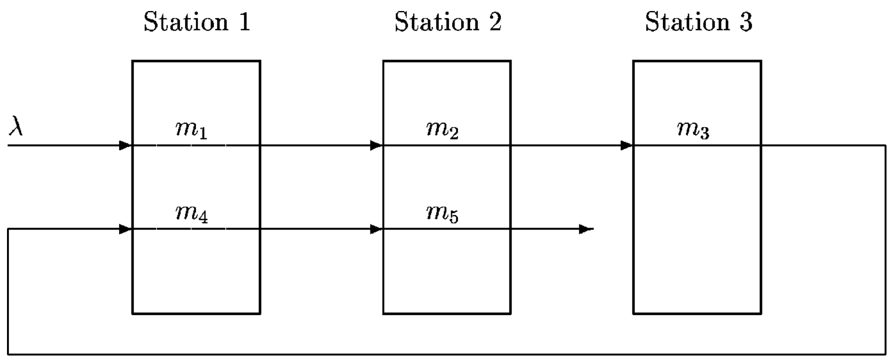

Figure 3. The five-class network obtained by deleting class 6 from the six-class fluid network.

We first show that the stability region of the network under all but one of the static buffer priority disciplines is determined by the usual traffic conditions at each station. Thus, stability under the remaining static buffer priority discipline $\pi_{\{4,2,6\}}$ implies stability under all static buffer priority disciplines. We then demonstrate a service time vector m that is not in the global stability region, but is in the stability region under the discipline $\pi_{\{4,2,6\}}$. This shows that the global stability region of a fluid network with more than two stations is determined by a richer family of disciplines than simply the static buffer priority disciplines.

We show that every fluid solution under a discipline that gives priority to class 1 over class 4 reduces to a fluid solution in the five-class network in figure 2 obtained by deleting class 1. Similarly, every fluid solution under a discipline that gives priority to class 3 over class 6 eventually reduces to a fluid solution in the five-class network in figure 3 obtained by deleting class 6.

We start by showing that the global stability regions of these two five-class subnetworks are defined by the usual traffic conditions at each station.

Lemma 6.2. The five-class three-station fluid network in figure 2 is globally stable so long as the traffic intensity at each station is less than one.

Proof. Consider the fluid network in figure 2. For a given $x = (x_1, \ldots, x_5)' > 0$, let

$$f_1\bigl(x, Q(t)\bigr) = x_3 Q_3^+(t),$$
$$f_2\bigl(x, Q(t)\bigr) = x_1 Q_1^+(t) + x_4 Q_4^+(t),$$
$$f_3\bigl(x, Q(t)\bigr) = x_2 Q_2^+(t) + x_5 Q_5^+(t),$$

where, as before, $Q_k^+(t) = \sum_{\ell=1}^{k} Q_\ell(t)$. If, for each non-idling fluid solution $(Q(\cdot), T(\cdot))$ of the network, f_1, f_2 and f_3 satisfy conditions (4.2)–(4.4), it follows from the proof of lemma 4.1 that the fluid network in figure 2 is globally stable.

Mimicking the proof of lemma 4.2, (4.2)–(4.4) hold if there is $x = (x_1, \ldots, x_5) > 0$ satisfying

$$\begin{aligned}
\lambda(x_1 + x_4) &< x_1\mu_1,\\
\lambda(x_1 + x_4) &< x_4\mu_4,\\
\lambda(x_2 + x_5) &< x_2\mu_2,\\
\lambda(x_2 + x_5) &< x_5\mu_5,\\
\lambda x_3 &< x_3\mu_3,\\
x_4 &\leqslant x_3,\\
x_5 &\leqslant x_4,\\
x_2 + x_5 &\leqslant x_1 + x_4,\\
x_3 &\leqslant x_2 + x_5.
\end{aligned}$$

Employing the techniques used in the proof of lemma 4.4, we conclude that there is $x > 0$ satisfying (6.25)–(6.25) if and only if

$$\lambda(m_1 + m_4) < 1, \qquad \lambda(m_2 + m_5) < 1, \qquad \lambda m_3 < 1.$$

This proves the lemma for the network in figure 2. □

The corresponding result for the network in figure 3 follows immediately from re-numbering the stations of the network in figure 2.

Corollary 6.3. The five-class three-station fluid network in figure 3 is globally stable so long as the traffic intensity at each station is less than one.

Lemma 6.4. The stability region for any non-idling discipline that gives priority to class 3 over class 6 is $\mathcal{D}_0$.

Proof. Consider $m \in \mathcal{D}_0$. Any fluid solution $(Q(\cdot), T(\cdot))$ under the priority discipline satisfies (2.1)–(2.5). In addition, $(Q(\cdot), T(\cdot))$ satisfies $\dot{T}_3(t) = 1$ for each regular point t such that $Q_3(t) > 0$. Therefore, $(Q_1(t), \ldots, Q_5(t))$ together with $(T_1(t), \ldots, T_5(t))$ is a fluid solution to the five-class fluid network of figure 3 and, by corollary 6.3, there exists $\delta > 0$ such that $(Q_1(t), \ldots, Q_5(t)) = 0$ for $t \geqslant \delta$. After δ, the input rate to buffer 6 is λ. If $Q_6(t) > 0$ for a regular point $t > \delta$, the departure rate d_6 from buffer 6 satisfies $\lambda m_3 + d_6 m_6 = 1$ [19, proposition 4.2]. Thus, $d_6 = \mu_6(1 - \lambda m_3)$, which is faster than the input rate λ. Hence, buffer 6 will be empty by

$$\frac{Q_6(0) + \lambda\delta}{\mu_6(1 - \lambda m_3) - \lambda}.$$

Therefore, m is in the stability region. □

Lemma 6.5. The stability region for any non-idling discipline that gives priority to class 1 over class 4 is $\mathcal{D}_0$.

Proof. Consider $m \in \mathcal{D}_0$. Any fluid solution $(Q(\cdot), T(\cdot))$ under the priority discipline satisfies (2.1)–(2.5). In addition, $(Q(\cdot), T(\cdot))$ satisfies $\dot{T}_1(t) = 1$ for each regular point t such that $Q_1(t) > 0$. Because $\lambda m_1 < 1$, $Q_1(t) = 0$ for $t \geqslant \delta_0 = Q_1(0)/(\mu_1 - \lambda)$. For notational convenience, we assume $Q_1(0) = 0$ and hence $\delta_0 = 0$. From (2.1)–(2.4), we have $\mu_1 T_1(t) = \lambda t$, and hence,

$$\begin{aligned}
Q_2(t) &= Q_2(0) + \lambda t - \mu_2 T_2(t),\\
Q_3(t) &= Q_3(0) + \mu_2 T_2(t) - \mu_3 T_3(t),\\
Q_4(t) &= Q_4(0) + \mu_3 T_3(t) - \mu_4 T_4(t),\\
Q_5(t) &= Q_5(0) + \mu_4 T_4(t) - \mu_5 T_5(t),\\
Q_6(t) &= Q_6(0) + \mu_5 T_5(t) - \mu_6 T_6(t),
\end{aligned}$$

and

$$\begin{aligned}
\dot{T}_2(t) + \dot{T}_5(t) &= 1 \qquad \text{if } Q_2(t) + Q_5(t) > 0,\\
\dot{T}_3(t) + \dot{T}_6(t) &= 1 \qquad \text{if } Q_3(t) + Q_6(t) > 0,\\
\lambda m_1 + \dot{T}_4(t) &= 1 \qquad \text{if } Q_4(t) > 0.
\end{aligned}$$

Let $\widetilde{T}_4 = T_4(t)/(1 - \lambda m_1)$, $\widetilde{m}_4 = m_4/(1 - \lambda m_1)$ and $\tilde{\mu}_4 = 1/\widetilde{m}_4$. Then, we have

$$\begin{aligned}
Q_2(t) &= Q_2(0) + \lambda t - \mu_2 T_2(t),\\
Q_3(t) &= Q_3(0) + \mu_2 T_2(t) - \mu_3 T_3(t),\\
Q_4(t) &= Q_4(0) + \mu_3 T_3(t) - \tilde{\mu}_4 \widetilde{T}_4(t),\\
Q_5(t) &= Q_5(0) + \tilde{\mu}_4 \widetilde{T}_4(t) - \mu_5 T_5(t),\\
Q_6(t) &= Q_6(0) + \mu_5 T_5(t) - \mu_6 T_6(t),
\end{aligned}$$

and

$$\begin{aligned}
\dot{T}_2(t) + \dot{T}_5(t) &= 1 \qquad \text{if } Q_2(t) + Q_5(t) > 0,\\
\dot{T}_3(t) + \dot{T}_6(t) &= 1 \qquad \text{if } Q_3(t) + Q_6(t) > 0,\\
\dot{\widetilde{T}}_4(t) &= 1 \qquad \text{if } Q_4(t) > 0.
\end{aligned}$$

Therefore, $(Q_2(t),\dots,Q_6(t))$ together with $(T_2(t)$, $T_3(t)$, $\widetilde{T}_4(t)$, $T_5(t)$, $T_6(t))$ is a fluid solution to the five-class fluid network of figure 2 with service times $(m_2, m_3, \widetilde{m}_4, m_5, m_6)$. Since $m \in \mathcal{D}_0$, we have

$$\begin{aligned}\lambda\widetilde{m}_4 &< 1,\\ \lambda(m_2+m_5) &< 1,\\ \lambda(m_3+m_6) &< 1.\end{aligned}$$

It follows from lemma 6.2 that $(Q_2(t),\dots,Q_6(t)) = 0$ for $t > \delta$ for some $\delta > 0$. □

Proof of theorem 2.9. Part (a). By lemma 6.4, $\mathcal{D}_\pi = \mathcal{D}_0$ for $\pi = \pi_{\{1,2,3\}}, \pi_{\{1,5,3\}}, \pi_{\{4,2,3\}}, \pi_{\{4,5,3\}}$. By lemma 6.5, $\mathcal{D}_\pi = \mathcal{D}_0$ for $\pi = \pi_{\{1,2,6\}}, \pi_{\{1,5,6\}}$. The static buffer priority discipline $\pi_{\{4,5,6\}}$ corresponds to the last-buffer-first-served priority discipline, whose stability region Dai and Weiss [19] showed to be $\mathcal{D}_0$.

Part (b). Let $\lambda = 1$. Hasenbein [25] proved that under the preemptive-resume priority discipline $\pi_{\{4,2,6\}}$ in the corresponding queueing network, classes 2, 4 and 6 constitute a *pseudostation*, in which at most two classes of jobs can be processed simultaneously. Assume the queueing network is initially empty. Let $(Q(\cdot), T(\cdot))$ be a fluid limit as taken in Dai [14]. Because classes 2, 4 and 6 constitute a pseudostation, we have

$$T_2(t) + T_4(t) + T_6(t) \leqslant 2t.$$

Assume that $m_2 + m_4 + m_6 > 2$. Let

$$\begin{aligned}g(t) &= m_2 Q_2^+(t) + m_4 Q_4^+(t) + m_6 Q_6^+(t)\\ &= g(0) + (m_2+m_4+m_6)t - \bigl(T_2(t)+T_4(t)+T_6(t)\bigr)\\ &\geqslant g(0) + \bigl[(m_2+m_4+m_6) - 2\bigr]t.\end{aligned}$$

It is clear that $g(t) \to \infty$ as $t \to \infty$. Because such a fluid limit is a fluid solution to equations (2.1)–(2.7), the fluid model is unstable under the discipline. The service time vector,

$$m = (0.1,\ 0.8,\ 0.1,\ 0.8,\ 0.1,\ 0.8),$$

for example, is in $\mathcal{D}_0$, but since $m_2 + m_4 + m_6 = 2.4 > 2$, $m \notin \mathcal{D}_\pi$.

Part (c). Let $\lambda = 1$ and consider the service time vector

$$m = (0.1,\ 0.8,\ 0.1,\ 0.45,\ 0.1,\ 0.45).$$

It is easy to check that

$$m_4 > m_3, \qquad \lambda m_2 + \lambda^2 m_4 m_6 = 1.0025 > 1$$

and so, by theorem 2.7, m is not in the global stability region.

It will be shown in the appendix that

$$x = (139, 139, 59, 63, 27, 27)$$

satisfies the linear constraints in (6.10)–(6.13) with $\varepsilon = 1$. Hence, $m \in \mathcal{D}_{\pi_{\{4,2,6\}}}$. □

Remark 6.6. For our three-station network, the pseudostation conditions as defined in Hasenbein [25] are

$$\begin{aligned} \lambda(m_1 + m_4) &< 1, \\ \lambda(m_2 + m_5) &< 1, \\ \lambda(m_3 + m_6) &< 1, \\ \lambda(m_2 + m_4 + m_6) &< 2. \end{aligned}$$

The proof of theorem 2.4 part (b) shows that these conditions are *not* sufficient to ensure (monotone) global stability for this network.

7. Concluding remarks

Dai and Vande Vate [17,18] showed that piecewise linear Lyapunov functions of the type used in this paper characterize the global stability region of two-station fluid networks and that these global stability regions are monotone. In this paper, we have shown that the global stability regions of networks with more than two stations need not be monotone, but piecewise linear Lyapunov functions do characterize the monotone global stability region for the three-station fluid network of figure 1. Unfortunately, analogous results for general networks appear difficult to obtain because the constraints on the coefficients of the Lyapunov functions are nonlinear in general. Further, since static buffer priority disciplines do not characterize the global stability region, the disciplines required to establish the necessity of proposed conditions may be rather complex. Down and Meyn [20] provided a way to linearize some similar nonlinear constraints. However, it does not appear that their linearization yields an equivalent problem.

Acknowledgement

We thank the anonymous referees for improving the exposition and streamlining a number of proofs.

Appendix

In this appendix, we complete part (c) proof of theorem 2.9 by showing that

$$x = (139, 139, 59, 63, 27, 27)$$

satisfies the linear constraints in (6.10)–(6.13) with $\varepsilon = 1$.

Recall that to generate the vectors $d^s \in \mathcal{T}_{\pi_{\{4,2,6\}}}$, we solve (6.3)–(6.9) for each of the possible cases. These cases reduce to the following three at each station:

1. The higher priority buffer has positive fluid level.

Table 2
Enumeration of 26 states: each state corresponds to a different set of highest priority non-empty buffers.

Case	Station A	Station B	Station C
1	None	None	3
2	None	None	6
3	None	5	None
4	None	5	3
5	None	5	6
6	None	2	None
7	None	2	3
8	None	2	6
9	1	None	None
10	1	None	3
11	1	None	6
12	1	5	None
13	1	5	3
14	1	5	6
15	1	2	None
16	1	2	3
17	1	2	6
18	4	None	None
19	4	None	3
20	4	None	6
21	4	5	None
22	4	5	3
23	4	5	6
24	4	2	None
25	4	2	3
26	4	2	6

2. Only the lower priority buffer has positive fluid level.

3. Both buffers are empty.

These three cases at each of the three stations lead to the 26 cases listed in table 2 (there is no need to consider the case in which all the buffers are empty).

If the solution d for a case (and the solution is unique for each case) does not satisfy

$$d_1 m_1 + d_4 m_4 \leqslant 1, \tag{A.1}$$

$$d_2 m_2 + d_5 m_5 \leqslant 1, \tag{A.2}$$

$$d_3 m_3 + d_6 m_6 \leqslant 1, \quad \text{and} \tag{A.3}$$

$$d_i \geqslant 0 \quad \text{for } i = 1, 2, \ldots, 6, \tag{A.4}$$

Table 3
The departure rates for all 6 classes in all the states of the three-station fluid network under the static priority discipline $\pi_{\{4,2,6\}}$. Each state is characterized by giving the highest priority non-empty buffer (if any) at each station as indicated in table 2.

Case	Departure rate
1	$d_2 = d_1,\ d_4 = d_3,\ d_6 = d_5,\ d_1 = \lambda,\ d_5 = 1/(m_3 + m_6),\ d_3 = d_5$
2	$d_3 = d_4 = 0,\ d_2 = d_1,\ d_6 = \mu_6,\ d_1 = \lambda,\ d_5 = 0$
3	$d_2 = d_1,\ d_4 = d_3,\ d_6 = d_5,\ d_1 = \lambda,\ d_3 = \lambda,\ d_5 = \mu_5(1 - \lambda m_2)$
4	$d_2 = d_1,\ d_4 = d_3,\ d_6 = d_5,\ d_1 = \lambda,\ d_5 = \mu_5(1 - \lambda m_2)$
5	$d_3 = d_4 = 0,\ d_2 = d_1,\ d_6 = \mu_6,\ d_1 = \lambda,\ d_5 = \mu_5(1 - \lambda m_2)$
6	$d_5 = d_6 = 0,\ d_2 = \mu_2,\ d_4 = d_3,\ d_3 = \mu_2,\ d_1 = \lambda$
7	$d_5 = d_6 = 0,\ d_2 = \mu_2,\ d_4 = d_3,\ d_3 = \mu_3,\ d_1 = \lambda$
8	$d_3 = d_4 = d_5 = 0,\ d_2 = \mu_2,\ d_6 = \mu_6,\ d_1 = \lambda$
9	$d_2 = d_1,\ d_4 = d_3,\ d_6 = d_5,\ d_1 = 1/(m_1 + m_4),\ d_3 = d_1,\ d_5 = d_1$
10	$d_2 = d_1,\ d_4 = d_3,\ d_6 = d_5,\ d_3 = 1/(m_3 + m_6),\ d_5 = d_4,\ d_1 = \mu_1(1 - d_3 m_4)$
11	$d_3 = d_4 = 0,\ d_2 = d_1,\ d_6 = \mu_6,\ d_1 = \mu_1,\ d_5 = 0$
12	$d_2 = d_1,\ d_4 = d_3,\ d_6 = d_5,\ d_1 = 1/(m_1 + m_4),\ d_3 = d_1,\ d_5 = \mu_5(1 - d_1 m_2)$
13	$d_1 m_1 + d_3 m_4 = 1,\ d_1 m_2 + d_5 m_5 = 1,\ d_3 m_3 + d_5 m_6 = 1,\ d_2 = d_1,\ d_4 = d_3,\ d_6 = d_5$
14	$d_3 = d_4 = 0,\ d_2 = d_1,\ d_6 = \mu_6,\ d_1 = \mu_1,\ d_5 = \mu_5(1 - \mu_1 m_2)$
15	$d_5 = d_6 = 0,\ d_2 = \mu_2,\ d_4 = d_3,\ d_3 = \mu_2,\ d_1 = \mu_1(1 - \mu_2 m_4)$
16	$d_5 = d_6 = 0,\ d_2 = \mu_2,\ d_4 = d_3,\ d_3 = \mu_3,\ d_1 = \mu_1(1 - \mu_3 m_4)$
17	$d_3 = d_4 = d_5 = 0,\ d_2 = \mu_2,\ d_6 = \mu_6,\ d_1 = \mu_1$
18	$d_1 = d_2 = 0,\ d_4 = \mu_4,\ d_6 = d_5,\ d_5 = \mu_4,\ d_3 = 0$
19	$d_1 = d_2 = 0,\ d_4 = \mu_4,\ d_6 = d_5,\ d_5 = \mu_4,\ d_3 = \mu_3(1 - \mu_4 m_6)$
20	$d_1 = d_2 = d_3 = 0,\ d_4 = \mu_4,\ d_6 = \mu_6,\ d_5 = \mu_4$
21	$d_1 = d_2 = 0,\ d_4 = \mu_4,\ d_6 = d_5,\ d_5 = \mu_5,\ d_3 = 0$
22	$d_1 = d_2 = 0,\ d_4 = \mu_4,\ d_6 = d_5,\ d_5 = \mu_5,\ d_3 = \mu_3(1 - \mu_5 m_6)$
23	$d_1 = d_2 = d_3 = 0,\ d_4 = \mu_4,\ d_6 = \mu_6,\ d_5 = \mu_5$
24	$d_1 = d_5 = d_6 = 0,\ d_2 = \mu_2,\ d_4 = \mu_4,\ d_3 = \mu_2$
25	$d_1 = d_5 = d_6 = 0,\ d_2 = \mu_2,\ d_4 = \mu_4,\ d_3 = \mu_3$
26	$d_1 = d_3 = d_5 = 0,\ d_2 = \mu_2,\ d_4 = \mu_4,\ d_6 = \mu_6$

then the corresponding state is not feasible, and hence, not in $\mathcal{T}_{\pi_{\{4,2,6\}}}$. Otherwise, we include d in $\mathcal{T}_{\pi_{\{4,2,6\}}}$. Table 3 shows the departure rates d in each case and table 4 shows the departure rates for the service times $m = (0.1, 0.8, 0.1, 0.45, 0.1, 0.45)$ used in part (c) proof of theorem 2.9. Table 5 shows both the rates of change in the buffer levels $\dot{Q}$ and the value of

$$\frac{\mathrm{d}f(x, Q(t))}{\mathrm{d}t} = \sum_{k=1}^{6} \dot{Q}_k x_k,$$

where $x = (139, 139, 59, 63, 27, 27)$, for each regular state. This demonstrates that $f(x, Q(t))$ is a linear Lyapunov function proving the network is stable under the static buffer priority discipline $\pi_{\{4,2,6\}}$.

Table 4

The departure rates for all 6 classes in all the states of the three-station fluid network with processing times $m = (0.1, 0.8, 0.1, 0.45, 0.1, 0.45)$ under the static priority discipline $\pi_{\{4,2,6\}}$. Each state is characterized by giving the highest priority non-empty buffer (if any) at each station as indicated in table 2. A state is feasible if the departure rates are non-negative and at most 100% of each server's time is allocated. Values preventing states from being feasible are indicated with boldfaced type.

	Departure rate						Busy fraction			
Case	d_1	d_2	d_3	d_4	d_5	d_6	A	B	C	Feasible
1	1.00	1.00	1.82	1.82	1.82	1.82	0.92	0.98	1.00	yes
2	1.00	1.00	0.00	0.00	0.00	2.22	0.10	0.80	1.00	yes
3	1.00	1.00	1.00	1.00	2.00	2.00	0.55	1.00	1.00	yes
4	1.00	1.00	1.00	1.00	2.00	2.00	0.55	1.00	1.00	yes
5	1.00	1.00	0.00	0.00	2.00	2.22	0.10	1.00	1.00	yes
6	1.00	1.25	1.25	1.25	0.00	0.00	0.66	1.00	0.13	yes
7	1.00	1.25	10.00	10.00	0.00	0.00	**4.60**	1.00	1.00	no
8	1.00	1.25	0.00	0.00	0.00	2.22	0.10	1.00	1.00	yes
9	1.82	1.82	1.82	1.82	1.82	1.82	1.00	**1.64**	1.00	no
10	1.82	1.82	1.82	1.82	1.82	1.82	1.00	**1.64**	1.00	no
11	10.00	10.00	0.00	0.00	0.00	2.22	1.00	**8.00**	1.00	no
12	1.82	1.82	1.82	1.82	−**4.55**	−**4.55**	1.00	1.00	−1.86	no
13	1.03	1.03	1.99	1.99	1.78	1.78	1.00	1.00	1.00	yes
14	10.00	10.00	0.00	0.00	−**70.00**	2.22	1.00	1.00	1.00	no
15	4.38	1.25	1.25	1.25	0.00	0.00	1.00	1.00	0.13	yes
16	−**35.00**	1.25	10.00	10.00	0.00	0.00	1.00	1.00	1.00	no
17	10.00	1.25	0.00	0.00	0.00	2.22	1.00	1.00	1.00	yes
18	0.00	0.00	0.00	2.22	2.22	2.22	1.00	0.22	1.00	yes
19	0.00	0.00	0.00	2.22	2.22	2.22	1.00	0.22	1.00	yes
20	0.00	0.00	0.00	2.22	2.22	2.22	1.00	0.22	1.00	yes
21	0.00	0.00	0.00	2.22	10.00	10.00	1.00	1.00	**4.50**	no
22	0.00	0.00	−**35.00**	2.22	10.00	10.00	1.00	1.00	1.00	no
23	0.00	0.00	0.00	2.22	10.00	2.22	1.00	1.00	1.00	yes
24	0.00	1.25	1.25	2.22	0.00	0.00	1.00	1.00	0.13	yes
25	0.00	1.25	10.00	2.22	0.00	0.00	1.00	1.00	1.00	yes
26	0.00	1.25	0.00	2.22	0.00	2.22	1.00	1.00	1.00	yes

Table 5
Rates of change in the buffer levels for the 17 feasible states in the three-station fluid network with processing times $m = (0.1, 0.8, 0.1, 0.45, 0.1, 0.45)$ under the static priority discipline $\pi_{\{4,2,6\}}$. The last column computes $\sum_{k=1}^{6} \dot{Q}_k x_k$ where $x = (139, 139, 59, 63, 27, 27)$. This shows that the network is stable under the discipline $\pi_{\{4,2,6\}}$.

Case	$\dot{Q}_1$	$\dot{Q}_2$	$\dot{Q}_3$	$\dot{Q}_4$	$\dot{Q}_5$	$\dot{Q}_6$	$\sum_k x_k \dot{Q}_k$
1	0.00	0.00	−0.82	0.00	0.00	0.00	−48.27
2	0.00	0.00	1.00	0.00	0.00	−2.22	−1.00
3	0.00	0.00	0.00	0.00	−1.00	0.00	−27.00
4	0.00	0.00	0.00	0.00	−1.00	0.00	−27.00
5	0.00	0.00	1.00	0.00	−2.00	−0.22	−1.00
6	0.00	−0.25	0.00	0.00	1.25	0.00	−1.00
8	0.00	−0.25	1.25	0.00	0.00	−2.22	−21.00
13	−0.03	0.00	−0.97	0.00	0.21	0.00	−55.05
15	−3.38	3.13	0.00	0.00	1.25	0.00	−1.00
17	−9.00	8.75	1.25	0.00	0.00	−2.22	−21.00
18	1.00	0.00	0.00	−2.22	0.00	0.00	−1.00
19	1.00	0.00	0.00	−2.22	0.00	0.00	−1.00
20	1.00	0.00	0.00	−2.22	0.00	0.00	−1.00
23	1.00	0.00	0.00	−2.22	−7.78	7.78	−1.00
24	1.00	−1.25	0.00	−0.97	2.22	0.00	−36.00
25	1.00	−1.25	−8.75	7.78	2.22	0.00	−1.00
26	1.00	−1.25	1.25	−2.22	2.22	−2.22	−101.00

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