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Navier Stokes Equations – and Dynamical Systems

CONTENTS

1. Introduction

There are many deep reasons why the Mathematical Analysis of the Navier-Stokes equations fits in the theory of Dynamical Systems.

At the level of first principles the Navier-Stokes equations can be deduced from the Boltzmann equation which is obtained from a Hamiltonian system describing the evolution of molecules of gaz. To do so, one takes in account the magnitude of N the Avogadro number of the order of 10^{24} and considers the Boltzmann-Grad limit $N \to \infty$.

On the other hand, fluids described by the incompressible Navier Stokes may exhibit very complicated chaotic or self organized structures when the Reynolds number turns out to be very large. Commonly these situations are called "turbulent" and the present challenge is the construction of equations that will be used to compute the evolution of some type of averaged quantities.

Therefore the Navier Stokes equations appear to be one of the main pieces of a sequence of equations:

$$
\begin{array}{c}
\text{Hamiltonian system of particles} \\
\Downarrow \\
\text{Boltzmann equation} \\
\Downarrow \\
\text{Navier Stokes equations} \\
\Downarrow \\
\text{Models of turbulence}
\end{array}
$$

each of them being deduced from the previous one by some averaging process where the notion of irreversibility is embedded.

Irreversibility at the level of the Boltzmann equation and its relation with the irreversibility at the level of the compressible Euler equation, the compressible or incompressible Navier Stokes equations, are by now well understood and will be recalled in section **3.3**; the solutions are related to the notion of semiflow and global attractors which can be extended from finite to infinite dimensional systems.

It is more difficult to understand how the Boltzmann equation can be obtained as the limit of the genuinely reversible system, which describes the flow at the level of molecular dynamics. As will be shown, this can be done by some averaging process where the self interaction of the molecules and therefore the non linearity of the problem plays a crucial role so that the limit is in agreement with the appearance of irreversibility. This will be explained in section **3.2**.

Much more difficult and unsolved questions arise when the macroscopic fluid becomes turbulent and when some type of averaging is necessary for quantitative or qualitative results. In spite of being the very end of the hierarchy, this step shares in common some points with the previous one.

It is an averaging process and the "turbulent model" starts to be efficient when the original Navier Stokes are outside the reach of direct numerical simulations.

In this averaging appears a problem of moments and of closure and the search for something that would play the role of the thermodynamical equilibrium.

However this is not easy for the following reason.

There is up to now no well defined notion of equilibrium and relaxation to this equilibrium with something that would play the role of the entropy.

The parameters that would lead to turbulent phenomena are not so clearly identified as in the previous step of the hierarchy. In some sense they are less universal and more local.

In this process the dynamical point of view is also essential:

The introduction of randomness requires the construction of a "canonical measure" on the set of solutions. This leads to the adaptation of the Birkhoff ergodic theorem to the Navier Stokes equations. The structure of the turbulent spectra which would play the role of the thermodynamical equilibrium has been the object of phenomenological studies initiated by Kolmogorov and Kraichnan (for the two dimensional flow) and it is only to the best of our knowledge, in the frame of statistical semiflows defined on a periodic box that some "spin-offs" of this theory can be proven in full rigor. The structure of the turbulent spectra also leads to the notion of degrees of freedom and exponential decay after the Kolmogorov or Kraichan cut-off wave number. Here also some counterpart of these notions can be proven in full rigor provided the global attractors or exponential attractors of the semiflow are introduced.

Eventually (this is the last chapter of this presentation) at very large scales one observes coherent structures (the classical examples are the Jupiter red spot or the anticyclone of the Açores). These structures are generated through turbulent processes but play the role of metastable thermodynamical equilibriums. Up to now there has been no dynamical derivation for their appearance and stability; however, some notions of entropy and "negative temperature" inherited from statistical mechanics are used and motivated by comparison with the evolution of point vortices which are a "canonical Hamiltonian" system.

As a consequence this contribution is organized as follow. In chapter **2** the basic mathematical properties of the Navier Stokes equations are presented. Relations between compressible and incompressible equations are given and the emphasis is put on the finite time stability (which very often in its mathematical formulation concerns the regularity and uniqueness of solutions).

One should keep in mind that local results for smooth solution go back to Lichtenstein (1927). The fact that these results cannot always be global in time is well understood on the example of the appearance of singularities (in particular shock waves) for the compressible Euler equation. On the other hand:

i) The existence of a global in time weak solution for the 3 dimensional Navier Stokes equation has been proven by Leray. His notion of weak solution (1934) preceded both the introduction of the Sobolev spaces (Sobolev 1936) and the distributions (Schwartz 1944). However, in spite of several interesting improvements, the question of the existence of a global in time smooth solution remains essentially open. An interesting instability result of Lions and DiPerna described in section **2.4** may give some clues to the reason why the regularity of the solution of the Navier Stokes equation is "hard" to prove.

ii) For the compressible Euler equation the only convenient global solution is the weak solution. Here also one should keep in mind that the only available result goes back to J. Glimm (1965) and that it has never been improved.

The existing and non existing results for these macroscopic equations which are at the center of the hierarchy give some indication on what could be proven above and below.

Chapter **3** is concerned with the hierarchy from the Hamiltonian system of particles to the macroscopic equation with an essential intermediate step of the introduction of the Boltzmann equation. First it is shown how to derive the Boltzmann equation from a Hamiltonian system of particles using the BBGKY hierarchy. It is important to observe in this section how the fact that the initial problem is non linear is in agreement with the appearance of an irreversible process with a nontrivial entropy from a reversible process. Then, following Hilbert, Chapman, Enskog and a series of more recent contributions, the relations between kinetic and macroscopic equations are explained. It is important to notice that most of the rigorous results are the counterpart of the classical results of the previous section for the Navier Stokes equation.

Chapter **4** is a short introduction to turbulence. To introduce the Reynolds stress tensor a classical model of turbulence (the k, ϵ model) is presented. Through the study of the Reynold stress tensor, with the use of the Wigner transform, a local notion of turbulent spectra appears. The necessity of introducing some randomness is compared with the use of defect measures.

In chapter **5** connection is made with dynamical systems to prove some of the basic properties of turbulent spectra. For sake of simplicity (many results of this section have been adapted to other configurations) one considers a flow in a periodic box with some time independent low frequency external force.

First the classical phenomenological derivation of the Kolmogorov and Kraichnan inertial range and dissipative wave numbers is given. Then the global attractor and some rigorous properties for the invariant measure are given and counterparts (in this setting) of the phenomenological results are proven–some in full rigor, others with natural hypotheses.

The comparison of the evolution of the flow with the solution of a finite dimensional dynamical system has led many authors to the introduction of the notion of "inertial manifold." This notion, which works well for a series of equations (for instance Kuramoto-Sivashinsky) as discussed in section 5.5, does not seem effective for the Navier-Stokes equation due to intermittency in turbulence. This section is concluded with a description of a more robust and flexible object: the exponential attractor, which with some generalization of the Mané projection theorem yields equivalent finite-dimensional "inertial" dynamical systems.

The last chapter concludes the description of the hierarchy by the introduction of objects which in some sense are even more coarser than the coherent structures. A short state of the art in conjunction with the dynamical system of interacting particles is given.

As a large part of the material of this contribution is classical it is useful to conclude the introduction with some references:

The up to date but classical theory on Navier Stokes equation can be found in the books of P. Constantin and C. Foias, "Navier-Stokes Equations" and the book of P.L. Lions, "Mathematics Topics in Fluid Mechanics, Volume 1, incompressible models."

The presentation of the $\epsilon - k$ model of Launder and Spalding follows the book of Mohammadi and Pironneau Analysis of the K-Epsilon Turbulence Model. Of course this is not the only (or the always more relevant) model. Besides the ideas given here many other approaches have been tried including the use of renormalization group (cf. Orzag and Yakhot [YO]). However the $k - \epsilon$ model seemed well adapted to the introduction of the problematic of turbulence.

The authors found in the technical report of Besnard, Harlow and Rauenzahn, *Spectral Transport Model for Turbulence* [BHRZ] the use of the Wigner transform for the analysis of the local turbulent spectra. In spite of the fact that this is a very natural approach it does not seem to have appeared anywhere else.

Section **5** borrows many ideas and most of the presentation to the review article of C. Foias *What do the Navier -Stokes equations tells us about turbulence* [Fo]. Eventually basic ideas and a systematic presentation on the notion of attractors and of inertial manifolds are an essential part of the books by A.V. Babin and M.I. Vishik, *Attractors of Evolution Equations,* and by Temam on *Infinite Dimensional Dynamical System in Mechanics and Physics* [BV4]. The notion of the exponential attractor itself appears in the more recent book by Eden, Foias et al., *Exponential attractor for dissipative evolution equations* [EFNT].

Up to now very few mathematical books have considered the question of coherent structure; however, most of the material of chapter **6** can be found in Chorin, *Vorticity and Turbulence* [Cho] or in Marchioro and Pulvirenti, *Mathematical theory of incompressible non viscous fluids*[MaPu].

In 87 and 99 years passed away to Scientists whose contributions, as we try to explain, were corner stones for the present theory: Jean Leray and Andrei Nikolaevich Kolmogorov. We think that their pictures should be present in this review article on Navier-Stokes equations.

Acknowledgements First the authors wish to thank profusely Professor Ciprian Foias for his encouragments and suggestions. Furthermore, as said above, part of this presentation owes much to his review article "What do the Navier-Stokes equations tell us about turbulence".

Several sections of this presentation result joint work or long discussion with friends and colleagues. For instance the chapter 3 is an upshot of a long term project of C. Bardos with Francois Golse and Dave Levermore, results on rotating fluids are in the core of a project of B. Nicolaenko with Anatoli Babin and Alex Mahalov. The chapter on coherent structures in this presentation follows discussions with Marie Farge who introduced these concepts in our community. Finally we ow to Uriel Frisch a general approach on turbulence. It is a pleasure for us to thank all of them.

Jean Leray
By courtesy of the French Academy of Sciences, "Service des Archives"

Andrei Nicolaevich Kolmogorov
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2. Euler and Navier Stokes Equations: scaling parameters, regularity and stability results, theorems and counterexamples

2.1. Introduction

The macroscopic description of the fluid is the cornerstone of the analysis of the hierarchy. It is at this level that the Reynolds number and Mach number are the most easily defined. Relations between compressible and incompressible equations are a clue to the understanding of the different types of limits of the kinetic equation as described in the next chapter. The present chapter is organized as follows. In the second section as an introduction, the relation between compressible and incompressible equations is derived at a formal level. Rigorous proof of convergence can be found in Klainerman and Majda [KM] or in Benabdallah-Lagha [Ben]. The next section is devoted to the entropy which is used to show that the compressible Euler equation is an hyperbolic system, to prove some results of uniqueness and finite speed of propagation. These properties will be used in the next section where physical sufficient conditions for loss of regularity are given.

The Section **2.4** is devoted to the incompressible Euler equation in 2 and 3 space variables. In 2 space variables the conservation of the vorticity along the trajectories of the flow is a precious information which gives in particular global (in time) regularity results. Nevertheless the question of the large time stability (in higher norms) remains open and this is in full agreement with the ideas exposed in the chapter **6**.

In 3 space variables, the local in time existence of a smooth solution can be obtained by an adaptation to pseudodifferential operators of the Cauchy-Kowalevski theorem and by now it is a very classical result. On the other hand the existence of a solution in the large and the possible loss of regularity remains a completely open problem. The difficulty of this problem can be illustrated by a very explicit example of instability due to P.L. Lions and R. DiPerna which is given.

This example is also used to illustrate the difficulty of analyzing the regularity of the weak solution of the 3 dimensional incompressible Navier-Stokes equations which is considered in the Section **2.5**, where some conditional results are given. For the authors, at present, the most striking one is the contribution of Constantin and Fefferman [CF]. They have shown that loss of regularity (or stability) is induced by strong oscillations in the direction of the vorticity.

This result should be combined with the a complementary point of view contained in a series of papers by Babin, Mahalov and Nicolaenko, motivated by the rotating Euler and Navier-Stokes equations in the atmosphere. These authors have studied the effect of the presence of a large Coriolis term (or large rotation frequency). They have shown that high-frequency oscillations induced by this term do stabilize the three dimensional Euler or Navier Stokes equation [BMN1-3]. Of course in this situation the Coriolis force is an external force. However it may appear that large vorticity could play the same role and in the end lead to regularity results for the classical Navier Stokes equations.

2.2. Compressible and incompressible equations

At the macroscopic level, the most universal equations of fluid dynamics are the compressible Navier Stokes equations. They involve, as unknowns, ρ, u, θ and p the density the velocity, the temperature and the pressure.

In this contribution emphasis is put on the notion of hierachy of equations therefore the state law which gives the pressure in term of temperature and density will be the Mariotte law for the perfect gases:

$$p = \frac{R\theta}{\mu}\,.$$

R is the *gas* constant and μ its molecular weight.

As it will be shown in the section 3.3 the evolution equation for a perfect gas are derived from the Boltzmann equation when the Knudsen number goes to zero. With a convenient scaling this limit produces the following equations:

$$\partial_t \rho_\epsilon + \nabla_x\!\cdot\!(\rho_\epsilon u_\epsilon) = 0\,, \tag{2.2.1}$$

$$\rho_\epsilon\big(\partial_t + u_\epsilon\!\cdot\!\nabla_x\big)u_\epsilon + \nabla_x p_\epsilon = \epsilon\nabla_x\!\cdot\![\nu\sigma(u_\epsilon)]\,, p_\epsilon = \rho_\epsilon\theta_\epsilon\,, \tag{2.2.2}$$

$$\tfrac{3}{2}\rho_\epsilon\big(\partial_t + u_\epsilon\!\cdot\!\nabla_x\big)\theta_\epsilon + \rho_\epsilon\theta_\epsilon\nabla_x\!\cdot\! u_\epsilon = \epsilon\tfrac{1}{2}\nu\sigma(u_\epsilon)\!:\!\sigma(u_\epsilon) + \epsilon\nabla_x\!\cdot\![\kappa\nabla_x\theta_\epsilon]\,. \tag{2.2.3}$$

The numbers $\epsilon\nu$ and $\epsilon\kappa$ are the viscosity and thermal diffusivity they are proportional to ϵ the Knudsen number which is the ratio between the mean free path (the mean distance travelled by a molecule of fluid between two successive collisions and the characteristic size of the domain where the interaction takes place). Observe that the ratio between the viscosity and the thermal diffusivity is an ϵ independent number. In fact it is one of the characteristic number of the fluid and it is called the Prandtl number; $\sigma(u)$ denotes the strain-rate tensor given by

$$\sigma_{ij}(u) = (u^i_{x_j} + u^j_{x_i}) - \tfrac{2}{3}\nabla_x\!\cdot\! u\,\delta_{ij}\,. \tag{2.2.4}$$

In a compressible fluid one also introduces the Mach number Ma which is the ratio of the bulk velocity to the sound speed and the Reynolds number Re which is a dimensionless reciprocal viscosity $\epsilon\nu$ of the fluid. These numbers in consistency with the derivation of the above equations from the Boltzmann equations (cf. section 3.3 and [LL]) are related by the formula

$$\epsilon = \frac{Ma}{Re}\,. \tag{2.2.5}$$

When the Reynolds number goes to infinity the equations (2.2.1)...(2.2.3) reduce to the compressible Euler equations

$$\partial_t \rho + \nabla_x\!\cdot\!(\rho u) = 0\,, \tag{2.2.6}$$

$$\rho\big(\partial_t + u\!\cdot\!\nabla_x\big)u + \nabla_x(\rho\theta) = 0\,, \tag{2.2.7}$$

$$\tfrac{3}{2}\rho\big(\partial_t + u\!\cdot\!\nabla_x\big)\theta + \rho\theta\nabla_x\!\cdot\! u = 0\,. \tag{2.2.8}$$

On the other hand the incompressible Navier Stokes equation can also be deduced from the above equations when the Mach number goes to zero. More precisely, consider in three space variables, for time of the order of $(\epsilon)^{-1}$ the solutions of the equations (2.2.1), (2.2.2), (2.2.3).

Assume that the velocity, the fluctuation of density and temperature are also of the order of ϵ, introduce the change of functions:

$$\rho_\epsilon = \rho_0 + \epsilon\tilde{\rho}_\epsilon(\epsilon t, x)\,, \quad u_\epsilon = \epsilon\tilde{u}_\epsilon(\epsilon t, x)\,, \quad \theta_\epsilon = \theta_0 + \epsilon\tilde{\theta}_\epsilon(\epsilon t, x)\,, \tag{2.2.9}$$

and observe that if these functions converge (for $\epsilon \to 0$) in a convenient topology their limit satisfy the following system of equations

$$\partial_t \tilde{u} + \tilde{u}\cdot\nabla_x \tilde{u} + \nabla_x p = \nu\Delta\tilde{u}\,, \qquad \nabla_x\cdot\tilde{u} = 0\,, \tag{2.2.10}$$

$$\nabla_x(\rho_0\tilde{\rho} + \theta_0\tilde{\theta}) = 0\,, \tag{2.2.11}$$

$$\tfrac{5}{2}(\partial_t\tilde{\theta} + \tilde{u}\cdot\nabla_x\tilde{\theta}) = \kappa_*\Delta\tilde{\theta}\,, \tag{2.2.12}$$

where (2.2.10) is a standard version of the incompressible Navier Stokes equation, (2.2.11) is the Boussinesq relation between the fluctuations of density and temperature and (2.2.12) is the equation for the temperature. For $\nu = 0$ the system becomes the incompressible Euler equation.

Most of the above equations are non linear and this is one of the main reasons why analytical solutions almost never exist. Therefore the analysis relies on estimates usually called a priori estimates and the connection between the different type of equations also explain if these estimates are difficult to obtain for some equation (E) it will also be difficult to obtain for any other family (E_ϵ) which in some sense converge to (E) $\epsilon-$independent estimates of the same type.

2.3. Entropy and the stability of the compressible Euler equations

Observe any smooth solution of (2.2.6), (2.2.7) and (2.2.8) satisfies the entropy relation

$$\partial_t\rho S + \nabla_x\cdot(u\rho S) = 0 \text{ with } \rho S = \rho\log\frac{\rho^{\frac{2}{3}}}{\theta} \tag{2.3.1}$$

Since S is a convex function of the principal variables

$$\rho, \rho u, \rho(\frac{|u|^2}{2} + \tfrac{3}{2}\theta)$$

one can show that the corresponding linearized system is hyperbolic. As a consequence one obtains the existence and stability of smooth solutions of the system (2.2.6), (2.2.7) and (2.2.8) for smooth initial data. One can also in the same situation prove the finite speed of propagation of localized perturbations of the constant state, and (cf. Sideris [S]) the appearance after a finite time of singularities. This correspond in particular to the generation of shock waves. In the presence of such singularities the relation (2.3.1) is no more valid and both on physical and mathematical ground it has to be replaced by the relation:

$$\partial_t \rho S + \nabla_x \cdot (u\rho S) \leq 0 \tag{2.3.2}$$

which describes the decay of entropy (observe that the mathematical and physical entropy are of opposed sign, this is due to the fact that mathematicians do prefer to consider convex functions). The entropy decay can also be used to prove a stability result between regular solutions and weak solutions which satisfy (2.3.2) (cf. Dafermos [Da]).

However from the mathematical point of view the situation is far from being satisfactory. The existence for all time of weak solutions has only been proved in one space variable by Glimm [G] in 1965 and in spite of tremendous efforts involving the best mathematicians of our generation the problem remains widely open. No progress has been made concerning the existence of global in time weak solution of the genuine compressible Euler equation since the work of Glimm. Furthermore one of the basic tools of this approach, in one space dimension, is the introduction of the space of functions with bounded variation. This approach seems quite natural to deal with shocks. Unfortunately it has been proven by Rauch [Ra], that no estimate of this type would be valid in higher dimension.

2.4. Stability and instability of the incompressible Euler equation

In the previous section it has been shown that the incompressible Navier Stokes or Euler ($\nu = 0$) equations (equation (2.2.10) above) are the incompressible limit of the corresponding compressible equations. The necessity to have at our disposal viscosity independent results lead to the consideration of the incompressible Euler equation which in 2 and 3 space variables is

$$\partial_t u + u \cdot \nabla_x u = -\nabla_x p,\ \nabla_x \cdot u = 0 \tag{2.4.1}$$

with, if a boundary is present, an "impermeability boundary condition"

$$u \cdot \vec{n} = 0$$

on the boundary of the domain ($\vec{n}$ is the outward normal to the boundary). However for sake of simplicity some of the present analysis is done for domains with no boundary (all space or space periodic solutions).

The relation $\nabla_x \cdot u = 0$ can be viewed as a constraint and p the pressure is in this point of view the Lagrange multiplier of this constraint.

With the introduction of the vorticity (or $\nabla \times u$) $\omega = \nabla \times u$ the above equation can also be written

$$\begin{gathered} \partial_t \omega + u \cdot \nabla_x \omega - \omega \cdot \nabla_x u = 0 \\ \nabla_x \cdot u = 0, \quad \text{curl } u = \omega, \quad u.\vec{n} = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{2.4.2}$$

The second line of (2.4.2) defines a -1 order pseudo-differential operator $\omega \mapsto K(\omega) = u$. This observation has important consequences both at the level of abstract geometry and analysis:

"i)" The expression

$$\{u, \omega\} = u \cdot \nabla_x \omega - \omega \cdot \nabla_x u$$

has the formal properties of a Poisson bracket and the Euler equation can be written in the form:

$$\dot{\omega} = -\{u, \omega\}\,.$$

Therefore it may have in some sense an hamiltonian structure.

In fact it is natural, and this will be used in others subsections, to introduce the trajectories of the particles of the flow (or the Lagrangian coordinates) defined by the relation:

$$\dot{x}(t, X) = u(t, x(t, X)),\ \ x(0) = X\,.$$

The mapping

$$G_t :\ X \mapsto x(t, X)$$

is a volume preserving (use the relation $\nabla_x u = 0$ and $u \cdot \vec{n} = 0$ on the boundary) isomorphism of the domain of definition Ω of the fluid.

On the other hand one can define in terms of energy the Riemannian distance between the identity and any volume preserving G_T isomorphism of Ω according to the formula:

$$E_G = \min \int_\Omega \int_0^T |\dot{G}(t, X)|^2 dX\,dt \tag{2.4.3}$$

with $G(t, .)$ ranging over the C^1 volume preserving transformation of Ω with the initial and final conditions:

$$G(0, X) = X,\ G(T, X) = G_T(X)\,.$$

A standard variational computation shows that if $(X, t) \mapsto G(t, X)$ is an extremal for the action given by (2.4.3) then

$$u(t, x) = \dot{G}(t, G^{-1}(t, x))$$

is a solution of the Euler equation. Systematic extensions of this point of view can be found in Abraham Marsden and Arnold Khesin ([AM], [AK]) and are used to characterize the stability of some stationary solutions.

"ii)" It is easy to see that, in d-dimension, for initial data in a enough regular Sobolev space (for instance $H^s(\mathbf{R}^d)$ with $s > \frac{d}{2}+1$) the problem (2.4.1) has a unique solution continuously defined in the same space for a finite time interval $(0, T)$ with

$$T \geq C \frac{1}{\|u(.,0)\|_{H^s(\mathbf{R}^d)}}\,.$$

Results of this type were already obtained by Lichtenstein in 1925 [Lic] (in a less elaborate language).

However the problem of the existence of a solution in the large is still widely open and there is no, at variance with the compressible case, a proof of the appearance of singularity or a good physical reason for such an event.

When the space dimension is equal to 2 the term $\omega\cdot\nabla_x u$ disappears from the equation (2.4.1). In fact $\omega =$ curl$u$ remains perpendicular to the plane where the fluid evolves. The equation (2.4.1) becomes:

$$\partial_t\omega(x,t) + u\cdot\nabla_x\omega(x,t) = 0 \qquad (2.4.4)$$

and ω is conserved along the trajectories of the particles of the fluid:

$$\dot{x} = u(x(t),t)$$

and remains bounded in L^∞ for all time. This is enough to prove the existence of a weak solution (cf. Yudovich [Y]). However the proof of the uniqueness in a convenient class is slightly more elaborated (cf. also [Y]) and to prove the persistence of the regularity of the solution with smooth initial data one has to face the following problem: The estimate

$$\omega = \nabla\times u \in L^\infty \qquad (2.4.5)$$

which comes from (2.4.4) is simply not enough to imply that

$$\nabla_x u \in L^\infty \qquad (2.4.5)$$

and (2.4.5) seems compulsory for any boot-strap argument for the proof of the regularity. However a more precise use of (2.4.4) gives according to Wolibner [Wo] a regularity result as follows:

Theorem 2.1 *Consider the solution of the 2d incompressible Euler equation in a bounded domain Ω of diameter L with an "impermeability boundary condition". Assume that the initial vorticity is in the Hölder space $C^{0,\alpha}(\Omega)$. Then one has the following uniform (in time) estimate:*

$$\|\nabla\times u(.,t)\|_{C^{0,\alpha(t)}} \le C\|\nabla\times u(.,0)\|_{C^{0,\alpha}} \qquad (2.4.6)$$

with $\alpha(t) = \alpha\exp\{-Ct\|\nabla\times u(.,0)\|_{L^\infty(\Omega)}\}$.

Proof Observe that the Green function of the Laplace operator with Dirichlet boundary condition is of the following form

$$G(x,z) = -\frac{1}{2\pi}\log|x-z| + \gamma(x,z) \qquad (2.4.7)$$

with $\gamma(x,y)$ being a smooth function and that

$$u(x,t) = \int_\Omega \nabla\times G(x,z)\nabla\times u(z,t)dz\,. \qquad (2.4.8)$$

¿From (2.4.7) and (2.4.8) one deduces the estimate:

$$|\nabla_x u(x) - \nabla_x u(y)| \le C|\nabla\times u|_{L^\infty(\Omega)}|x-y|\log\left(\frac{|x-y|}{D}\right) \qquad (2.4.9)$$

with D denoting the diameter of Ω. Since the vorticity is constant along the trajectories of the flow one can use in the relation (2.4.9) the estimate

$$|\nabla \times u(.,t)|_{L^\infty} = |\nabla \times u(.,0)|_{L^\infty} \tag{2.4.10}$$

and for the Hölder norm of the *curl*, the estimate:

$$\begin{aligned} &\frac{|\nabla \times u(x(t),t) - \nabla \times u(y(t),t)|}{|x(t)-y(t)|^{\alpha(t)}} \\ &= \frac{|\nabla \times u(x(0),t) - \nabla \times u(y(0),t)|}{|x(0)-y(0)|^{\alpha}} \Big(\frac{|x(0)-y(0)|^{\alpha}}{|x(t)-y(t)|^{\alpha(t)}} \Big) \end{aligned} \tag{2.4.11}$$

where $x(t)$ and $y(t)$ denotes Lagrangian coordinates as introduced above. With

$$\rho(t) = |x(t) - y(t)|\,,$$

uses the estimate (2.4.9), (2.4.10) and by comparison with the solution of the differential equation

$$\frac{\dot{\rho}(t)}{D} = \pm C \frac{\rho(t)}{D} \log(\frac{\rho(t)}{D}) \tag{2.4.12}$$

obtains:

$$\Big(\frac{|x(0)-y(0)|}{D}\Big)^{e^{-Ct|\nabla\times u|_{L^\infty(\Omega)}}} \geq \frac{|x(t)-y(t)|}{D} \geq \Big(\frac{|x(0)-y(0)|}{D}\Big)^{e^{Ct|\nabla\times u|_{L^\infty(\Omega)}}}\,. \tag{2.4.13}$$

Since D is the diameter of the domain Ω where the particles live one has

$$\frac{|x(0)-y(0)|}{D} < 1$$

and therefore the first term of (2.4.13) goes very rapidly to 1 and the last one very rapidly to zero when $t \to \infty$. Eventually one deduces form (2.4.13) that:

$$\frac{|x(0)-y(0)|^{\alpha}}{|x(t)-y(t)|^{\alpha(t)}} \leq D^{\alpha - \alpha(t)} \tag{2.4.14}$$

which gives (2.4.6) and by classical Hölder estimates for elliptic operators:

$$|\nabla_x u|_{L^\infty(\Omega)} \leq C \frac{e^{Ct|\nabla\times u|_{L^\infty(\Omega)}}}{\alpha} |\nabla \times u(.,t)|_{C(\alpha e^{-Ct|\nabla\times u|_{L^\infty(\Omega)}})} \leq |\nabla \times u(.,0)|_{C^\alpha}\,. \tag{2.4.15}$$

Now this relation can be used to prove that for any finite time the solution remains in the same regularity class as the initial data.

Remark 2.2 The estimates (2.4.15) do not prevent the measure of the regularity to deteriorate very rapidly for $t \to \infty$ even according to the rate

$$|u(t)|_{C^{1+\alpha}} \equiv e^{e^{Ct}}\,.$$

Such a behavior is not incompatible with the following facts:

i) An example due to Bahouri and Chemin [BaCh] of a flow with an initial vorticity in L^∞ (but not in an Hölder class) shows that (2.4.14) is optimal.

ii) The singular behavior (for $t \to \infty$) of the Hölder norm of the curl implies that due to the corresponding loss of compactness the omega limit set of the family $\nabla \times u(x,t)$ which exist for the weak$^*L^\infty$ topology may not be approached in the strong L^p norm. Such an observation would be consistent with the appearance of some coherent structures as described in the chapter **6**.

As said above in three space variable the problem is locally in time well posed in $H^s(\mathbf{R}^3)$ for $s > 3$ (or in $C^{1,\alpha}$). In fact it seems much more "physical" to believe that eventually the loss of regularity would be governed by the "sup norm" of the vorticity. It turns out that such a result is true and has been proven by Beale, Kato and Majda [BKM] with an extension of the method of the proof of the Theorem 2.1 For sake of simplicity the proof is done for periodic solutions defined in the "flat torus" $\mathbf{R}^3\backslash\mathbf{Z}^3$, extension to bounded domain or to the whole space are just technical.

Theorem 2.3 (Beale, Kato, Majda): *Let $u \in C^0([0,T[; H^3(\mathbf{R}^3\backslash\mathbf{Z}^3))$ be a solution of the three dimensional incompressible Euler equation. Suppose that there exists a time T_* such that the solution cannot be continued up to $T = T_*$ and assume that T_* is the first such time. Then one has for $\omega(x,t) = \nabla \times u(x,t)$,*

$$\int_0^{T_*} \|\omega(t)\|_{L^\infty}\, dt = \infty\,, \tag{2.4.16}$$

and in particular

$$\limsup_{t \uparrow T_*} \|\omega(t)\|_{L^\infty} = \infty. \tag{2.4.17}$$

Proof Start from the standard estimates in $H^3(\mathbf{R}^3\backslash\mathbf{Z}^3)$

$$\tfrac{1}{2}\frac{d}{dt}\|u\|^2_{H^3} \le C\|\nabla_x u\|_{L^\infty}\|u\|^2_{H^3}\,. \tag{2.4.18}$$

Then as in the proof of the Theorem 2.1 introduce the Green function $G(x,y)$ of the Laplacian, defined on the function with mean value zero and observe that one has:

$$u(x,t) = \int_{\mathbf{R}^3\backslash\mathbf{Z}^3} \nabla \times G(x,y)\,\omega(y,t)\,dy\,. \tag{2.4.19}$$

Furthermore for $x \neq y$ $G(x,y)$ is very smooth (analytic) and for x near y one has :

$$G(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|} + \gamma(x,y) \tag{2.4.20}$$

with $\gamma(x,y)$ smooth. As a consequence one can prove the following estimate:

$$\|\nabla_x u\|_{L^\infty} \le C\log(1 + \|u\|_{H^3})\|\omega\|_{L^\infty}\,. \tag{2.4.21}$$

Therefore one deduces from (2.4.18) and (2.4.21) and for $\|u\|^2_{H^3} > 1$ an estimate of the following type:

$$\partial_t \|u\|^2_{H^3} \leq C\|\omega(t)\|_\infty \|u\|^2_{H^3} \log \|u\|^2_{H^3} \tag{2.4.22}$$

which gives by integration:

$$\|u(t)\|_{H^3} \leq \left(\|u(t_0)\|_{H^3}\right)^{e^{C\int_{t_0}^t \|\omega(s)\|_{L^\infty}\, ds}}. \tag{2.4.23}$$

and proves the theorem.

Eventually the fact that existence of a regular solution is a difficult problem is illustrated by the following

Theorem 2.4. (P.-L. Lions and R. DiPerna):

i) For each $1 < p < \infty$, $t > 0$ and each $\epsilon > 0$, $\delta > 0$ there exists a smooth periodic solution of the 3d periodic incompressible Euler equation

$$\partial_t u + u \cdot \nabla_x u = -\nabla_x p, \quad \nabla_x \cdot u = 0 \tag{2.4.24}$$

which satisfies the estimates:

$$\|u(0)\|_{W^{1,p}} \leq \epsilon \quad \text{and} \quad \frac{1}{\delta} \leq \|u(t)\|_{W^{1,p}}\,. \tag{2.4.25}$$

ii) There exist solutions of the periodic 3d incompressible Euler equation with a vorticity linearly increasing in time, according to the formula:

$$\|\nabla \times u(t)\|_{L^\infty} \geq t\|\nabla \times u(0)\|^2_{L^\infty}\,. \tag{2.4.26}$$

iii) There exists no continuous smooth function, $\phi(t,s)$ independent of the Reynolds number ν such that one has for smooth solution of the 3 space periodic Navier Stokes equation the estimate:

$$\|u(t)\|_{W^{1,p}} \leq \phi(t, \|u(0)\|_{W^{1,p}})\,. \tag{2.4.27}$$

Proof Let $u_1^0(x_2)$ be a x_2 dependent smooth periodic function and similarly let $u_3^0(x_1,x_2)$ be a (x_1,x_2) dependent smooth periodic function. Then

$$\begin{aligned} U(x_1,x_2,x_3) &= (u_1(x_1,x_2,x_3), u_2(x_1,x_2,x_3), u_3(x_1,x_2,x_3)) \\ &= (u_1^0(x_2), 0, u_3^0(x_1 - tu_1^0(x_2), x_2)) \end{aligned} \tag{2.4.28}$$

is a periodic smooth solution of the 3 dimensional Euler equation with constant pressure.

Now introduce $\theta \in [0,1]$ such that

$$\theta \neq \tfrac{1}{2}, \quad 1 \leq \frac{1}{1-\theta}, \quad 1 \leq p < \frac{2}{3(1-\theta)}\,. \tag{2.4.29}$$

Pick two ϵ-dependent families:

$u_1^{0,\epsilon}(x_2)$ uniformly smooth away from zero and behaving near 0 like $(\epsilon^2+x_2^2)^{\theta/2}$ and $u_3^{0,\epsilon}(x_1,x_2)$ uniformly smooth away 0 behaving near 0 like $(\epsilon+x_1^2+x_2^2)^{\theta-1/2}$. Consider the corresponding solution of the Euler equation constructed according to the recipes (2.4.28)

$$U^\epsilon(x_1,x_2,x_3)=(u_1^\epsilon(x_1,x_2,x_3),u_2^\epsilon(x_1,x_2,x_3),u_3^\epsilon(x_1,x_2,x_3))\,. \tag{2.4.30}$$

Explicit computation show that on one hand $U(0)$ is uniformly bounded in $W^{1,p}$ and that on the other hand, for any $t\in\mathbf{R}$, $t\neq 0$ and uniformly for ϵ small enough one has:

$$\begin{aligned}
&\int_{\mathbf{R}^3/\mathbf{Z}^3}|\partial_{x_2}u_3^\epsilon(t)|^p\,dx_1\,dx_2\,dx_3=\\
&\int_{\mathbf{R}^2/\mathbf{Z}^2}|t\partial_{x_1}u_3^{0,\epsilon}(x_1-tu_1^{0,\epsilon}(x_2),x_2)(u_1^{0,\epsilon'}(x_2)-\partial_{x_2}u_3^{0,\epsilon}(x_1-tu_1^{0,\epsilon}(x_2),x_2)|^p\,dx_1\,dx_2\\
&\geq Ct^p\int_{\mathbf{R}^3/\mathbf{Z}^3}|\partial_{x_1}u_3^{0,\epsilon}(x_1,x_2)(u_1^{0,\epsilon'}(x_2))|^p\,dx_1\,dx_2\\
&\geq Ct^p\int_{|x|\leq\delta}\frac{|x_1|^p}{(\epsilon^2+|x|^2)^{(\frac{3}{2}-\theta)p}}\frac{|x_2|^p}{(\epsilon^2+x_2^2)^{(1-\frac{\theta}{2})p}}\,dx_1\,dx_2\,.
\end{aligned} \tag{2.4.31}$$

Now when ϵ goes to zero the right hand side of (2.4.32) behaves like the integral

$$\int_0^\delta\frac{1}{r^{3(1-\theta)p}}r\,dr$$

and therefore goes to infinity for $p\geq\frac{2}{3}\frac{1}{(1-\theta)}$.

The same method can be used to prove the item ii) with ϵ fixed and t going to infinity. Finally if a function $\phi(t,s)$ independent of the viscosity would satisfy (2.4.27) then letting the Reynolds number go to infinity in the Navier Stokes equation one would contradict the item i) and the proof is complete.

The item iii) gives some evidence to the difficulty of proving the existence of smooth solutions for the $3d$ Navier Stokes equation and could introduce the next section.

2.5. Existence and regularity results for the 3d Navier Stokes equation. The weak solution of J. Leray

In this section it is assumed that the viscosity is non zero (finite Reynolds number) and therefore the equations of the motion in $\mathbf{R}^3$ are

$$\partial_t u+u\cdot\nabla_x u-\nu\Delta_x u=-\nabla_x p,\quad \nabla_x\cdot u=0 \tag{2.5.1}$$

Assuming that the solution is smooth and multiplying (2.5.1) by u one obtains the local balance of energy

$$(\partial_t+u\cdot\nabla_x-\nu\Delta_x)\tfrac{1}{2}|u|^2+\nu|\nabla_x u|^2+\nabla_x\cdot(pu)=0 \tag{2.5.2}$$

which by integration from 0 to t and over $\mathbf{R}^3$ produces the a-priori estimate.

$$\frac{1}{2}\int_{\mathbf{R}^3} |u(x,t)|^2\,dx + \nu\int_0^t \|u\|^2_{H^1(\mathbf{R}^3)}(s)\,ds = \frac{1}{2}\int_{\mathbf{R}^3} |u(x,0)|^2\,dx\,. \tag{2.5.3}$$

The presence of the term

$$\nu\int_0^t \|u(s)\|^2_{H^1(\mathbf{R}^3)}(s)\,ds$$

plus some weak time regularity ensure enough compactness property to prove, with initial data

$$u_0(x) \in L^2(\mathbf{R}^2),\ \nabla_x\!\cdot u_0 = 0,$$

the existence of a weak solution

$$u(.,t) \in C_w(\mathbf{R}_+; L^2(\mathbf{R}^3)) \cap L^2(\mathbf{R}_+; H^1(\mathbf{R}^3));\ \nabla_x\!\cdot u = 0\,. \tag{2.5.4}$$

In (2.5.4), $C_w(\mathbf{R}_+; L^2(\mathbf{R}^3))$ denotes the space of function defined in $\mathbf{R}_+$ with value in $L^2(\mathbf{R}^3))$ and continuous with respect to the L^2 weak topology. This is the basic result of Leray (1934). However, as known, even for smooth initial data it has not been possible to prove that the solution will be smooth for all time. Furthermore the class of solutions constructed by Leray is not regular enough to afford a proof of uniqueness (in the same class); also it is not regular enough to show that it satisfies the conservation of energy (2.5.3), a fortiori it is not known if it satisfies the local balance of energy (2.5.2).

The following comments are usually made:

1) The instability theorem of P.L. Lions and R. DiPerna proven above shows that uniform estimates (with respect to ν) are not available and suggests that the dependence of the regularity with respect to the viscosity may be difficult to control.

2) Some simple regularity results can easily be obtained, for instance:

i) If the initial data belongs to the space $H^k(\mathbf{R}^3)$ with $k > 5/2$ then the solution is smooth during a finite time $]0,T[$ with T independent of ν.

ii) If at the time t_1 the weak solution (which belongs to $C^0(\mathbf{R}_+; L^2(\mathbf{R}^3))$) is in H^1 then it is smooth on a interval t_1, T_ν, with $T_\nu > t_1 > 0$ depending on ν and on the norm of $u(t_1)$ in the space H^1 .

iii) The conjunction of the point ii) with the fact that u is in $L^2(\mathbf{R}_t^+; H^1(\mathbf{R}^3))$ implies that the solution will become eventually smooth for t large enough (how large, up to now, is an open problem).

This also implies that the set of points where the weak solution may be singular is small. In fact this idea already was present in the original work of Leray and was refined by several authors. On its present form this refinement culminate with the work of Caffarelli, Kohn and Nirenberg [CaKNi] where it is shown that if singularities exists for the Leray solution they should be contained in a set of Hausdorff measure smaller or equal to 1 in $\mathbf{R}^3 \times \mathbf{R}_t^+$. Later on Sohr and von Wahl [SW] proved for the pressure associated to the Leray solution, the estimate:

$$p \in L^{\frac{5}{3}}(\Omega \times]0,T[)$$

which allowed Fanghua Lin [Lin] to produce a simpler proof of the result of [CaKni].

3) With the item ii) the smoothness is realized if one shows that the weak solution is bounded in

$$L^\infty(\mathbf{R}_t^+, H^1(\mathbf{R}^3)).$$

Observe that with the divergence free condition one has

$$\int_{\mathbf{R}^3} |\nabla_x u(x)|^2\, dx = \int_{\mathbf{R}^3} |\nabla \times u|^2\, dx\,. \tag{2.5.5}$$

The right hand side of (2.5.5) is usually called the enstrophy and from the Navier Stokes equation one deduces for $\omega = \nabla \times u$ the equation:

$$\partial_t \omega + u \cdot \nabla_x \omega - \omega \cdot \nabla_x u - \nu \Delta_x \omega = 0\,. \tag{2.5.6}$$

The energy estimates implies the estimate:

$$\nu \int_0^T \int_{\mathbf{R}^3} |(\omega \cdot \nabla_x u)(x,t)|\, dx\, dt \le C|u_0(x)|^2_{L^2(\mathbf{R}^3)}\,. \tag{2.5.7}$$

Introduce the direction ξ of the vorticity defined by

$$\begin{aligned} &\xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|} \quad \text{if} \quad \omega(x,t) \ne 0\,,\\ &\xi(x,t) = 0 \quad \text{if} \quad \omega(x,t) = 0\,. \end{aligned} \tag{2.5.8}$$

Multiplying the equation (2.5.6) by $\xi(x,t)$ one obtains (formal computation is done first then a rigorous proof can be obtained by a regularization process):

$$\begin{aligned} &(\partial_t + u \cdot \nabla_x - \nu \Delta_x)|\omega| + \nu |\omega|^{-3} |\nabla_x \xi|^2\\ &\quad + \sum_{ikl} (\omega_i^2 \delta_{kl} - \omega_k \omega_l) \partial_i \omega_k \partial_i \omega_l \le |\omega \cdot \nabla_x u|\,. \end{aligned} \tag{2.5.9}$$

In agreement with the convexity of the function $\omega \mapsto |\omega|$ the last term of the left hand side of (2.5.9) is non negative and with the energy estimate this give the bound:

$$\int |\omega(x,t)|\, dx \le C \int_{\mathbf{R}^3} \{|u_0(x)|^2 + |\omega_0(x)|\}\, dx\,. \tag{2.5.11}$$

Observe that among the quantities which have been shown to be uniformly bounded are

$$\sup_{t>0} \int_{\mathbf{R}^3} |\nabla \times u(x,t)|\, dx \quad \text{and} \quad \int_0^\infty \Big(\int_{\mathbf{R}^3} |\nabla \times u(x,t)|^2\, dx\Big)\, dt$$

while the typical one which is missing for global regularity is

$$\sup_{t>0} \int_{\mathbf{R}^3} |\nabla \times u(x,t)|^2\, dx \le C\,.$$

The gap is not that big but seems very difficult to fill.

4) It was already observed in by Serrin [Se] and by Kaniel and Shinbrot [KS] that in dimension 3 the supplementary hypothesis:

$$u(x,t) \in L^s(0,T;(L^r(\Omega))^3),\ \frac{2}{s}+\frac{3}{r}\leq 1,\ r>3$$

was sufficient to ensure the persistence of regularity and the uniqueness. The marginal case is $s=2$ and $r=\infty$, result which has been recently improved by Kozono and Taniuchi [KT]. Introducing the space BMO (cf. [St]) which contains L^∞ they have shown, in the spirit of the Beale Kato Majda theorem (theorem 2.4) that the condition

$$\int_0^T \|u(.,t)\|^2{}_{BMO}dt<\infty$$

was enough to ensure the persistence of regularity up to the time T.

In the same spirit it should also be observed that another sufficient condition for the uniqueness and regularity have been obtained for solutions with value in $L^3(\mathbf{R}^3)$. Since the transformation:

$$(u(x,t),p(x,t))\mapsto(\lambda u(\lambda x,\lambda^2 t),\lambda^2 p(\lambda x,\lambda t))$$

preserves both the solution of the Navier Stokes equation and the norm of u in $L^\infty(0,\infty,L^3(\mathbf{R}^3)$ this space seems to play a crucial role for the analysis of the problem. Such point of view introduced by Kato and Ponce [KP] and Weissler [We] involved several contributions from Calderon [Cal], Cannone [Can] and culminated with the work of G. Furioli, G. Lemarrié et E. Terraneo [FLT]. Once again the gap between uniform bound in L^3 and L^2 seems small.

Eventually the derivation of the relation (2.3.10) let appears some relation between the regularity of the solution and the regularity of the direction of the vorticity and in fact, more precisely, one has:

Theorem 2.5. (Constantin and Fefferman):

i) For any Leray solution of the 3d Navier Stokes equation one has:

$$\int_0^T dt\int_{\{(x,t);|\omega(x,t)|>\Omega\}}|\nabla_x\xi(x,t)|^2\,dx\,dt\leq\frac{C}{\nu\Omega}\int_{\mathbf{R}^3}\{|u_0(x)|^2+|\nabla\times u_0(x)|\}\,dx\,. \quad (2.3.11)$$

ii) Assume that the direction of the vorticity of the weak solution u is uniformly lipschitz with respect to x when the modulus of this vorticity is large; this means that there exist two positive finite constants C and ρ such that one has

$$\begin{aligned}\forall(x,y,t)\in(\mathbf{R}^3)^2\times\mathbf{R}^+_t\ \{|\omega(x,t)|>\Omega \text{ and } |\omega(y,t)|>\Omega\}\\ \Rightarrow\\ |\sin\phi(x,y,t)|\leq\frac{|x-y|}{\rho}\end{aligned} \quad (2.5.12)$$

with $\phi(x,y,t)$ denoting the angle of the two vectors $\omega(x,t)$ and $\omega(y,t)$. Then the vorticity, if bounded for $t=0$ in $L^2(\mathbf{R}^3)$ remains bounded in the same space for $t\geq 0$ and therefore the solution is "regular".

Remark 2.6. The significance of the relation (2.5.11) is: in regions of high vorticity the direction of the vorticity is regular in an averaged sense but uniformly with respect to the initial data and with a $\frac{1}{\nu}$ dependence with respect to the viscosity.

The significance of the assertion ii) is that singularities (or loss of control of the regularity) to appear need both large (in modulus) vorticity and large oscillations of the direction of this vorticity.

Proof As above the proof is made with a-priori estimates which are later used for rigorous proof with the introduction of some regularization process. Here the emphasis is put on the a-priori estimates.

First, observe that the relation (2.5.11) is just a direct consequence of the estimate (2.5.10). Second, to make the proof simpler and to focus on the key point it is assumed that the estimate (2.5.12) is valid not only at points (x,y) where the vorticity is greater than Ω but everywhere. Releasing this is hypothesis implies the introduction of terms which are quadratic with respect to the ω instead of being cubic and which can be easily handled.

Therefore one writes for the formal estimate:

$$\partial_t \int_{\mathbf{R}^3} |\omega(x,t)|^2 + \nu \int_{\mathbf{R}^3} |\nabla_x \omega(x,t)|^2\, dx \leq \int_{\mathbf{R}^3} |(\omega\nabla_x u, \omega)|\, dx\,. \tag{2.5.13}$$

The last term of the right hand side is cubic with respect to ω. In fact it involves the strain matrix:

$$S(x,t) = \{\tfrac{1}{2}(\nabla_x u + (\nabla_x u)^*)\}(x,t) = S(\omega)(x,t)$$

where appears the direction

$$\xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|}$$

of the vorticity. With

$$\hat{y} = \frac{y}{|y|} \text{ and } M(\hat{y},\omega) = \tfrac{1}{2}[\hat{y}\otimes(\hat{y}\times\omega) + (\hat{y}\times\omega)\otimes\hat{y}]$$

one has

$$S(x,t) = \frac{3}{4\pi}\,\text{P.V.}\int M(\hat{y},\omega(x+y,t))\frac{dy}{|y|^3}$$

and

$$\begin{aligned}|(\omega\nabla_x u,\omega)| &= \Big|\frac{3}{4\pi}\int_{\mathbf{R}^3}(\hat{y},\omega(x,t))(\hat{y},\omega(x+y,t),\omega(x,t))\,dy\Big| \\ &= |\omega(x,t)|^2\frac{3}{4\pi}\Big|\int_{\mathbf{R}^3}(\hat{y},\xi(x,t))(\hat{y},\omega(x+y,t),\xi(x,t))\Big|\frac{dy}{|y|^3}\,.\end{aligned} \tag{2.5.14}$$

It is in this last term that the hypothesis is used because

$$|(\hat{y},\xi(x,t))(\hat{y},\omega(x+y,t),\xi(x,t))| \le |\omega(x+y,t)|\sin(\xi(x+y,t),\xi(x,t))| \le C|y| \quad (2.5.15)$$

and therefore in the equation (2.5.14) the order of the singularity has been reduced. This equation becomes:

$$|(\omega\nabla_x u\omega)| \le C|\omega(x,t)|^2 \int_{\mathbf{R}^3} |\omega(x+y,t)| \frac{dy}{|y|^2} = C|\omega(x,t)|^2 I(x,t) \quad (2.3.16)$$

for which the following estimate can be easily obtained:

$$\begin{aligned} \|I(x,t)\|_{L^2} &= \Big[\int_{\mathbf{R}^3} dx \Big(\int_{\mathbf{R}^3} |\omega(x+=y,t)| \frac{dy}{|y|^2}\Big)^2\Big]^{\frac{1}{2}} \\ &\le C\big(\|\omega(.,t)\|_{L^1}\big)^{\frac{2}{3}}\big(\|\omega(.,t)\|_{L^2}\big)^{\frac{1}{3}} . \end{aligned} \quad (2.5.17)$$

With the Cauchy-Schwartz relation and the Gagliardo-Nirenberg inequality which is presently used in the following form:

$$\Big(\int_{\mathbf{R}^3} |\omega(x)|^4\, dx\Big)^{\frac{1}{2}} \le C\Big(\int_{\mathbf{R}^3} |\nabla_x\omega(x)|^2\, dx\Big)^{\frac{3}{4}}\Big(\int_{\mathbf{R}^3} |\omega(x)|^2\, dx\Big)^{\frac{1}{4}}$$

one obtains :

$$\begin{aligned} \int_{\mathbf{R}^3} |(\omega\nabla_x u\omega)|\, dx &\le C\int_{\mathbf{R}^3} |\omega(x,t)|^2 I(=x,t)\, dx \\ &\le \Big(\int_{\mathbf{R}^3} |\omega(x)|^4\, dx\Big)^{\frac{1}{2}}\|I(x,t)\|_{L^2} \\ &\le \Big(\int_{\mathbf{R}^3} |\nabla_x\omega(x)|^2\, dx\Big)^{\frac{3}{4}}\Big(\int_{\mathbf{R}^3} |\omega(x)|^2\, dx\Big)^{\frac{1}{4}}\big(\|\omega(.,t)\|_{L^1}\big)^{\frac{2}{3}}\big(\|\omega(.,t)\|_{L^2}\big)^{\frac{1}{3}} . \end{aligned} \quad (2.5.18)$$

Eventually the uniform estimate on the L^1 of the curl is used (cf. (2.5.10)), giving:

$$\begin{aligned} \int_{\mathbf{R}^3} |(\omega\nabla_x u\omega)|\, dx &\le \Big(\int_{\mathbf{R}^3} |\nabla_x\omega(x)|^2\, dx\Big)^{\frac{3}{4}}\Big(\int_{\mathbf{R}^3} |\omega(x)|^2\, dx\Big)^{\frac{1}{4}}\big(\|\omega(.,t)\|_{L^2}\big)^{\frac{1}{3}} \\ &= C\Big(\int_{\mathbf{R}^3} |\nabla_x\omega(x)|^2\, dx\Big)^{\frac{3}{4}}\Big(\int_{\mathbf{R}^3} |\omega(x)|^2\, dx\Big)^{\frac{5}{2}} \\ &\le \frac{\nu}{2}\int_{\mathbf{R}^3} |\nabla_x\omega(x)|^2\, dx + C\nu^{-3}\Big(\int_{\mathbf{R}^3} |\omega(x,t)|^2\, dx\Big)^{\frac{5}{3}} . \end{aligned} \quad (2.5.19)$$

The term

$$\int_{\mathbf{R}^3} |\nabla_x\omega(x)|^2\, dx$$

is in the equation (2.5.13) balanced by the viscosity and therefore one obtains for the enstrophy the relation:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^3}|\omega(x,t)|^2\,dx+\frac{1}{2}\int_{\mathbf{R}^3}|\nabla_x\omega(x,t)|^2\,dx \\ \le C\Big(\int_{\mathbf{R}^3}|\omega(x,t)|^2\,dx\Big)\Big(\int_{\mathbf{R}^3}|\omega(x,t)|^2\,dx\Big)^{\frac{2}{3}}. \tag{2.5.20}$$

Since one has

$$\int_0^T\Big(\int_{\mathbf{R}^3}|\omega(x,t)|^2\,dx\Big)^{\frac{2}{3}}dt\le T^{\frac{1}{3}}\Big(\int_0^T\int_{\mathbf{R}^3}|\omega(x,t)|^2\,dx\,dt\Big)^{\frac{2}{3}} \tag{2.5.21}$$

one concludes the proof with the estimate of energy and the Gronwall lemma.

Remark 2.7 In fact the uniform lipschitz condition can be relaxed what really matter is an estimate of the type

$$|(\hat{y},\omega(x+y,t),\xi(x,t))|\le C|\omega(x+y,t)||y| \tag{2.5.22}$$

which is much weaker (valid for two vectors of opposite direction).

The above results should be compared with the analysis done by Babin Mahalov and Nicolaenko [BMN1,2] who, motivated by the geophysical applications, consider the Navier Stokes equation with a large Coriolis force:

$$\partial_t u+u\cdot\nabla_x u+2\Omega_0 e_3\times u-\nu\Delta_x u+\nabla_x p=\mathbf{F},\,\nabla_x\!\cdot u=0 \tag{2.5.23}$$

In (2.3.23) e_3 is the vertical unit vector and Ω_0 is the frequency of the background rotation which introduces a Coriolis force which is assumed to be large compared to the other parameters of the flow. A detailed analysis is done in the case of a periodic domain or stress-free boundary conditions.

In fact the linearized version of the equation (2.5.23) was studied (cf Arnold and Khezin [AK]) by Sobolev who started from an analysis done by Poincaré and who applied it to the description of the behaviour of fuel tanks of rotating projectiles. The work of Sobolev was done in Kazan around 1942 and by that time classified. It was declassified in [Sob]. In the extension of this analysis to the genuine non linear equation (2.5.23) Babin Mahalov and Nicolaenko used a sharp Fourier analysis involving small denominators and a Diophantine condition on the incommensurability of the condition on the domain geometrical parameters.

It is shown in [BMN1], [BMN2] and [BMN3] that the solutions of the 3-D Euler or Navier Stokes equation of uniformly rotating fluids can be decomposed into the sum of the following terms : a solution of the 2-D Euler (or Navier Stokes) system with vertically averaged initial data, a vector field explicitly expressed in terms of phase and a small remainder term. In the course of the proof [BMN1] [BMN2] have obtained the following stability-regularity results:

i) Whatever the size of the smooth initial data the life span of the corresponding regular solution of the Euler equation is ensured to go to infinity when Ω_0 goes to infinity.

ii) For non zero but fixed viscosity ν, whatever the size of the smooth initial data, the corresponding classical Leray solution of the 3-D Navier-Stokes system becomes smooth (for $\hat{T} \le t < \infty$) for Ω_0 large enough. This is true for all domain geometrical parameters. Specifically:

Theorem 2.8. *Let $\nu > 0$, $\alpha > 1/2$, $\|\mathbf{U}(0)\|_\alpha \le M_\alpha$ and*

$$\sup_T \int_T^{T+1} \|\mathbf{F}\|^2_{\alpha-1} dt \le M^2_{F_\alpha} \quad \forall T. \tag{2.5.24}$$

Let $\Omega_0 \ge \Omega_0^* \ (M_\alpha, M_{F_\alpha}, \nu)$. Then solutions of rotating Navier-Stokes equations for any periodic domain parameters are regular for all t and

$$\|u(t)\|_\alpha \le M'_\alpha \quad \forall t \ge 0. \tag{2.5.25}$$

Theorem 2.9. *Let $\nu > 0$ and conditions of Theorem 2.6 hold. Let $\|u(0)\|_0 \le M_0$, $\hat{T} = \|u(0)\|_0^2/\nu$. Then, for every fixed $\Omega_0 \ge \Omega'_0$ with $\Omega'_0 = \Omega'_0(M_{F_\alpha})$ and for any weak solution $u(t)$ of rotating Navier-Stokes equations which is defined on $[0, \hat{T}]$ and satisfies the classical energy inequality on $[0, \hat{T}]$, the following is true: $u(t)$ can be extended to $0 < t < +\infty$ and it is regular for $\hat{T} \le t < +\infty$; it belongs to H_1 and $\|u(t)\|_1 \le C_1(M_{F1}, \nu)$ for every $t \ge \hat{T}$.*

In particular, Theorem 2.8 relies on the global regularity of a "2-1/2 dimensional" limit nonlinear Navier-Stokes equation as $\Omega_0 \to \infty$, [BMN2].

These results are not conditional, in contrast to the work of Constantine et al., with the following remark. In the rotating equation (2.5.23) when the vorticity Ω_0 is large, but bounded, the highly oscillatory (in time) solution is regular. One can show that (2.5.23) is equivalent to a Navier-Stokes equation without Coriolis term, with a base steady flow $(-\Omega_0 y, +\Omega_0 x, 0)$ of vorticity $2\Omega_0 e_3$, plus spatially periodic perturbations of vorticity ω_1. Then if $|\omega_1|$ is not too large with respect to $|\Omega_0|$, with $|\Omega_0| \gg 1$, one proves [BMN2] that the corresponding Navier-Stokes system stays smooth for all times (ω_1 is not a small perturbation). Note that the perturbed ω_1-flow is genuinely 3 dimensional. The technique of bootstrapping regularity of solutions of 3-dimensional Navier-Stokes equations by perturbation from limit equations has been done in various contexts: thin domains [RS], helical flows [MLT?]. In these previous works, limit equations are 2-D Navier-Stokes equations for which global regularity is well known. In [BMN2], for the first time, the limit equations are genuinely 3-dimensional, but with restricted wave-number interactions in $B(u, u)$ ("2-1/2 dimensional"). Their global existence is nontrivial and requires dyadic decomposition methods and Littlewood-Paley theory [St].

Similar results for more general Boussinesq equations of geophysics can be found in [BMN4-8].

3. Hierarchy of Equations

3.1. Introduction

As said in the general introduction one shows that, with the Boltzmann equation, the Navier-Stokes and Euler equations can be derived from a genuine Hamiltonian system of N interacting particles.

This Hamiltonian system is the beginning of the hierarchy. The end of the hierarchy is the introduction of turbulent models which are in some cases constructed with a statistic description of the fluid. In this case some basic properties of dynamical systems like the ergodic hypothesis are involved and some "magic" numbers like the Kolmogorov exponent appears.

Eventually one should observe that the different steps of the hierarchy share in common several features like the evolution of the notion of entropy and the recurrent use of moments or averages.

3.2 The Boltzmann-Grad Limit

The purpose of this section is to provide a rapid overview of the derivation of the Boltzmann equation from molecular dynamics.

The points that should be emphasized are the following:

i) In this problem there are two natural parameters N the number of particles and σ the radius of these particles. N is a very large number and σ (expressed in common units, such as centimeters) is very small; Consider a rarefied gas in a box whose volume is 1cm^3 at room temperature and atmospheric pressure. Then $N \simeq 10^{20}$, $\sigma = 10^{-8}\text{cm}$ and $N\sigma^2 = 1\text{m}^2$ is a sizable quantity. Therefore it is natural to consider situations where

$$\lim_{N\to\infty,\,\sigma\to 0} N\sigma^2$$

is strictly positive and finite. This gives the mean free path and the Knudsen number.

ii) The transition from a reversible problem to a irreversible problem is made by an averaging process which takes in account the self interaction of the particles of the media.

The direction of the time appears because one obtains an equation for averaged quantities at time $t > 0$ only keeping information on averaged quantities at time $t = 0$. The same construction should be possible for negative time but would lead to a Boltzmann equation with negative term that would therefore increase the mathematical entropy.

iii) On one hand the nonlinearity helps because at variance with diffusion approximation of reversible linear systems (as presented in Chapter 2) the entropy at the level of the Boltzmann equation (a quantity which naturally decays) is the limit of a quantity which, due to the non linearity, is not conserved by the molecular dynamics. Therefore this does not contradict strong convergence results.

iv) On the other hand non-linearity creates also a limitation on the obtaining of rigorous results based on strong convergence because such results would be, when the kinetic limit is involved, in contradiction with the instabilities of the compressible and incompressible Euler equations described in the previous section

At present there are two types of rigorous results. Both involve regular quantities and therefore they should be kept away from the limits leading to singular solutions of the compressible Euler equation and there are two ways to do so.

The first one (Lanford [La]) is to start from very regular initial data and prove the results for a very small time that would avoid the time where the compressible Euler equation may present singularities. The second one is to consider a dilute gas (Illner and Pulvirenti [IP]) in an infinite media which will never behave like the solution of the compressible Euler equation and therefore global-in-time convergence proof can be obtained in this case. However since this regime does not lead to the Fluid dynamics equations it should not be considered as a pertinent step for our hierarchy.

Below only a formal proof is given following the Section 2.2 of Cercignani, Illner and Pulvirenti [CIP]. The convergence proofs quoted above can also be found in this book. In any case for the derivation it is both natural and compulsory to invoke the BBGKY hierarchy named after Bogoliubov, Born, Green, Kirkwood and Yvon. This point of view was discovered by Yvon in 1935 and rediscovered independently eleven years later by Kirkwood, Born and Green on one side and by Bogoliubov on the other. It is in the construction of the BBGKY hierarchy that the Boltzmann entropy (which as explained in the previous section is simply related to the macroscopic entropy) appears as a limit process which is not in contradiction to the derivation of a reversible system from an irreversible one.

The starting point is the consideration of a family of N particles of radius σ which evolve freely in the whole space and interact through elastic collisions.

More precisely if two particles with incoming velocity (ξ, ξ_*) and centers x and x_* collide (i.e. $|x - x_*| = \sigma$) then the outgoing velocities (ξ', ξ'_*) compatible with the conservation of mass momentum and energy are given (in term of the incoming velocities) by the formula:

$$\begin{aligned} &\xi' = \xi - \omega[\omega\cdot(\xi-\xi_*)] \\ \text{with} \quad &\omega = \frac{x-x^*}{|x-x_*|}\,, \\ &\xi'_* = \xi_* + \omega[\omega\cdot(\xi-\xi_*)]\,. \end{aligned} \tag{3.2.1}$$

For obvious reason the above problem is called the *hard sphere model.* Other models are based on mass points interacting with central forces. However in this case the Boltzmann Grad limit is more difficult to obtain and to the best of our knowledge it is up to now only done with the introduction of a ad hoc *cutt off* (cf [Ce] page 59).

N is the Avogrado number and it is of the order of 10^{24} and σ is of the order of 10^{-8} meters while $N\sigma^2$ is of the order of 1 square meter. This means that the Boltzmann equation should be derived from the molecular dynamics by letting N go to infinity, σ to zero and letting $N\sigma^2$ go to a positive finite constant which is the inverse of the Knudsen number.

It is convenient to denote by

$$z^s = \{z_1, z_2, ..., z_s\} = \{(x_1, \xi_1), (x_2, \xi_2), ..., (x_s, \xi_s)\} \in (\mathbf{R}^3 \times \mathbf{R}^3)^s\,,$$

the variables of the s dimensional phase space and by $P^s(z_1, z_2, ..., z_s)$ the probability of having jointly the s particles at the point $\{x_1, x_2, ..., x_s\}$ with velocity $\{\xi_1, \xi_2, ..., \xi_s\}$.

It is assumed that these functions are symmetric with respect to their arguments (observe that if such a property is true at some time it is conserved for all time by the collision rule given by (3.2.1)) and are equal to zero on one hand for $s > N$ and on the other hand for

$$z \notin \Lambda^s, \quad \Lambda^s = \{z^s \text{ such that } \forall 1 \le i, j \le s, \, (x_i \neq x_j) \, |x_i - x_j| > \sigma\}. \tag{3.2.2}$$

The meaning of equation (3.2.2) is that it is assumed that the particles do not penetrate one into the others. Furthermore one has

$$P^s(z^s) = \int_{(\mathbf{R}^3 \times \mathbf{R}^3)^{N-s}} P^N(z_1, z_2, ..., z_s, z_{s+1}, ..., z_N) \prod_{(s+1) \le j \le N} dz_j \,. \tag{3.2.3}$$

Since the particles evolve freely in Λ^N one has

$$\partial_t P^N(x, v, t) + \sum_1^N \xi_i \partial_{x_i} P^N(x, v, t) = 0 \,. \tag{3.2.4}$$

This equation has to be accompanied by suitable initial and boundary conditions.

To take into account the fact that the shocks between the particles are elastic it is assumed that the distribution P^N is invariant under the transformation induced by such shocks, i.e., $P^N(z) = P^N(z')$ for any pair (z, z') defined by the following relations:

i) There exists a pair (i, j), $i \neq j$ such that $|x_i - x_j| = \sigma$ (i.e., $z \in \partial\Lambda^N$).

ii) z' is given in terms of z by the formula

$$\begin{aligned} &z' = \{z_1, z_2, ..., (x_i, \xi_i - \omega_{ij}(\omega_{ij} \cdot V_{ij}), ..., (x_i, \xi_j + \omega_{ij}(\omega_{ij} \cdot V_{ij}), ..., z_N\}, \\ &V_{ij} = \xi_i - \xi_j, \quad \omega_{ij} = \frac{x_i - x_j}{|x_i - x_j|}, \end{aligned} \tag{3.2.5}$$

which is simply the translation at the level of probability distribution of the formula (3.2.1).

For the initial data it is assumed that $P^N(z_1, z_2, ..., z_N, 0)$ is a probability density which is invariant under any permutations of the variables z_i. Even better it is assumed that this probability is factorized:

$$P^N(z_1, z_2, ..., z_N, 0) = \prod_{1 \le j \le N} f(z_j), \quad f(z_j) \ge 0, \int_{\mathbf{R}^3 \times \mathbf{R}^3} f(z) dz = 1. \tag{3.2.6}$$

Following [CIP] the equation (3.2.4) is integrated over Λ^N with respect to the variables

$$z_j, \quad (s+1) \le j \le N \,,$$

and one obtains:

$$\partial_t P^s + \int \sum_{i=1}^{s} \xi_i \partial_{x_i} P^N \prod_{(s+1)\le j \le N} dz_j + \int \sum_{k=(s+1)}^{N} \xi_k \partial_{x_k} P^N \prod_{(s+1)\le j\le N} dz_j = 0\,. \qquad (3.2.7)$$

Since the boundary of the integration domain is characterized by the relation $|x_i - x_j| = \sigma$ it depends (even for $i \le s$) upon x_i. Therefore one obtains for the second term of the left hand side of (3.2.7)

$$\int \xi_i \partial_{x_i} P^N \prod_{(s+1)\le j\le N} dz_j = \xi_i \partial_{x_i} P^s - \sum_{k=s+1}^{N} \int P^{(s+1)} \xi_i \cdot \omega_{ik} d\sigma_{ik} d\xi_k \qquad (3.2.8)$$

with ω_{ik} denoting the outer normal to the sphere of radius σ and center x_k and $d\sigma_{ik}$ being the surface element on the same sphere. The second integral term of (3.2.8) is easier to compute because it involves the integration of a derivative taken with respect to one of the integration variables. One obtains:

$$\begin{aligned}\int \xi_k \partial_{x_k} P^N \prod_{(s+1)\le j\le N} dz_j &= \sum_{i=1}^{s} \int P^{(s+1)} \xi_k \cdot \omega_{ik}\, d\sigma_{ik}\, d\xi_k \\ &+ \sum_{i=s+1, i\ne k}^{N} \int P^{(s+2)} \xi_k \cdot \omega_{ik}\, d\sigma_{ik}\, d\xi_k\, dx_i\, d\xi_i\,.\end{aligned} \qquad (3.2.9)$$

Therefore with (3.2.8) and (3.2.9) the following relation is deduced from (3.2.7):

$$\begin{aligned}\partial_t P^s + \sum_{i=1}^{s} \xi_i \partial_{x_i} P^s &= \sum_{i=1}^{s} \sum_{k=s+1}^{N} \int P^{(s+1)} V_{ik} \cdot \omega_{ik} d\sigma_{ik} d\xi_k \\ &+ \tfrac{1}{2} \sum_{i=s+1, i\ne k}^{N} \int P^{(s+2)} V_{ki} \cdot \omega_{ik} d\sigma_{ik} d\xi_k\, dx_i d\xi_i\, dx_i\,.\end{aligned} \qquad (3.2.10)$$

In the above equation $V_{ik} = \xi_i - \xi_k$ is the relative velocity of the ith particle with respect to kth particle. The relations $\omega_{ik} = -\omega_{ki}$ have been used to replace $\xi_k \cdot \omega_{ik}$ by $\frac{1}{2} V_{ik} \cdot \omega_{ik}$ Observe that with the boundary condition $P^N(z) = P^N(z')$ with z' given by (3.1.4), the last term of the right hand side of (3.2.10) is identically zero.

Observe also that the first integral in the right hand side of (3.2.10) is the same no matter what the value of the dummy index k is. This index can be abolished; x_*, ξ_* is written in place of x_k, ξ_k and (3.2.10) is transformed into:

$$\partial_t P^s + \sum_{i=1}^{s} \xi_i \partial_{x_i} P^s = (N-s) \sum_{i=1}^{s} \int P^{(s+1)} V_i . \omega_i \, d\sigma_i \, d\xi_*\,. \qquad (3.1.11)$$

In (3.2.11) the arguments of $P^{(s+1)}$ are $(z_1, z_2, ..., z_s, z_* = (x_*, \xi_*))$ while V_i and n_i are defined by the relation:

$$V_i = \xi_i - \xi_*, \quad \omega_i = \frac{x_i - x_*}{\sigma} \qquad (3.2.12).$$

We separate the contribution of the two hemispheres S^i_+ and S^i_- respectively defined by $V_i \cdot \omega_i > 0$ and $V_i \cdot \omega_i < 0$. In addition we remark the relation $d\sigma_i = \sigma^2 d\omega_i$ where $d\omega_i$ is the surface element on the unit sphere described by ω_i and eventually we obtain

$$\begin{aligned}\partial_t P^s + \sum_{i=1}^{s} \xi_i \partial_{x_i} P^s = (N-s)\sigma^2 \sum_{i=1}^{s} \Big(\int_{\mathbf{R}^3} \int_{S_+} P^{(s+1)'} |V_i.\omega_i| \, d\omega_i \, d\xi_* \\ - \int_{\mathbf{R}^3} \int_{S_-} P^{(s+1)} |V_i.\omega_i| \, d\omega_i \, d\xi_* \Big)\end{aligned} \qquad (3.2.13)$$

where $P^{(s+1)'}$ means that in $P^{(s+1)}$ the argument ξ_i and ξ^* are replaced by the following ones:

$$\begin{aligned}\xi'_i &= \xi_i - \omega_i(\omega_i \cdot V_i)\,, \\ \xi'_* &= \xi_* + \omega_i(\omega_i \cdot V_i)\,.\end{aligned} \qquad (3.2.14)$$

At this point a choice in the direction of the time has been made because the velocities after the shock have been express in term of the velocities before the shock. The above integrals can be changed into an a single integral by changing ω_i into $-\omega_i$ in the second integral. The index i in ω_i can be dropped provided the argument x_* in the second integral of the ith term is replaced by

$$x_* = x_i - \omega\sigma$$

while x_* is replaced by $x_i + \omega\sigma$ in the first integral. These computation leads to a system of N equations for N unknown P^s:

$$\partial_t P^s_N + \sum_{i=1}^{s} \xi_i \partial_{x_i} P^s_N = (N-s)\sigma^2 \sum_{i=1}^{s} \int_{\mathbf{R} \times S^2} \big(P_N^{(s+1)'} |V_i.\omega_i| - P_N^{(s+1)} |V_i.\omega_i|\big) d\omega_i d\xi_* \quad (3.2.15)$$

which is called the BBGKY hierarchy.

The Boltzmann limit is obtained by letting N go to infinity and σ to zero with the condition:

$$\lim N\sigma^2 = \epsilon^{-1}$$

and the initial data:

$$P^N = \prod_{1 \le j \le N} f(z_j), \quad f(z_j) \ge 0, \ \int f(z_j) dz_j = 1\,. \qquad (3.2.16)$$

If we assume (this is one of the main object of the contributions of Landord and Illner and Pulvirenti) that the convergence (for s fixed and N going to infinity)

$$\lim_{N \to \infty} P^s_N = P^s$$

holds for any s in a convenient topology, we can deduce from (3.2.15) an infinite set of equations (for $1 \leq s < \infty$)

$$\partial_t P^s + \sum_{i=1}^{s} \xi_i \partial_{x_i} P^s = \frac{1}{\epsilon} \sum_{i=1}^{s} \int_{\mathbf{R}^3 \times S^2} \left(P_N^{(s+1)'} |V_i.\omega_i| - P_N^{(s+1)} |V_i.\omega_i| \right) d\omega_i d\xi_* \qquad (3.2.17)$$

which is called the Boltzmann hierarchy.

This derivation is completed by the three following statements.

" i)" Introduce the Boltzmann equation for hard spheres which will also be considered in the next section:

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\epsilon} C(F) \,, \qquad (3.2.18)$$

$$F(0, x, v) = F^{in}(x, v) \geq 0 \,, \qquad (3.2.19)$$

where the collision operator $C(F)$ is quadratic and given (with an abuse of language) by

$$C(F) = C(F, F) = \iint (F_1' F' - F_1 F) \, |(v_1 - v) \cdot \omega| \, d\omega \, dv_1 \,. \qquad (3.2.20)$$

with the F, F_1, F' and F_1' appearing in the integrand understood to mean $F(t, x, \cdot)$ evaluated at the velocities v, v_1, v' and v_1' respectively, where the primed velocities are defined by

$$v' = v + \omega \, \omega \cdot (v_1 - v) \,, \qquad v_1' = v_1 - \omega \, \omega \cdot (v_1 - v) \,, \qquad (3.2.21)$$

for any given $(v, v_1, \omega) \in \mathbf{R}^{\mathrm{D}} \times \mathbf{R}^{\mathrm{D}} \times \mathbf{S}^{\mathrm{D}-1}$. Use the fact that for smooth initial data close to an absolute Maxwellian (ρ_* and θ_* are constant)

$$M = \frac{\rho_*}{(2\pi\theta_*)^{\frac{3}{2}}} \exp(-\tfrac{1}{2}|v|^2/\theta_*) \,. \qquad (3.2.22)$$

the Boltzmann equation (3.2.18) has a unique smooth solution (defined at least during a finite time).

" ii)" For smooth initial data the Boltzmann hierarchy has a unique solution (defined at least for a small time).

" iii)" If $F(z, t) = F(x, \xi, t)$ is the corresponding solution of the Boltzmann equation then the unique solution of the Boltzmann hierarchy with initial data given by

$$P^s(z_1, z_2, z_3, \ldots z_s) = \prod_{1 \leq i \leq s} f(z_i) \qquad (3.2.23)$$

is given by the same factorization:

$$P^s(z_1, z_2, z_3, \ldots z_s, t) = \prod_{1 \leq i \leq s} f(z_i, t) \,. \qquad (3.2.24)$$

The item i) is now classical in theory of Boltzmann equation (Ukai [U] or Nishida and Imai [NI]).

The item ii) has been solved by Lanford and Illner and Pulvirenti under weaker hypothesis than the one needed to prove the convergence of the P_N^s. The item iii) is obtained by direct inspection.

As a consequence under convenient hypothesis the function $P_N^s(z_1, ..., z_n, t)$ converge to a function

$$P^s(z_1, ..., z_n, t) = \prod_{1\le i\le s} f(z_i, t)$$

with $f(z,t)$ solution of the Boltzmann equation. This implies in particular that the function $P^1(z,t)$ converges to the solution of the Boltzmann equation and that the P_N^s factorize at the limit. Such a property is called *propagation of chaos.*

It is important to notice that the above convergence does not contradict the appearance of the decay of entropy for solution of the Boltzmann equation. On one hand it should be observed that for any solution of the Liouville equation and for any function F one has

$$\frac{d}{dt}\Big(\int_{(\mathbf{R}^3\times\mathbf{R}^3)^N} F(P^N(z_1, z_2, ...z_N, t)) \prod_{1\le i\le N} dz_i\Big) = 0\,. \tag{3.2.25}$$

On the other hand (this is just obtained by Fubini Theorem and change of variables) for any solution of the Boltzmann equation one has

$$\frac{d}{dt}\Big(\int_{(\mathbf{R}^3\times\mathbf{R}^3)} f\log f(z,t)dz\Big) \le 0 \tag{3.2.26}$$

with a strict inequality whenever f is not an absolute maxwellian.

Eventually with (3.2.25) one has

$$\begin{aligned}
&\int_{(\mathbf{R}^3\times\mathbf{R}^3)} f(z,t)\log f(z,t)dz \le \int_{(\mathbf{R}^3\times\mathbf{R}^3)} f(z,0)\log f(z,0)dz\\
&= \frac{1}{N}\int\int_{(\mathbf{R}^3\times\mathbf{R}^3)^N} (P^N(z_1, z_2, ...z_N, 0))\log(P^N(z_1, z_2, ...z_N, 0)) \prod_{1\le i\le N} dz_i\\
&= \frac{1}{N}\int\int_{(\mathbf{R}^3\times\mathbf{R}^3)^N} (P^N(z_1, z_2, ...z_N, t))\log(P^N(z_1, z_2, ...z_N, t)) \prod_{1\le i\le N} dz_i\,.
\end{aligned} \tag{3.2.27}$$

However,

$$\int_{\mathbf{R}^3\times\mathbf{R}^3} f(z,t)\log f(z,t)\,dz$$

is not the limit of

$$H^N(P^N)(t) = \frac{1}{N}\int\int_{(\mathbf{R}^3\times\mathbf{R}^3)^N} (P^N(z_1, ..., z_N, t))\log(P^N(z_1, ..., z_N, t)) \prod_{1\le i\le N} dz_i$$

but of

$$H^1(P^1)(t) = \int\int_{(\mathbf{R}^3\times\mathbf{R}^3)^N} (P^1_N(z,t))\log(P^1_N(z,t))dz\,.$$

And according to the Proposition 3.1 (below) one has $H^1(P^1)(t) \leq H^N(P^N)(t)$ with strict inequality unless $P^1(t)$ is factorized. Therefore the strong convergence of $P^1(t)$ does not contradict the decay of entropy of the solution of the Boltzmann equation. In fact one observes also that the factorization property given for P^N at time $t=0$ is immediately lost for $t>0$ but is recovered for the limit of $P^s_N(t)$ (s, $t>0$ fixed and N going to infinity).

Proposition 3.1 *Suppose that P^N is a symmetric probability density on the phase space and that P^s are the s particles distribution function associated with P^N . Then*

$$H^1(P^1) \leq H^N(P^N)$$

with equality if and only if

$$P^N(z) = \prod_{i=1}^{N} P^1(z_i)$$

for almost all z.

Proof Note that for $x, y \geq 0$ one has (with the right hand side equal to zero for $y=0$ and to $-\infty$ for $y>0$ and $x=0$):

$$x - y \geq y \log\frac{x}{y}\,.$$

Therefore one has

$$0 = \Big(\big(\prod_{i=1}^{N} P^1(z_i)\big) - P^N(z)\Big)dz \geq \int P^N(z)\log\Big(\frac{\prod_{i=1}^N P^1(z_i)}{P^N(z)}\Big)\,dz \qquad (3.2.28)$$

and this relation is equivalent to the relation:

$$\int P^N(z)\log P^N(z)\,dz \geq N.H^1(P^1)$$

with equality only when the middle term of (3.2.28) is zero, i.e; when factorization occurs.

Remark 3.2 Rigorous proofs contain many more ingredients however two points should be stressed.

1) Since the Boltzmann equation is quadratic it involves only binary collision therefore an important step is to prove that other events can be excluded and this is a direct consequence of the following theorem (cf. [CIP], page 65): "The set of points that are led into a multiple collision under forward or backward evolution of the dynamical system and the set of points such that where there is a cluster of collision instants under forward and backward evolution are of measure zero in the phase space."

2) The solution of the BBGKY hierarchy can be written in an integrated form (or weak form) leading first to a formal series expansion:

$$P_N^s(z)(z^s,t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n \\ S_\sigma(t-t_1) Q_{s+1}^\sigma S_\sigma(t_1-t_2) ... Q_{s+n}^\sigma S_\sigma(t_n) P_N^{s+n}(z^s,0) \tag{3.2.29}$$

with $P_N^s = 0$ for $s > N$, with S_σ and Q_{s+1}^σ operators describing the advection and the collisions. It is in the uniform estimate of the right hand side of (3.2.29) that the hypothesis on the data or on the small time intervals of validity does appear.

3.3 The Fluid dynamics limits

Continuing the hierarchy we return to the equation (3.2.18):

$$\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon} C(F_\epsilon) \,. \tag{3.3.1}$$

whith $C(F)$ denoting the collision operator given by (3.2.20) and

$$\epsilon = \lim_{N \to \infty, \sigma \to 0} N\sigma^2 \,.$$

The equation (3.3.1) (cf [Go] for details in the following discussion) shows that

$$\frac{1}{\epsilon} \iint F_1 \, |(v_1 - v) \cdot \omega| \, d\omega \, dv_1$$

is homogenous to a frequency. If the variation of F in t and in x are not too fast, this frequency is the reciprocal of the averaged time between two successive collisions undergone by the same typical particle under the distribution F moving with speed v.

However this frequency depends on the particle distribution itself, which makes it difficult to use this expression as a tool to discriminate between various qualitative behavior of this particle distribution. Rather, pick an averaged macroscopic density ρ_* an averaged temperature θ_* and choose a macroscopic lenght scale λ_* (for instance the size of the domain where the flow takes place or the average size of $\partial_x F/F$ at $t = 0$. Then rewrite the equation (3.3.1) for the dimensionless number density

$$\hat{F} = \frac{\theta_*^{\frac{3}{2}}}{\rho_*} F$$

in term of the dimensionless time, space and velocity variable:

$$\hat{t} = \frac{t\sqrt{\theta_*}}{\lambda_*}, \ \frac{\hat{x}}{\lambda_*}, \ \hat{v} = \frac{v}{\sqrt{\lambda_*}}$$

and obtain:

$$\partial_{\hat{t}}\hat{F} + \hat{v}\nabla_{\hat{x}}\hat{F} = \frac{1}{K_n}C(\hat{F})\,. \tag{3.\hat{3}.1}$$

where K_n the (dimensionless) Knudsen number is the ratio of the collision mean free time to the macroscopic time scale.

All hydrodynamic limits of the Boltzmann equation consist in considering this dimensionless form and in discussing the limit as K_n (and possibly other parameters) tend to 0. Physically, this means that a great number of collision take place in the gas per unit of (macroscopic) observation time.

Observe that this is not in contradiction with the usual phrase about the Boltzmann equation which applies to gases in a state of low density because one has:

$$K_n \simeq \frac{N\sigma^2}{\lambda^*}$$

The analysis done below applies also to other collision operators which are introduced either for physical or numerical reason. Therefore it is interesting to select the basic properties of the operator which are used at different steps:

i) conservation properties and an entropy relation that implies that the equilibria are Maxwellian distributions for the zero-th order limit;

ii) the derivative of $C(F)$ satisfies a formal Fredholm alternative with a kernel related to the conservation properties of (i).

Properties (i) are sufficient to derive the compressible Euler equations from equation (3.2.1) (Theorem 3.3). The compressible Euler equations also arise as the leading order dynamics from a systematic expansion of F in ϵ.

Properties (ii) are used to obtain the Navier-Stokes equations; they depend on a more detailed knowledge of the collision operator. The compressible Navier-Stokes equations arise as corrections to those of Euler at the next order in the Chapman-Enskog expansion. This expansion shows that in a compressible gaz the Knudsen and Reynold number are of the same order. To recover directly from the Boltzmann equation the incompressible Navier- Stokes equation one also introduces the Mach number Ma which is the ratio of the bulk velocity to the sound speed and the Reynolds number Re which is a dimensionless reciprocal viscosity of the fluid. These numbers (cf. [LL]) satisfy the relation

$$\epsilon = \frac{Ma}{Re}\,. \tag{3.3.2}$$

Therefore when ϵ goes to zero, to obtain a fluid dynamical limit with a finite Reynolds number, the Mach number must vanish too.

The compressible Euler limit

The integral of any scalar or vector valued function $f(v)$ with respect to the variable v is denoted by $\langle f\rangle$;

$$\langle f\rangle = \int_{\mathbf{R}^3} f(v)\,dv\,. \tag{3.3.3}$$

Use of Fubini theorem and change of variable to show that the operator C satisfies the conservation properties

$$\langle C(F)\rangle = 0\,, \quad \langle vC(F)\rangle = 0\,, \quad \langle |v|^2 C(F)\rangle = 0\,. \tag{3.3.4}$$

which are the simple translation at the level of the function F of the corresponding properties for the system of particles of the previous section. As a consequence one has the following conservation laws:

$$\begin{aligned} \partial_t\langle F\rangle + \nabla_x\cdot\langle vF\rangle = 0\,, \\ \partial_t\langle vF\rangle + \nabla_x\cdot\langle v\otimes v\, F\rangle = 0\,, \\ \partial_t\langle \tfrac{1}{2}|v|^2 F\rangle + \nabla_x\cdot\langle v\tfrac{1}{2}|v|^2 F\rangle = 0\,. \end{aligned} \tag{3.3.5}$$

Similarly it has been observed in the previous section that $\langle C(F)\log F\rangle$ is non-positive, this implies the local entropy inequality

$$\partial_t\langle F\log F\rangle + \nabla_x\cdot\langle vF\log F\rangle = \langle C(F)\log F\rangle \le 0\,. \tag{3.3.6}$$

A more detailed analysis shows that

$$\langle C(F)\log F\rangle = 0 \tag{3.3.7}$$

implies that F is a Maxwellian:

$$F = \frac{\rho}{(2\pi\theta)^{3/2}}\exp\Big(-\tfrac{1}{2}\frac{|v-u|^2}{\theta}\Big)\,. \tag{3.2.8}$$

The parameters ρ, u and θ introduced at the right side of (3.3.8) are related to the fluid dynamics moments giving the mass, momentum and energy densities:

$$\langle F\rangle = \rho\,, \qquad \langle vF\rangle = \rho u\,, \qquad \langle \tfrac{1}{2}|v|^2 F\rangle = \rho(\tfrac{1}{2}|u|^2 + \tfrac{3}{2}\theta)\,.$$

They are called respectively the (mass) density, velocity and temperature of the fluid. In the compressible Euler limit these variables are shown to satisfy the system of compressible Euler equations (3.3.11 below).

The main obstruction to proving the validity of this fluid dynamical limit is the fact that, as said in the section (2.4), the solutions of the compressible Euler equations generally become singular after a finite time. Therefore any global (in time) convergence proof cannot rely on uniform regularity estimates. A reasonable assumption would be that the limiting distribution exists and that the relevant moments converge pointwise.

Theorem 3.3 *Let $F_\epsilon(t,x,v)$ be a sequence of nonnegative solutions of the equation*

$$\partial_t F_\epsilon + v\cdot\nabla_x F_\epsilon = \frac{1}{\epsilon}C(F_\epsilon)\,, \tag{3.3.9}$$

such that as ϵ goes to zero, F_ϵ converges almost everywhere to a nonnegative function F. Moreover, assume that the moments

$$\langle F_\epsilon \rangle \,, \qquad \langle vF_\epsilon \rangle \,, \qquad \langle v \otimes v\, F_\epsilon \rangle \,, \qquad \langle v|v|^2 F_\epsilon \rangle \,,$$

converge in the sense of distributions to the corresponding moments

$$\langle F \rangle \,, \qquad \langle vF \rangle \,, \qquad \langle v \otimes v\, F \rangle \,, \qquad \langle v|v|^2 F \rangle \,;$$

the entropy densities and fluxes converge in the sense of distributions according to

$$\lim_{\epsilon \to 0} \langle F_\epsilon \, \log F_\epsilon \rangle = \langle F \, \log F \rangle \,, \quad \lim_{\epsilon \to 0} \langle vF_\epsilon \, \log F_\epsilon \rangle = \langle vF \, \log F \rangle \,;$$

while the entropy dissipation rates satisfy

$$\limsup_{\epsilon \to 0} \langle C(F_\epsilon) \, \log F_\epsilon \rangle \leq \langle C(F) \, \log F \rangle \,.$$

Then the limit $F(t,x,v)$ is a Maxwellian distribution,

$$F(t,x,v) = \frac{\rho(t,x)}{(2\pi\theta(t,x))^{3/2}} \, \exp\left(-\tfrac{1}{2} \frac{|v - u(t,x)|^2}{\theta(t,x)} \right) \,, \tag{3.3.10}$$

where the functions ρ, u and θ solve the compressible Euler equations,

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0 \,, \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x (\rho \theta) &= 0 \,, \\ \partial_t (\rho (\tfrac{1}{2}|u|^2 + \tfrac{3}{2}\theta)) + \nabla_x \cdot (\rho u (\tfrac{1}{2}|u|^2 + \tfrac{5}{2}\theta)) &= 0 \end{aligned} \tag{3.3.11}$$

and satisfy the entropy inequality,

$$\partial_t \Big(\rho \log \Big(\frac{\rho^{2/3}}{\theta} \Big) \Big) + \nabla_x \cdot \Big(\rho u \log \Big(\frac{\rho^{2/3}}{\theta} \Big) \Big) \leq 0 \,. \tag{3.3.12}$$

Proof Multiplying (3.3.9) by $\epsilon(1 + \log F_\epsilon)$ and integrating over v gives the entropy relation

$$\epsilon \Big(\partial_t \langle F_\epsilon \log F_\epsilon \rangle + \nabla_x \cdot \langle vF_\epsilon \log F_\epsilon \rangle \Big) = \langle C(F_\epsilon) \log F_\epsilon \rangle \,. \tag{3.3.13}$$

Letting ϵ go to zero in (3.3.13) and using the convergence assumptions of the theorem regarding the entropic quantities shows that the limiting distribution F must satisfy

$$0 \leq \limsup_{\epsilon \to 0} \langle C(F_\epsilon) \, \log F_\epsilon \rangle \leq \langle C(F) \log F \rangle \,. \tag{3.2.14}$$

But the entropy dissipation rate of $C(F)$ is non-positive by assumption, so (3.3.14) implies $\langle C(F) \log F \rangle = 0$. The characterization of equilibria (3.3.8) then gives that for almost every (t,x) the distribution F is a solution of the equation $C(F)=0$ and is a Maxwellian distribution with the form (3.3.10).

The system of local conservation laws

$$\begin{aligned} \partial_t\langle F_\epsilon\rangle + \nabla_x\cdot\langle vF_\epsilon\rangle &= 0\,, \\ \partial_t\langle vF_\epsilon\rangle + \nabla_x\cdot\langle v\otimes v\,F_\epsilon\rangle &= 0\,, \\ \partial_t\langle \tfrac{1}{2}|v|^2F_\epsilon\rangle & \\ +\nabla_x\cdot\langle v\tfrac{1}{2}|v|^2F_\epsilon\rangle &= 0\,, \end{aligned} \tag{3.3.15}$$

is not closed. Each of these equations for the determination of the time derivative of a moment involves the knowledge of a higher order moment. However, if the convergence assumptions of the theorem regarding these moments are used, one can pass to the limit of ϵ going to zero and replace F_ϵ by F, as given by (3.3.10), in these equations. A system of five equations for the five unknowns $\{\rho, u_1, u_2, u_3, \theta\}$ is obtained which is the compressible Euler system (3.3.11).

Finally, utilizing the entropy dissipation property

$$\langle C(F_\epsilon)\log F_\epsilon\rangle \le 0\,, \tag{3.3.16}$$

equation (3.3.9) leads to the inequality

$$\partial_t\langle F_\epsilon\log F_\epsilon\rangle + \nabla_x\cdot\langle vF_\epsilon\log F_\epsilon\rangle \le 0\,. \tag{3.3.17}$$

Once again using the convergence hypothesis of the theorem regarding the entropy densities and fluxes along with the form of F given by (3.3.10), this inequality gives the classical entropy inequality (3.3.12).

The compressible Navier-Stokes limit

In the derivation of the compressible Euler limit the main ingredient turned out to be the identification of the equilibrium points of the collision operator. Such points, as observed, are the maxwellian:

$$M_{(\rho,u,\theta)} = \frac{\rho}{(2\pi\theta)^{3/2}}\exp\left(-\tfrac{1}{2}\frac{|v-u|^2}{\theta}\right)\,. \tag{3.3.18}$$

As said above the Navier Stokes equation is derived as an higher order approximation, either as it is done in the present section as a second order approximation for the solution of the Boltzmann equation or as it will be done in the next section for fluctuations near an absolute maxwellian. Therefore the properties of the Frechet derivative (involved in higher order expansion) appear in the present section. Denote by M the absolute maxwellian $\rho=1,\ u=0\ \theta=0)$ and consider perturbations of the form:

$$F = M(1+f)$$

with f in the Hilbert space L^2_M defined by the scalar product

$$(f|g)_M = \langle fg \rangle_M = \int f(v)g(v)\,M(v)\,dv\,. \qquad (3.3.19)$$

The linear and quadratic operators L and Q are defined according to the formula:

$$\frac{1}{M}C(M(1+f)) = \frac{2}{M}C(Mf,M) + \frac{1}{M}C(Mf,Mf) = Lf + Q(f,f) \qquad (3.3.20)$$

Direct computation shows that L is given by the expression:

$$Lf = \iint M_1(f_1' + f' - f_1 + f)\,|(v_1 - v)\cdot\omega|\,d\omega\,dv_1\,. \qquad (3.3.21)$$

It is a self adjoint Fredholm operator in the space L^2_M. Its kernel is the 5 dimensional space spanned by the functions $\{1, v_1, v_2, v_3, |v|^2\}$. Furthermore it is a non positive operator.

The vector or tensor valued functions

$$A(v) = (\tfrac{1}{2}|v|^2 - \tfrac{5}{2})v\,, \quad B(v) = v\otimes v - \tfrac{1}{3}|v|^2 I\,. \qquad (3.3.22)$$

are orthogonal to the kernel of L; therefore the equations

$$L(A') = A\,, \qquad L(B') = B\,, \qquad (3.3.23)$$

have unique solutions in $Ker(L)^-$.

The rotationally invariance of the collision operator implies that these solutions are given by the formula:

$$A'(v) = -\alpha(|v|)A(v) \ \text{ and } B'(v) = -\beta(|v|)B(v) \qquad (3.3.24)$$

with α and β denoting two positive functions (cf Chapman and Cowling [ChCo] for their explicit computation) and the formulas:

$$\nu_* = \tfrac{1}{10}\langle\beta(v)|B(v)|^2\rangle,\ \kappa_* = \tfrac{1}{3}\langle\alpha(v)|A(v)|^2\rangle \qquad (3.3.25)$$

define two numbers which in some sense are the “universal” viscosity and heat conductivity.

A function $H_\epsilon(t,x,v)$ is said to be an approximate solution of order p to the kinetic equation (3.3.1) if

$$\partial_t H_\epsilon + v\cdot\nabla_x H_\epsilon = \frac{1}{\epsilon}C(H_\epsilon) + O(\epsilon^p)\,, \qquad (3.3.26)$$

where $O(\epsilon^p)$ denotes a term bounded by ϵ^p in some convenient norm. An approximate solution of order 2 will be constructed in the form

$$H_\epsilon = M_\epsilon(1 + \epsilon g_\epsilon + \epsilon^2 w_\epsilon)\,, \qquad (3.3.27)$$

where $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ solve the compressible Navier-Stokes equations with dissipation of the order ϵ (denoted $CNSE_\epsilon$):

$$\partial_t \rho_\epsilon + \nabla_x \cdot (\rho_\epsilon u_\epsilon) = 0\,, \tag{3.3.28}$$

$$\rho_\epsilon \big(\partial_t + u_\epsilon \cdot \nabla_x\big) u_\epsilon + \nabla_x(\rho_\epsilon \theta_\epsilon) \epsilon \nabla_x \cdot [\nu_* \theta_\epsilon^{\frac{1}{2}} \sigma(u_\epsilon)]\,, \tag{3.3.29}$$

$$\tfrac{3}{2}\rho_\epsilon \big(\partial_t + u_\epsilon \cdot \nabla_x\big)\theta_\epsilon + \rho_\epsilon \theta_\epsilon \nabla_x \cdot u_\epsilon = \epsilon \tfrac{1}{2} \nu_* \theta_\epsilon^{\frac{1}{2}} \sigma(u_\epsilon) : \sigma(u_\epsilon) + \epsilon \nabla_x \cdot [\kappa_* \theta_\epsilon^{\frac{1}{2}} \nabla_x \theta_\epsilon]\,. \tag{3.3.30}$$

The Chapman-Enskog derivation can be formulated according to the following theorem.

Theorem 3.4 *Assume that $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ solve the $CNSE_\epsilon$. Then there exist g_ϵ and w_ϵ in $Ker(L_{(\rho_\epsilon, u_\epsilon, \theta_\epsilon)})^-$ such that H_ϵ, given by (3.3.27), is an approximate solution of order 2 to equation (3.3.1). Moreover, g_ϵ is given by the formula*

$$g_\epsilon = -\tfrac{1}{2}\rho_\epsilon^{-1}\theta_\epsilon^{-\frac{1}{2}}\beta(|V|)B(V) : \sigma(u_\epsilon) - \rho_\epsilon^{-1}\theta_\epsilon^{-\frac{1}{2}}\alpha(|V|)\frac{A(V) \cdot \nabla_x \theta_\epsilon}{\sqrt{\theta_\epsilon}}\,. \tag{3.3.31}$$

Proof. In the computation below the subscript ϵ is omitted in the notation of the local maxwellian $M_{(\rho_\epsilon, u_\epsilon, \theta_\epsilon)}$, in the variable V and in the linearized collision operator $L_{(\rho_\epsilon, u_\epsilon, \theta_\epsilon)}$.

Setting the form (3.3.27) for an approximate solution of order two into (3.3.26) yields the formula

$$\begin{aligned} \frac{(\partial_t + v \cdot \nabla_x)M}{M} &+ \epsilon \frac{(\partial_t + v \cdot \nabla_x)(Mg)}{M} \\ &= L(g) + \epsilon\big(L(w) + \tfrac{1}{2}Q(g,g)\big)\,. \end{aligned} \tag{3.3.32}$$

A direct derivation of (3.3.18) gives the formulas

$$\partial_u M = V \frac{1}{\sqrt{\theta}} M\,, \qquad \partial_\theta M = (\tfrac{1}{2}|V|^2 - \tfrac{3}{2})\frac{1}{\theta} M\,;$$

utilizing these shows that the contribution of the first term on the left side of (3.3.32) is given by the formula

$$\frac{(\partial_t + v \cdot \nabla_x)M}{M} = \frac{(\partial_t + v \cdot \nabla_x)\rho}{\rho} + V \cdot \frac{(\partial_t + v \cdot \nabla_x)u}{\sqrt{\theta}} + (\tfrac{1}{2}|V|^2 - \tfrac{3}{2})\frac{(\partial_t + v \cdot \nabla_x)\theta}{\theta}\,. \tag{3.3.33}$$

The $CNSE_\epsilon$ are used to replace the time derivatives of the functions ρ, u and θ by expressions involving only spatial derivatives. This introduces terms of order ϵ, corresponding to the right side of the equations (3.3.29) and (3.3.30), into (3.3.33):

$$\frac{(\partial_t + v \cdot \nabla_x)M}{M} = \tfrac{1}{2}B(V) : \sigma(u) + A(V) \cdot \frac{\nabla_x \theta}{\sqrt{\theta}} + \epsilon R\,, \tag{3.3.34}$$

with

$$R = V\cdot\frac{(\nabla_x\cdot[\nu_*\sigma(u)])}{\rho\sqrt{\theta}} + (\tfrac{1}{3}|V|^2-1)\frac{\frac{1}{2}\nu_*\sigma(u):\sigma(u)+\nabla_x\cdot[\kappa_*\nabla_x\theta]}{\rho\theta}\,. \tag{3.3.35}$$

¿From (3.3.32) and (3.3.34) it follows that the term of order one with respect to ϵ has to be given by the formula (3.3.31). To complete the proof one must show the existence of a function w that cancels the term of order one in (3.3.32). This amounts to proving the existence of a solution to the equation

$$L(w) = R + \frac{(\partial_t + v\cdot\nabla_x)(Mg)}{M} - \tfrac{1}{2}Q(g,g)\,. \tag{3.3.36}$$

Such a solution exists if and only if the right side of (3.3.36) is orthogonal to the kernel of L and this (details of the computation are omitted (cf [BGL1])) turns out to be realized when $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ are solution of the $CNSE_\epsilon$

Remark 3.5 Analysis of the above computation shows that the existence of an expansion of the form

$$M_\epsilon(1+\epsilon g_\epsilon + \epsilon^2 w_\epsilon)$$

for a solution of the Boltzmann equation with $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ solution of "some" compressible Navier Stokes equation is possible if and only the viscosity ν and the thermal diffusivity κ are given by the formulas:

$$\nu = \nu_*\epsilon\theta_\epsilon^{\frac{1}{2}}\,, \quad \text{and } \kappa = \kappa_*\epsilon\theta_\epsilon^{\frac{1}{2}} \tag{3.3.37}$$

¿From the formula (3.3.37) one deduces two important facts. First the ratio of the viscosity and the thermal diffusivity is an "absolute" number (independent of the Knudsen number and of the temperature) given by:

$$Pr = \frac{\nu_*}{\kappa_*}$$

and therefore defined by the collision operator. Second when the Knudsen number goes to zero the viscosity goes also to zero at the same rate. Therefore it does not seem possible to derive directly a Navier Stokes equation with a finite Reynolds number from the Boltzmann equation. The way to do it is to consider that the Mach number is also of the order of ϵ and then according to the relation $\epsilon = Ma/Re$ one obtains at the limit the incompressible Navier Stokes equation and this is the object of the next section:

The incompressible Navier-Stokes limit

¿From formula $\epsilon = Ma/Re$, one deduces that in order to obtain a fluid dynamics regime (corresponding to a vanishing Knudsen number) with a finite Reynolds number, the Mach number must vanish and to realize distributions with a small Mach number it is natural to consider them as perturbations about a given absolute Maxwellian (constant in space and time). By the proper choice of Galilean frame and dimensional units this absolute Maxwellian can be taken to have velocity equal to 0, and density and temperature equal to 1; it will be denoted by M. The initial data $F_\epsilon(0,x,v)$ is assumed to be close to M where the order of the distance will be measured with the Knudsen number. Furthermore, if the flow is to be incompressible, the kinetic energy of the flow in the acoustics modes must be smaller than that in the rotational modes. Since the acoustics modes vary on a faster time scale than rotational modes, they may be suppressed by assuming that the initial data is consistent with motion on a slow time scale; this scale separation will also be measured with the Knudsen number. Thus, solutions F_ϵ to the equation

$$\epsilon\, \partial_t F_\epsilon + v\cdot\nabla_x F_\epsilon = \frac{1}{\epsilon} C(F_\epsilon)\,, \tag{3.3.38}$$

are sought in the form

$$F_\epsilon = M(1+\epsilon g_\epsilon)\,. \tag{3.3.39}$$

and one has the

Theorem 3.6. *Let $F_\epsilon(t,x,v)$ be a sequence of nonnegative solutions to the scaled kinetic equation (3.3.38) such that, when it is written according to formula (3.3.39), the sequence g_ϵ converges in the sense of distributions and almost everywhere to a function g as ϵ goes to zero. Furthermore, assume that the moments*

$$\langle g_\epsilon\rangle_M\,, \qquad \langle v g_\epsilon\rangle_M\,, \qquad \langle v\otimes v g_\epsilon\rangle_M\,, \qquad \langle v|v|^2 g_\epsilon\rangle_M\,,$$
$$\langle L^{-1}(A(v))\otimes v g_\epsilon\rangle_M\,, \qquad \langle L^{-1}(A(v))\, Q(g_\epsilon, g_\epsilon)\rangle_M\,,$$
$$\langle L^{-1}(B(v))\otimes v g_\epsilon\rangle_M\,, \qquad \langle L^{-1}(B(v))\, Q(g_\epsilon, g_\epsilon)\rangle_M$$

converge in $D'(R_t^+\times R_x^3)$ to the corresponding moments

$$\langle g\rangle_M\,, \qquad \langle v g\rangle_M\,, \qquad \langle v\otimes v g\rangle_M\,, \qquad \langle v|v|^2 g\rangle_M\,,$$
$$\langle L^{-1}(A(v))\otimes v g\rangle_M\,, \qquad \langle L^{-1}(A(v))\, Q(g, g)\rangle_M\,,$$
$$\langle L^{-1}(B(v))\otimes v g\rangle_M\,, \qquad \langle L^{-1}(B(v))\, Q(g, g)\rangle_M\,,$$

Then the limiting g has the form

$$g = \rho + v\cdot u + (\tfrac{1}{2}|v|^2 - \tfrac{3}{2})\theta\,, \tag{3.3.40}$$

where the velocity u is divergence free and the density and temperature fluctuations, ρ and θ, satisfy the Boussinesq relation

$$\nabla_x\cdot u = 0\,, \qquad \nabla_x(\rho+\theta) = 0\,. \tag{3.3.41}$$

Moreover, the functions ρ, u and θ are weak solutions of the equations

$$\partial_t u + u\cdot\nabla_x u + \nabla_x p = \nu_*\Delta u, \qquad \tfrac{5}{2}\big(\partial_t\theta + u\cdot\nabla_x\theta\big) = \kappa_*\Delta\theta\,, \tag{3.3.42}$$

with ν_* , κ_* given by (3.3.25) and p denoting the pressure which as usual in the incompressible case is the Lagrange multiplier of the constrain $\nabla_x\cdot u = 0$.

Proof of Theorem 3.6 Setting (3.3.39) into (3.3.38) gives

$$\epsilon\,\partial_t g_\epsilon + v\cdot\nabla_x g_\epsilon = \frac{1}{\epsilon}L(g_\epsilon) + \tfrac{1}{2}Q(g_\epsilon, g_\epsilon)\,. \tag{3.3.43}$$

Multiplying this by ϵ, letting ϵ go to zero, and using the moment convergence assumption, yields the relation

$$L(g) = 0\,. \tag{3.3.44}$$

This implies that g belongs to the kernel of L and thus can be written according to the formula (3.3.40).

The derivation of (3.3.41) starts from the equations for conservation of mass and momentum associated with (3.3.43):

$$\epsilon\,\partial_t\langle g_\epsilon\rangle_M + \nabla_x\cdot\langle v g_\epsilon\rangle_M = 0\,, \tag{3.3.45}$$

$$\epsilon\,\partial_t\langle v g_\epsilon\rangle_M + \nabla_x\cdot\langle v\otimes v g_\epsilon\rangle_M = 0\,. \tag{3.3.46}$$

Letting ϵ go to zero above (understanding the limit to be in the sense of distributions) gives the relations

$$\nabla_x\cdot\langle v g\rangle_M = 0\,, \qquad \nabla_x\cdot\langle v\otimes v g\rangle_M = 0\,.$$

When g is replaced by the right side of (3.3.40) these become (3.3.41).

The limiting momentum equation is obtained from

$$\partial_t\langle v g_\epsilon\rangle_M + \frac{1}{\epsilon}\nabla_x\cdot\langle v\otimes v g_\epsilon\rangle_M = 0 \tag{3.3.46}$$

by first separating the flux tensor into its tracefree and diagonal parts:

$$\partial_t\langle v g_\epsilon\rangle_M + \frac{1}{\epsilon}\nabla_x\cdot\langle(v\otimes v - \tfrac{1}{3}|v|^2 I)g_\epsilon\rangle_M + \frac{1}{\epsilon}\nabla_x\langle\tfrac{1}{3}|v|^2 g_\epsilon\rangle_M = 0\,. \tag{3.3.47}$$

This is best thought of as being in the form

$$\partial_t\langle v g_\epsilon\rangle_M + \frac{1}{\epsilon}\nabla_x\cdot\langle B(v) g_\epsilon\rangle_M + \nabla_x p_\epsilon = 0\,, \tag{3.3.48}$$

where the pressure is given by $p_\epsilon = \epsilon^{-1}\langle\frac{1}{3}|v|^2 g_\epsilon\rangle_M$. In the same spirit, the limiting temperature equation is obtained by combining the density and energy equations for (3.3.38) as

$$\partial_t\langle(\tfrac{1}{2}|v|^2 - \tfrac{5}{2})g_\epsilon\rangle_M + \frac{1}{\epsilon}\nabla_x\cdot\langle A(v) g_\epsilon\rangle_M = 0\,. \tag{3.3.49}$$

Utilization of the moment convergence assumption and the limiting form of g given by (3.3.40) provides the evaluation of the distribution limits

$$\begin{gathered}\lim_{\epsilon\to 0}\partial_t\langle v g_\epsilon\rangle_M = \partial_t\langle v g\rangle_M = \partial_t u\,,\\ \lim_{\epsilon\to 0}\partial_t\langle(\tfrac{1}{2}|v|^2 - \tfrac{5}{2})g_\epsilon\rangle_M = \partial_t\langle(\tfrac{1}{2}|v|^2 - \tfrac{5}{2})g\rangle_M = \tfrac{5}{2}\partial_t\theta\,.\end{gathered} \tag{3.3.50}$$

As is classical (since the contribution of Leray) in most treatments of the incompressible Navier-Stokes equations, the pressure term that appears on the right side of (3.3.48) will be eliminated upon integrating the equation against a divergence free test function.

To complete the proof of the Theorem 3.6, the limits of the moments $\epsilon^{-1}\langle B(v)g_\epsilon\rangle_M$ in (3.3.48) and $\epsilon^{-1} = \langle A(v)g_\epsilon\rangle_M$ in (3.3.49) have to be estimated. Start from the identities (recall that L is self-adjoint)

$$\langle A(v)g_\epsilon\rangle_M = \langle L^{-1}(A(v))\, L(g_\epsilon)\rangle_M \,, \quad \langle B(v)g_\epsilon\rangle_M = \langle L^{-1}(B(v))\, L(g_\epsilon)\rangle_M \,,$$

and eliminate $L(g_\epsilon)$ using equation (3.3.43),

$$\epsilon\, \partial_t g_\epsilon + v\cdot\nabla_x g_\epsilon = \frac{1}{\epsilon} L(g_\epsilon) + \tfrac{1}{2} Q(g_\epsilon, g_\epsilon)\,.$$

The convergence assumptions of the theorem then imply that the limiting moments may be evaluated by

$$\begin{aligned}
\lim_{\epsilon\to 0} \frac{1}{\epsilon}\langle A(v)g_\epsilon\rangle_M &= \lim_{\epsilon\to 0} \epsilon\nabla_x\cdot\langle v\otimes L^{-1}(A(v))\, g_\epsilon\rangle_M \\
&\quad - \lim_{\epsilon\to 0} \tfrac{1}{2}\langle L^{-1}(A(v))\, Q(g_\epsilon, q_\epsilon)\rangle_M \,, \\
\lim_{\epsilon\to 0} \frac{1}{\epsilon}\langle B(v)g_\epsilon\rangle_M &= \lim_{\epsilon\to 0} \nabla_x\cdot\langle v\otimes L^{-1}(B(v))\, g_\epsilon\rangle_M \\
&\quad - \lim_{\epsilon\to 0} \tfrac{1}{2}\langle L^{-1}(B(v))\, Q(g_\epsilon, g_\epsilon)\rangle_M \,;
\end{aligned} \tag{3.3.51}$$

The limiting form (3.3.40) and the Boussinesq relation (3.3.41) imply that

$$\begin{aligned}
\nabla_x\cdot\langle v\otimes L^{-1}(A(v))\, g\rangle_M &= \langle L^{-1}(A(v))\otimes v(\tfrac{1}{2}|v|^2 - \tfrac{5}{2})\rangle_M \cdot\nabla_x\theta \\
&= -\langle \alpha(|v|)\, A(v)\otimes A(v)\rangle_M \cdot\nabla_x\theta\,.
\end{aligned} \tag{3.3.52}$$

This expression gives the thermal diffusion term appearing in the second equation of systems (3.3.42). Even more directly, the limiting form (3.3.40) implies

$$\begin{aligned}
\nabla_x\cdot\langle v\otimes L^{-1}(B(v))\, g\rangle_M &= \langle L^{-1}(B(v))\otimes(v\otimes v)\rangle_M : \nabla_x u \\
&= -\langle \beta(|v|)\, B(v)\otimes B(v)\rangle_M : \nabla_x u\,.
\end{aligned} \tag{3.3.53}$$

After applying a divergence, this expression gives the viscous term appearing in the first equation of systems (3.3.42).

Next, consider the moments $\langle L^{-1}(A(v))\, Q(g,g)\rangle_M$ and $\langle L^{-1}(B(v))\, Q(g,g)\rangle_M$; these may be evaluated by using the fact that $C(F)$ vanishes for all Maxwellians . The first and second differentials of $M_{(\rho,u,\theta)}$ computed at the point $(1,0,1)$ are

$$dM = M\left(d\rho + v\cdot du + (\tfrac{1}{2}|v|^2 - \tfrac{3}{2})d\theta\right)\,, \tag{3.3.54}$$

$$\begin{aligned}
d^2M &= M\left(d\rho + v\cdot du + (\tfrac{1}{2}|v|^2 - \tfrac{3}{2})d\theta\right)^2 \\
&\quad + M\left(d^2\rho + v\cdot d^2u + (\tfrac{1}{2}|v|^2 - \tfrac{3}{2})d^2\theta\right)\,.
\end{aligned} \tag{3.3.55}$$

Comparison of (3.3.54) with the limiting form (3.3.40) shows that a correct choice of parametrization leads to $dM = Mg$ and $d^2M = Mg^2$. Twice deriving the formula that states Maxwellians are equilibria for the collision operator then gives

$$\begin{aligned} 0 = d^2C(M) &= D^2C(M):(dM \vee dM) + DC(M)\cdot d^2M \\ &= D^2C(M):(Mg \vee Mg) + DC(M)\cdot(Mg^2)\,. \end{aligned} \tag{3.3.56}$$

Applying the definitions of L and Q, this becomes simply

$$Q(g,g) = -L(g^2)\,. \tag{3.3.57}$$

Using relation (3.3.57) and the self-adjointness of L, the desired moments are found to be

$$\tfrac{1}{2}\langle L^{-1}(A(v))\,Q(g,g)\rangle_M = -\tfrac{1}{2}\langle L^{-1}(A(v))\,L(g^2)\rangle_M = -\tfrac{1}{2}\langle A(v)\,g^2\rangle_M = -\tfrac{5}{2}u\,\theta\,, \tag{3.3.58}$$

$$\tfrac{1}{2}\langle L^{-1}(B(v))\,Q(g,g)\rangle_M = -\tfrac{1}{2}\langle L^{-1}(B(v))\,L(g^2)\rangle_M = -\tfrac{1}{2}\langle B(v)\,g^2\rangle_M = -B(u)\,. \tag{3.3.59}$$

Formula (3.3.58) gives the term $u\cdot\nabla_x\theta$ while (3.3.59) gives the term $u\cdot\nabla_x u$. The proof of Theorem 3.6 is now complete.

Remark 3.7

Any proof concerning the fluid dynamical limit for a kinetic model will, as a by-product, give an existence proof for the corresponding macroscopic equation. However, up to now no new result has been obtained by this type of method. Uniform regularity estimates would likely be needed for obtaining the limit of the nonlinear term. These estimates, if they exist, must be sharp because, as explained in the previous chapter, the solutions of the compressible nonlinear Euler equations become singular after a finite time and the solutions of the incompressible Euler equation (if not singular) may exhibit serious instabilities.

In agreement with these observations and in the absence of boundary layers (full space or periodic domain), the following theorems are proved and were quoted in the previous chapter:

i) Existence and uniqueness of the solution to the compressible, or incompressible Navier Stokes equation, for a finite time that depends on the size of the initial data, provided this initial data is smooth enough (say in H^s with $s > 3/2$). This time of existence is in both cases independent of ϵ and when ϵ goes to zero the solutions converge respectively to the solution of the compressible Euler equations or to the solution of the incompressible Navier Stokes equation.

ii) Global (in time) existence of a smooth solution to the compressible or the incompressible Navier Stokes equations provided the initial data is small in a convenient norm enough with respect to the viscosity.

These points have their counterparts at the level of the Boltzmann equation and at the level of the macroscopic limit of the corresponding solutions:

i) Existence and uniqueness (under stringent regularity assumptions) during a finite time independent of the Knudsen number, was proved by Nishida [N] (cf. also Caflisch

[Ca1]). When the Knudsen number goes to zero this solution converges to a local thermodynamics equilibrium solution governed by the compressible Euler equations.

ii) Existence of a global in time smooth solution of the Boltzmann equation provided the fluctuation (with respect to an absolute maxwellian) of the initial data is small enough compared to the inverse of the Knudsen number.

The above consideration can be adapted to the rescaled equations

$$\epsilon \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon} C_B(F_\epsilon) \tag{3.3.60}$$

with initial data of the form

$$F_\epsilon = M(1 + \epsilon^r g_\epsilon). \tag{3.3.61}$$

It is easy to adapt the result of Nishida and to prove that with an initial data which is a smooth fluctuation of an absolute Maxwellian there will exist a finite time say, $T = T^*$ such that on the interval $(0, T^*)$ the statement of Theorem 3.6 (which corresponds to $r = 1$ in (3.3.61)) can be rigorously proven, furthermore in this case if g_ϵ is small enough at $t = 0$ the above results holds for $T^* = \infty$ (cf [BU]). Similarly for $r > 1$ it is possible to show (cf. Bellouquid [Be]) that the solution of the Boltzmann equation will be smooth for all time and will converge to the solution of the Stokes equation.

Using a method with many similarities to Leray's, R. DiPerna and P.-L. Lions [dPL1] have proved the global existence of a weak solution to a class of normalized Boltzmann equations, their so-called renormalized solution. This solution exists without assumptions concerning the size of the initial data with respect to the Knudsen number.

In this case it is natural to conjecture that the DiPerna-Lions renormalized solutions of the Boltzmann equation converge (for all time and with no restriction on the size of the initial data) to a Leray solution of the incompressible Navier-Stokes equations. However, no complete proof is available and some partial results may be found in [BGL3] and [BGL4](in particular for the Stokes limit with $r > 1$) in (3.3.61).

4. Turbulence and turbulence modelling

4.1 Introduction and the example of the $k - \epsilon$ model

Phenomena described by the Navier Stokes equation, may become, in particular for very large Reynolds numbers extremely complicated (as said in the introduction the world "turbulence" which is never completely defined is used in these situations). In the mean time the persistence of the divergence free condition and the fact that the energy remains bounded implies that it is the vorticity which becomes in some places very important both in size and in variation of its direction. This gives to the trajectories of the fluid some important averaging effects which correspond in the case of finite dimensional system to a complex system (notes, written by Leonardo da Vinci, quoted by several authors seem to indicate that he had already an intuition of this complexity.)

Therefore a first natural approach is the assumption that what we observe can be described by a statistical turbulence. Namely it is assume that the velocity of the fluid is a random variable given by the formula:

$$u(x,t,\omega) = U(x,t) + \tilde{u}(x,t,\omega) \tag{4.4.1}$$

with $\tilde{u}(x,t,\omega)$ denoting a random variable of mean value zero and that it is only the knowledge of the averaged value that will be important for applications. Equation for this averaged value would be some super Navier Stokes equation and would play for the Navier Stokes equation the role played by the Navier Stokes equation itself for the Boltzmann equation or by the Boltzmann equation for the equation of a system of N molecules.

A theoretical reason for the study of such equation would be the idea that serious mathematical progress is obtained first in coarser descriptions, because this level contains as a limit the more detailed one; for instance no progress on the proof of the regularity of the $3d$ Navier-Stokes equations was ever derived from the mathematical analysis of the Boltzmann equation and at variance the results on the Boltzmann equation can be viewed as adaptation (even if some of them are highly non trivial) of known results on the Navier-Stokes equations. Therefore one may think that progress in the understanding of turbulence may be a compulsory step in solving the classical open problems for the $3d$ Navier-Stokes equations like the existence of smooth solution in the large.

Inserting the right hand side of (4.4.1) in the Navier and denoting by $\langle . \rangle$ the average with respect to the random variable ω gives the equation:

$$\partial_t U + U \cdot \nabla_x U - \nu \Delta U + U \nabla_x U + \nabla_x \langle \tilde{u} \otimes \tilde{u} \rangle = -\nabla_x P\,, \tag{4.1.2}$$

which contains a "closure" problem because $\langle \tilde{u} \otimes \tilde{u} \rangle$ which is called the *Reynolds stress tensor* is not expressed in term of U.

However by a change in the pressure the Reynolds stress tensor can be always chosen to be traceless and therefore if one assumes

i) That this stress tensor depends only on $\nabla_x U$

and

ii) That the mapping $\nabla_x U \mapsto \langle \tilde{u} \otimes \tilde{u} \rangle$ is invariant under galilean transformations (isotropy assumption) one finds out that this Reynolds tensor is indeed proportional to $\nabla_x U +^t \nabla_x U$ i.e.

$$\langle \tilde{u} \otimes \tilde{u} \rangle = \nu_T(x,t)(\nabla_x U +^t \nabla_x U) \tag{4.1.3}$$

The scalar $\sigma(x,t)$ depends on the time and the position and is hopefully positive. This correspond to the introduction of a turbulent viscosity. In spite of the absence of complete rigorous derivation some rule are used for practical computations. The most common one being probably the $k-\epsilon$ model introduced by Landauer and Spalding in 1972 [LS] and widely used in numerical simulations. The basic idea is that the turbulent diffusion depends only on the fluctuation of energy (at variance with other part of the subject and other section of this monograph here the turbulent energy is denoted k and not e) and the fluctuation of enstrophy

$$k = \frac{1}{2}\langle|\tilde{u}|^2\rangle,\ \epsilon = \frac{\nu}{2}\langle|\nabla_x\tilde{u}|^2\rangle$$

then a dimension analysis gives for $\nu_T(x,t)$ an expression of the form

$$\nu_T = c_\nu\frac{k^2}{\epsilon}\,.$$

To determine the functions k and ϵ one introduces an equation for $\tilde{u}$ by subtracting the equation (4.1.2) from the basic Navier Stokes equation with solution $U+\tilde{u}$ this gives the equations:

$$\partial_t\tilde{u} + \tilde{u}\cdot\nabla_x U + (U+\tilde{u})\nabla_x\tilde{u} - \nu\Delta\tilde{u} - \nabla_x R = -\nabla_x\tilde{p}\,, \tag{4.1.4}$$

and for $\omega = \nabla\times u$

$$\partial_t\tilde{\omega} + \tilde{u}\cdot\nabla_x(\nabla\times U) + (U+\tilde{u})\nabla_x\tilde{\omega} - -(\nabla\times U+\tilde{\omega})\nabla_x\tilde{u} - \tilde{\omega}\nabla_x U - \nu\Delta\tilde{u} = -\nabla\times\nabla_x R\,, \tag{4.1.5}$$

The equation (4.1.4) is multiplied by $\tilde{u}$ and the equation (4.1.5) is multiplied by $\tilde{\omega}$. Basic assumptions (with up to now no rigorous justifications) are made concerning the approximation of the terms

$$\partial_t\langle\psi\rangle + \langle(U+\tilde{u})\nabla_x\psi\rangle$$

by terms of the form

$$\partial_t\langle\psi\rangle + U\nabla_x\langle\psi\rangle - \nu_T\Delta\langle\psi\rangle$$

according to the convection of a passive scalar by a random field. Eventually an ergodicity hypothesis is used to replace random averages by spatial averages when needed and the following system is obtained.

$$\begin{aligned}
&\partial_t U + U\nabla_x U + \nabla_x P - \nu\Delta_x U - \nabla_x\big(c_\nu[\frac{k^2}{\epsilon}(\nabla_x U +^t \nabla_x U)]\big) = 0\,,\\
&\partial_t k + U\nabla_x k - \frac{c_\nu k^2}{2\epsilon}|\nabla_x U + (\nabla_x U)^t|^2 - \nabla_x\cdot[c_\nu\frac{k^2}{\epsilon}\nabla_x k] + \epsilon = 0\,,\\
&\partial_t\epsilon + U\nabla_x\epsilon - \frac{c_1 k}{2}|\nabla_x U + (\nabla_x U)^t|^2 - \nabla_x\cdot[c_3\frac{k^2}{\epsilon}\nabla_x\epsilon] + c_2\frac{\epsilon^2}{k} = 0\,.
\end{aligned} \tag{4.1.6}$$

with c_i denoting several positive constants which are determined either by experiment or by phenomenological considerations.

It is known that there exist some cases where the above derivation is not valid (in particular near the walls). Even if the most convenient hypothesis are assumed, many important gaps remains in the proof of the validity of the $k-\epsilon$ model.

i) It is assumed that the velocity of the fluid is a random variable $u(x,t,\omega)$. This seems reasonable keeping in mind a generalization of the Birkhoff ergodic theorem to the Navier Stokes flow. However this ergodic theorem (to be stated) requires the existence of a nontrivial invariant probabilistic measure and the definition of such a measure is (for many reasons) a widely open problem, some partial results having been obtained by Foias [F1-2] Foias and Prodi [FP] and by Fursikov, Vishik et al. cf [VF1-3] and [EF]). Some of these results will appear in the next chapter.

ii) There is no universal parameter like the Knudsen number or the Mach number, and since macroscopic phenomena are involved the Reynolds number which would be the best candidate to measure the relaxation to turbulence is a local quantity. In fact it is the fluid self interaction which is responsible of the relaxation to "turbulence".

iii) There no evident rigorous formulation that would play the role of the thermodynamical equilibrium and no trend to relaxation like the decay of entropy at the level of the Boltzman equation.

4.2 Wigner Transform and Defect measures for the Reynolds tensor

Since there is no complete mathematical theory that even in some particular cases would produce an expression for the turbulent Reynolds tensor. Several ideas may be used; many of them do have in common the introduction of the two points spatial or temporal correlation function:

$$\langle \tilde{u}(x+\frac{r}{2},t)\otimes\tilde{u}(x-\frac{r}{2},t)\rangle \text{ or } \langle \tilde{u}(x,t+\frac{s}{2})\otimes\tilde{u}(x,t-\frac{s}{2})\rangle$$

such quantities are the object of many experimental measurements which do involve Fourier transform, which for instance in the spatial configuration is:

$$\langle \hat{R}(x,k,t)\rangle = \int_{\mathbf{R}^3} e^{-r\cdot k}\langle \tilde{u}(x+\frac{r}{2},t)\otimes\tilde{u}(x-\frac{r}{2},t)\rangle\,dr \tag{4.2.1}$$

With the inverse Fourier transform one deduces the relation:

$$\langle \tilde{u}(x,t)\otimes\tilde{u}(x,t)\rangle = \Big(\frac{1}{2\pi}\Big)^3\int_{\mathbf{R}^3}\langle \hat{R}(x,k,t)\rangle\,dk\,. \tag{4.2.2}$$

In fact the two above formulas turn out to be the Wigner transform and its reciprocal. Along this line it is important to keep in mind the fact that the Wigner transform provides (for the energy) some type of local high frequency expansion.

The tensor valued function $\hat{R}(x,k,t)$, or its average plays for the Navier Stokes equation the role assumed by the thermal equilibrium (Maxwellian for instance) at other level of the hierarchy.

Since it involves only the fluctuation one may assume that it is invariant under Galilean transformation and this would lead for instance in 3 dimensions, to the formula:

$$\hat{R}(x,k,t) = \frac{E(|k|,x,t)}{4\pi|k|^2}(I-\frac{k\otimes k}{|k|^2}) \tag{4.2.3}$$

where $E(|k|,x,t)$ is a scalar function called the *energy spectra of turbulence.*

Of course with the assumptions that $R(x,k,t)$ is invariant under Galilean transformations and depends only on

$$k = \tfrac{1}{2}\langle|\tilde{u}|^2\rangle,\ \ \epsilon = \frac{\nu}{2}\langle|\nabla_x\tilde{u} + \nabla_x\tilde{u}^T|^2\rangle$$

and on the tensor

$$(\nabla_x U +^t \nabla_x U)$$

one recovers the formula:

$$R(x,t) \simeq \frac{k^2}{\epsilon}(\nabla_x U +^t \nabla_x U) \tag{4.2.4}$$

In spite of its long history, the statistical approach does not seem to be compulsory to introduce turbulent effect and it is important to observe that all the issues raised for statistic family of solutions do have their counterpart when one considers the family of defect measure of a sequence u_n of deterministic solutions of Euler or Navier-Stokes equations which are uniformly bounded in energy

$$\sup_{t>0,n}\int_\Omega |u_n(x,t)|^2\,dx \le C < \infty \tag{4.2.5}$$

and which do not uniformly satisfy other "regularity" estimates. More precisely only the energy estimate

$$\tfrac{1}{2}\int_\Omega |u_n(x,t)|^2\,dx + \nu_n\int_0^t\int_\Omega \|\nabla_x u_n(x,s)\|^2\,dx\,ds \le \tfrac{1}{2}\int_\Omega |u(x,0)|^2\,dx \tag{4.2.6}$$

remains valid and it is assumed that the viscosity ν_n is either zero or goes to zero. In this case, modulo the extraction of a subsequence, u_n converges "weakly" to a function $U(x,t)$ which satisfies also the estimate (4.2.5). However due to the likely lack of compactness it may happen that

$$\lim_{n\to\infty} u_n(x,t)\otimes u_n(x,t) \ne U(x,t)\otimes U(x,t) \tag{4.2.7}$$

and the difference

$$\begin{aligned} R(x,t) &= \lim_{n\to\infty} u_n(x,t)\otimes u_n(x,t) - U(x,t)\otimes U(x,t) \\ &= \lim_{n\to\infty} (u_n(x,t) - U(x,t))\otimes(u_n(x,t) - U(x,t)) \end{aligned} \tag{4.2.8}$$

is a measure valued positive tensor which is zero only in case of strong convergence. It is called the defect measure and has been introduced for related purpose by several authors (cf for instance [G] and [Tar]). The limit $U(x,t)$ is the solution of the "turbulent" equation:

$$\partial_t U + U\cdot\nabla_x U + \nabla_x R + \nabla_x P = 0,\ \nabla_x\cdot U = 0\,. \tag{4.2.9}$$

with R being like in the random case the Reynolds stress tensor.

This tensor may be present:

When $u_n(x,t)$ is a sequence of Leray solutions of the Navier-Stokes equations in $\mathbf{R}^3$ with the same given regular initial data and a sequence of viscosities ν_n which goes to zero with n. Because in this situation the global existence of a smooth solutions of the Euler equations remains an open problem.

It is almost surely present in the following situations:

" i)" When $u_n(x,t)$ is a sequence of solutions of the Euler equations (or of the Navier-Stokes equations with viscosity going to zero) with initial data $u_n(0,t)$ uniformly bounded in L^2 but not in a more regular space (for instance when the initial data exhibit large oscillations).

" ii)" When both large time behavior and zero viscosity limit are simultaneously considered.

" iii)" Even for finite time, for the solutions $u_n(x,t)$ of the Navier-Stokes equations in a bounded domain Ω when the viscosity ν_n goes to zero and when a viscous boundary condition

$$u_n(x,t) = 0 \text{ for } x \in \partial\Omega \tag{4.2.10}$$

is prescribed. It this situation a boundary layer appears near the boundary but due to the non linearity of the problem this boundary layer (at variance with what happens for linear problems) may propagate inside the domain. It has been recently proven by Asano and Caflisch and Sammartino (cf. [CS]) that such a phenomena is not present, but only for small time and analytic initial data.

Assuming that in all of the above cases the weak limit $U(x,t)$ of the sequence $u_n(x,t)$ is a smooth function, the following "conjectures", which are the deterministic counterpart of the "folklore" of statistical turbulence, should be studied.

One introduces the sequence of functions

$$\tilde{u}_n = u_n - U$$

which converges weakly to zero and which plays the role of the fluctuation and its Wigner transform

$$\hat{R}_n(x,k,t) = \int_{\mathbf{R}^3} e^{-r\cdot k}((\tilde{u}_n(x+\frac{r}{2},t)\otimes((\tilde{u}_n(x-\frac{r}{2},t))\,dr,\ \hat{R}(x,k,t) = \lim_{n\to\infty}\hat{R}_n(x,k,t)\,. \tag{4.2.10}$$

which is the analogous of (4.2.1).

The local turbulent energy and turbulent enstrophy could be defined as

$$e = \tfrac{1}{2}\lim_{n\to\infty}|u_n - U|^2;\ \epsilon = \tfrac{1}{2}\lim_{n\to\infty}\nu|\nabla\times(u_n - U)|^2$$

It this context exactly as in the case of the random solution one may assume (and may in the future in some case prove) a galilean invariance hypothesis which gives:

$$\hat{R}(x,k,t) = \frac{E(|k|,x,t)}{4\pi|k|^2}(I - \frac{k\otimes k}{|k|^2}) \tag{4.2.11}$$

leading as above to the introduction of the *turbulence spectrum*. Furthermore the galilean invariance implies that the Reynolds tensor itself can be as above written as

$$R(x,t) = -\nu_T(x,t)(\nabla_x U +^t \nabla_x U)$$

The next step in this analysis should be to prove that the scalar $\nu_T(x,t)$ is non negative. This does not result from the positivity of the defect measure R itself. Explicit model of this weak convergence can be constructed for the dispersive limit of the KdV equation or for the non linear Schrodinger equation and show that the appearance of analogous phenomena (positive diffusion) are possible but not systematic.

4.3 The Kolmogorov Kraichnan Theory

Another approach to the closure problem is the direct analysis of the turbulence spectra $E(x,t,k)$ introduced above under the galilean invariance hypothesis. This program was initiated by Kolmogorov in 1941 and stimulated many further researches.

With the following assumption:

There exists a region (called the "inertial range" $0 \le k_1 \le |k| \le k_1$ where $E(|k|,x,t)$ depends only on $|k|$ and on

$$\epsilon = \frac{d}{dt}\langle|\tilde{u}(x,t)|^2\rangle$$

a dimensional analysis gives the formula

$$E(k) = C(x,t)\epsilon^{\frac{2}{3}}|k|^{-\frac{5}{3}} \tag{4.3.1}$$

with $C(x,t)$ a dimensional number.

This is the famous Kolmogorov law. It is independent of the equation of motion; no mechanical explanation for its validity in three dimension has yet been offered here. It is well verified both by physical and numerical experiments. (quoted from Chorin [Cho] page 52)) and furthermore it will lead to an analysis of the degree of freedom of the fluid.

In fact the intrinsic nature of the turbulent spectra should be present in the case where the action of the macroscopic part of the fluid U on the stress tensor $\langle\tilde{u}\otimes\tilde{u}\rangle$ is replaced by the action of an external force f on random fluctuations of mean value zero. It turns out that in this configuration, with a fluid evolving in a $D = 2$ or $D = 3$ periodic domain the analysis is both relevant and simpler. For formal and rigorous results it will use the dynamical aspect of the Navier Stokes flow and this is the object of the next chapter.

5 Invariant measures, Attractors, and evaluation of the number of degree of freedom of the flow

5.1 Introduction and formal derivations

In this section the solutions of the Navier Stokes equations in a bounded domain with a time independent forcing term are considered. This forcing term can be either distributed in the domain or located on the boundary. However, for sake of simplicity, only the case of an internal force acting on a fluid defined in a periodic domain

$$\Omega = [0, L]^D$$

is described. Therefore, the equations are

$$\partial_t u + u \cdot \nabla_x u - \nu \Delta u + \nabla_x p = f, \nabla_x \cdot u = 0\,,\ u(x,0) = u_0\,. \tag{5.1.1}$$

Using the Galilean invariance one can assume without loss of generality that

$$\int_\Omega f(x)dx = 0\,,\ \text{and}\ \int_\Omega u(x)dx = 0\,. \tag{5.1.2}$$

The phase space H is defined as the L^2-completion of smooth periodic divergence free functions satisfying (5.1.2) and P denotes the orthogonal projection of $(L^2([0,L])^D)^D)$ onto H (Leray Projection). The following standard notations are used:

$$\begin{aligned} &Au = -\Delta u,\ B(u,v) = P[(u \cdot \nabla_x v)]\,,\\ &(u,v) = \frac{1}{L^d}\int_{[0,L]^d} u(x)\cdot v(x)dx\,, \quad ((u,v)) = \frac{1}{L^d}\int_{[0,L]^d} \nabla_x u(x) : \nabla_x v(x)dx\,,\\ &|u| = (u,u)^{\frac{1}{2}},\ \|u\| = ((u,v))^{\frac{1}{2}}\,. \end{aligned} \tag{5.1.3}$$

The operator A is selfadjoint positive and the domain of $A^{\frac{1}{2}}$ coincides with the space $V = H \cap H^1(\Omega)^d$, $\|u\|^2 = |A^{\frac{1}{2}} u|^2$. The quantities $\frac{1}{2}|u|^2$ and $\|u\|^2$ represent the kinetic energy and the enstrophy per unit mass of the flow described by u. Eventually the following identities are recalled:

$$\begin{aligned} &(B(u,v),v) = 0,\ \text{if}\ D = 2,3\,,\\ &(B(u,u),Av) + (B(u,v) + B(v,u),Au) = 0\ \text{if}\ D = 2\,. \end{aligned} \tag{5.1.4}$$

Ignoring in the present section the difficulties related to our incomplete knowledge of the regularity and uniqueness of the solutions of the Navier Stokes equation and making the convenient regularity hypothesis, one describes the solutions of (5.1.1) with the introduction of a non linear semiflow

$$u(t,x) = S(t)u_0(x) \tag{5.1.5}$$

and defines the global attractor $\mathcal{A}$ as follows: the global attractor $\mathcal{A}$ for the semiflow $\{S(t)\}_{t\geq 0}$ is a compact set in the space H, $\mathcal{A}\subset H$ such that

$$S(t)\mathcal{A}=\mathcal{A}\quad \forall t\geq 0 \tag{5.1.6}$$

and $\mathcal{A}$ attracts all bounded sets of H, i.e., for all $\mathcal{B}\subset H$ bounded, for all $\epsilon>0$, there exists $T_1=T_1(\epsilon,\mathcal{B})$ such that, for $t\geq T_1(\epsilon,\mathcal{B})$, $S(t)\mathcal{B}$ is included in an ϵ-neighborhood of $\mathcal{A}$.

When the viscosity is large enough (small Reynolds numbers) $\mathcal{A}$ is reduced to the unique solution of the time independent equation

$$u\cdot\nabla_x u-\nu\Delta u+\nabla_x p=f,\quad \nabla_x\cdot u=0\,. \tag{5.1.7}$$

However, as is the case in finite dimensional models (like the Lorentz attractor [Lo]), which is constructed as the simplest Galerkin approximation of the Boussinesq equation), the structure of the complexity of $\mathcal{A}$ increases with the Reynolds number.

The first steps in this process are described by adaptation to the Navier Stokes equation of the standard bifurcation theory and could be found for instance (with other references in Chossat and Iooss [ChI]). Then the trend toward the complexity of the attractor should be understood by the introduction of a *cascade* of bifurcations. However rigorous construction of bifurcations after the second order seem to be out of the scope of our present knowledge and are in any case very different from the analysis in a turbulent regime which is the goal of the present section.

Observe that the complexity depends on the viscosity ν, the size of the box L and the magnitude of the driving force f; therefore, it should be described in terms of a dimensional number depending on these three quantities. Such a number is called the generalized Grashoff number. In dimension 2 it is given by the formula:

$$G=\frac{L^3|f|}{\nu^2}=\frac{L^3}{\nu^2}\Big(\int_\Omega |f(x)|^2dx\Big) \tag{5.1.8}$$

and in dimension 3 it is convenient to replace the above definition by the formula:

$$G=\frac{L^2|A^{-\frac{1}{2}}f|}{\nu^2}=\frac{L^2}{\nu^2}\Big(\int_\Omega(-\Delta^{-1}f)(x)\cdot f(x))dx\Big)^{\frac{1}{2}}\,. \tag{5.1.9}$$

For large Grashoff number the fluid should become ergodic and define an intrinsic probability measure.

More precisely, one assumes the following ergodicity hypothesis **Herg**:

There exists a unique probability measure μ on the phase space H invariant under the action of the Navier Stokes semiflow such that, for almost any initial data $u_0\in H$ and any functional $\Phi(u)$ representing some physical quantity associated with the fluid flow u,

$$\langle\Phi(u_0)\rangle=\lim_{T\to\infty}\Big(\frac{1}{T}\int_0^T\Phi(S(t)u_0)dt\Big)=\int_H\Phi(u_0)d\mu(u_0)\,. \tag{5.1.10}$$

Using the Fourier series representation:

$$u(x) = \sum_{k \in Z^D} a_k(u) e^{\frac{2\pi}{L} k \cdot x} \tag{5.1.11}$$

one can define, for any $0 < \kappa_1 < \kappa_2$, u_{κ_1, κ_2}

$$u_{\kappa_1, \kappa_2}(x) = \sum_{\kappa_1 \le |\kappa| < \kappa_2} a_k(u) e^{\frac{2\pi}{L} k \cdot x} \tag{5.1.12}$$

and assume **Herg-is** (ergodicity and isotropy hypothesis) the existence of a positive function $E(\kappa)(\ge 0)$ such that one has:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |(S(t) u_{0\kappa_1, \kappa_2)}|^2 dt = \int_H |u_{0\kappa_1, \kappa_2)}|^2 d\mu(u_0) = \int_{\kappa_1}^{\kappa_2} E(\kappa) d\kappa. \tag{5.1.13}$$

As in the previous chapter, the function $E(\kappa)$ is called *the energy spectrum of the turbulent flow produced by f.* It gives an intrinsic (if not rigorous, see below) definition of an object which adapts to the present context the definition given in (4.2.3). The fact that Fourier series representation is used instead of Fourier transform creates no problem, and the fact that the function E can be defined in term of the modulus of the wave number corresponds to the isotropy hypothesis made in section (4.2). Therefore, the question raised there can be addressed in the present context and leads to formal and in some cases rigorous results.

Observe that one has immediately:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \|(S(t) u_{0\kappa_1, \kappa_2)}\|^2 dt = \int_H \|u_{0\kappa_1, \kappa_2)}\|^2 d\mu(u_0) = \int_{\kappa_1}^{\kappa_2} \kappa^2 E(\kappa) d\kappa \tag{5.1.14}$$

and

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |(A S(t) u_{0\kappa_1, \kappa_2)}|^2 dt = \int_H |A u_{0\kappa_1, \kappa_2)}|^2 d\mu(u_0) = \int_{\kappa_1}^{\kappa_2} \kappa^4 E(\kappa) d\kappa \tag{5.1.15}$$

5.2 Kolmogorov and Kraichnan inertial range.

This section is devoted to the construction of the inertial range. The argument is inspired by the classical ideas of Kolmogorov and Kraichnan and we follow the exposition done in [Fo]. It is assumed that $f = f_{0,\kappa_0}$ where κ_0 is of the order of $2\pi/L$ (the lowest wave number).

For κ_1 and κ_2 given, the following notations are used:

$$v_< = u_{0,\kappa_1} \; v = u_{\kappa_1, \kappa_2)}, \; v_> = u_{\kappa_2, \infty} \, . \tag{5.2.1}$$

First the case $D = 2$ is considered; therefore, with (5.1.4) one deduces from the energy balance equation (for $\kappa_1 > \kappa_0$) the relation:

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\|v\|^2 = -\nu|Av|^2 &- (B(v_<, v_<), Av) + ((B(v,v) + (B(v_>, v) + B(v, v_>)), Av_<) \\ &+ ((B(v,v) + (B(v_<, v) + B(v, v_<)), Av_>) - (B(v_>, v_>), Av)\,.\end{aligned} \tag{5.2.2}$$

Taking the average in the sense of (5.1.10) one obtains:

$$\begin{aligned}0 = -\nu\langle|Av|^2\rangle &- \langle((B(v_<, v_<), Av)) + (B(v + v_>, v + v_>), Av_<)\rangle \\ &+ \langle(B(v + v_<, v + v_<), Av_>) - \langle(B(v_>, v_>), A(v_< + v))\rangle\,.\end{aligned} \tag{5.2.3}$$

Now assume that at these wave numbers κ and 2κ the enstrophy is in average carried only from low wave numbers to high wave numbers (Kraichnan's cascading scenario), then one has

$$\langle(B(v + v_>, v + v_>), Av_<)\rangle \simeq 0 \text{ and } \langle\langle(B(v_>, v_>), A(v_< + v))\rangle \simeq 0 \tag{5.2.4}$$

so that

$$\nu\langle|Av|^2\rangle \simeq -\langle((B(v_<, v_<), Av))\rangle + \langle(B(v + v_<, v + v_<), Av_>)\rangle\,. \tag{5.2.5}$$

Thus, as long as

$$\nu\langle|Av|^2\rangle = \nu\int_\kappa^{2\kappa} \xi^4 E(\xi)d\xi \simeq \nu\kappa^5 E(\kappa) \tag{5.2.6}$$

is small compared to the two terms of the right hand side of (5.2.5), one has

$$\langle((B(v_<, v_<), Av))\rangle \simeq \langle(B(v + v_<, v + v_<), Av_>)\rangle\,. \tag{5.2.7}$$

The left-hand side represents the mean enstrophy/ mass passed per unit time from the component with wave number less than κ to wave numbers living in $[2\kappa, 4\kappa)$. Let η denote the constant dissipation of enstrophy resulting from Kraichnan scenario. Let also κ_1 be the smallest wave number from which the Kraichnan scenario is valid. Then writing (5.2.6) for $\kappa = 2^j\kappa_1, (j = 1, 2...)$ and summing up, one obtains:

$$\nu\langle|Au_{\kappa_1,\infty}|^2\rangle \simeq -\langle((B(u_{0,\kappa_1}, u_{0,\kappa_1}), Au_{\kappa_1,2\kappa_1}))\rangle \simeq \eta \tag{5.2.8}$$

so with the assumptions that $\kappa_1 \simeq \kappa_0 \simeq 2\pi/L,\ \ , L >> 1$,

$$\nu\langle|Au_{\kappa_1,\infty}|^2\rangle \simeq \nu\int_{\kappa_1}^{\infty} \xi^4 E(\xi)d\xi \simeq \nu\int_{\frac{2\pi}{L}}^{\infty} \xi^4 E(\xi)d\xi = \nu\langle|Au|^2\rangle\,. \tag{5.2.9}$$

This last quantity is the dissipation (due to viscosity) of the enstrophy / mass per unit time of the whole fluid flow. As long as

$$\nu\kappa^5 E(\kappa) << \eta\,, \tag{5.2.10}$$

the component with wave number in $[\kappa, 2\kappa)$ is just transferring enstrophy with the constant rate $\simeq \eta$ from lower wave numbers to higher wave numbers.

Following Kraichnan [KR1] and Frisch, Nelkin and Sulem [FNS] we inject into the above formulas a phenomenological description of turbulence. To start, observe that the wave number κ has the physical dimension of $(\text{length})^{-1}$ and the component with wave number κ is considered to represent eddies of linear size about $\frac{1}{\kappa}$. Thus, as a function of x, the component $u_{\kappa,2\kappa}$ is thought to represent the system of eddies of linear size $\in (1/2\kappa, 1/\kappa]$. The transfer of enstrophy is considered to be produced by the breaking of the eddies into eddies of linear size $\leq 1/2\kappa$. This breaking is assumed to occur after the eddy travels a distance comparable to its linear size. Since the energy/ mass of the eddies with linear size $\in (1/2\kappa, 1/\kappa]$ is in average about

$$\int_{\kappa}^{2\kappa} E(\xi)d\xi \simeq \kappa E(\kappa)\,, \tag{5.2.11}$$

the average velocity of those eddies is about

$$V_\kappa \simeq (\kappa E(\kappa))^{\frac{1}{2}}\,. \tag{5.2.12}$$

Therefore the time necessary for the eddies to travel their linear size is about

$$t_\kappa \simeq (1/\kappa)/V_\kappa = 1/(\kappa V_\kappa) \simeq 1/\big(\kappa^{\frac{3}{2}} E(\kappa)^{\frac{1}{2}}\big) \tag{5.2.13}$$

On the other hand, the enstrophy / mass of the eddies with linear size $\in (1/2\kappa, 1/\kappa]$ is

$$\int_{\kappa}^{2\kappa} \xi^2 E(\xi)d\xi \simeq \kappa^3 E(\kappa)\,. \tag{5.2.14}$$

According to the breaking mechanism the mean dissipation of the enstrophy /mass per unit of time should thus be

$$\eta \simeq \frac{\kappa^3 E(\kappa)}{t_\kappa} \simeq \kappa^3\big(\kappa E(\kappa)\big)^{\frac{3}{2}} \tag{5.2.15}$$

which gives

$$E(\kappa) \simeq \frac{\eta^{\frac{2}{3}}}{\kappa^3}\,. \tag{5.2.16}$$

According to the previous arguments, one expects (5.2.16) as long has $\kappa > \kappa_0$ and (5.2.10) hold. Using (5.2.16) one finds that (5.2.10) is equivalent to $\kappa^2 << \eta^{\frac{1}{3}}/\nu$, that is:

$$\kappa << \kappa_\eta = \big(\frac{\eta}{\nu^3}\big)^{\frac{1}{6}}. \tag{5.2.17}$$

The equation (5.2.17) defines the *Kraichnan dissipation wave number* [Kr1]. For larger wave numbers the viscosity forces become dominant.

In the three dimensional case, $D = 3$, the role played by the enstropy in the preceding argument is taken over by the energy. Consequently, one starts with

$$\int_{\kappa}^{2\kappa} \xi^2 E(\xi) d\xi \simeq -\langle (B(v_<, v_<), v) \rangle + \langle (B(v_< + v, v_< + v), v_v) \rangle \tag{5.2.18}$$

and then writes:

$$\langle (B(v_<, v_<), v) \rangle \simeq -\langle (B(v_< + v, v_< + v), v_v) \rangle = \simeq \epsilon = \nu \langle \|u\|^2 \rangle \tag{5.2.19}$$

provided that

$$\nu \kappa^3 E(\kappa) << \epsilon \tag{5.2.20}$$

Here ϵ represents the mean dissipation of energy /mass per unit of time. The Kraichnan mechanism now leads to the estimate

$$\epsilon \simeq \frac{\kappa E(\kappa)}{t_\kappa} \simeq \big(\kappa E(\kappa)\big)^{\frac{3}{2}} \kappa \,, \tag{5.2.21}$$

that is,

$$E(\kappa) \simeq \frac{\epsilon^{\frac{2}{3}}}{\kappa^{\frac{5}{3}}} \tag{5.2.22}$$

for

$$\nu \kappa^{\frac{4}{3}} \epsilon^{\frac{2}{3}} << \epsilon; \tag{5.2.23}$$

that is,

$$\kappa << \kappa_\epsilon = \big(\frac{\epsilon}{\nu^3}\big)^{\frac{1}{4}} \,. \tag{5.2.24}$$

This number κ_ϵ is called the *Kolmogorov wave number* . The spectra given by (5.2.23) and (5.2.16) are respectively the *Kolmogorov spectrum for turbulence* and the *Kraichnan spectrum for 2D turbulence.*

The wave number where the mechanism described above holds (in or $2D$ is the *inertial range of turbulence.* The empirical evidence for the existence of the Kolmogorov inertial range of turbulence is much stronger than that for the existence of the Kraichnan inertial range of turbulence; this may be due to the fact that we have at our disposal more experiments in $D = 3$ than in $D = 2$. Furthermore both the phenomenological theory and the rigorous mathematical analysis described below indicate for $D = 2$ the existence of a logarithmic correction.

5.3. Kolmogorov -Kraichnan waves numbers and asymptotic Degrees of Freedom

In almost all cases the use of the primitive Navier Stokes equation for the computation of a flow Direct Navier Stokes Simulation DNS introduces a discretization which involves a finite number of degrees of freedom. Evaluation of the order of magnitude of this number is the first step of the computation. It turns out that there are several approaches for this evaluation, some based on the heuristic argument as a continuation of the above discussion and others based on some more mathematically tractable objects like the notion of attractors. At the present, to the best of the knowledge of the writers, no formal mathematical derivation of the relation between the different approaches exists; derivations should come from a better understanding of the ergodic aspect of the theory. However surprisingly (or not surprisingly?) the different approaes lead to very similar estimates. The heuristic approach is discussed below as a continuation of the previous section, and the more mathematical approach will be one of the main the objects of section 5.4.

The mean dissipation of energy /mass per unit of time which appears in the evaluation of the Kolmogorov scaling law and of the Kolmogorov dissipation wave number can be evaluated with the following heuristic argument:

If

$$e = \tfrac{1}{2}\int_0^\infty E(\kappa)d\kappa \tag{5.3.1}$$

is the average of the energy /mass in a turbulent fluid flow with an average dissipation rate

$$\epsilon = \nu \int_0^\infty \kappa^2 E(\kappa)d\kappa, \tag{5.3.2}$$

then $t_\epsilon = e/\epsilon$ should represent the characteristic time for the dissipation of energy and the characteristic mean velocity should be $U = \sqrt{2e}$. The corresponding length $l = Ut_\epsilon$ can be viewed as the average distance travelled by the turbulent eddies until they dissipate. So

$$\epsilon \simeq \frac{U^2}{t_\epsilon} = \frac{U^3}{l}. \tag{5.3.3}$$

This is the Kolmogorov estimate for energy dissipation in a turbulent fluid flow. Since U and L are the characteristic velocity and length for the flow one introduces the Reynolds number

$$Re = \frac{UL}{\nu} \tag{5.3.4}$$

and obtains with (5.2.24), (5.3.3) and (5.3.4) the following formula for the Kolmogorov wave number:

$$L\kappa_\epsilon \simeq \left(Re\right)^{\frac{3}{4}}. \tag{5.3.5}$$

Above it has been observed and used that in $D = 2$ the enstrophy dissipation

$$\nu = \langle |Au|^2 \rangle \tag{5.3.6}$$

has to be considered instead of the energy dissipation. However the same analysis leads in this case to the same formula:

$$L\kappa_\eta \simeq \left(Re\right)^{\frac{3}{4}} \tag{5.3.7}$$

The dissipation length is therefore given by

$$
\begin{aligned}
l_\eta &= \frac{1}{\kappa_\eta} \text{ for } D = 2 \\
l_\epsilon &= \frac{1}{\kappa_\epsilon} \text{ for } D = 3\,.
\end{aligned}
\tag{5.3.8}
$$

(cf. Foias [Fo]). Since structures of size less than l_η (resp. l_ϵ) correspond to wave numbers which are in the dissipative range, they are rapidly annihilated by viscous effects and therefore are of no dynamical consequence. On the other hand, any eddy of size l_η (resp l_ϵ) will be tracked at some grid point. One expects that the degrees of freedom of a $2D$ resp. $3D$ flow should be at most about

$$
\begin{aligned}
\Big(\frac{L}{l_\eta}\Big)^2 &= \big(L\kappa_\eta\big)^2 \text{ for } D = 2 \\
\Big(\frac{L}{l_\epsilon}\Big)^3 &= \big(L\kappa_\epsilon\big)^3 \text{ for } D = 3
\end{aligned}
\tag{5.3.9}
$$

or with (5.3.5) and (5.3.7) $\big(Re\big)^{\frac{3}{2}}$ for $D = 2$ and $\big(Re\big)^{\frac{9}{4}}$ for $D = 3$.

Assuming that the non linear semiflow has a compact global attractor $\mathcal{A}$ one could use the fractal dimension of this attractor as an alternate definition of the number of freedom of the turbulent flow. Recall that the fractal dimension of a compact subset $\mathcal{A}$ of a Hilbert space H is defined by the formula:

$$
d_M(\mathcal{A}) = \lim_{\delta \to 0_+} \sup \frac{\log N_\delta(\mathcal{A})}{\log \frac{1}{\delta}} \tag{5.3.10}
$$

where, for $\delta > 0$, N_δ is the smallest number of balls of radii equal to δ needed to cover $\mathcal{A}$. The fractal dimension can be ∞ even if its Hausdorff dimension is 0 (cf. [BEFN]). Moreover (cf. [EFNT, FO, BEFN]), if $d_M(\mathcal{A}) < \infty$ there is a dense set of orthogonal projections P (Mañe's projection) in H of rank $\leq 2d_M(\mathcal{A}) + 1$ with a Hölder continuous pseudo inverse

$$
P^{-1} : \mathcal{A} \to H,\ P \circ P^{-1} = I_{\mathcal{A}}\,. \tag{5.3.11}
$$

Therefore, the fractal dimension is a better indicator than the Hausdorff dimension of the number of parameters necessary to describe a set as well as the dynamics it may carry. This observation is particularly important for the *exponential attractor* which is introduced in section **5.5**.

Furthermore it is appropriate to mention that exponential attractors are probably more relevant than the global attractor $\mathcal{A}$ itself. These are outgrowths of $\mathcal{A}$, still with fractal dimension but attracting all solutions at an exponential rate. Moreover, the estimates for their fractal dimension are as sharp as the one for $d_M(\mathcal{A})$.

The notion of the number of determining nodes should be halfway between the concept of degrees of freedom according to Landau and Livschitz and the dimension of the attractor. The points of a fixed finite set $\mathcal{F}$ in the domain of the fluid are called determining nodes whenever, for any two solutions u, v of the Navier Stokes equations, the convergence on $\mathcal{F}$ implies the global convergence of these solutions, i.e.:

$$\lim_{t\to\infty}(u(t,a)-v(t,a)) = 0 \text{ (in } \mathbf{R}^D)\, \forall a \in \mathcal{F} \;\Rightarrow\; \lim_{t\to\infty}|u(t,.)-v(t,.)|_H = 0 \qquad (5.3.12)$$

Eventually it is worth mentioning that a theorem of Takens [Ta] asserts that generically one node should suffice, but it is not known if the Navier Stoke is generic in the sense of Takens.

For dimension of the attractor and for the number of determining modes, rigorous results are available. They are not as precise for determining modes as for the dimension of the attractor (cf. [JT] and [CDT]) but in any case they are more tractable than the heuristic estimates of Kolmogorov and Kraichnan.

Finite fractal dimension for the attractor raises the following deeper questions, which are partially answered in Section 5.5:

(i) Can we imbed the attractor in a smooth finite-dimensional manifold;

(ii) Are the dynamics on the attractor equivalent to the dynamics of a finite differential dynamical system (also called "inertial dynamical system") on such a finite-dimensional manifold.

5.4 Mathematical tools for rigorous results

In this section the dynamical system point of view is systematically used to produce some estimates on the number of degrees of freedom. As said above the approach differs from the historical approach of Kolmogorov and Kraichnan but the results are in extremely good agreement.

At first glance a complete justification of the above analysis should include at least

i) The use of regular semi flow

ii) The existence and uniqueness of the probability measure μ on the phase flow satisfying the hypothesis **Herg** (cf. formula (5.1.9)).

However:

i) As discussed at length in previous sections the existence of a "better" solution than the Leray weak solution is for $D = 3$ still an open problem.

ii) Furthermore, nobody has ever come close in proving even for $D = 2$ the existence of a probability measure μ satisfying the hypothesis **Herg** and even more the hypothesis **Herg-is**.

To partly overcome these difficulties one proceeds as follows:

1) As usual, weak solutions of (5.1.1) are considered for $D = 3$. However, classical energy estimates show the existence of an absorbing ball B *bounded in* H such that for any solution $u(t)$ of (5.1.1) there exist a t_0 which depends on the solution such that:

$$t > t_0 \Rightarrow u(t) \in B. \qquad (5.4.1)$$

All these solutions always converge in the weak topology of H to a maximal weakly compact set $\mathcal{A} \subset B$ with the property that if $u_0 \in \mathcal{A}$ then there exists a (weak) solution u defined for $-\infty < t < \infty$ bounded in H such that $u(0) = u_0$. By definition this set is the *global attractor* of the weak semiflow associated to the equation (5.1.1). It contains a dense open (for the weak topology of H) subset $\mathcal{A}_{\rm reg}$ with the property that for $u_0 \in \mathcal{A}_{\rm reg}$ the solution u with this initial condition (at $t = 0$) is unique and analytic on a time interval $0 \in (t_1, t_2)$. In the case $D = 2$ B is a bounded subset of V and therefore is compact in H, $\mathcal{A}$ coincides with $\mathcal{A}_{\rm reg}$ and is the usual global attractor of the dissipative evolution equation. For precise results in dimension 3 the extra hypothesis $\mathcal{A} \equiv \mathcal{A}_{\rm reg}$ or equivalently $B \subset V$ will be necessary.

2)"Weak stationary statistical solutions of the Navier Stokes equation" are defined as probability measures μ on H which satisfy the following relations:

$$\int \|u\|^2 \mu(du) < \infty, \quad \mu(\mathcal{A}) = 1 \tag{5.4.2}$$

and

$$\int [\nu((u, \Phi'(u)) + (B(u,u) - f, \Phi'(u)]\mu(du) = 0 \tag{5.4.3}$$

for all test functionals $\Phi : H \mapsto \mathbf{R}$ which are Gateaux differentiable in H at any point $u \in V$ with derivative Φ' bounded on subsets of V. In dimension 2 stationary statistical solutions coincide with probability (Borel) measures which are invariant for the non linear semiflow, i.e.,

$$\int \Phi(S(t)u) d\mu = \int \Phi(u) d\mu, \ \forall t > 0\,.$$

Rigorous mathematical treatment of this notion appeared in Foias [Fo] and were developed by several authors; in particular it was shown first by Foias [F1] [F2] that such measures are solutions of the Hopf equation. The latter was also studied by Fursikov Vishik [FV1-3] and Fursikov Ehmanuilov [FE]. Finally one observes that the support of the measure is the global attractor $\mathcal{A}$.

3) Eventually the notion of a generalized limit is used and denoted $\mathrm{Lim}_{T\to\infty}$.

With this notion the following result is a soft hybrid of both Birkhoff ergodic theorem and Krylov-Bogoliubov theory:

Theorem 5.1 *For any (weak) solution u defined on $(0, \infty)$ with initial data at $t = 0$, there exists a stationary statistical solution μ such that*

$$Lim_{T\to\infty} \frac{1}{T} \int_0^T \Phi(u(t)) dt = \int \Phi(u) d\mu. \tag{5.4.4}$$

With this statement one can easily and rigorously prove still with energy estimates the following

Theorem 5.2 *In dimension $D = 2$ the set*

$$Range(\eta) = \{Lim_{T\to\infty} \frac{1}{T} \int_0^T \nu |AS(t)u_0|^2 dt, u_0 \in H\} \tag{5.4.5}$$

coincides with the set

$$\int \nu |A(u)|^2 \mu(du)$$

where μ runs over all probability (Borel) measures invariant for the semiflow $S(t)$, $t > 0$.

Since the driving force f is assumed to be localized to the low frequency modes, the only quantities which characterize the flow are the size of the "box" L, the viscosity ν and the L^2 norm of f, and this leads to the introduction of a dimensional number called the *generalized Grashoff numbers* constructed with these quantities:

$$\begin{aligned} G &= \frac{L^2|f|}{\nu^2} = \frac{L^2}{\nu^2}\Big(\int_{[0,L]^2} f(x)\cdot f(x)dx\Big)^{\frac{1}{2}} \text{ for } D = 2\,, \\ G_* &= \frac{L^2|A^{-\frac{1}{2}}f|}{\nu^2} = \frac{L^2}{\nu^2}\Big(\int_{[0,L]^2} (-\Delta)^{-1} f(x)\cdot f(x)dx\Big)^{\frac{1}{2}} \text{ for } D = 3 \end{aligned} \tag{5.4.5}$$

and rigorous estimates on

$$\frac{1}{T}\int_0^T \nu|A(S(t)u_0)_{\kappa,\infty}|^2 dt, \text{ for } D = 2 \text{ and } \frac{1}{T}\int_0^T \nu\|(S(t)u_0)_{\kappa,\infty}\|^2 dt \text{ for } D = 3$$

leads (cf. [FMT] to almost rigorous evaluation of Kraichnan and Kolmogorov wave numbers in term of the Grashoff number:

$$\begin{aligned} C_0 G^{\frac{1}{6}} L^{-1} &\le \kappa_\eta \le C_1 G^{\frac{1}{3}} L^{-1} D = 2 \\ C_0 G_*^{\frac{1}{4}} L^{-1} &\le \kappa_\eta \le C_1 G_*^{\frac{1}{2}} L^{-1} D = 3. \end{aligned} \tag{5.4.6}$$

In (5.4.6) C_0 and C_1 denote universal constants; for $D = 2$ the ergodicity hypothesis **Herg** is assumed and in dimension $D = 3$ a regularity hypothesis is added.

For a large enough wave number (after Kolmogorov or Kraichnan dissipation wave number) the dissipation effect dominates, and this should imply an *exponential decay* for the turbulent spectra (of course with the assumptions that $f = f_{0,\kappa_0}$ and that $G >> \kappa_0 L$).

Then $\langle \|u_{\kappa,\infty}\|^2\rangle$ for $\kappa >> \kappa_\eta$ and $\langle |u_{\kappa,\infty}|^2\rangle$ for $\kappa >> \kappa_\epsilon$ are very small due mainly to the viscous dissipations, and these averages should behave like the Fourier components of the linear equation

$$\frac{du}{dt} + \nu A u = 0 \text{ for } t > 0 \tag{5.4.7}$$

leading to an expression of the form:

$$\begin{aligned} E(\kappa) &\simeq C_1 \exp -C_2\Big(\frac{\kappa}{\kappa_\eta}\Big)^2 \text{ for } \kappa >> \kappa_\eta \text{ for } D = 2 \\ E(\kappa) &\simeq C_1 \exp -C_2\Big(\frac{\kappa}{\kappa_\epsilon}\Big)^2 \text{ for } \kappa >> \kappa_\epsilon \text{ for } D = 3\,. \end{aligned} \tag{5.4.8}$$

Such results are not proven (and in some cases may be false) (cf. [SR] and [Ma]). However it is important to notice that weaker (not too weak) forms of (5.4.8) can be obtained at least for $D = 2$ with full mathematical rigor.

Phenomenological analysis and numerical experiments lead to the idea that (5.4.8) should be replaced by estimates of the form

$$\begin{aligned} E(\kappa) &\simeq C_1 E(\kappa_\eta)(\kappa/\kappa_\eta)^\alpha \exp -\beta((\frac{\kappa}{\kappa_\eta}) \text{ for } \kappa >> \kappa_\eta \text{ for } D = 2 \\ E(\kappa) &\simeq C_1 E(\kappa_\epsilon)(\kappa/\kappa_\epsilon)^\alpha \exp -\beta(\frac{\kappa}{\kappa_\epsilon}) \text{ for } \kappa >> \kappa_\epsilon \text{ for } D = 3. \end{aligned} \tag{5.4.9}$$

For the case $D = 2$ recall the "phenomenological relation" (cf. (5.2.16))

$$E(\kappa) \simeq \frac{\eta^{\frac{2}{3}}}{\kappa_\eta^3} = \kappa_\eta \nu^2.$$

The proof of (5.4.9) with $D = 2$ is equivalent to the obtention of the estimate:

$$\int |A^{\alpha/2} u_{\kappa,\infty}|^2 \mu(du) \simeq C G^{\frac{2}{3}(1+\alpha)} \frac{\nu^2}{L^{2(1+\alpha)}} e^{-\beta\kappa L G^{-\frac{1}{3}}}\,. \tag{5.4.10}$$

The approach is based on Gevrey spaces (it follows [FT3] and it seems that the potential of the method is not yet fully exploited). It starts from the relation:

$$\int |A^{\alpha/2} u_{\kappa,\infty}|^2 \mu(du) \le e^{-2\theta L\kappa} \int |A^{\frac{\alpha}{2}} e^{\theta L A^{\frac{1}{2}}} u|^2 \mu(du) \text{ with } \theta > 0\,. \tag{5.4.11}$$

A slight improvement of the proof in [Cha] leads to the estimate:

$$\| e^{\frac{1}{G(\log c_0 G + 1)} \frac{L}{2\pi} A^{\frac{1}{2}}} u \|^2 \le C_1 G^2 \frac{\nu^2}{L^4} \; \forall u \in \mathcal{A} \tag{5.4.12}$$

under the assumption that

$$C_1 G \ge 1 \text{ and } G \ge L\kappa_0\,. \tag{5.4.13}$$

By integrating (5.4.12) with respect to μ, using (5.4.11) one obtains for $\alpha = 1$

$$\int |A^{\frac{1}{2}} u_{\kappa,\infty}|^2 \mu(du) \le c_0 G^2 \frac{\nu^2}{L^4} e^{-\frac{1}{G(\log G + d)} \frac{L\kappa}{\pi}}\,. \tag{5.4.14}$$

Observe that except for the constants (they should have a uniform dependence on the Grasshoff number) an estimate of the type (5.4.9) for $D = 2$ has been proven.

Eventually the weakest form of (5.4.9) is the analysis of the *power spectrum of the velocity at a given point*, namely the expression:

$$P(\omega) = \lim_{T\to\infty} \frac{1}{T} \Big| \int_0^2 e^{-i\omega t} (S(t)u_0)_j(x_0) dt \Big|^2, \quad j = 1, 2, 3. \tag{5.4.15}$$

Once again, for $D=2$, it can be rigorously proven that $m(d\omega)=P(\omega)d\omega$ defines a positive Borel measure on $\mathbf{R}$ such that

$$\int_{-\infty}^{\infty} e^{-\delta_0|\omega|}m(d\omega)\leq CG^2\frac{\nu^2}{L^2}(\log^+ CG+1)$$
$$\delta_0=C\frac{L^2}{\nu}G^{-2}(\log^+ cG+1)^{-1}. \qquad (5.4.16)$$

The Kraichan-Kolmogorov approach gave, in section (5.2), an estimate of the number of degrees of freedom in terms of the Reynolds number as $(Re)^{\frac{3}{2}}$ for $D=2$ and $(Re)^{\frac{9}{4}}$ for $D=3$. The dimension of the attractor gives an alternative way of measuring the number of degrees of freedom of the flow. Even if this approach is completely different, what is striking is the fact that it leads to the same type of estimates if one observes that the Reynolds number which appears in this derivation is bounded by the Grashoff number:

$$Re\leq CG^{\frac{2}{3}}\,.$$

First results for the attractor in dimension 2 were obtained by O. Ladyzhenskaia [LA1-4] and by Foias and Temam [FT2].

More precisely, in dimension 2 the global attractor is perfectly defined (with no extra hypothesis), is compact in H and its fractal dimension can be estimated in terms of the Grashoff number according to the formula:

$$d_M(\mathcal{A})\leq c_0G^{\frac{2}{3}}(\log(c_1G)+1)^{\frac{1}{3}}. \qquad (5.4.17)$$

This estimate was obtained by [CFT2] in (1988). Observe that (5.4.17) differs from the Kraichnan estimate by a logarithmic term which cannot be present when directly derived by the arguments of section (5.3.). However, remarkably, in a follow up of [CFT], K. Okhitani has shown [Ok] that a more careful analysis within Kraichnan heuristic framework does yield the logarithmic corrective term. More recently, V.X. Liu [LI] presented a proof that $d_M(\mathcal{A})\geq c_1G^{\frac{2}{3}}$ with, for the driving force, a well chosen eigenvector of A.

For $D=3$ one assumes that the attractor $\mathcal{A}$ is a bounded subset of V (consequence of conjectured but non proven regularity results).

Then one defines the quantities:

$$\tilde{\epsilon}=\sup_{u_0\in\mathcal{A}}\Big(\lim_{T\to\infty}\Big[\frac{1}{T}\int_0^T\nu L^{-3}\int_\Omega(\nabla_x u:\nabla_x u(x,t))^{\frac{5}{4}}dxdt\Big]^{\frac{4}{5}}\Big) \qquad (5.4.18)$$

and

$$\kappa_{\tilde{\epsilon}}\Big(\frac{\tilde{\epsilon}}{\nu^3}\Big)\,.$$

With a slight improvement (made possible by [EFT]) of the result in [CFT] one shows, that

$$d_M(\mathcal{A}) \le c_0(L\kappa_\epsilon), \tag{5.4.19}$$

however with $\tilde{\epsilon}$ larger than the one given by the Kolmogorov scaling law.

The proofs of (5.4.17) or (5.4.19) use some of the basic tools of dynamical systems extended to infinite dimensional spaces, and therefore this justifies that a short description of the proof of (5.4.17) be given below.

Sketch of proof of the estimate of the fractal dimension of the $D = 2$ attractor

Inspired by methods of finite dimensional dynamical systems, one first introduces the derivative $DS(t, u_0)$ of the flow $S(t)$ as the solution of the equation:

$$\begin{gathered} \partial_t\phi - \nu\Delta\phi + \phi\nabla_x(S(t)u_0) + (S(t)u_0)\nabla_x\phi + \nabla_x p = 0, \;\; \nabla_x \cdot \phi = 0, \\ \phi(0, x) = \xi \in H, \;\; (DS(t, u_0)\xi) = \phi(t, x)\,. \end{gathered} \tag{5.4.20}$$

The operator $DS(t, u_0)$ is compact in V and one can introduce the infinite sequence

$$\alpha_1(t, u_0) \ge \alpha_2(t, u_0).... \ge \alpha_n(t, u_0) \ge ... \ge 0.$$

of eigenvalues for the self adjoint positive operator $(DS^*(t, u_0) \circ DS(t, u_0))^{\frac{1}{2}}$. *Classical Lyapunov numbers* would be defined as:

$$\lambda_n(u_0) = \lim_{t\to\infty} \{\alpha_n(t, u_0)\}^{\frac{1}{t}}, \text{ and } \mu_n(u_0) = \log \lambda_n(u_0) \tag{5.4.21}$$

However such pointwise Lyapunov numbers may not exist (since we do not know the existence of a canonical ergodic measure μ on $\mathcal{A}$), therefore one uses a topological version of *uniform (global) Lyapunov numbers* introduced in [CF1] and expanded in [EFT]. Let:

$$P_k(t, u_0) = \alpha_1(t, u_0)\alpha_2(t, u_0)\cdots\alpha_k(t, u_0); \tag{5.4.22}$$

$$\pi_k(t) = \sup\{P_k(t, u_0) : u_0 \in \mathcal{A}\}; \tag{5.4.23}$$

because of the subexponential identity $\pi_k(t + s) \le \pi_k(t)\pi_k(s)$, it can be shown that the following limit exists:

$$\Pi_k = \lim_{t\to\infty} (\pi_k(t))^{1/t}. \tag{5.4.24}$$

One can then define recursively the uniform Lyapunov numbers Λ_k, $k = 1, 2, \ldots$;

$$\Lambda_1 = \Pi_1, \quad \Lambda_1\Lambda_2 = \Pi_2, \ldots, \Lambda_1 \ldots \Lambda_k = \Pi_k, \ldots \tag{5.4.25}$$

and the uniform (global) Lyapunov exponents are defined by:

$$\mu_m = \log \Lambda_m, \quad m \ge 1; \tag{5.4.26}$$

equivalently:

$$\mu_1 + \mu_2 + \cdots + \mu_k = \lim_{t\to\infty} \tfrac{1}{t} \log \pi_k(t). \qquad (5.4.27)$$

These exponents converge to $-\infty$ as $k \to \infty$. Then, with some further hypotheses on the uniform differentiability of $S(t)$ with respect to u_0 in $\mathcal{A}$, one uses classical fractal geometry arguments ([CF], [DO], [EFT]) to cover $\mathcal{A}$ by iterations of increasingly refined families of balls; each ball of radius ϵ centered at some u_0 is deformed by $DS(t;u_0)$ into an ellipsoid whose principal axes are $\alpha_1(t,u_0)\epsilon, \ldots, \alpha_n(t,u_0)\epsilon, \ldots$. The scaling laws of such coverings yield estimates on the Hausdorff and Fractal dimensions of $\mathcal{A}$ according to:

Theorem 5.3 ([CF1], [T], [EFT]) If for some $n \geq 1$

$$\mu_1 + \mu_2 + ... + \mu_{n+1} < 0; \qquad (5.4.28)$$

then

$$\mu_{n+1} < 0, \ \frac{(\mu_1 + \mu_2 + ... + \mu_n)}{|\mu_{n+1}|} < 1 \qquad (5.4.29)$$

and

i) The Hausdorff dimension of $\mathcal{A}$ is less than or equal to

$$n + \frac{(\mu_1 + \mu_2 + ... + \mu_n)_+}{|\mu_{n+1}|}. \qquad (5.4.30)$$

ii) The fractal dimension of $\mathcal{A}$ is less than or equal to

$$\max_{1\leq j\leq n} \left(j + \frac{(\mu_1 + \mu_2 + ... + \mu_j)_+}{|\bar{\mu}_{n+1}|}\right), \qquad (5.4.31)$$

where

$$\bar{\mu}_{n+1} = \limsup_{t\to\infty} \tfrac{1}{t} \log \left[\sup_{u_0\in\mathcal{A}} \alpha_{n+1}(t;u_0) \right]. \qquad (5.4.32)$$

Next introduce an m-dimensional volume element in V spanned by m independent elements $\xi_1, \xi_2, ..., \xi_m$; denote $U_j(t) = DS(t,u_0)\xi_j$ and observe that a variant of the classical Liouville theorem gives:

$$|U_1(t) \wedge ... \wedge U_m(t)|_{\wedge^m V} \leq |\xi_1(t) \wedge ... \wedge \xi_m(t)|_{\wedge^m V} \exp\left(\int_0^t \mathrm{Tr}(DS(\tau,u_0) \circ Q_m(\tau)) d\tau \right) \qquad (5.4.33)$$

In (5.4.33) $Q_m(\tau))$ denotes the projector on the space spanned by $U_1(\tau), ... \wedge U_m(\tau)$. For the Lyapunov exponent describing the evolution of the "volume element" one shows the relation:

$$\mu_1 + \mu_2 + ... + \mu_m \leq q_m = \limsup_{t\to\infty} \sup_{u_0\in\mathcal{A}} \sup_{\xi_i\in V, \|\xi\|\leq 1, 1\leq i\leq m} \left(\frac{1}{t} \int_0^t \mathrm{Tr}(DS(\tau,u_0) \circ Q_m(\tau)) d\tau \right). \qquad (5.4.34)$$

To estimate the quantity

$$\int_0^t \mathrm{Tr}(DS(\tau, u_0) \circ Q_m(\tau))d\tau$$

one introduces an orthonormal (in V) basis of $Q_m(\tau))V$, $\{\phi_j(\tau)\}$ and uses the relation:

$$Tr(DS(\tau, u_0) \circ Q_m(\tau)) = \sum_{1 \le j \le m} (DS(\tau, u_0)\phi_j, A\phi_j) = \sum_{1 \le j \le m} \{-\nu |A\phi_j|^2 + B(\phi_j, \phi_j), Au)\}\,. \tag{5.4.35}$$

With the properties of the quadratic advection operator B in dimension 2 one has:

$$\sum_{1 \le j \le m} (B(\phi_j, \phi_j), Au) \le |\rho|^{\frac{1}{2}}_{L^\infty(\Omega)} |\sigma|^{\frac{1}{2}}_{L^2(\Omega)} |\Delta u|_{L^{\frac{3}{4}}(\Omega)} \tag{5.4.36}$$

with

$$\rho(x) = \sum_{1 \le j \le m} |\phi_j(x)|^2,\ \sigma(x) = \sum_{1 \le j \le m} |\nabla \phi_j(x)|^2\,. \tag{5.4.37}$$

The proof is completed with the two following estimates:

$$\sum_{1 \le j \le m} |A\phi_j(x)|^2 \ge C\lambda_1 m^2 \tag{5.4.38}$$

where λ_1 is the first non zero eigenvalue of Stokes operator A and

$$|\rho|_{L^\infty(\Omega)} \le C\big(1 + \log\big(\frac{1}{\lambda_1} \sum_{1 \le j \le m} |A\phi_j(x)|^2\big) \tag{5.4.39}$$

which comes from the log-singularity of the Green function in two space variables and which in a weaker form is due to Brezis and Gallouet [BG] (alternative proofs of (5.4.39) can be found in Lieb [Lie] and Constantin [Co]). This is the very estimate which is responsible for the log correction in (5.4.17)

5.5 Exponential Attractors.

As observed in the previous section, the viscous effects make the fluid dependent on a finite number of degrees of freedom and therefore there are good reasons to develop the analogy with finite dimensional dynamical systems and even to try to reduce the Navier Stokes flow to a flow on a finite dimensional manifold. According to this idea several authors ([FST], [CFT]) proposed the notion of inertial manifold closely related to the eigenmode decomposition of the linear operator A.

Consider a semi flow $S(t)$, in a Hilbert space H, generated by an evolution equation of the form

$$u_t + \nu A(u) + B(u) = F \tag{5.5.1}$$

where A is (as is the case for the Navier Stokes equation) a linear self adjoint operator and B a lower order non linear operator. Introduce an orthogonal decomposition of the Hilbert space H into the space spanned by the first N eigenvectors of A and its orthogonal complement. Denote by P_N and Q_N the two corresponding projections and observe that the equation (5.5.1) is then decomposed into a system of two equations according to the formula:

$$\begin{aligned} p = P_N u, \quad q =& Q_N u, \\ p_t + \nu A(p) + P_N(B(u)) =& P_N f \\ q_t + \nu A(q) + Q_N(B(u)) =& Q_N f. \end{aligned} \tag{5.5.2}$$

Then one says that the above spectral decomposition defines an inertial manifold $\mathcal{M}$ if there exists a Lipschitz map $\Phi : P_N H \to Q_N H$ with the following properties:

$$\mathcal{M} = \text{ Graph } (\Phi) = \{(p, \Phi(p)),\ p \in P_N H\} \tag{5.5.3}$$

$$S(t)\mathcal{M} \subset \mathcal{M} \ \forall t > 0 \tag{5.5.4}$$

which attracts uniformly exponentially all solutions of the equation (5.5.1).

The existence of the inertial manifold would imply not only that the dynamics are finite dimensional but also that it is completely described by the evolution of the first N eigenmodes (in periodic configuration of the first N Fourier modes).

The existence of such an invariant manifold has been proved for several equations like the Kuramoto-Sivashinsky [FNST], the Ginzburg Landau equations [CFT] or a hyper-dissipative version of the Navier Stokes equation (cf. [CF2] and [MP]). However, it has never been established for the genuine Navier Stokes equation even in two space variables (published claims are incorrect) and besides technical difficulties this fact can explained as follows:

A consequence of the existence of an inertial manifold is that the higher order modes (of order greater than N) are completely driven by the lower order modes; in the "folklore" of the field they are "slaved modes" and this property seems to be in contradiction with current phenomenological theories of turbulent intermittencies.

Recalling that such theories rely on averaged properties: spectral modes of arbitrarily large frequency and non small amplitude may appear intermittently in physical space with a small probability and such occurrence makes impossible a rigorous description of the dynamics of infinite dimensional system by a N modes dynamical system.

Therefore, Eden, Foias et al. [EFNT] have proposed a more physical and more robust (under perturbations) notion which is the *exponential attractor*:

Definition 5.4 *Let $\{S(t)\}_{t\geq 0}$ be a Lipschitz continuous semiflow with a positively invariant compact set X, $X \subset H$, $S(t)X \subset X$ for every $t \geq 0$. A compact set $\mathcal{M}_0$ is called* *an exponential attractor for* $S(t)$ *if*

$$\begin{aligned}
&i)\ \mathcal{A} = \cap_{t>0}(S(t)X) \subset \mathcal{M}_0\,,\\
&ii)\ S(t)(\mathcal{M}_0) \subset \mathcal{M}_0\ \forall t \geq 0\,,\\
&iii)\ \mathcal{M}_0 \text{ has finite fractal dimension } d_F(\mathcal{M}_0)\\
&iv)\ \text{dist } (S(t)X_0, \mathcal{M}_0) \leq Ce^{-\beta t} \text{ for convenient constants } C \text{ and } \beta\,.
\end{aligned} \tag{5.5.5}$$

Exponential attractors are fractal objects which not only contain the ultimate attractors but capture important slow scale transient dynamics. Clearly exponential attractors are not unique; by definition any two exponential attractors are exponentially attracted to each other. The major difference between exponential attractors and the global attractor is that the latter may only attract at an algebraically slow rate (there are examples to that effect, [Kos]). The major difference between inertial manifolds and exponential attractors is that the latter do not assume any global slaving of small scales. As a consequence, the exponential attractors can deal with cases where an exponential convergence is not restricted within a smooth manifold structure. As far as the theory goes, it might well be a fractal set. The physical relevance of exponential attractors for Navier-Stokes turbulence is discussed in [EFNS].

The existence of an exponential attractor, its dimension and the value of the constants C and β appearing in (5.5.5, (iv)) can be obtained by an iterative covering process (cf. [EFNT]) from a dichotomy principle, called the squeezing property.

Because of its importance, we recall its definition.

Definition 5.4 *Discrete squeezing property* (DSP). *In the context of Definition 5.1 one will say that a semiflow* $S(t)$ *satisfies the weak discrete squeezing property if there exists an orthogonal projection* P_N *of rank* N *and a positive time* t_* *such that the relation*

$$|P_N(S(t_*)u_0 - P_N(S(t_*)v_0| < |Q_N(S(t_*)u_0 - Q_N(S(t_*)v_0| \tag{5.5.6}$$

implies the relation

$$|(S(t_*)u_0 - (S(t_*)v_0| < \delta|u_0 - v_0| \ \ \text{with } 0 \leq \delta < 1. \tag{5.5.7}$$

The condition (5.5.6) can be rephrased as "when the difference between two solutions is mainly concentrated in small scale modes" and the consequence (5.5.7) means that the difference is contracted in time during some past (from $t = 0$ to t_*). Strong versions of the squeezing property go back to [FT1] and O. Ladyzhenskaïa [La2].

For the proof of the (DSP) property a convenient tool is the quantities:

$$\lambda(t, u, v) = \frac{\|u(t) - v(t)\|^2}{|u(t) - v(t)|^2} = \frac{\|w(t)\|^2}{|w(t)|^2} \tag{5.5.8}$$

and

$$\mu(t, u, v) = \frac{|A[u(t) - v(t)]|^2}{\|u(t) - v(t)\|^2} = \frac{|A[w(t)]|^2}{\|w(t)\|^2} \tag{5.5.9}$$

defined for two solutions $u(t)$, $v(t)$ with $w(t) = u(t) - v(t)$. Now, for the $D = 2$ Navier Stokes equation, the squeezing property is deduced from the energy estimate:

$$\frac{d}{dt}|w(t)|^2 + \nu\lambda(t,u,v)|w(t)|^2 \leq c\frac{G^2\nu}{L^2}|w(t)|^2 \tag{5.5.10}$$

which leads to:

$$|w(t_*)|^2 \leq \exp[(-C\nu N + C_1\frac{G^2\nu}{L^2})t_*] \times |w(0)|^2\,. \tag{5.5.11}$$

The same estimates can also be obtained for the enstrophy.

Exponential attractors constructed with the DSP have fractal dimensions higher (as a function of the Grashoff number) than the estimates of $d_F(\mathcal{A})$ for the global attractor which relies on Lyapunov number techniques. Eden et al. ([EFN] [EFNT]) give an alternative construction of exponential attractors based on the concept of outer Lyapunov exponents and outer Lyapunov dimension. The outer Lyapunov exponents are defined as in the beginning of Section 5.5, but with the "sup over $u_0 \in \mathcal{A}$" replaced by "sup over $u_0 \in X$" in Eqn. (5.4.23) and (5.4.32), where X is the positively invariant compact set of the semiflow $S(t)$. The outer Lyapunov dimension d_{0L} of $\mathcal{M}_0$ is given by a formula identical to (5.4.31), with the μ_i replaced by outer Lyapunov exponents. In principle,

$$d_F(\mathcal{A}) \leq d_{0L}(\mathcal{M}_0)\,; \tag{5.5.12}$$

but in terms of the practical estimates which both use the trace operator formulas (5.4.34), the two dimensions above are indistinguishable. In that sense, such exponential attractors have optimal outer Lyapunov dimension.

Eventually one recovers also for the fractal dimension of the exponential attractor (which contains the global attractor) an estimate in

$$G^{\frac{2}{3}}(\log G + 1)^{\frac{1}{3}} \tag{5.5.13}$$

in agreement with (5.4.17).

Recently, Le Dung and Nicolaenko [LDN] have demonstrated that exponential attractors are objects as universal as global attractors for dissipative infinite dimensional dynamical systems: no squeezing properties, nor fine structure of Lyapunov exponents are required. They extend the theory of exponential attractors from the Hilbert space setting to the Banach space setting. The only requirements are for the semiflow to be C^1 in some absorbing ball and for the linearized semiflow at every point inside the absorbing ball to split into the sum of a compact operator plus a contraction. In some sense, [LDN] establish a global exponential dichotomy for infinite-dimensional dissipative dynamical systems; however, the exponential attractor $\mathcal{M}_0$ is not in general a smooth manifold.

Let E be a Banach space, $U \subset E$ an open set and $S : U \to E$ a C^1 map. We consider the discrete dynamical system $\{S^n\}_{n=1}^{\infty}$ generated by S.

We start with the assumption that there is a compact connected subset $X \subset U$ and $S : X \to X$ and S possesses a universal (global) topological attractor $\mathcal{A}$ which is a compact, connected set given by

$$\mathcal{A} = \bigcap_{n=1}^{\infty} S^n(X)\,. \tag{5.5.14}$$

We denote by $\mathcal{L}(E)$ the space of bounded linear maps from E into itself. For a given positive real λ we denote by $\mathcal{L}_\lambda(E)$ the set of maps $L \in \mathcal{L}(E)$ such that L can be decomposed as $L = K + C$ with K compact and $\|C\| < \lambda$. Here $\|C\|$ denotes the norm of the operator C.

The main result of [LDN] is the following

Theorem 5.5 *If there exists $\lambda \in (0,1)$ such that $D_x S(x) \in \mathcal{L}_\lambda(E)$ for all $x \in X$, then the discrete dynamical system $\{S^n\}_{n=1}^\infty$ possesses an exponential attractor.*

Define S as the map induced by Poincaré sections of a Lipschitz continuous semiflow $S(t)$, $t \geq 0$, at the time $t = T^*$ for some $T^* > 0$; that is, $S := S(T^*)$. We consider the *discrete semigroup* $\{S^n\}_{n\geq 0}$ generated by S. Once the existence of exponential attractors for the discrete case is proved, the result for the continuous case follows in a standard manner (e.g. see [EFNT]). We have

Theorem 5.6 *Let X be an absorbing set for a continuous semiflow $S(t)$. Suppose that there is a $T^* > 0$ such that $S = S(T^*)$ satisfies the condition of Theorem 5.4. Assume further that the map $F(x,t) = S(t)x$ is Lipschitz from $[0,t] \times X$ into X for any $T > 0$. Then the flow $\{S(t)\}_{t\geq 0}$ admits an exponential attractor $\mathcal{M}$.*

An immediate consequence of the above is the existence of exponential attractors for the fast-rotating 3D Navier Stokes equations (2.5.23) investigated in [BMN1], [BMN2]. This is the only known rigorous result of its kind for genuinely 3-D Navier-Stokes-like equations.

In the absence of an inertial manifold one would like to address the following question: Is there a natural way of reconstructing the dynamical system without recourse to the underlying equation?

Once the existence of an exponential attractor of an infinite dimensional dynamical system is established, the next stage is to unravel the dynamics on this set. A natural way is to show that the infinite dimensional dynamical system is inertially equivalent to some finite dimensional one:

Definition 5.7 [EFNT] [Chapter 10] *Two dynamical systems are inertially equivalent if:*

i) *they have a common exponential attractor*

ii) *the dynamics on that exponential attractor coincide.*

First, one can imbed the fractal exponential attractor $\mathcal{M}$ into an Euclidean manifold with a Mañé Projection P which admits a continuous pseudo-inverse when restricted to $P\mathcal{M}$; note that Mañé's projections are dense:

Theorem 5.8 (Modified Mañé's theorem, [BEFN]) *Let H be a separable Hilbert space, Y a fractal compact subset of H such that $d_F(Y) = D$. If P_0 is an orthogonal projection with* rank $\tilde{N} \geq [2D+1]$, *then for every $\delta \in (0,1)$ there exists an orthogonal projection $P = P(\delta)$ such that*

$$\|P - P_0\| \leq \delta \; (\text{Ker } P) \cap Y = \{0\}\,. \tag{5.5.15}$$

The procedure of constructing a finite dynamical system which is inertially equivalent to an infinite dimensional one can be roughly described as follows ([EFNT, Chapter 10]).

First we start out with a dissipative dynamical system associated to a PDE written in the evolution form $du/dt = F(u)$, $u(0) = u_0$, and project the evolution equation on $\mathcal{M}$ via Mañé's projection P onto a system of ODE's on an Euclidean space of dimension $\tilde{N} = [2D + 1]$. On $P\mathcal{M}$, this dynamical system is well defined by:

$$\frac{dx}{dt} = PF\{(P|_{\mathcal{M}})^{-1}x(t)\}$$

$$x(0) = Pu(0)\,. \tag{5.5.16}$$

The next step is to extend that dynamical system to a generalized dynamical system defined *everywhere* in $R^{\tilde{N}}$. The solutions of the generalized system of ODE's so obtained may not be unique and differentiable (for the definition and construction of such a generalized system, see [EFNT, Chapter 10]. However, one can show that the solutions exist globally in time and are attracted exponentially to $P\mathcal{M}$. It is possible to show that the projected system of ODE's generates a generalized dynamical system on the Euclidean space; the continuity points of that system form a dense G_δ subset of $R^{\tilde{N}}$.

The next step is to lift the generalized dynamical system back to the infinite dimensional space by the lifting P^{-1}. Unfortunately, without further properties on the inverse of the Mañé projection, we cannot proceed with such a lifting to obtain a dynamical system which admits the set $\mathcal{M}$ as an exponential attractor in a Banach space context. Remarkably, this is true in a Hilbert space.

It was shown in [EFNT, Chapter 10] that this lifting is possible if a Hölder-Mañé projection theorem can be established; that is, if one can show that there is a Mañé projection P whose inverse is Hölder continuous on $P\mathcal{M}$. Recently, in [FO] such a theorem is proven by Foias and Olsen for the case of infinite dimensional Hilbert spaces. We remark here (see [EFNT, Appendix A]) that there are counterexamples where P^{-1} cannot be Lipschitz so that the best result can only be where P^{-1} is Hölder continuous.

Theorem 1.1 in [FO] considers a real Hilbert space H and $X \subset H$ such that $d_F(X) < m/2$, where d_F denotes the fractal dimension. Then for any orthogonal projection P of rank m and $\delta > 0$ there is an orthogonal projection $\tilde{P}$ such that $\|P - \tilde{P}\| < \delta$ and $\tilde{P}|_X$ has Hölder inverse.

Combining the discussion in [CFNT, Chapter 10], [FO, Theorem 1.1] and Theorem 5.4 [LDN], we can conclude that

Theorem 5.9 *For a Hilbert space H, let the semiflow $S(t)$ satisfy the conditions of Theorem 5.4 and $\mathcal{M}$ be an exponential attractor for $S(t)$. Then $S(t)$ admits an inertially equivalent generalized dynamical system in H of dimension $[2d_F(\mathcal{M}) + 1]$.*

Here, $[d_F(\mathcal{M})]$ denotes the largest integer which is less than or equal to $d_F(\mathcal{M})$. There is reasonable hope to extend the above result to inertially equivalent dynamical systems which are locally of the Caratheodory-type on local pieces of smooth manifolds.

6 Coherent Structures in two space variables

Physical observations, illustrate by the figure 1 and numerical simulations illustrated by the figure 2 and studied for for instance in Farge et al [KF], [FCH], [FGMPW], [FSK] and Marcus [Mar], show the persistence in two spaces variables of coherent structures the most classical one being the Jupiter red spot (cf. Ingersol and Ingersol and Cuong [In], [InC]) or the anticyclone of the Acores. In both cases the problems are $3d$ but due to the smallness of the thickness of the atmosphere they are mostly driven by two dimensional dynamic as said in section 2.5 (cf equation (2.5.23) and related works [BMN1] for instance).

The Jupiter Red Spot
by courtesy of the Jet Propulsion Laboratory Pasadena

A Coherent structure generated by a numerical simulation
by courtesy of Marie Farge and Nicholas Kevlahan

These coherent structure present an alternative for what would the thermodynamical equilibrium for turbulent flow.

The mathematical construction of these structures relies up to now on the minimization of some type of entropy which would be related to conserved quantities *unless the flow becomes turbulent*. In this sense these structures are related to the theory of turbulence. However there are by themselves very regular. It is the transient regime and not the structures themselves which is related to turbulence. Eventually question of a mathematical relation between these objects and the one described in the previous sections (global attractors, turbulent energy spectra etc..) seems to be completely open.

Consider the solutions of the $2d$ Euler equations in an open set Ω with boundary $\partial\Omega$ and impermeability boundary condition

$$u \cdot \vec{n} = 0 \text{ on } \partial\Omega \tag{6.1.1}$$

with $\vec{n}$ denoting the outward normal. Of course the condition (6.1.1) is omitted when there is no boundary or in particular when $\Omega = \mathbf{R}^2$ and in the periodic case $\Omega = \mathbf{R}^2 \backslash \mathbf{Z}^2$. Finally for the sake of clarity Ω is assumed to be simply connected (even if some interesting examples for the theory do appear in non simply connected domains like the annulus cf. [CLMP]).

6.1 Stability of stationary solutions

The geometry of 2 dimensional incompressible Euler equation is characterized by two following fact.

i) As already said in section 2.3 the vorticity is conserved along the trajectories of the flow:

$$\partial_t \omega + u \nabla_x \omega = 0 \tag{6.1.2}$$

and

ii) With the divergence free condition and the impermeability boundary condition (when some boundary $\partial\Omega$ is present) the existence of a scalar stream function Ψ such that one has

$$u(x,t) = \nabla^{-}\Psi(x,t)\,. \tag{6.1.3}$$

The current and vorticity corresponding to a vector field u will be denoted in this section Ψ_u and ω_u and if there is no risk of confusion the indices u will be omitted.

The first consequence of (6.1.2) and (6.1.3) is that the stationary solutions are characterized by the fact that the gradient of their current and their vorticity are everywhere colinear and that gives (no proof is needed) the:

Theorem 6.1 *A divergence free vector field $u^*(x_1, x_2)$ is a stationary (time independent) solution of the $2d$ Euler equation if and only if there exists a real (in general multivalued valued) function G_{u^*} which relates the current Ψ^* and the vorticity ω^* according to the formula:*

$$\Psi^* = G(\omega^*) = G(-\Delta_x \Psi^*)\,. \tag{6.1.3}$$

Remark 6.2 *The above theorem gives a criteria used in numerical codes or physical experiments to detect if a solution of the Euler equation comes close to a stationary state: The plot of the points $\Psi(x,t), \nabla\times u(x,t)$ for t fixed and $x\in\Omega$ should form a graph.*

The second consequence of the Euler equation itself and of the relation (6.1.2) is the

Proposition 6.3 *or any real valued function ϕ the quantity*

$$H(u) = \frac{1}{2}\int_\Omega |u(x,t)|^2dx + \int_\Omega \phi(\nabla\times u(x,t))dx \tag{6.1.4}$$

is conserved whenever u is a smooth solution of the Euler equation. In particular if u^* is a stationary solution the quantity:

$$H(u) - H(u*) = \frac{1}{2}\int_\Omega (|u(x,t)|^2 - |u^*(x)|^2)dx + \int_\Omega \phi((\nabla\times u(x,t)) - \phi(\omega^*))dx \tag{6.1.5}$$

is also conserved.

Remark 6.4 *As shown by a basic example due to Scheffer [Sche] and Shnirelman [Shni] the conservation properties are not always true they require some regularity which are in particular ensured when the vorticity belongs to $L^\infty(\Omega)$. This will be assumed in all this section.*

The energy which appears in (6.1.4) is given in term of the vorticity by the formula:

$$E = \frac{1}{2}\int_\Omega |u(x,t)|^2dx = \frac{1}{2}\int_\Omega ((-\Delta)^{-1}\omega(.,t))(x)\omega(.,t))dx \tag{6.1.6}$$

with $(-\Delta)^{-1}$ denoting the inverse of the Laplacian with Dirichlet boundary condition, or in terms of Green function:

$$((-\Delta)^{-1}\omega)(x) = \int_\Omega V(x,y)\omega(y)dy\,. \tag{6.1.7}$$

Observe that one has:

$$\begin{aligned}\int_\Omega u^*(x)(u(x,t) - u^*(x)) &= \int_\Omega \nabla_x^\perp\Psi^*(x)\cdot(\nabla_x^\perp\Psi^*(x) - \nabla_x^\perp\Psi(x,t))dx\\ &= \int_\Omega G(\omega^*)(\omega-\omega^*)dx\end{aligned} \tag{6.1.8}$$

Eventually with ϕ in (6.1.5) such that

$$\phi'(s) = -G(s)\,, \tag{6.1.9}$$

$$\begin{aligned}H(u) - H(u*) &= \frac{1}{2}\int_\Omega |u(x,t) - u^*(x)|^2dx - \int_\Omega \phi'(\omega^*)(\omega-\omega^*)dx\\ &+ \int_\Omega \phi((\omega(x,t)) - \phi(\omega^*))dx\\ &= \frac{1}{2}\int_\Omega |u(x,t) - u^*(x)|^2dx + \frac{1}{2}\int_\Omega \phi''(\xi(x,t))(\omega-\omega^*)^2dx\end{aligned} \tag{6.1.10}$$

where $\xi(x,t)$ is a real number which depends on the values of $\omega(x,t)$ and of $\omega^*(x)$. The right hand side of (6.1.9) plays the role of a Liapounov functional and one has

Corollary 6.5 *Any stationary solution of the Euler equation is stable both for positive and negative time for the $H^1(\Omega)$ norm, under perturbations with uniformly bounded vorticity, if one of the two conditions are satisfied:*

i) The function G' with G appearing in (6.1.3) is strictly convex.

ii) There exists a large enough constant C such that

$$-G'(s) \geq C\,.$$

Proof: The result is a consequence of the existence of two strictly positive constants α and β such that one has

$$\begin{aligned}\alpha(\|u(t,.)-u^*(.)\|^2_{L^2(\Omega)} + \|\omega(t,.)-\omega^*(.)\|^2_{L^2(\Omega)}) &\leq |H(u)-H(u^*)| \\ &\leq \beta(\|u(t,.)-u^*(.)\|^2_{L^2(\Omega)} + \|\omega(t,.)-\omega^*(.)\|^2_{L^2(\Omega)})\end{aligned}\,. \qquad (6.1.11)$$

The existence of β is always ensured by the conservation of the L^∞ norm of the vorticity. The existence of α is trivial in the case i). In the second case consider the quantity

$$H(u^*)-H(u)$$

and use the Poincaré inequality to bound

$$\|u(t,.)-u^*(.)\|^2_{L^2(\Omega)}$$

by

$$-\tfrac{1}{2}\int_\Omega \phi"(\xi(x,t))(\omega-\omega^*)^2\,dx\,.$$

The above theorem due to Arnold extends a series of results on linear stability obtained already in the last century by Rayleigh and others. On the other hand it is important to observe that any stationary state which satisfies the hypothesis of the above corollary is stable in the $H^1(\Omega)$ norm both in the future and in the past. This implies that such a solution cannot be an attractor in the future for this norm; However this observation does not prevent the same solution to be an attractor in a weaker norm. And eventually the notion used in this chapter may differ from the one introduced before. One could try to find a stationary (may be unstable) solution with the property that "most" (in a convenient way) solutions would come very often in an arbitrarily small neighborhood of this solution. Therefore the criteria proposed here will differ from the one given in previous section:

6.2 Criteria for attractor

In this section are described some classical criteria for the ω limit set of a family of solution of the $2d$ Euler equation. First acting as mathematician we give the recipes and then try to justify them. As in the theory of turbulence the reader should keep in mind the fact there are no up to now dynamical proof of the validity of these recipes. The arguments given are borrowed from other fields of physics, mostly statistical mechanic.

Considered here are families of solutions u_ϵ with initial data, current and vorticity: $u^0_\epsilon, \Psi^0_\epsilon, \omega^0_\epsilon$ and the limit points of the sequence $\omega_\epsilon(x,t)$, for $t \to \infty$, and for ω^0_ϵ converging to ω^0 in $L^\infty(\Omega)$ weak* are analyzed.

Observe that

i) Weak* $L^\infty(\Omega)$ convergence to a stationary state (u^*, ω^*) satisfying the hypothesis of the corollary 6.4 does not contradict the fact that this stationary solution is stable both in the past and in the future (the topologies are different).

ii) Even when the initial data converge in a very strong norm no uniform (with respect to time) estimate are available (cf. remark 2.1 of section 2.4)) the only thing which is sure is that the curl remains uniformly in time bounded in $L^\infty(\Omega)$. Let

$$\epsilon_i \to 0 \text{ and } t_i \to \infty$$

such that $u_\epsilon(.,t_i)$ converges to a stationary solution u^* in Weak* $L^\infty(\Omega)$ then the following identities are true (the index i being omitted in what follow):

$$\lim \int_\Omega \|u_\epsilon(x,t_i)\|^2 dx = \int_\Omega \|u^0(x)\|^2 dx \tag{6.2.1}$$

and

$$\lim \int_\Omega \omega_{\pm\epsilon}(x,t) dx = \int_\Omega \omega^0{}_\pm(x) dx \tag{6.2.2}$$

(the sign $\pm$ refers to the absolute value of the positive and negative part of the vorticity). On the other hand for a genuinely non linear function F, due to the lack of compactness, one may have:

$$\lim \int_\Omega F(\omega_\epsilon(x,t)) dx \neq \int_\Omega F(\omega^0(x)) dx\,. \tag{6.2.3}$$

However for a convex function F the relation

$$\lim \int_\Omega F(\omega_\epsilon(x,t)) dx \geq \int_\Omega F(\omega^0(x)) dx \tag{6.2.4}$$

remains always valid. Therefore according to the intuition one should introduce the "entropy" and defines as a "good guess " the natural stationary solution as the one which minimizes the quantity

$$\int_\Omega |(\omega^*(x))| \log(|(\omega^*(x))| dx$$

under the constraints:

$$\int_\Omega \|u^*(x)\|^2 dx = \int_\Omega \|u^0(x)\|^2 dx \tag{6.2.5}$$

and

$$\int_\Omega \omega^*_\pm(x) dx = \int_\Omega \omega^0_\pm(x) dx \tag*{(6.2.6)$_\pm$}$$

A simple variational computation which uses in particular the formula (6.1.6): shows that any solution of this minimization problem should satisfy the equation:

$$-\Delta\psi(x) = c_+ e^{-\beta\psi} - c_- e^{\beta\psi},\ \psi = 0 \text{ on } \partial\Omega \tag{6.2.7}$$

In (6.2.7) $c_\pm$ are the two Lagrange multipliers of the two constraints (6.2.6)$\pm$ while β is the Lagrange multiplier of the constraint (6.2.5). The above equation is called (according to the scientists who introduced it in the field) the Joyce Montgomery equation and it has been widely studied (ref [JM]). In the special case where the vorticity is of constant sign it is reduced to the so called mean field equation:

$$-\Delta\psi = c\frac{\exp(-\beta\psi)}{\int_\Omega \exp(-\beta\psi)dx}\,. \tag{6.2.8}$$

In the absence of dynamical proofs, numerical simulations and experiments have been done producing excellent agreement with the "attractor " computed with the above recipe.

A more detailed construction has been proposed by Robert and Sommeria [Ro], [Ro-So], Miller et al [MiWeCr] and others (the initial idea probably going back to Linden-Bell [LiB]) with the purpose of preserving all conserved quantities of the form

$$\int_\Omega f(\omega_\epsilon(x,t))dx\ \forall f$$

and therefore not to exclude in some cases strong convergence for $t_i \to \infty$.

It is described below.

The starting point is the introduction of a family of solutions with initial vorticity ω_ϵ^0 uniformly bounded in $L^\infty(\Omega)$:

$$\forall x \in \Omega\ \ -\infty < -q \le \omega_\epsilon^0(x) \le q < \infty \tag{6.2.9}$$

converging to ω^0 in weak* $L^\infty(\Omega)$. Up to the extraction of a subsequence such convergence is characterized by a Young measure $d\nu(y)_x$ and one has:

$$\text{weak}^* \lim_{\epsilon\to\infty} f(\omega_\epsilon(x)) = \int_{-q}^{q} f(y)d\nu(y)_x \tag{6.2.10}$$

The strong convergence being characterized by the points where $d\nu(y)_x = \delta_{y(x)}$. With (6.2.10) one can defined a measure of mass 1 with support on the interval $]-q,q[$ according to the formula:

$$\int f(y)d\pi_0(y) = \frac{1}{|\Omega|}\int_\Omega dx \int_{\mathbf{R}} f(y)d\nu(y)_x \tag{6.2.11}$$

and introduce the "reference measure" measure $d\sigma = dx \otimes d\pi_0$:

$$\int_{\Omega\times\mathbf{R}} f(x,y)d\sigma_O = \int_\Omega dx \int_{-q}^{q} f(x,y)d\pi_0(y) \tag{6.2.12}$$

Now the recipe goes as follow:

Among the measures which are absolutely continuous with respect to $d\sigma$:

$$d\mu(x,y) = \rho(x,y)d\sigma$$

select the one which minimize the so called Kullback entropy:

$$K(d\mu) = \int_\Omega dx \int_{-q}^{q} \rho(x,y)\log(\rho(x,y))d\sigma$$

under the followings constraints:

i) A consequence of the definition of ρ with Young measures:

$$\begin{aligned} &\int_{\mathbf{R}} \rho(x,y)d\nu(y)_x = 1, \quad dx - \text{a.e} \\ &\int_\Omega \rho(x,y)dx = |\Omega|, \quad \int_\Omega d\pi_0 dx - \text{a.e}\,. \end{aligned} \tag{6.2.13}$$

ii) The conservation of real valued functions of the vorticity:

$$\forall f, \int_\Omega\int_{\mathbf{R}} f(y)\rho(x,y)d\sigma = \int_{\mathbf{R}} f(y)d\pi_0(y) \tag{6.2.14}$$

iii) The constraint of conservation of energy:

$$\begin{aligned} \tfrac{1}{2}\int_\Omega ((-\Delta)^{-1}\omega)(x).\omega(x)dx &= \tfrac{1}{2}\int_\Omega ((-\Delta)^{-1}\omega^0)(x).\omega^0(x)dx \\ \text{with } \omega(x) &= \int_{-q}^{q} y\rho(x,y)d\nu(y)_x \end{aligned} \tag{6.2.15}$$

As in the Joyce Montgomery equation a variational computation is easily done and implies that the function $\rho(x,y)$ is a solution of the following system:

$$\begin{aligned} \rho(x,y) &= \frac{e^{\alpha(y)-\beta y\psi(x)}}{\int_{\mathbf{R}} e^{\alpha(y)-\beta y\psi(x)}d\nu(y)_x} \\ -\Delta\psi(x) &= \frac{\int_{\mathbf{R}} ye^{\alpha(y)-\beta y\psi(x)}d\nu(y)_x}{\int_{\mathbf{R}} e^{\alpha(y)-\beta y\psi(x)}d\nu(y)_x} \end{aligned} \tag{6.2.16}$$

In the above system the function $\alpha(y)$ is the multiplier of the constraints (6.2.14) while the number β is the Lagrange multiplier of the energy constraint (6.2.15). These equations are called the Miller-Robert equations.

The second equation of (6.2.16) has to be complemented by the boundary condition $\psi = 0$ on $\partial\Omega$ and then, since it is on the form

$$-\Delta\psi = G(\psi) \tag{6.2.17}$$

it defines the current of a stationary state.

Writing the second equation (6.2.16) in the form

$$\begin{aligned} Z(x) &= \int_{\mathbf{R}} e^{\alpha(y)-\beta y\psi(x)} d\nu(y)_x \\ -\Delta\psi(x) &= \frac{-1}{\beta}\frac{d}{d\psi}\log Z \end{aligned} \tag{6.2.18}$$

using classical rules of differentiation and Cauchy Schwartz estimates on e proves that for $\beta > 0$ the function G is strictly decreasing and strictly increasing for $\beta < 0$. Therefore for $\beta > 0$ it satisfies the stability criteria of Arnold (Corollary 5.4) and it cannot be an attractor for strong norms. Such a contradiction may not be present when β is negative and small. This same remark applies to the mean fields equation and this emphasizes the importance of the case $\beta < 0$ which (cf. below) can be interpreted in the frame of statistical physic as a negative temperature. As a conclusion the Miller Robert solution (which preserves all the conserved quantities) with $\beta < 0$ seems to be the best candidate for a strong attractor.)

Both the Joyce-Montgomery mean Field equation and the Miller-Robert equation (with the boundary condition $\psi = 0$ on $\partial\Omega$ have been the object of intensive study (cf. [CLMP] and others..). As far as existence an uniqueness is involved the increasing monotonicity of the function $G(\psi)$ simplifies the analysis of the problem:

$$-\Delta\psi = G(\psi),\ \psi_{|\partial\Omega} = 0 \tag{6.2.19}$$

One has the following

Theorem 6.6. *For $\beta > 0$ both the mean field equation and the Miller Robert equation do have a unique solution.*

The proof of this theorem is by now classical one could look at the book [MaPu] or the papers [CLMP] and [Kie]for details and references. Cagliotti-Lions-Marchioro and Pulvirenti and Kiessling have also studied with some details the case of $\beta < 0$ and found much different situations including existence and uniqueness in the case of the ball for $\beta > -8\pi$, non existence in star shaped domains for β small enough (with Pohozaev identity)

6.3 Some heuristic justification for the construction of the attractors

As said above there is no up to now mechanical justification of the introduction of the solutions defined by the equations of Joyce and Montgomery Miller Robert et al... The arguments given rely on the analysis of some special type of solution and some limit process. Along this line the construction of Miller and Robert can be related to a notion of "concentration" of stationary states and a construction starting with piecewise constant initial vorticity. At variance the initial construction for the mean field equation was initiated by Onsager with the introduction of point vortices and a limit process for the corresponding Gibbs measure [On]. Once again in relation with dynamical systems and for a rapid introduction of the notion of negative temperature we shortly review Onsager approach and its further extensions.

The first idea is the introduction of solutions of the two dimensional Euler equation as finite sum of say N vortex points located at the points:

$$x_i(t) = (x_{i_1}(t), x_{i_2}(t)).$$

with intensity $\tilde{a}(v)_i$.

To do so the Green function of the Laplacian is decomposed into its smooth and singular part according to the formula:

$$V(x,y) = -\frac{1}{2\pi}\log|x-y] + \tilde{\gamma}(x,y) \tag{6.3.1}$$

and $\frac{1}{2}\tilde{\gamma}(x,x)$ is denoted $\gamma(x)$. Next one introduces the Hamiltonian

$$H(x_1, x_2, ..., x_N) = -\frac{1}{4\pi}\sum_{ij=1;i\neq j}^{N} a_i a_j V(x_i, x_j) + \sum_{i=1}^{N} a_i \gamma(x_i) \tag{6.3.2}$$

and the corresponding Hamiltonian system defined in Ω^N

$$\begin{aligned} a_i\frac{d}{dt}x_{i_1} &= \partial_{x_{i_2}} H \\ a_i\frac{d}{dt}x_{i_3} &= -\partial_{x_{i_1}} H \end{aligned} \tag{6.3.3}$$

The main difficulty in the analysis of the above systems comes from the log singularity of the Hamiltonian, this is the reason why in the definition of this Hamiltonian the constraint $i \neq j$ is prescribed and that for $t = 0$ all the x_i are assumed to be different. Then one can show that (6.3.3) has a local in time solution which remains in Ω. However this system may collapse in a finite time if two points collides. But if all the intensities a_i have the same sign the conservation of the Hamiltonian implies global existence for the solution of (6.3.3). The connection of the above system with the solutions of the Euler is therefore not easy to establish and it is illustrated at best by the following result due to Marchioro and Pulvirenti [MaPu] page 165 which is quoted with no proof.

Theorem 6.7 *Denote by $\Xi_\epsilon(x)$ the characteristic function of the ball of center x and radius ϵ, introduce N points $x_i \in \Omega$, assume that ϵ is small enough to ensure that all the balls of radius ϵ and center x_i are small enough and contained in Ω and consider the vorticity $\omega_\epsilon(x,t)$ of the uniquely defined solution of the Euler equation with initial vorticity*

$$\omega_\epsilon(x,0) = \epsilon^{-2}\sum_{i=1}^{N} a_i \Xi_\epsilon(x_i) \tag{6.3.4}$$

then as long (with respect to time t) as the system (6.3.2) does not develop collapses one has, of course in the sense of distributions:

$$\lim_{\epsilon\to 0}\omega_\epsilon(x,t) = \sum_{i=1}^{N} a_i \delta(x_i(t)) . \tag{6.3.5}$$

The justification of the mean field equation which correspond to non negative vorticity is done with the introduction of the Gibbs measure associated to the Hamiltonian system (6.3.3) which is formally an invariant measure for the Euler equation.

$$\mu^{\alpha,\tilde\beta,N}(dx_1dx_2...dx_N) = Z_{\alpha,\tilde\beta}(N)^{-1}e^{-\tilde\beta\alpha^2H(x_1,x_2,...x_N)}dx_1dx_2...dx_N\,. \qquad (6.3.6)$$

Since μ is defined in term of the Hamiltonian H of the system it is invariant; $Z_{\alpha,\tilde\beta}(N)^{-1}$ is a normalizing constant which is given by:

$$Z_{\alpha,\tilde\beta}(N)^{-1} = \int_{\Omega^N} e^{-\tilde\beta\alpha^2H(x_1,x_2,...x_N)}dx_1dx_2...dx_N\,. \qquad (6.3.7)$$

and which has to be finite. Indeed one has:

Lemma 6.8 $Z_{\alpha,\tilde\beta}(N)^{-1}<\infty$ *if and only if* $\tilde\beta\in\left(-\frac{8\pi}{\alpha^2N},\infty\right)$. *Moreover, in this range of "temperature," the following estimates hold:*

$$Z_{\alpha,\tilde\beta}(N)^{-1}\le C(\tilde\beta,N\alpha,|\Omega|)^N \qquad (6.3.8)$$

with $C(\tilde\beta,N\alpha,|\Omega|)^N$ *a constant depending only on the product* $\tilde\beta$, $N\alpha$ *and on* $|\Omega|$.

This lemma is quoted from [CLMP] (cf. also [Kie]) where the proof, obtained with standard estimates, can be found.

As in the derivation of the Boltzmann equation in section (3.2) the limit of

$$\mu^{\alpha,\tilde\beta,N}(dx_1dx_2...dx_N)$$

is considered when $N\to\infty$ and $\alpha\to 0$ with the introduction of the "marginals:"

$$\mu_j^{\alpha,\tilde\beta,N}(dx_1dx_2...dx_N) = dx_1dx_2...dx_j\int_{\Omega^{N-j}}dx_{j+1}dx_{j+2}...dx_NZ_{\alpha,\tilde\beta}(N)^{-1}e^{-\tilde\beta\alpha^2H(x_1,x_2,...x_N)} \qquad (6.3.9)$$

and the relations

$$\tilde\beta=\beta N,\ \beta \text{ (fixed) and } \alpha=\frac{1}{N}\,. \qquad (6.3.10)$$

¿From the above lemma one deduces (cf. also [CLMP] and [Kie] for proofs and details) the following:

Theorem 5.9 *Assume that* Ω *is a simply connected domain, let* $\beta\in(-8\pi,\infty)$, *and assume that the equation*

$$-\Delta\psi=\frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}dx} \qquad (6.3.11)$$

has a unique solution (condition automatically fulfilled for $\beta>0$*) then in the sense of measures one has*

$$\lim_{\tilde\beta=\beta N,\alpha=\frac{1}{N},N\to\infty}\mu_j^{\alpha,\tilde\beta,N}(dx_1dx_2...dx_N)=\prod_{i=1}^{j}\psi(x_i)\,. \qquad (6.3.12)$$

Observe that as in the derivation of the Boltzmann equation a factorization process related to the minimization of some entropy appears in the proof.

As a conclusion once again one should observe the following facts:

i) The above derivation contains no mechanics.

ii) On the other hand a justification of the relevance of the equation (6.3.11) may come from the following interpretation of the Theorem 5.9. (quoted from [MaPu] page 262

"What is expected to happen is the following. The vortices are distributed according to the Gibbs distribution. When N is large they fluctuate very little. With very large probability they arrange themselves to form the solution of the mean field equation."

iii) As shown by Majda and Holen [MaHo] the two above constructions (Onsager Joyce and Montgomery on one side and Miller Robert Sommeria on the other side) produce the same solution if and only if the density $\rho(x,y)$ given by (6.2.16) is *statistically sharp* i.e. if one has:

$$\rho(x,y)d\nu(y)_x = \delta_{\omega(x)} \tag{6.3.13}$$

with $\omega(x) = -\Delta\psi(x)$ given by the second equation of (6.2.16).

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