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# On reliability estimation based on fuzzy lifetime data

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**Abstract**. Real liefetime data are never precise numbers but more or less non-precise, also called fuzzy. This kind of imprecision is connected with all measurement results of continuous variables, therefore also with time observations. Imprecision is different from errors and variability. Therefore estimation methods for reliability characteristics have to be adapted to the situation of fuzzy lifetimes in order to obtain realistic results.

**Keywords:** Bayesian inference, Fuzzy lifetimes, Reliability estimation

# 1. Introduction

All observations and measurements of continuous variables are not precise numbers but more or less non-precise. This imprecision is different from variability and errors. Therefore also lifetime data are not precise numbers but more or less fuzzy. The best up-to-date mathematical model for this imprecision are so-called *non-precise numbers*.

Besides censored data, which are a special type of non-precise data, there were several approaches to include fuzziness into reliability research. Some key refernces are (Onisawa et al., 1995), (Cai, 1996), and (Ross et al., 2002).

But censoring models are not sufficient for all situations in life testing, for example for continuous degradation processes, where the boundaries of censoring intervals are non-precise, also called fuzzy.

If the variability of life time data is large compared to the imprecision of the individual life times, the fuzziness can be neglected without loss of essential information. A different situation is given when the fuzziness is essential compared to the variability. In this situation neglecting fuzziness can produce unrealistic results. Moreover looking at values of test statistics, which become fuzzy in case of fuzzy data by the propagation of fuzziness via the extension principle (Klir et al., 1995) this fuzziness is expressing the uncertainty of data in a quantitative way. This shows whether it is necessary or not to take additional data in order to obtain a justified decision. Classical statistical procedures and Bayesian inference is not taking care of this.

**Definition 1:** A non-precise number $x^\star$ is characterized by a so-called *characterizing function* $\xi(\cdot)$ obeying the following:

(1) $\xi : I\!R \longrightarrow [0,1]$

(2) the support of $\xi(\cdot)$, i.e. $\{x \in I\!R \colon \xi(x) > 0\}$ is a bounded subset of $I\!R$

(3) for all $\delta \in (0,1]$ the $\delta$-cut $C_\delta\left[\xi(\cdot)\right]$ defined by $C_\delta\left[\xi(\cdot)\right] := \{x \in I\!R \colon \xi(x) \geq \delta\}$ is a finite union of compact intervals, i.e. $C_\delta\left[\xi(\cdot)\right] = \bigcup_{j=1}^{k_\delta} \left[a_{\delta,j}, b_{\delta,j}\right]$

**Remark 1:** Non-precise numbers are special fuzzy subsets of the set $I\!R$ of real numbers. But they are more general than so-called *fuzzy numbers* $y^\star$, for which all $\delta$-cuts have to be non-empty compact intervals, i.e. $C_\delta\left[y^\star\right] = [a_\delta, b_\delta]$. Fuzzy numbers are essential for so-called fuzzy probability distributions, especially for fuzzy probability densities.

An important question is how to obtain the characterizing function of a fuzzy lifetime.

**Example 1:** If the end of a lifetime is given by the degradation of a characteristic quantity, i.e. $q(t), t \geq 0$ with $\lim_{t \to \infty} q(t) = 0$, then the characterizing function $\xi(\cdot)$ of the lifetime $t^\star$ is given, using the derivative

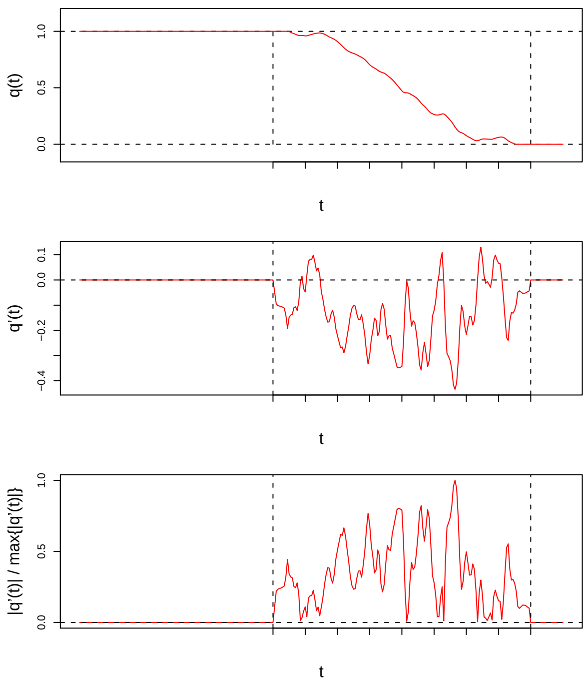
$$q'(t) = \frac{d}{dt}\, q(t)$$

in the following way:

$$\xi(t) := \frac{|q'(t)|}{\max\left\{\, |q'(t)| : t \geq 0 \right\}} \qquad \forall \quad t \geq 0$$

This is a special kind of taking the scaled rate of change of the function $q(\cdot)$ which is taken in information science to quantify fuzzy transitions from one state to another.
In figure 1 an example for $q(\cdot)$ and the corresponding characterizing function $\xi(\cdot)$ is given.

Figure 1: Construction of the characterizing function of a fuzzy lifetime




**Remark 2:** By the possible oscillations of the derivative $q'(\cdot)$ the result can be a non-precise number but not a fuzzy number.

## 2. Generalized classical parametric reliability estimation

In case of a parametric lifetime model $X \sim f(\cdot \mid \theta)$, $\theta \in \Theta$ and fuzzy lifetime data $t_1^\star, \cdots, t_n^\star$ classical estimators $\vartheta(t_1, \cdots, t_n)$ for $\theta$ can be generalized to the situation of fuzzy data by application of the so-called *extension principle* from the theory of fuzzy sets, which formalizes the propagation of imprecision (Klir at al., 1995). This is the following:

Let $g : M \to N$ be an arbitrary function, i.e. $g(x) \in N \qquad \forall \quad x \in M$.

For an arbitrary fuzzy element $x^\star$ in $M$, i.e. $x^\star$ is a fuzzy subset of $M$ with membership function $\zeta(\cdot)$, the membership function $\eta(\cdot)$ of the fuzzy value $g(x^\star)$ is defined by

$$\eta(y) = \left\{ \begin{array}{ll} \sup\left\{\zeta(x)\colon x \in M, g(x) = y\right\} & \text{if} \quad g^{-1}(\{y\}) \neq \emptyset \\ 0 & \text{if} \quad g^{-1}(\{y\}) = \emptyset \end{array} \right\} \quad \forall \quad x \in M.$$

**Remark 3:** For $M = N = R$, continuous function $g(\cdot)$, and fuzzy number $x^\star$ the resulting fuzzy value $g(x^\star)$ is a fuzzy number in the sense of the definition above. For details compare (Viertl, 1996).

Statistical estimators for reliability characteristics are usually functions of a sample $x_1, \cdots, x_n$. Therefore it is necessary to generalize functions of $n$ variables. In order to make this possible a fuzzy sample $x_1^\star, \cdots, x_n^\star$ has to be combined into a fuzzy element of the sample space. In the standard situation of number-valued data $x_1, \cdots, x_n$ this combination is trivial: Let $M$ be the observation space of a stochastic quantity $X$, for lifetimes $M = [0, \infty)$. Then the sample space is the Cartesian product of $n$ copies of the observation space, i.e. $M \times \cdots \times M = M^n$. In this situation the combination of the sample $x_1, \cdots, x_n$ with $x_i \in M$ is just the $n$-tuple $(x_1, \cdots, x_n) \in M^n$.

In case of a fuzzy sample $x_1^\star, \cdots, x_n^\star$, i.e. the observations $x_i^\star$ are fuzzy elements in the observation space $M$, the combination is in no way trivial because the so-called combined fuzzy sample has to be a fuzzy element $\underline{x}^\star$ of the sample space $M^n$. The $n$-tuple $(x_1^\star, \cdots, x_n^\star)$ is a vector of fuzzy elements of $M$ but not a fuzzy element in $M^n$. But it is necessary to form a fuzzy element of the sample space in order to apply the extension principle for generalizing estimators. This combination of the fuzzy sample in order to obtain the so-called *fuzzy combined sample* $\underline{x}^\star$ is possible using so-called *triangular norms* (shortly called *$t$-norms*). For details on $t$-norms compare (Klement at al., 2000).

**Definition 2:** Using the notation $\underline{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ a $n$-dimensional fuzzy vector $\underline{x}^\star$ is a fuzzy subset of $\mathbb{R}^n$ whose membership function $\zeta(\cdot, \cdots, \cdot)$ obeys the following:

(1) $\zeta : \mathbb{R}^n \longrightarrow [0, 1]$

(2) $\text{supp}\big[\zeta(\cdot, \cdots, \cdot)\big]$ is a bounded subset of $\mathbb{R}^n$

(3) for all $\delta \in (0, 1]$ the so-called $\delta$-cut $C_\delta\left[\underline{x}^\star\right] := \left\{(x_1, \cdots, x_n) \in \mathbb{R}^n := \zeta(x_1, \cdots, x_n) \geq \delta\right\}$ is non-empty and a finite union of simply connected compact subsets of $\mathbb{R}^n$

A function $\zeta(\cdot, \cdots, \cdot)$ of $n$ real variables obeying the conditions in the definition is called *vector-characterizing function.*

Using a $t$-norm $T$, the fuzzy sample can be combined into a fuzzy vector which is a fuzzy element of the sample space. The vector-characterizing function $\zeta(\cdot, \cdots, \cdot)$ of the combined fuzzy sample $\underline{x}^\star$ is given by its values $\zeta(x_1, \cdots, x_n)$ in the following way: Let $\xi_1(\cdot), \cdots, \xi_n(\cdot)$ be the corresponding characterizing functions of the fuzzy observations $x_1^\star, \cdots, x_n^\star$. Then

$$\zeta(x_1, \cdots, x_n) := T\Big(\xi_1(x_1), \cdots, \xi_n(x_n)\Big) \qquad \forall \quad (x_1, \cdots, x_n) \in \mathbb{R}^n.$$

This vector-characterizing function is the basis for the generalization of estimators.

**Remark 4:** For the combination of fuzzy samples usually the following $t$-norm is used.

$$T(z_1, \cdots, z_n) := \min\left\{z_1, \cdots, z_n\right\} \qquad \forall \quad z_i \in [0, 1]$$

This is motivated by the shape of the $\delta$-cuts of the combined fuzzy sample:

$$C_\delta\big[\zeta(\cdot, \cdots, \cdot)\big] = \mathop{\times}_{i=1}^{n} C_\delta\big[\xi_i(\cdot)\big] \qquad \forall \quad \delta \in (0, 1].$$

That is, the $\delta$-cuts of the combined fuzzy sample are the Cartesian products of the $\delta$-cuts of the fuzzy observations $x_i^\star$. For more details see (Viertl, 2006).

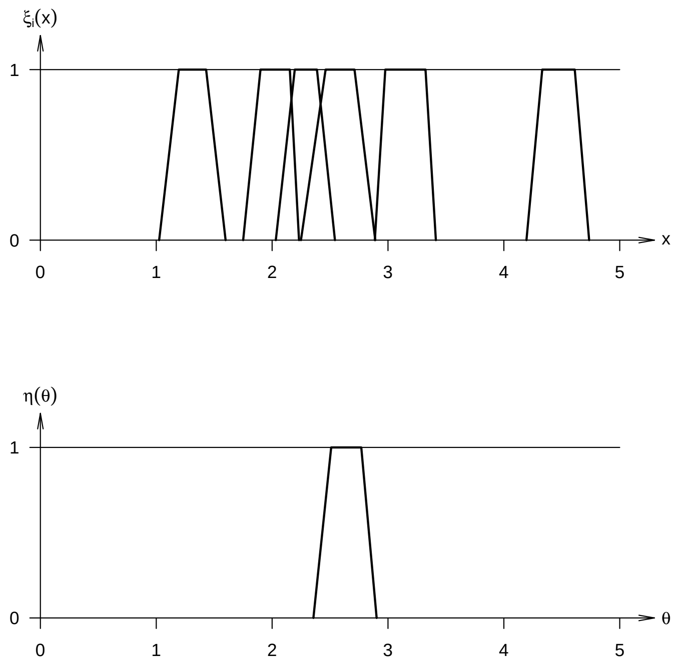
It is possible to generalize estimators $\vartheta(t_1, \cdots, t_n) = \hat{\theta} \in \Theta$ to the situation of fuzzy lifetime data $t_1^\star, \cdots, t_n^\star$.

First the fuzzy lifetimes $t_1^\star, \cdots, t_n^\star$ have to be combined to form a fuzzy vector $\underline{t}^\star$ in the sample space $[0, \infty)^n$. Denoting the vector-characterizing function of $\underline{t}^\star$ by $\zeta(\cdot, \cdots, \cdot)$, the extension principle is applied. The memberhip function $\eta(\cdot)$ of the generalized fuzzy estimate $\hat{\theta}^\star = \vartheta(t_1^\star, \cdots, t_n^\star)$ is given by its values $\eta(\theta)$, using the notation $\underline{t} = (t_1, \cdots, t_n) \in [0, \infty)^n$:

$$\eta(\theta) = \left\{ \begin{array}{ll} \sup \left\{ \zeta(\underline{t}) : \underline{t} \in I\!R^n, \vartheta(\underline{t}) = \theta \right\} & \text{if } \vartheta^{-1}\big(\{\theta\}\big) \neq \emptyset \\ 0 & \text{if } \vartheta^{-1}\big(\{\theta\}\big) = \emptyset \end{array} \right\} \quad \forall \;\; \theta \in \Theta.$$

In figure 2 a fuzzy sample and the generalized estimate $\hat{\theta}^\star$ of the expected lifetime $\theta$ are depicted. The characterizing functions $\xi_i(\cdot)$ in this example are assumed to be of trapezoidal shape. In this situation also the characterizing function $\eta(\cdot)$ of the generalized estimate for the expectation is of trapezoidal shape.

Figure 2: Fuzzy lifetimes and corresponding estimate




Confidence estimations of reliability characteristics can be generalized to the situation of fuzzy data in the following way: Let $X \sim P_\theta$, $\theta \in \Theta$ be a lifetime model. For classical lifetime data $t_1, \cdots, t_n$ let $\kappa(\cdot, \cdots, \cdot)$ be a confidence function with confidence level $1 - \alpha$. That is for observed data $t_1, \cdots, t_n$, $\kappa(t_1, \cdots, t_n) = \Theta_{1-\alpha} \subset \Theta$ is a confidence set.

For a fuzzy sample $t_1^\star, \cdots, t_n^\star$ of lifetimes a generalized confidence set is obtained which is a fuzzy subset of $\Theta$ whose membership function $\varphi(\cdot)$ can be obtained by using again the fuzzy combined sample $\underline{t}^\star$ from above. Let $\zeta(\cdot, \cdots, \cdot)$ be the vector-characterizing function of the combined fuzzy sample. Then the values $\varphi(\theta)$ of the membership function for the fuzzy confidence set $\Theta_{1-\alpha}^\star$ are given, using the notation $\underline{t} = (t_1, \cdots, t_n) \in [0, \infty)^n$, by

$$\varphi(\theta) = \left\{ \begin{array}{ll} \sup \left\{ \zeta(\underline{t}) : \underline{t} \in [0, \infty)^n, \theta \in \kappa(\underline{t}) \right\} & \text{if } \exists \; \underline{t} : \theta \in \kappa(\underline{t}) \\ 0 & \text{if } \nexists \; \underline{t} : \theta \in \kappa(\underline{t}) \end{array} \right\} \quad \forall \quad \theta \in \Theta.$$

This generalized concept reduces to the standard concept of confidence regions in case of idealized lifetime data $t_1, \cdots, t_n$. In that situation for $\varphi(\cdot)$ the indicator function of the standard confidence set $\kappa(\underline{t}) = \kappa(t_1, \cdots, t_n)$ is obtained.

## 3. Generalized Bayesian reliability estimation

For parametric lifetime models $X \sim f(\cdot \mid \theta)$, $\theta \in \Theta$ and a-priori distribution $\pi(\cdot)$ of $\tilde{\theta}$, and classical data $x_1, \cdots, x_n$ the a-posteriori distribution $\pi(\cdot \mid x_1, \cdots, x_n)$ is obtained by Bayes' theorem, i.e. the values $\pi(\theta \mid x_1, \cdots, x_n)$ of the a-posteriori density are given by

$$\pi(\theta \mid x_1, \cdots, x_n) = \frac{\pi(\theta) \cdot \ell(\theta; x_1, \cdots, x_n)}{\int\limits_{\Theta} \pi(\theta) \cdot \ell(\theta; x_1, \cdots, x_n)\, d\theta} \qquad \forall \quad \theta \in \Theta,$$

where $\ell(\cdot; x_1, \cdots, x_n)$ is the likelihood function.

There are two kinds of fuzzy information in this context: Fuzzy data and fuzzy a-priori information.

Fuzzy data are already described above.

Fuzzy a-priori information can be modelled by so-called *fuzzy probability densities*. These are generalizations of classical probability densities.

Let $(M, \mathcal{A})$ be a measurable space and $f^\star(\cdot)$ a function which assigns to every $x \in M$ a fuzzy number $f^\star(x)$ in the following way: Denoting by $\left[\underline{f}_\delta(x), \overline{f}_\delta(x)\right]$ the $\delta$-cut of $f^\star(x)$ for every $\delta \in (0, 1]$, the real valued functions $\underline{f}_\delta(\cdot)$ and $\overline{f}_\delta(\cdot)$ are assumed to be integrable and there exists a density function $g\colon M \to [0, \infty)$ such that $g(x) \in \left[\underline{f}_1(x), \overline{f}_1(x)\right] \;\; \forall\, x \in M$, and $\int\limits_M g(x)dx = 1$.

A fuzzy valued function $f^\star(\cdot)$ obeying these conditions is called *fuzzy probability density*.

**Remark 5:** Fuzzy probability densities are generalizations of classical probability densities $f(\cdot)$. Assigning to $x \in M$ the indicator function $I_{\{f(x)\}}(\cdot)$ fulfills the definition.

Fuzzy probability distributions are a more realistic formalization of a-priori knowledge than classical probability distributions.

Probabilities of events $A \in \mathcal{A}$ are calculated using a special kind of integration which assigns fuzzy numbers $P^\star(A)$ as probabilites, i.e.

$$(P) \int_A f^\star(x)dx.$$

This integration is defined using the so-called $\delta$-level functions $\underline{f}_\delta(\cdot)$ and $\overline{f}_\delta(\cdot)$. For details see (Viertl, 2006).

It is necessary to generalize Bayes' theorem to the situation of fuzzy a-priori distributions and fuzzy lifetime data. This is possible in the following way:

For continuous lifetime model $X \sim f(\cdot \mid \theta)$, $\theta \in \Theta$ with continuous parameter $\theta$ the a-priori density is a fuzzy density function $\pi^\star(\cdot)$ with $\delta$-level functions $\underline{\pi}_\delta(\cdot)$ and $\overline{\pi}_\delta(\cdot)$ $\forall\, \delta \in (0, 1]$ respectively.

In order to obtain the likelihood function $\ell(\cdot; t_1^\star, \cdots, t_n^\star)$ for fuzzy lifetime data $t_1^\star, \cdots, t_n^\star$ the extension principle, based on the combined fuzzy sample $\underline{t}^\star$ is applied.

The characterizing function $\eta_\theta(\cdot)$ of the fuzzy value $\ell^\star(\theta; t_1^\star, \cdots, t_n^\star)$ is given by

$$\eta_\theta(y) = \left\{ \begin{array}{ll} \sup\left\{\zeta(\underline{t})\colon \ell(\theta; \underline{t}) = y\right\} & \text{if } \; \ell^{-1}(\{y\}) \neq \emptyset \\ 0 & \text{if } \; \ell^{-1}(\{y\}) = \emptyset \end{array} \right\} \; \forall\, y \in \mathbb{R}.$$

Using the $\delta$-level functions of $\pi^\star(\cdot)$ and $\ell^\star(\cdot; t_1^\star, \cdots, t_n^\star)$, i.e. $\underline{\pi}_\delta(\cdot)$, $\overline{\pi}_\delta(\cdot)$, $\underline{\ell}_\delta(\cdot; \underline{t}^\star)$ , $\overline{\ell}_\delta(\cdot; \underline{t}^\star)$ the fuzzy a-posteriori density $\pi^\star(\cdot \mid t_1^\star, \cdots, t_n^\star)$ is calculated via its $\delta$-level functions $\underline{\pi}_\delta(\cdot \mid \underline{t}^\star)$ and $\overline{\pi}_\delta(\cdot \mid \underline{t}^\star)$ in the following way:

$$\left.\begin{array}{lcl} \overline{\pi}_\delta(\theta \mid \underline{t}^\star) & = & \dfrac{\overline{\pi}_\delta(\theta) \cdot \overline{\ell}_\delta(\theta; \underline{t}^\star)}{\int\limits_\Theta \frac{1}{2}\left[\underline{\pi}_\delta(\theta) \cdot \underline{\ell}_\delta(\theta; \underline{t}^\star) + \overline{\pi}_\delta(\theta) \cdot \overline{\ell}_\delta(\theta; \underline{t}^\star)\right] d\theta} \\ \text{and} & & \\ \underline{\pi}_\delta(\theta \mid \underline{t}^\star) & = & \dfrac{\underline{\pi}_\delta(\theta) \cdot \underline{\ell}_\delta(\theta; \underline{t}^\star)}{\int\limits_\Theta \frac{1}{2}\left[\underline{\pi}_\delta(\theta) \cdot \underline{\ell}_\delta(\theta; \underline{t}^\star) + \overline{\pi}_\delta(\theta) \cdot \overline{\ell}_\delta(\theta; \underline{t}^\star)\right] d\theta} \end{array}\right\} \forall\ \delta \in (0, 1]$$

An example with graphical details is given in (Viertl, 2006).

**Remark 6:** The essential information from the a-priori distribution and the data is contained in the a-posteriori distribution. Here the a-posteriori distribution is a fuzzy probability distribution, especially a fuzzy probability density on the parameter space $\Theta$.

The fuzzy a-posteriori distribution can be used in different ways. The first is to calculate *generalized HPD-regions.*

Let $\Theta_{H,1-\alpha}$ be a standard highest probability density region, short HPD-region for $\theta$ (Viertl, 1996) with coverage probability $1 - \alpha$. Then the generalized (fuzzy) HPD-region $\Theta^\star_{H,1-\alpha}$ based on fuzzy lifetime data $t^\star_1, \cdots, t^\star_n$ is characterized by its membership function $\varphi(\cdot)$ whose values are defined by

$$\varphi(\theta) := \left\{\begin{array}{ll} \sup\{\zeta(\underline{t})\colon \theta \in \Theta_{H,1-\alpha}\} & \text{if } \exists\, \underline{t}\colon \theta \in \Theta_{H,1-\alpha} \\ 0 & \text{if } \nexists\, \underline{t}\colon \theta \in \Theta_{H,1-\alpha} \end{array}\right\} \ \forall\ \theta \in \Theta.$$

The generalized HPD-region $\Theta^\star_{H,1-\alpha}$ is a typical example for a fuzzy subset of $\Theta$.

Another application of a-posteriori distributions are predictive distributions for the lifetime based on observed data.

Adapting the standard predictive density

$$p(x \mid t_1, \cdots, t_n) = \int\limits_\Theta f(x \mid \theta)\ \pi\ (\theta \mid t_1, \cdots, t_n)\, d\theta \qquad \forall \quad x \in [0, \infty)$$

to the situation of fuzzy lifetime data $t^\star_1, \cdots, t^\star_n$ is possible using an integration operation for fuzzy valued functions. This integration is based on $\delta$-level functions. For a fuzzy valued function $g^\star(\cdot)$ defined on $\Theta$ all $\delta$-level functions $\underline{g}_\delta(\cdot)$ and $\overline{g}_\delta(\cdot)$ are assumed to be integrable. Then the generalized integral $I^\star = \rlap{\,-}\int\limits_A g^\star(\theta) d\theta$ is defined via its $\delta$-level functions in the following way (Viertl, 1999).

Calculating the classical integrals

$$\underline{I}_\delta = \int\limits_A \underline{g}_\delta(\theta) d\theta \quad \text{and} \quad \overline{I}_\delta = \int\limits_A \overline{g}_\delta(\theta) d\theta \quad \forall \quad \delta \in (0, 1]$$

we obtain $\delta$-cuts $\left[\underline{I}_\delta, \overline{I}_\delta\right]$ $\forall\ \delta \in (0, 1]$ of a fuzzy number $I^\star$ with characterizing function $\psi(\cdot)$, which is given by the presentation lemma for fuzzy numbers by

$$\psi(y) := \max\left\{\delta \cdot I_{\left[\underline{I}_\delta, \overline{I}_\delta\right]}(y) \colon \delta \in [0, 1]\right\} \quad \forall\ y \in I\!R.$$

The fuzzy number $I^\star$ with characterizing function $\psi(\cdot)$ is the generalized integral.

For details on the representation lemma (Viertl, 1996).

The predictive density can be generalized using the generalized integral. The so-called *fuzzy predictive density* $p^\star(\cdot \mid t^\star_1, \cdots, t^\star_n)$ is defined by its values

$$p^\star(x \mid t^\star_1, \cdots, t^\star_n) = \rlap{\,-}\int\limits_\Theta f(x \mid \theta) \pi^\star(\theta \mid t^\star_1, \cdots, t^\star_n)\, d\theta \qquad \forall \quad x \in [0, \infty),$$

where the generalized integral is the one defined above.

**Remark 7:** The fuzzy predictive density is a fuzzy probability density.

## 4. Conclusion

Generalized estimation procedures for reliability characteristics based on fuzzy lifetime data are necessary and possible. In this paper generalized parametric procedures are presented including fuzzy point estimators and generalized Bayesian procedures. Generalized confidence sets are presented in (Viertl, 2006). It is also possible to generalize statistical tests based on fuzzy data, compare (Filzmoser et al., 2004). Moreover also nonparametric estimation of the reliability function based on fuzzy lifetime data is possible (Viertl, 1996). Further research is necessary for the analysis of accelerated life tests if fuzzy life time data are obtained from these accelerated tests.

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